Commensurators of surface braid groups

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Abstract. We prove that if g and n are integers at least two, then the abstract commensurator of the braid group with n strands on a closed orientable surface of genus g is naturally isomorphic to the extended mapping class group of a compact orientable surface of genus g with n boundary components.

1. Introduction.

For a positive integer n and a manifold M, we define $PB_n(M)$ to be the *pure braid group* of n strands on M, i.e., the fundamental group of the space of ordered distinct n points in M. Let S be a connected, compact and orientable surface of genus g with p boundary components. Let $\bar{S}$ denote the closed surface obtained by attaching disks to all boundary components of S. We define $P(S)$ as the kernel of the homomorphism $\iota\colon \mathrm{PMod}(S) \to \mathrm{Mod}(\bar{S})$ associated with the inclusion of S into $\bar{S}$, where $\mathrm{PMod}(S)$ denotes the pure mapping class group for S, and $\mathrm{Mod}(\bar{S})$ denotes the mapping class group for $\bar{S}$ (see Section 2.3 for a definition of these groups). As discussed in Section 4.1 of [**2**], the Birman exact sequence tells us that if $g \geq 2$ and $p \geq 1$, then $PB_p(\bar{S})$ is identified with $P(S)$.

Automorphisms of $P(S)$ are studied in [**1**], [**11**] and [**22**]. The aim of this paper is to describe any isomorphism between finite index subgroups of $P(S)$. The same problem is also considered for the subgroup, denoted by $P_s(S)$, of $P(S)$ generated by all HBC twists and all HBP twists about separating HBPs in S. We denote by $\mathrm{Mod}^*(S)$ the *extended mapping class group* for S, i.e., the group of isotopy classes of homeomorphisms from S onto itself, where isotopy may move points of the boundary of S. Since $P(S)$ and $P_s(S)$ are normal subgroups of $\mathrm{Mod}^*(S)$, the conjugation by each element of $\mathrm{Mod}^*(S)$ defines an automorphism of $P(S)$ and of $P_s(S)$. The following theorem says that any isomorphism between finite index subgroups of $P(S)$ and of $P_s(S)$ can be described in this way.

Theorem 1.1.   *Let S be a connected, compact and orientable surface of genus g with p boundary components. We assume $g \geq 2$ and $p \geq 2$. Then the following assertions hold*:

(i) *Let Γ_1 and Γ_2 be finite index subgroups of $P(S)$, and let $f\colon \Gamma_1 \to \Gamma_2$ be an isomorphism. Then there exists an element γ of $\mathrm{Mod}^*(S)$ with $f(x) = \gamma x \gamma^{-1}$ for any $x \in \Gamma_1$.*

(ii) *Let Λ_1 and Λ_2 be finite index subgroups of $P_s(S)$, and let $h\colon \Lambda_1 \to \Lambda_2$ be an isomorphism. Then there exists an element λ of $\mathrm{Mod}^*(S)$ with $h(y) = \lambda y \lambda^{-1}$ for any $y \in \Lambda_1$.*

For a group Γ, we define $F(\Gamma)$ to be the set of isomorphisms between finite index subgroups of Γ. We say that two elements f, h of $F(\Gamma)$ are equivalent if there exists a finite index subgroup of Γ on which f and h are equal. The composition of two elements $f\colon \Gamma_1 \to \Gamma_2$, $h\colon \Lambda_1 \to \Lambda_2$ of $F(\Gamma)$ given by $f \circ h\colon h^{-1}(\Gamma_1 \cap \Lambda_2) \to f(\Lambda_2 \cap \Gamma_1)$ induces the product operation on the quotient set of $F(\Gamma)$ by this equivalence relation. This makes it into the group called the *abstract commensurator* of Γ and denoted by $\mathrm{Comm}(\Gamma)$.

Since $P(S)$ and $P_s(S)$ are normal subgroups of $\mathrm{Mod}^*(S)$, the homomorphisms

$$\boldsymbol{i}\colon \mathrm{Mod}^*(S) \to \mathrm{Comm}(P(S)), \quad \boldsymbol{i}_s\colon \mathrm{Mod}^*(S) \to \mathrm{Comm}(P_s(S))$$

are defined by conjugation. Theorem 1.1 shows that if $g \geq 2$ and $p \geq 2$, then $\boldsymbol{i}$ and $\boldsymbol{i}_s$ are surjective and thus isomorphisms by Lemma 2.2.

For a positive integer n and a manifold M, we define $B_n(M)$ as the *braid group* of n strands on M, i.e., the fundamental group of the space of non-ordered distinct n points in M. The group $PB_n(M)$ is identified with a subgroup of $B_n(M)$ of index $n!$. We note that if Γ is a group and if Λ is a finite index subgroup of Γ, then the natural homomorphism from $\mathrm{Comm}(\Lambda)$ into $\mathrm{Comm}(\Gamma)$ is an isomorphism. We therefore obtain the following:

Corollary 1.2.   *Let g and n be integers at least two. Let M be a connected, closed and orientable surface of genus g. Then $\mathrm{Comm}(B_n(M))$ and $\mathrm{Comm}(PB_n(M))$ are isomorphic to $\mathrm{Mod}^*(S)$, where S is a connected, compact and orientable surface of genus g with n boundary components.*

Let us mention surfaces excluded in Theorem 1.1.

- If $p = 0$, then both $P(S)$ and $P_s(S)$ are trivial.
- If $g \geq 2$ and $p = 1$, then $P(S)$ is isomorphic to $\pi_1(\bar{S})$ by the Birman exact sequence, and $P_s(S)$ is identified with a subgroup of the commutator subgroup of $\pi_1(\bar{S})$ and thus isomorphic to a non-abelian free group.

- If $g = 0$, then we have $P(S) = P_s(S) = \mathrm{PMod}(S)$.
- If $g = 1$, then we have $P(S) = \mathscr{I}(S)$ and $P_s(S) = \mathscr{K}(S)$, where $\mathscr{I}(S)$ is the Torelli group for S, and $\mathscr{K}(S)$ is the Johnson kernel for S (see [15] for a definition of these groups).

When $g = 0, 1$, a description of any isomorphism between finite index subgroups of $P(S)$ and of $P_s(S)$ is therefore given in [17] and [15], respectively.

As studied in [13], [17] and [18], the complex of curves for S plays an important role in the computation of the abstract commensurator of $\mathrm{Mod}^*(S)$. Afterward, in [4], [5], [6], [15] and [19], automorphisms and the abstract commensurators of certain subgroups of $\mathrm{Mod}^*(S)$ are understood through the study of appropriate variants of the complex of curves. To prove Theorem 1.1, we follow this strategy and introduce two simplicial complexes $\mathscr{CP}(S)$ and $\mathscr{CP}_s(S)$ associated to S, inspired by a work due to Irmak-Ivanov-McCarthy [11]. Vertices of those simplicial complexes are defined as isotopy classes of certain simple closed curves in S and pairs of them. Simplices are defined in terms of disjointness of curves in the same manner as the definition of the complex of curves. We then have the natural action of $\mathrm{Mod}^*(S)$ on $\mathscr{CP}(S)$ and on $\mathscr{CP}_s(S)$. Theorem 1.1 can be deduced by combining the following two assertions: If $g \geq 2$ and $p \geq 2$, then

(1) any isomorphism between finite index subgroups of $P(S)$ (resp. $P_s(S)$) induces an automorphism of $\mathscr{CP}(S)$ (resp. $\mathscr{CP}_s(S)$); and
(2) any automorphism of $\mathscr{CP}(S)$ (resp. $\mathscr{CP}_s(S)$) is induced by an element of $\mathrm{Mod}^*(S)$.

Assertion (1) is proved in Theorem 7.13, and assertion (2) is proved in Corollary 6.3. In the proof of assertion (1), we present an algebraic characterization of certain elements of $P(S)$ associated to vertices of $\mathscr{CP}(S)$, based on [11].

This paper is organized as follows. In Section 2, we provide basic terminology and the definition of simplicial complexes and groups discussed above. In the final subsection of Section 2, we present an outline of the proof of assertion (2) given throughout Sections 3–6. In Section 7, we prove assertion (1). Moreover, we show that any injective homomorphism from a finite index subgroup of $P_s(S)$ into $P(S)$ induces a superinjective map from $\mathscr{CP}_s(S)$ into $\mathscr{CP}(S)$. This result is used in our subsequent paper [16], where we conclude that any injective homomorphism from a finite index subgroup of $P(S)$ into $P(S)$ is the conjugation by an element of $\mathrm{Mod}^*(S)$ if $g \geq 2$ and $p \geq 2$. The same conclusion is also proved for $P_s(S)$.

2.   Preliminaries.

2.1.   Notation and terminology.

Unless otherwise stated, we assume a surface to be connected, compact and orientable. Let $S = S_{g,p}$ be a surface of genus g with p boundary components. A simple closed curve in S is said to be *essential* in S if it is neither homotopic to a single point of S nor isotopic to a component of ∂S. When there is no confusion, we mean by a curve in S either an essential simple closed curve in S or the isotopy class of it. A curve α in S is said to be *separating* in S if $S \setminus \alpha$ is not connected. Otherwise α is said to be *non-separating* in S. These properties depend only on the isotopy class of α. We mean by a *holed sphere* a surface of genus zero with non-empty boundary.

Hole-bounding curves (HBC).

A curve α in S is called a *hole-bounding curve* (*HBC*) in S if α is separating in S and cuts off a holed sphere from S. When $g \geq 1$, if the holed sphere cut off by α from S contains exactly k components of ∂S, then we call α a *k-HBC* in S. Note that we have $2 \leq k \leq p$.

Hole-bounding pairs (HBP).

A pair $\{\alpha, \beta\}$ of curves in S is called a *hole-bounding pair* (*HBP*) in S if

- α and β are disjoint and non-isotopic;
- either α and β are non-separating in S or α and β are separating in S and are not an HBC in S; and
- $S \setminus (\alpha \cup \beta)$ is not connected and has a component of genus zero.

We note that if $g \geq 2$, then the component of genus zero in the last condition, denoted by Q, uniquely exists. In this case, if Q contains exactly k components of ∂S, then we call the pair $\{\alpha, \beta\}$ a *k-HBP* in S. Note that we have $1 \leq k \leq p$. An HBP in S is said to be *non-separating* in S if both curves in it are non-separating in S. Otherwise it is said to be *separating* in S (see Figure 1).

We define $V(S)$ as the set of isotopy classes of essential simple closed curves in S. We denote by $i \colon V(S) \times V(S) \to \mathbf{Z}_{\geq 0}$ the *geometric intersection number*, i.e., the minimal cardinality of the intersection of representatives for two elements of $V(S)$. Let $\Sigma(S)$ denote the set of non-empty finite subsets σ of $V(S)$ with $i(\alpha, \beta) = 0$ for any $\alpha, \beta \in \sigma$. For an element σ of $\Sigma(S)$, we mean by a *representative* of σ the union of mutually disjoint representatives of elements of σ. We extend the function i to the symmetric function on the square of $V(S) \sqcup \Sigma(S)$ with $i(\alpha, \sigma) = \sum_{\beta \in \sigma} i(\alpha, \beta)$ and $i(\sigma, \tau) = \sum_{\beta \in \sigma, \gamma \in \tau} i(\beta, \gamma)$ for any $\alpha \in V(S)$ and $\sigma, \tau \in \Sigma(S)$. We say that two elements α, β of $V(S) \sqcup \Sigma(S)$ are *disjoint* if $i(\alpha, \beta) = 0$. Otherwise, we say that α and β *intersect*. We say that two elements α, β of $V(S)$ *fill* S if
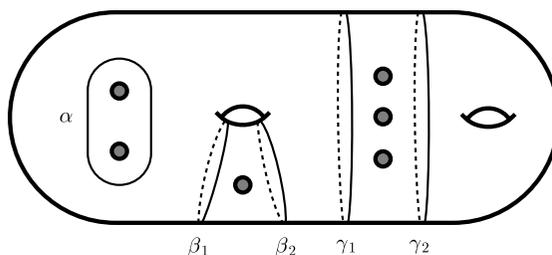
Figure 1. α is an HBC, $\{\beta_1, \beta_2\}$ is a non-separating HBP, and $\{\gamma_1, \gamma_2\}$ is a separating HBP.

there exists no element of $V(S)$ disjoint from both α and β.

Let σ be an element of $\Sigma(S)$. We denote by S_σ the surface obtained by cutting S along all curves in σ. When σ consists of a single curve α, we denote it by S_α. Each component of S_σ is often identified with a complementary component of a tubular neighborhood of a one-dimensional submanifold representing σ in S. Let R be a component of S_σ. The set $V(R)$ is then identified with a subset of $V(S)$.

2.2. Simplicial complexes associated to a surface.

Let S be a surface. We recall two complexes of curves and then introduce complexes of HBCs and HBPs.

Complex $\mathscr{C}(S)$.

We define $\mathscr{C}(S)$ as the abstract simplicial complex such that the sets of vertices and simplices of $\mathscr{C}(S)$ are $V(S)$ and $\Sigma(S)$, respectively, and call it the *complex of curves* for S.

This complex was introduced by Harvey [**9**]. The following theorem says that for almost all surfaces S, any automorphism of $\mathscr{C}(S)$ is generally induced by an element of $\mathrm{Mod}^*(S)$. This fact is fundamental in the study of the abstract commensurators of various subgroups of the mapping class group.

THEOREM 2.1 ([**13**], [**17**], [**18**]). *Let $S = S_{g,p}$ be a surface with $3g+p-4 > 0$. Then the following assertions hold*:

(i) *If $(g,p) \neq (1,2)$, then any automorphism of $\mathscr{C}(S)$ is induced by an element of $\mathrm{Mod}^*(S)$.*

(ii) *If $(g,p) = (1,2)$, then any automorphism of $\mathscr{C}(S)$ preserving vertices which correspond to separating curves in S is induced by an element of $\mathrm{Mod}^*(S)$.*

Let $\bar{S}$ be the closed surface obtained from S by attaching disks to all boundary components of S. Let $\mathscr{C}^*(\bar{S})$ be the simplicial cone over $\mathscr{C}(\bar{S})$ with its cone point

$*$. Namely, $\mathscr{C}^*(\bar{S})$ is the abstract simplicial complex such that the set of vertices of $\mathscr{C}^*(\bar{S})$ is the disjoint union $V(\bar{S}) \sqcup \{*\}$; and the set of simplices of $\mathscr{C}^*(\bar{S})$ is

$$\Sigma(\bar{S}) \cup \big\{ \sigma \cup \{*\} \mid \sigma \in \Sigma(\bar{S}) \cup \{\emptyset\} \big\}.$$

We have the simplicial map $\pi \colon \mathscr{C}(S) \to \mathscr{C}^*(\bar{S})$ associated with the inclusion of S into $\bar{S}$. Note that $\pi^{-1}(\{*\})$ consists of all HBCs in S. An HBP in S is identified with an edge of $\mathscr{C}(S)$. For each edge $\{\alpha, \beta\}$ of $\mathscr{C}(S)$, it is an HBP in S if and only if we have $\pi(\alpha) = \pi(\beta) \neq *$. Two disjoint HBPs a, b in S are said to be *equivalent* in S if $\pi(a) = \pi(b)$.

Complex $\mathscr{C}_s(S)$.

Let $V_s(S)$ denote the set of all elements of $V(S)$ whose representatives are separating in S. We define $\mathscr{C}_s(S)$ as the full subcomplex of $\mathscr{C}(S)$ spanned by $V_s(S)$ and call it the *complex of separating curves* for S.

This complex (for closed surfaces) appears in [4], [5], [6] and [19]. Automorphisms of $\mathscr{C}_s(S)$ are studied in [4], [5] and [15]. We now introduce two simplicial complexes whose vertices are HBCs and HBPs in S, inspired by the work due to Irmak-Ivanov-McCarthy [11] characterizing elements of $P(S)$ associated with HBCs and HBPs in S algebraically.

Complexes $\mathscr{C}\mathscr{P}(S)$ and $\mathscr{C}\mathscr{P}_s(S)$.

Let $V_c(S)$ denote the set of all elements of $V(S)$ whose representatives are HBCs in S. Let $V_p(S)$ denote the set of all elements of $\Sigma(S)$ whose representatives are HBPs in S.

We define $\mathscr{C}\mathscr{P}(S)$ as the abstract simplicial complex such that the set of vertices is the disjoint union $V_c(S) \sqcup V_p(S)$, and a non-empty finite subset σ of $V_c(S) \sqcup V_p(S)$ is a simplex of $\mathscr{C}\mathscr{P}(S)$ if and only if any two elements of σ are disjoint. We call elements of $V_c(S)$ and $V_p(S)$ *HBC-vertices* and *HBP-vertices* of $\mathscr{C}\mathscr{P}(S)$, respectively.

We define $\mathscr{C}\mathscr{P}_s(S)$ as the full subcomplex of $\mathscr{C}\mathscr{P}(S)$ spanned by all vertices whose representatives are either an HBC in S and or a separating HBP in S.

We note that if S is of genus zero, then $\mathscr{C}\mathscr{P}(S) = \mathscr{C}\mathscr{P}_s(S) = \mathscr{C}(S) = \mathscr{C}_s(S)$. If S is of genus one, then $\mathscr{C}\mathscr{P}(S)$ is equal to the Torelli complex studied in [4] and [15], and the equality $\mathscr{C}\mathscr{P}_s(S) = \mathscr{C}_s(S)$ holds.

Superinjective maps.

Let X and Y be any of the four simplicial complexes introduced above. We denote by $V(X)$ and $V(Y)$ the sets of vertices of X and Y, respectively. Note that

a map $\phi\colon V(X) \to V(Y)$ defines a simplicial map from X into Y if and only if $i(\phi(a), \phi(b)) = 0$ for any $a, b \in V(X)$ with $i(a, b) = 0$. We mean by a *superinjective map* $\phi\colon X \to Y$ a simplicial map $\phi\colon X \to Y$ satisfying $i(\phi(a), \phi(b)) \neq 0$ for any $a, b \in V(X)$ with $i(a, b) \neq 0$. One can check that any superinjective map from X into Y is injective, along the proof of Lemma 3.1 in [**10**] proving that any superinjective map from $\mathscr{C}(S)$ into itself is injective.

2.3. Surface braid groups.

Let S be a surface. The *mapping class group* $\mathrm{Mod}(S)$ for S is defined as the subgroup of $\mathrm{Mod}^*(S)$ consisting of all isotopy classes of orientation-preserving homeomorphisms from S onto itself. The *pure mapping class group* $\mathrm{PMod}(S)$ for S is defined as the subgroup of $\mathrm{Mod}^*(S)$ consisting of all isotopy classes of homeomorphisms from S onto itself that preserve an orientation of S and each component of ∂S as a set. We refer to [**14**] for fundamentals of these groups.

For each $\alpha \in V(S)$, let $t_\alpha \in \mathrm{PMod}(S)$ denote the (*left*) *Dehn twist* about α. The Dehn twist about an HBC is called an *HBC twist*. For each HBP $\{\alpha, \beta\}$ in S, the elements $t_\alpha t_\beta^{-1}$, $t_\beta t_\alpha^{-1}$ of $\mathrm{PMod}(S)$ are called *HBP twists* about the HBP $\{\alpha, \beta\}$.

Let $\bar{S}$ be the closed surface obtained by attaching disks to all boundary components of S. We then have the surjective homomorphism

$$\iota\colon \mathrm{PMod}(S) \to \mathrm{Mod}(\bar{S})$$

associated with the inclusion of S into $\bar{S}$. We define $P(S)$ to be $\ker \iota$. The group $P(S)$ is known to be generated by all HBC twists and all HBP twists in S (see Section 4.1 in [**2**]). We define $P_s(S)$ to be the subgroup of $P(S)$ generated by all HBC twists and all HBP twists about separating HBPs in S.

LEMMA 2.2. *Let $S = S_{g,p}$ be a surface with $g \geq 2$ and $p \geq 1$. Then the following assertions hold*:

(i) *The actions of $\mathrm{Mod}^*(S)$ on $\mathscr{CP}(S)$ and on $\mathscr{CP}_s(S)$ are faithful.*
(ii) *The homomorphisms*

$$\boldsymbol{i}\colon \mathrm{Mod}^*(S) \to \mathrm{Comm}(P(S)), \quad \boldsymbol{i}_s\colon \mathrm{Mod}^*(S) \to \mathrm{Comm}(P_s(S))$$

defined by conjugation are injective.

PROOF. Let x be an element of $\mathrm{Mod}^*(S)$ fixing any vertex of $\mathscr{CP}_s(S)$. Pick $\alpha \in V_s(S) \setminus V_c(S)$. We choose separating HBPs $\{\alpha, \beta_1\}$, $\{\alpha, \beta_2\}$ in S with $\beta_1 \neq \beta_2$. Since x fixes these HBPs, it fixes α. Thus, x fixes any element of $V_s(S)$. For each non-separating curve γ in S, we choose separating curves δ_1, δ_2 in S disjoint from

γ and filling S_γ. Since x fixes δ_1 and δ_2, it fixes γ. It follows that x fixes any element of $V(S)$ and is thus neutral. Assertion (i) is proved.

To prove assertion (ii), it suffices to show that $\boldsymbol{i}_s$ is injective. Pick $y \in \mathrm{Mod}^*(S)$ with $\boldsymbol{i}_s(y)$ neutral. There exists a finite index subgroup of $P_s(S)$ such that y commutes any element of it. For any separating HBP $\{\alpha, \beta\}$ in S, we then have a non-zero integer n with $y(t_\alpha t_\beta^{-1})^n y^{-1} = (t_\alpha t_\beta^{-1})^n$. Thus, y fixes $\{\alpha, \beta\}$. Similarly, y fixes any HBC in S. By assertion (i), y is neutral. Assertion (ii) is proved. $\qquad\square$

LEMMA 2.3. *Let S be a surface of genus at least one. Pick $\sigma \in \Sigma(S)$ and let D_σ be the subgroup of $\mathrm{PMod}(S)$ generated by all Dehn twists about curves in σ. Then the following assertions hold*:

(i) *$D_\sigma \cap P(S)$ is generated by all Dehn twists about HBCs in σ and all HBP twists about HBPs of two curves in σ.*
(ii) *$D_\sigma \cap P_s(S)$ is generated by all Dehn twists about HBCs in σ and all HBP twists about separating HBPs of two curves in σ.*

In particular, $P_s(S)$ contains no non-zero power of the HBP twist about a non-separating HBP in S.

PROOF.   Assertion (i) holds because any element of $P(S)$ induces the neutral element of $\mathrm{Mod}(\bar{S})$. We define $\mathscr{K}(S)$ to be the group generated by all Dehn twists about separating curves in S, and call it the *Johnson kernel* for S. The group $P_s(S)$ is contained in $\mathscr{K}(S)$. Theorem 6.1 (ii) in [**15**] shows that $D_\sigma \cap \mathscr{K}(S)$ is generated by Dehn twists about curves in $\sigma \cap V_s(S)$. Assertion (ii) thus follows because any element of $P_s(S)$ induces the neutral element of $\mathrm{Mod}(\bar{S})$. $\qquad\square$

2.4. Plan of Sections 3–6.

Let $S = S_{g,p}$ be a surface with $g \geq 2$ and $p \geq 2$. We explain an outline to prove that any automorphism ϕ of $\mathscr{CP}(S)$ is induced by an element of $\mathrm{Mod}^*(S)$. This conclusion will be obtained by constructing an automorphism Φ of $\mathscr{C}(S)$ inducing ϕ.

In Section 3, we prove that

- ϕ preserves vertices corresponding to HBCs in S, non-separating HBPs in S and separating HBPs in S, respectively; and
- for any two disjoint HBPs b_1, b_2 in S containing a common curve, the two HBPs $\phi(b_1)$, $\phi(b_2)$ also contain a common curve.

For each $\alpha \in V(S)$, we define $\Phi(\alpha) \in V(S)$ as follows. If α is an HBC in S, then we put $\Phi(\alpha) = \phi(\alpha)$. Otherwise, choosing two disjoint and distinct HBPs in S, say b_1 and b_2, containing α, we define $\Phi(\alpha)$ to be the common curve of the

two HBPs $\phi(b_1)$ and $\phi(b_2)$. Sections 4 and 5 are devoted to showing that this is well-defined. In Section 6, the map Φ is shown to be an automorphism of $\mathscr{C}(S)$ and is thus induced by an element of $\mathrm{Mod}^*(S)$ by Theorem 2.1. In a similar way, for any automorphism ψ of $\mathscr{CP}_s(S)$, we construct a map Ψ from $V_s(S)$ into itself and show it to be an automorphism of $\mathscr{C}_s(S)$ and thus induced by an element of $\mathrm{Mod}^*(S)$ by Theorem 1.2 in [**15**].

3. Basic properties of automorphisms of $\mathscr{CP}(S)$.

For an automorphism ϕ of $\mathscr{CP}(S)$, we show that ϕ preserves topological properties of HBCs and HBPs corresponding to vertices of $\mathscr{CP}(S)$. In Section 3.1, simplices of $\mathscr{CP}(S)$ of maximal dimension are completely described. It is obvious that such simplices are preserved by ϕ. In Section 3.2, using this fact, we show that ϕ preserves HBC-vertices and HBP-vertices, respectively, and preserves more detailed information on these vertices.

3.1. Simplices of $\mathscr{CP}(S)$ of maximal dimension.
We say that two disjoint curves in S are *HBP-equivalent* in S if they either are equal or form an HBP in S. This defines an equivalence relation on each simplex σ of $\mathscr{C}(S)$ with $\sigma \cap V_c(S) = \emptyset$. For each simplex σ of $\mathscr{C}(S)$, we mean by an *HBP-equivalence class* in σ an equivalence class in $\sigma \setminus V_c(S)$ with respect to this equivalence relation. Before describing simplices of $\mathscr{CP}(S)$ of maximal dimension, we give several elementary observations on HBP-equivalence.

LEMMA 3.1. *Let $S = S_{g,p}$ be a surface with $g \geq 2$ and $p \geq 1$, and let $b \in \Sigma(S)$ be a simplex such that $|b| \geq 2$, $b \cap V_c(S) = \emptyset$ and any two curves of b are HBP-equivalent in S. Then the following assertions hold*:

(i) *Either all curves of b are non-separating in S or they are separating in S.*

(ii) *If all curves of b are non-separating in S, then each component of S_b contains exactly two curves of b as boundary components, and exactly one component of S_b is of positive genus.*

(iii) *If all curves of b are separating in S, then there exist exactly two components of S_b of positive genus. Moreover, those two components contain exactly one curve of b as a boundary component, and any other component of S_b contains exactly two curves of b as boundary components.*

PROOF. Assertion (i) follows from the definition of HBPs. Assertions (ii) and (iii) are verified by induction on $|b|$. □

LEMMA 3.2. *Let $S = S_{g,p}$ be a surface with $g \geq 2$ and $p \geq 1$, and let $b, c \in \Sigma(S)$ be simplices such that*

- $|b| \geq 2$, $b \cap c = \emptyset$ and $i(b, c) = 0$;
- b consists of non-separating curves in S; and
- each of b and c is an HBP-equivalence class in the simplex $b \cup c$.

Then any curve of c is contained in the component of S_b of positive genus.

PROOF.   This lemma follows from the fact that any curve in a component of S_b of genus zero is either an HBC in S or HBP-equivalent to a curve in b.   □

LEMMA 3.3.   Let $S = S_{g,p}$ be a surface with $g \geq 2$ and $p \geq 1$. Pick a separating curve α in S which is not an HBC in S, and pick a simplex $c \in \Sigma(S)$ such that

- $|c| \geq 2$, $\alpha \notin c$ and $i(\alpha, c) = 0$; and
- c is an HBP-equivalence class in the simplex $\{\alpha\} \cup c$.

Then there exists a component Q of S_α such that c belongs to $\Sigma(Q)$.

PROOF.   Pick two distinct curves γ, δ of c. Let Q and R be the components of S_α with $\gamma \in V(Q)$ and $\delta \in V(R)$. If all curves of c are non-separating in S, then γ and δ are non-separating in Q and R, respectively. Since $S_{\{\gamma,\delta\}}$ is not connected, we have $Q = R$. If all curves of c are separating in S, then for each curve β of c, since β is not an HBC in S and not equivalent to α, the curve β has to separate the component of S_α containing β into two components of positive genus. Since any two curves in c are HBP-equivalent, all curves of c are contained in the same component of S_α.   □

LEMMA 3.4.   Let $S = S_{g,p}$ be a surface with $g \geq 2$ and $p \geq 2$, and let σ be a simplex of $\mathscr{CP}(S)$ consisting of HBC-vertices. Then the inequality $|\sigma| \leq p - 1$ holds, and the equality can be attained.

PROOF.   We can find a simplex τ of $\mathscr{CP}(S)$ consisting of $p-1$ HBC-vertices. The inequality in the lemma is verified by induction on p.   □

We now describe simplices of $\mathscr{CP}(S)$ of maximal dimension in the following:

PROPOSITION 3.5.   Let $S = S_{g,p}$ be a surface with $g \geq 2$ and $p \geq 1$. Then we have

$$\dim(\mathscr{CP}(S)) = \binom{p+1}{2} - 1.$$

Moreover, for any simplex σ of $\mathscr{CP}(S)$ of maximal dimension, there exists a unique simplex $s = \{\beta_1, \beta_2, \ldots, \beta_{p+1}\}$ of $\mathscr{C}(S)$ such that
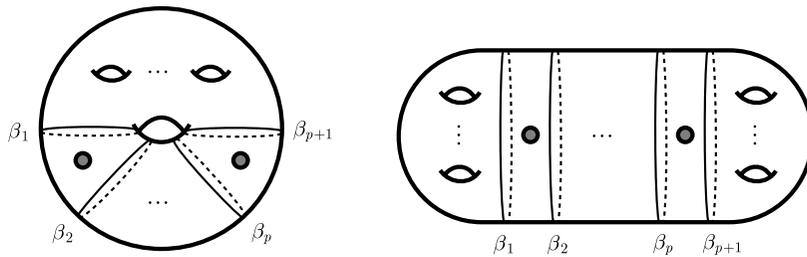
Figure 2.   Simplices of $\mathscr{C}\mathscr{P}(S)$ of maximal dimension.

(a) *any two curves in s are HBP-equivalent; and*
(b) *σ consists of all HBPs of two curves in s.*

Before proving this proposition, we note that if S is a surface of genus one, then $\mathscr{C}\mathscr{P}(S)$ is equal to the Torelli complex $\mathscr{T}(S)$ studied in [**15**], where simplices of $\mathscr{T}(S)$ of maximal dimension are described. Proposition 3.4 in [**15**] thus implies that if $S = S_{1,p}$ is a surface with $p \geq 2$, then we have

$$\dim(\mathscr{C}\mathscr{P}(S)) = \binom{p}{2} - 1.$$

PROOF OF PROPOSITION 3.5.   The $p + 1$ curves in S described in Figure 2 are mutually HBP-equivalent. It follows that the dimension of $\mathscr{C}\mathscr{P}(S)$ is not smaller than the right hand side of the equality in the proposition.

In what follows, we show the equality by induction on p. The equality obviously holds if $p = 1$. We assume $p \geq 2$. Pick a simplex σ of $\mathscr{C}\mathscr{P}(S)$ of maximal dimension. We define the simplex of $\mathscr{C}(S)$, denoted by

$$s = \big\{\alpha_1, \ldots, \alpha_k, \beta_{11}, \ldots, \beta_{1m_1}, \beta_{21}, \ldots, \beta_{lm_l}\big\},$$

so that

- $\{\alpha_1, \ldots, \alpha_k\}$ is the collection of HBCs of σ;
- $\{\beta_{11}, \ldots, \beta_{1m_1}, \beta_{21}, \ldots, \beta_{lm_l}\}$ is the collection of curves in HBPs of σ; and
- for each $j = 1, \ldots, l$, the set $b_j = \{\beta_{j1}, \ldots, \beta_{jm_j}\}$ is an HBP-equivalence class in s.

Since $\dim \sigma$ is maximal, σ contains all HBPs of two curves in each b_j. We hence obtain the equality

$$|\sigma| = k + \sum_{j=1}^{l} \binom{m_j}{2}.$$

CLAIM 3.6.   *We have $k = 0$.*

PROOF.   We assume $k \geq 1$. For each $j = 1, 2, \ldots, k$, let n_j be the integer with α_j an n_j-HBC in S, and put $n = \max\{n_1, n_2, \ldots, n_k\}$. Exchanging indices if necessary, we may suppose that α_1 is an n-HBC in S. Let $\partial_1, \ldots, \partial_n$ be the boundary components of S contained in the component of S_{α_1} of genus zero. The maximality of $\dim \sigma$ and n implies that there exists an n-HBP $b = \{\gamma_1, \gamma_2\}$ in σ such that the component of S_b of genus zero, denoted by Q, contains $\partial_1, \ldots, \partial_n$. We note that $\sigma \cap V(Q)$ consists of HBCs in S and that the inequality $|\sigma \cap V(Q)| \leq n-1$ holds. Remove all curves in $\sigma \cap V(Q)$ from σ and add curves $\gamma_3, \gamma_4 \ldots, \gamma_{n+1} \in V(Q)$ to σ such that for each $j = 3, 4, \ldots, n+1$, γ_j is HBP-equivalent to γ_1 and γ_2 as a curve in S. This new collection of curves is denoted by $s' \in \Sigma(S)$ and associates the simplex σ' of $\mathscr{CP}(S)$ consisting of all HBCs in s' and all HBPs of two curves in each HBP-equivalence class in s'. We thus obtain the inequality

$$|\sigma'| \geq |\sigma| - |\sigma \cap V(Q)| + \frac{(n+1)n}{2} - 1 \geq |\sigma| + \frac{n(n-1)}{2} > |\sigma|,$$

where the last inequality holds since we have $n \geq 2$. This contradicts the maximality of $\dim \sigma$. $\square$

CLAIM 3.7.   *We have $l = 1$.*

PROOF.   Assume $l \geq 2$. We deduce a contradiction in the following two cases: (i) b_1 consists of separating curves in S; and (ii) b_1 consists of non-separating curves in S. Let R be the component of S_{b_1} containing all curves in b_2, which exists by Lemmas 3.2 and 3.3. Let p_1 denote the number of boundary components of S contained in R. Since b_1 and b_2 are not equivalent, the inequality $1 \leq p_1 \leq p - 1$ holds, and the genus of R is positive.

In case (i), we may assume that β_{11} is the element of b_1 corresponding to a component of ∂R. Moreover, after exchanging indices, we may assume that there exists an integer l' with $2 \leq l' \leq l$ such that

- for each $j = 2, \ldots, l'$, all curves in b_j are contained in R; and
- for each $j = l'+1, \ldots, l$, all curves in b_j are contained in a component of S_{b_1} distinct from R.

We mean by a *handle* a surface homeomorphic to $S_{1,1}$. Let R' denote the surface obtained from R by attaching a handle to the component of ∂R corresponding

to β_{11}. The number of boundary components of R' is then equal to p_1. For each $j = 2, \ldots, l'$, any curve of b_j can be regarded as a curve in R' via the inclusion of R into R', and any two curves of b_j form an HBP in R'. We denote by $\sigma_{R'}$ the simplex of $\mathscr{CP}(R')$ consisting of all HBPs in R' associated with two curves in b_j for $j = 2, \ldots, l'$. Since the genus of R' is at least two, we obtain

$$|\sigma_{R'}| = \sum_{j=2}^{l'} \binom{m_j}{2} \leq \binom{p_1 + 1}{2}$$

by the hypothesis of the induction. We remove all curves in $b_2, b_3, \ldots, b_{l'}$ from s and add to s curves $\delta_1, \delta_2, \ldots, \delta_{p_1} \in V(R)$ which are mutually disjoint and HBP-equivalent to β_{11} as curves in S. This new collection of curves is denoted by $s_1 \in \Sigma(S)$. Let σ_1 be the simplex of $\mathscr{CP}(S)$ consisting of all HBPs of two curves in HBP-equivalence classes of s_1. We then obtain the equality

$$|\sigma_1| = \binom{m_1 + p_1}{2} + \sum_{j=l'+1}^{l} \binom{m_j}{2}$$

and the inequality

$$|\sigma_1| - |\sigma| = \binom{m_1 + p_1}{2} + \sum_{j=l'+1}^{l} \binom{m_j}{2} - \sum_{j=1}^{l} \binom{m_j}{2}$$

$$\geq \binom{m_1 + p_1}{2} - \binom{p_1 + 1}{2} - \binom{m_1}{2} = (m_1 - 1)p_1 > 0.$$

This contradicts the maximality of $\dim \sigma$.

Let us consider case (ii). Without loss of generality, we may assume that β_{11} and β_{12} are the two elements of b_1 corresponding to components of ∂R. It follows from Lemma 3.2 that for each $j = 2, \ldots, l$, any curve of b_j is contained in R as an essential one. Let R'' be the surface obtained by identifying β_{11} with β_{12}, whose genus is at least two. For each $j = 2, \ldots, l$, any curve of b_j can be regarded as a curve in R'' via the natural map from R into R'', and any two curves of b_j form an HBP in R''. We denote by $\sigma_{R''}$ the simplex of $\mathscr{CP}(R'')$ consisting of all HBPs in R'' associated with two curves in b_j for $j = 2, \ldots, l$. Along an argument of the same kind as in case (i), we can deduce a contradiction. $\qquad\square$

Since any HBP-equivalence class in a simplex of $\mathscr{C}(S)$ consists of at most $p+1$ curves, Claims 3.6 and 3.7 imply the equality in the proposition. Existence and

uniqueness of the simplex $s = \{\beta_1, \beta_2, \ldots, \beta_{p+1}\}$ in the proposition also follow. $\square$

3.2.   Topological properties preserved by ϕ.

The following three lemmas are obtained as a consequence of Proposition 3.5.

LEMMA 3.8.   *Let* $S = S_{g,p}$ *be a surface with* $g \geq 2$ *and* $p \geq 2$, *and let* $\phi \colon \mathscr{CP}(S) \to \mathscr{CP}(S)$ *be a superinjective map. Then the following assertions hold*:

(i) *The map* ϕ *preserves HBPs in* S.
(ii) *If* b_1 *and* b_2 *are disjoint and equivalent HBPs in* S, *then* $\phi(b_1)$ *and* $\phi(b_2)$ *are also equivalent.*

PROOF.   Let b be an HBP in S, and let σ be a simplex of $\mathscr{CP}(S)$ of maximal dimension containing b as its vertex. Since ϕ is injective, the equality $|\phi(\sigma)| = |\sigma|$ holds, and $\phi(\sigma)$ is thus a simplex of maximal dimension. By Proposition 3.5, each vertex of $\phi(\sigma)$ is an HBP in S. Assertion (i) is proved.

Let b_1 and b_2 be disjoint and equivalent HBPs in S. By Proposition 3.5, we can find a simplex τ of $\mathscr{CP}(S)$ of maximal dimension containing b_1 and b_2 as its vertices. Since the dimension of $\phi(\tau)$ is maximal in $\mathscr{CP}(S)$, all vertices of $\phi(\tau)$ are equivalent by Proposition 3.5. Assertion (ii) is proved. $\square$

LEMMA 3.9.   *Let* $S = S_{g,p}$ *be a surface with* $g \geq 2$ *and* $p \geq 2$. *Then any automorphism of* $\mathscr{CP}(S)$ *preserves HBCs in* S.

PROOF.   The lemma follows because for any automorphism ϕ of $\mathscr{CP}(S)$, ϕ and ϕ^{-1} preserve HBPs in S by Lemma 3.8 (i). $\square$

A verbatim proof shows the following:

LEMMA 3.10.   *Let* $S = S_{g,p}$ *be a surface with* $g \geq 2$ *and* $p \geq 2$, *and let* $\psi \colon \mathscr{CP}_s(S) \to \mathscr{CP}_s(S)$ *be a superinjective map. Then the following assertions hold*:

(i)   *The map* ψ *preserves HBPs in* S.
(ii) *If* b_1 *and* b_2 *are disjoint, equivalent and separating HBPs in* S, *then* $\psi(b_1)$ *and* $\psi(b_2)$ *are also equivalent.*
(iii) *If* ψ *is an automorphism of* $\mathscr{CP}_s(S)$, *then* ψ *preserves HBCs in* S.

To show that an automorphism of $\mathscr{CP}(S)$ preserves more topological information, let us introduce the following terminology.

DEFINITION 3.11.   Let S be a surface, and let σ be a simplex of $\mathscr{CP}(S)$ consisting of HBP-vertices. We say that σ is *rooted* if there exists a curve α in S

contained in any HBP of σ. In this case, if $|\sigma| \geq 2$, then α is uniquely determined and called the *root curve* of σ.

Rooted simplices were introduced in [**15**] for the Torelli complex analogously.

LEMMA 3.12. *Let $S = S_{g,p}$ be a surface with $g \geq 2$ and $p \geq 2$. Then any superinjective map from $\mathscr{CP}(S)$ into itself preserves rooted simplices. Moreover, the same conclusion holds for any superinjective map from $\mathscr{CP}_s(S)$ into itself.*

PROOF. Let $\phi\colon \mathscr{CP}(S) \to \mathscr{CP}(S)$ be a superinjective map. We note that the maximal dimension of a rooted simplex of $\mathscr{CP}(S)$ is equal to $p-1$. Let σ be a rooted simplex of $\mathscr{CP}(S)$ consisting of HBP-vertices $a_1, a_2, \ldots, a_p$. It suffices to prove that $\phi(\sigma)$ is rooted.

For each $j = 1, 2, \ldots, p$, there exists an HBP b_j in S with

$$i(a_j, b_j) \neq 0 \quad \text{and} \quad i(a_k, b_j) = 0 \quad \text{for any } k \in \{1, \ldots, p\} \setminus \{j\}.$$

Since ϕ is superinjective, for each $j = 1, 2, \ldots, p$, we have

$$i(\phi(a_j), \phi(b_j)) \neq 0 \quad \text{and} \quad i(\phi(a_k), \phi(b_j)) = 0 \quad \text{for any } k \in \{1, \ldots, p\} \setminus \{j\}.$$

For each $j = 1, 2, \ldots, p$, there thus exists a curve $c_j \in \phi(a_j)$ with $i(c_j, \phi(b_j)) \neq 0$.

We note that for each $k \in \{1, \ldots, p\} \setminus \{j\}$, the HBP $\phi(a_k)$ does not contain c_j. Let c_0 be the curve of $\phi(a_1)$ distinct from c_1. It then follows that $c_0, c_1, \ldots, c_p$ are mutually distinct, disjoint and HBP-equivalent in S. Proposition 3.5 implies that the simplex of $\mathscr{CP}(S)$ consisting of all HBPs of two of $c_0, c_1, \ldots, c_p$ is of maximal dimension. The simplex $\phi(\sigma)$ thus consists of p pairs of two of $c_0, c_1, \ldots, c_p$. For each $j = 1, \ldots, p$, since $\phi(a_j)$ contains c_j and does not contain c_k for any $k \in \{1, \ldots, p\} \setminus \{j\}$, we obtain the equality $\phi(a_j) = \{c_0, c_j\}$.

Along an argument of the same kind, we can prove the assertion for any superinjective map from $\mathscr{CP}_s(S)$ into itself. $\square$

LEMMA 3.13. *Let $S = S_{g,p}$ be a surface with $g \geq 2$ and $p \geq 2$, and let $\phi\colon \mathscr{CP}(S) \to \mathscr{CP}(S)$ be a superinjective map. Pick a simplex σ of $\mathscr{CP}(S)$ of maximal dimension, and let $\{\alpha_0, \alpha_1, \ldots, \alpha_p\}$ denote the collection of curves in HBPs of σ. Then there exists a collection of curves in S, $\{\beta_0, \beta_1, \ldots, \beta_p\}$, satisfying the equality*

$$\phi(\{\alpha_j, \alpha_k\}) = \{\beta_j, \beta_k\}$$

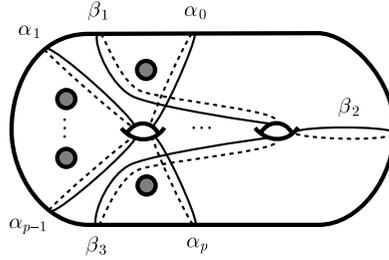*for any distinct $j, k = 0, 1, \ldots, p$. Moreover, the same conclusion holds for any su-*

Figure 3.

perinjective map from $\mathscr{CP}_s(S)$ into itself and any simplex of $\mathscr{CP}_s(S)$ of maximal dimension.

Proof. For each $j = 0, 1, \ldots, p$, we define σ_j to be the rooted subsimplex of σ of dimension $p-1$ that consists of all HBPs in σ containing α_j. Let β_j denote the root curve of the rooted simplex $\phi(\sigma_j)$. Note that we have $\beta_j \neq \beta_k$ for any $j \neq k$ because the maximal dimension of a rooted simplex of $\mathscr{CP}(S)$ is equal to $p-1$. For any $j \neq k$, since $\phi(\{\alpha_j, \alpha_k\})$ contains β_j and β_k, we have the equality $\phi(\{\alpha_j, \alpha_k\}) = \{\beta_j, \beta_k\}$.

Along an argument of the same kind, we can prove the assertion for any superinjective map from $\mathscr{CP}_s(S)$ into itself. □

Applying observations so far on rooted simplices, we prove the following:

Lemma 3.14. *Let $S = S_{g,p}$ be a surface with $g \geq 2$ and $p \geq 2$. Then any automorphism of $\mathscr{CP}(S)$ preserves non-separating HBPs in S and separating HBPs in S, respectively.*

Proof. Let ϕ be an automorphism of $\mathscr{CP}(S)$. It suffices to prove that ϕ preserves non-separating HBPs in S. Pick a simplex σ of $\mathscr{CP}(S)$ of maximal dimension consisting of non-separating HBPs in S. Let $s = \{\alpha_0, \alpha_1, \ldots, \alpha_p\}$ be the set of non-separating curves in S such that

- σ consists of all pairs of two curves in s; and
- for each $j = 0, \ldots, p-1$, $\{\alpha_j, \alpha_{j+1}\}$ is a 1-HBP in S.

As described in Figure 3, we can find non-separating curves β_1, β_2 and β_3 in S such that

- β_1, β_2 and β_3 are mutually disjoint, distinct and HBP-equivalent in S;
- $i(\beta_2, s) = i(\beta_1, s \setminus \{\alpha_0\}) = i(\beta_3, s \setminus \{\alpha_p\}) = 0$;
- $i(\beta_1, \alpha_0) \neq 0$ and $i(\beta_3, \alpha_p) \neq 0$;

- β_2 is not HBP-equivalent to any curve of s; and
- both $\{\beta_1, \beta_2\}$ and $\{\beta_2, \beta_3\}$ are 1-HBPs in S.

By Lemma 3.13, there exist curves γ_j, δ_k in S with

$$\phi(\{\alpha_{j_1}, \alpha_{j_2}\}) = \{\gamma_{j_1}, \gamma_{j_2}\}, \quad \phi(\{\beta_{k_1}, \beta_{k_2}\}) = \{\delta_{k_1}, \delta_{k_2}\}$$

for any distinct $j_1, j_2 = 0, \ldots, p$ and any distinct $k_1, k_2 = 1, 2, 3$. We put $t = \{\gamma_0, \ldots, \gamma_p\}$. Since $\{\delta_1, \delta_2\}$ intersects γ_0 and is disjoint from any curve in $t \setminus \{\gamma_0\}$ and since $\{\delta_2, \delta_3\}$ intersects γ_p and is disjoint from any curve in $t \setminus \{\gamma_p\}$, we see that δ_2 is disjoint from any curve in t. Since $\{\delta_1, \delta_2\}$ is not equivalent to any HBP of two curves in $t \setminus \{\gamma_0\}$, using Lemmas 3.2 and 3.3, we see that $\{\delta_1, \delta_2\}$ is a 1-HBP in S and that there exists a $(p-1)$-HBP of two curves in $t \setminus \{\gamma_0\}$ such that the other curves in it are contained in the holed sphere cut off by that $(p-1)$-HBP from S. Similarly, $\{\delta_2, \delta_3\}$ is a 1-HBP in S, and there exists a $(p-1)$-HBP of two curves in $t \setminus \{\gamma_p\}$ such that the other curves in it are contained in the holed sphere cut off by that $(p-1)$-HBP from S. It thus follows that $\{\gamma_0, \gamma_p\}$ is a p-HBP in S.

We now assume that each γ_j is a separating curve in S. If δ_2 lies in the component of $S_{\{\gamma_0, \gamma_p\}}$ of positive genus that contains γ_0 as a boundary component, then the HBP $\{\delta_2, \delta_3\}$ cannot intersect γ_p, kept disjoint from any curve in $t \setminus \{\gamma_p\}$, by Lemma 3.3. This is a contradiction. In a similar way, we can deduce a contradiction if we assume that δ_2 lies in the component of $S_{\{\gamma_0, \gamma_p\}}$ of positive genus that contains γ_p as a boundary component. $\qquad\square$

LEMMA 3.15.   *Let $S = S_{g,p}$ be a surface with $g \geq 2$ and $p \geq 2$. Then for any integers j, k with $2 \leq j \leq p$ and $1 \leq k \leq p$, any automorphism of $\mathscr{CP}(S)$ preserves j-HBCs in S and k-HBPs in S, respectively. Moreover, the same conclusion holds for any automorphism of $\mathscr{CP}_s(S)$.*

PROOF.   Let ϕ be an automorphism of $\mathscr{CP}(S)$. Pick a j-HBC α in S with $2 \leq j \leq p$, and suppose that $\phi(\alpha)$ is a j'-HBC in S with $2 \leq j' \leq p$. Let R (resp. R') be the component of S_α (resp. $S_{\phi(\alpha)}$) of positive genus, which is a surface of genus g with $p - j + 1$ (resp. $p - j' + 1$) boundary components. We note that each HBP in R can be identified with an HBP in S via the inclusion of R into S. The same thing holds for R'. Let σ be a simplex of $\mathscr{CP}(R)$ of maximal dimension, which is identified with a simplex of $\mathscr{CP}(S)$ consisting of HBP-vertices. Since each HBP in $\phi(\sigma)$ is disjoint from the HBC $\phi(\alpha)$, we can identify $\phi(\sigma)$ with a simplex of $\mathscr{CP}(R')$. We then obtain the inequality

$$\binom{p-j+2}{2} = |\sigma| = |\phi(\sigma)| \leq \binom{p-j'+2}{2},$$

which implies $j \geq j'$. The same argument for ϕ^{-1} shows $j \leq j'$. We thus conclude the equality $j = j'$.

We prove that ϕ preserves k-HBPs in S for each integer k with $1 \leq k \leq p$ by induction on p. If $p = 2$, then ϕ preserves 2-HBPs in S because ϕ preserves 2-HBCs in S and because any HBP in S disjoint from a 2-HBC in S is a 2-HBP in S. It then follows that ϕ preserves 1-HBPs in S.

We next assume $p \geq 3$. Pick a simplex σ of maximal dimension in $\mathscr{CP}(S)$ consisting of non-separating HBPs in S. Let $s = \{\alpha_0, \ldots, \alpha_p\}$ denote the collection of curves in HBPs of σ so that $\{\alpha_j, \alpha_{j+1}\}$ is a 1-HBP in S for each $j = 0, \ldots, p-1$. By Lemma 3.13, there exist curves $\beta_0, \ldots, \beta_p$ in S with $\phi(\{\alpha_j, \alpha_k\}) = \{\beta_j, \beta_k\}$ for any distinct $j, k = 0, \ldots, p$. Choose two distinct 2-HBCs γ_1, γ_2 in S contained in the holed sphere cut off by $\{\alpha_0, \alpha_2\}$ from S. We now apply the hypothesis of the induction to the component of S_{γ_1} of positive genus. It then follows that each of $\{\beta_0, \beta_2\}$ and $\{\beta_j, \beta_{j+1}\}$ for any $j = 2, \ldots, p-1$ is a 1-HBP in the component of $S_{\phi(\gamma_1)}$ of positive genus and that $\{\beta_0, \beta_p\}$ is a $(p-1)$-HBP in that component.

Suppose that $\{\beta_j, \beta_{j+1}\}$ is a 2-HBP in S for some $j = 2, \ldots, p-1$. We choose a curve α in S such that $\{\alpha_0, \alpha\}$ is an HBP in S; and α intersects α_{j+1} and is disjoint from α_k for any $k \in \{0, \ldots, p\} \setminus \{j+1\}$. Note that $\{\alpha_0, \alpha\}$ is disjoint from γ_1 and γ_2. On the other hand, $\phi(\gamma_1)$ and $\phi(\gamma_2)$ fill the holed sphere cut off by the 2-HBP $\{\beta_j, \beta_{j+1}\}$ in S. Since the 2-simplex of $\mathscr{CP}(S)$ consisting of $\{\alpha_0, \alpha\}$, $\{\alpha_0, \alpha_1\}$ and $\{\alpha_0, \alpha_2\}$ is rooted, the HBP $\phi(\{\alpha_0, \alpha\})$ contains β_0. Another curve of $\phi(\{\alpha_0, \alpha\})$ intersects β_{j+1} and thus intersects $\phi(\gamma_1)$ or $\phi(\gamma_2)$. This is a contradiction.

We thus proved that $\{\beta_0, \beta_2\}$ is a 2-HBP in S and that $\{\beta_j, \beta_{j+1}\}$ is a 1-HBP in S for each $j = 2, \ldots, p-1$. It follows that $\{\beta_0, \beta_1\}$ and $\{\beta_1, \beta_2\}$ are 1-HBPs in S. For each $k = 1, \ldots, p$, the map ϕ therefore preserves non-separating k-HBPs in S.

If non-separating HBPs are replaced by separating HBPs in the above argument, then we can prove that ϕ preserves separating k-HBPs in S for each k. A verbatim proof shows that any automorphism of $\mathscr{CP}_s(S)$ preserves j-HBCs in S for each j, and then shows that it also preserves separating k-HBPs in S for each k. $\qquad\square$

4.   Construction of Φ in the case $p = 2$.

Let $S = S_{g,2}$ be a surface with $g \geq 2$. For an automorphism ϕ of $\mathscr{CP}(S)$, we define a map $\Phi\colon V(S) \to V(S)$ as in Section 2.4, which is shown to be well-defined throughout this section. In Sections 4.1 and 4.2, we study pentagons in $\mathscr{CP}(S)$ and hexagons in $\mathscr{CP}_s(S)$, respectively. They are used to show that Φ is well-defined on the set of non-separating curves in S and on the set of separating curves in S which are not HBCs in S, respectively. In Section 4.3, we prove that Φ is well-defined.
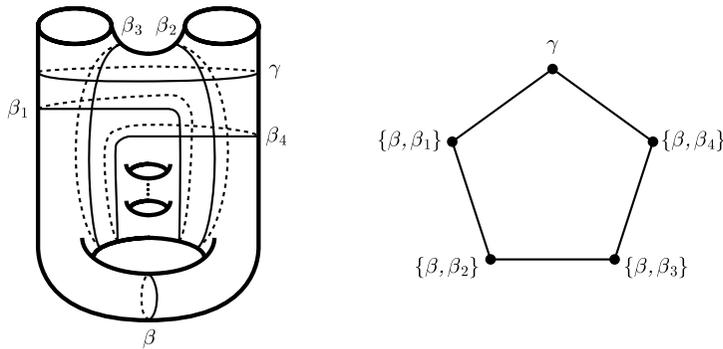
Figure 4.   A pentagon in $\mathscr{CP}(S)$.

4.1.   Pentagons in $\mathscr{CP}(S)$.

We mean by a *pentagon* in $\mathscr{CP}(S)$ the full subgraph of $\mathscr{CP}(S)$ spanned by five vertices $v_1, \ldots, v_5$ with $i(v_k, v_{k+1}) = 0$ and $i(v_k, v_{k+2}) \neq 0$ for each k mod 5 (see Figure 4). In this case, let us say that the pentagon is defined by the 5-tuple $(v_1, \ldots, v_5)$.

Let ϕ be an automorphism of $\mathscr{CP}(S)$, and pick a non-separating curve α in S. To define $\Phi(\alpha) \in V(S)$, we choose two disjoint and distinct HBPs $a_1 = \{\alpha, \alpha_1\}$ and $a_2 = \{\alpha, \alpha_2\}$ in S and have to show that the root curve of the edge $\{\phi(a_1), \phi(a_2)\}$ of $\mathscr{CP}(S)$ depends only on α. We connect any two edges of $\mathscr{CP}(S)$ consisting of two HBPs in S containing α by a sequence of pentagons in $\mathscr{CP}(S)$ such that

- each pentagon in that sequence is equal to the one described in Figure 4 up to a homeomorphism of S; and
- any two successive pentagons in that sequence share at least two HBPs.

In Lemma 4.7, considering the image of this sequence of pentagons via ϕ, we obtain the aforementioned assertion.

LEMMA 4.1.   *Let $S = S_{g,2}$ be a surface with $g \geq 2$. For each $k = 1, 2, 3$, let $a_k = \{\alpha, \alpha_k\}$ be a non-separating HBP in S such that $\{a_1, a_2\}$ and $\{a_2, a_3\}$ are edges of $\mathscr{CP}(S)$. Then there exists a sequence of pentagons in $\mathscr{CP}(S)$, $\Pi_1, \Pi_2, \ldots, \Pi_n$, satisfying the following: For each $k = 1, 2, \ldots, n$,*

- (i) *up to a homeomorphism of S, Π_k is equal to the pentagon in Figure 4 that is defined by a 5-tuple consisting of a 2-HBC, a 2-HBP, a 1-HBP, a 1-HBP and a 2-HBP in this order;*
- (ii) *any of the four HBPs of Π_k contains α;*
- (iii) *we have $a_1 \in \Pi_1$, $a_3 \in \Pi_n$ and $a_2 \in \Pi_k$; and*
- (iv) *if $k < n$, then Π_k and Π_{k+1} share at least two HBPs.*

Proof.   We first deal with the case where a_2 is a 2-HBP and subsequently the case where a_2 is a 1-HBP.

Let us assume that a_2 is a 2-HBP. Since any distinct two 2-HBPs in S intersect, a_1 and a_3 are both 1-HBPs. We can find a pentagon Π in $\mathscr{CP}(S)$ containing a_1 and a_2 as its vertices and equal to the one in Figure 4 up to a homeomorphism of S. Let b denote the 2-HBC in Π, and let Q denote the component of S_{a_2} of genus zero, which is homeomorphic to $S_{0,4}$. The set $V(Q)$ then contains α_1 and α_3. Let $h \in \mathrm{Mod}(S)$ be the half twist about b exchanging the two components of ∂S. Let $x \in \mathrm{Mod}(S)$ be the Dehn twist about α_1. Define Γ to be the subgroup of $\mathrm{Mod}(S)$ generated by h and x. Since a_2 is fixed by Γ, there exists a natural homomorphism $p\colon \Gamma \to \mathrm{Mod}(Q)$.

We denote by $\mathrm{Mod}(Q; \alpha, \alpha_2)$ the subgroup of $\mathrm{Mod}(Q)$ consisting of all elements that fix each of the two components of ∂Q corresponding to α and α_2. We show the equality $p(\Gamma) = \mathrm{Mod}(Q; \alpha, \alpha_2)$. The element $p(h)$ is a half twist about $b \in V(Q)$, and $p(x)$ is the Dehn twist about $\alpha_1 \in V(Q)$. We thus have the inclusion $p(\Gamma) < \mathrm{Mod}(Q; \alpha, \alpha_2)$. Since the Dehn twists about b and α_1 generate $\mathrm{PMod}(Q)$ and since $p(h)$ exchanges the two components of ∂S, we obtain the inclusion $\mathrm{Mod}(Q; \alpha, \alpha_2) < p(\Gamma)$. Our claim follows.

We put $H = \{h^{\pm 1}, x^{\pm 1}\}$. We note that α_1 and α_3 lie in $V(Q)$ and that α and α_2 are contained in distinct components of Q_{α_1} (resp. Q_{α_3}) as boundary components. It follows that α_1 and α_3 lie in the same orbit for the action of $\mathrm{Mod}(Q; \alpha, \alpha_2)$ on $V(Q)$. We can thus find $h_1, \ldots, h_n \in H$ with $\alpha_3 = p(h_1) \cdots p(h_n)\alpha_1$. Let us consider the following sequence of pentagons,

$$\Pi, \ h_1\Pi, \ h_1 h_2 \Pi, \ \ldots, \ h_1 h_2 \cdots h_n \Pi.$$

This sequence satisfies conditions (i), (ii) and (iii) in the lemma. We now prove that it satisfies condition (iv). Let c_1 and c_2 denote the two vertices of Π except a_1, a_2 and b so that c_j is a j-HBP for each $j = 1, 2$. It then follows that Π is the 5-tuple consisting of the five vertices b, a_2, a_1, c_1, c_2 in this order. Since a_2 and c_2 are disjoint from b, the three vertices a_2, c_2 and b are fixed by h. Since a_2 and c_1 are disjoint from a_1, the three vertices a_2, a_1 and c_1 are fixed by x. Our sequence of pentagons therefore satisfies condition (iv).

We next suppose that a_2 is a 1-HBP. Let R denote the component of S_{a_2} of positive genus, which is homeomorphic to $S_{g-1,3}$. Note that α_1 and α_3 are curves in R. We prove the existence of a sequence of pentagons in the lemma in the following three cases: (a) a_1 and a_3 are both 1-HBPs; (b) a_1 and a_3 are both 2-HBPs; and (c) one of a_1 and a_3 is a 1-HBP and another is a 2-HBP.

(a) Suppose that a_1 and a_3 are both 1-HBPs. Pick a pentagon Π containing a_1 and a_2 as its vertices and equal to the one in Figure 4 up to a homeomorphism
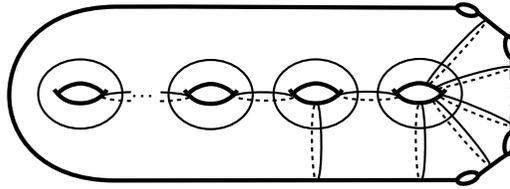
Figure 5.   PMod is generated by Dehn twists about these curves.

of S. Let b, c and d denote the 2-HBC of Π, the 2-HBP of Π disjoint from a_2 and the other 2-HBP of Π, respectively. It then follows that Π is defined by the 5-tuple consisting of the five vertices b, c, a_2, a_1 and d in this order. We put $c = \{\alpha, \gamma\}$.

Recall that in general, if X is a surface of positive genus described in Figure 5, then the set of Dehn twists about curves in Figure 5 generates $\mathrm{PMod}(X)$ (see [**8**]). One can then find a set of curves in R, denoted by U, satisfying the following two conditions (see Figure 6):

- The set of Dehn twists about curves in U generates $\mathrm{PMod}(R)$;
- There exists a curve δ_1 in U with $i(\alpha_1, \delta_1) \neq 0$ and $i(\alpha_1, \delta) = 0$ for any $\delta \in U \setminus \{\delta_1\}$; and
- There exists a curve δ_2 in U with $i(\gamma, \delta_2) \neq 0$ and $i(\gamma, \delta) = 0$ for any $\delta \in U \setminus \{\delta_2\}$.

Let T denote the subset of $\mathrm{PMod}(S)$ consisting of all Dehn twists about curves in U and their inverses, where curves in R are naturally identified with curves in S. Let Λ be the subgroup of $\mathrm{PMod}(S)$ generated by T, and define $q \colon \Lambda \to \mathrm{PMod}(R)$ as the natural homomorphism. Since α_1 and α_3 are both 2-HBCs in R cutting off a pair of pants containing α and a component of ∂S as boundary components, they lie in the same orbit for the action of $\mathrm{PMod}(R)$ on $V(R)$. Choose $g_1, \dots, g_m \in T$ with $\alpha_3 = q(g_1) \cdots q(g_m) \alpha_1$. The sequence of pentagons,

$$\Pi, \ g_1 \Pi, \ g_1 g_2 \Pi, \ \dots, \ g_1 g_2 \cdots g_m \Pi,$$

then satisfies conditions (i), (ii) and (iii) in the lemma. Note that a_1 is fixed by each element of T except t_{δ_1} and its inverse and that c is fixed by each element of T except t_{δ_2} and its inverse. Since a_2 is fixed by Λ, the above sequence of pentagons satisfies condition (iv).

(b) Suppose that a_1 and a_3 are both 2-HBPs. Pick a pentagon Π containing a_1 and a_2 as its vertices and equal to the one in Figure 4 up to a homeomorphism of S. Let c denote the 1-HBP of Π disjoint and distinct from a_2, and put $c = \{\alpha, \gamma\}$. Along an argument similar to that in case (a), we obtain a desired sequence of pentagons (see also Figure 6).
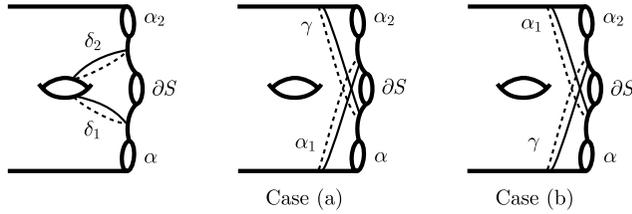
Figure 6.

(c) Finally, we assume that one of a_1 and a_3 is a 1-HBP and another is a 2-HBP. We may assume that a_1 is a 1-HBP and a_3 is a 2-HBP. Choose a pentagon Π containing a_1 and a_2 as its vertices and equal to the one in Figure 4 up to a homeomorphism of S. We denote by b, c and d the 2-HBC of Π, the 2-HBP of Π disjoint from a_2 and the other 2-HBP of Π, respectively. The pentagon Π is then defined by the 5-tuple consisting of b, c, a_2, a_1 and d in this order. We put $c = \{\alpha, \gamma\}$. Note that γ and α_3 are in the same orbit for the action of $\mathrm{PMod}(R)$ on $V(R)$. Along an argument of the same kind as in case (a) where α_1 and α_3 are in the same orbit for the action of $\mathrm{PMod}(R)$ on $V(R)$, we obtain the lemma. $\square$

Let us introduce terminology used in the subsequent proposition. Let $X = S_{g,p}$ be a surface with $p \geq 2$.

- Given two distinct components ∂_1 and ∂_2 of ∂X and a separating curve α in X, we say that α *separates* ∂_1 and ∂_2 if ∂_1 and ∂_2 are contained in distinct components of X_α.
- Let $\partial_1, \ldots, \partial_k$ be pairwise distinct components of ∂X. We say that an HBC β in X *encircles* $\partial_1, \ldots, \partial_k$ if β cuts off from S a holed sphere containing $\partial_1, \ldots, \partial_k$ and homeomorphic to $S_{0,k+1}$.

PROPOSITION 4.2.   *Let $X = S_{g,p}$ be a surface with $g \geq 1$ and $p \geq 4$, and let ∂_1 and ∂_2 be two distinct components of ∂X. Then the full subcomplex $\mathscr{D}$ of $\mathscr{C}(X)$ spanned by all vertices that correspond to HBCs in X separating ∂_1 and ∂_2 is connected.*

PROOF.   The proof is based on Lemma 2.1 of [20], which presents a technique to prove connectivity of a simplicial complex on which $\mathrm{PMod}(X)$ acts. Pick two distinct components ∂_3 and ∂_4 of ∂X other than ∂_1 and ∂_2.

CLAIM 4.3.   *For any HBC α in X separating ∂_1 and ∂_2, there exists a path in $\mathscr{D}$ connecting α to a vertex corresponding to an HBC in X encircling ∂_2 and ∂_3.*

PROOF.   If the component of X_α containing ∂_2 contains ∂_3, then one can find an HBC in X disjoint from α and encircling ∂_2 and ∂_3. Otherwise, one can find a path in $\mathscr{D}$, α, α_1, α_2, α_3, α_4, such that α_1 encircles ∂_1 and ∂_3; α_2 encircles ∂_1, ∂_3 and ∂_4; α_3 encircles ∂_1 and ∂_4; and α_4 encircles ∂_2 and ∂_3. $\qquad\square$

CLAIM 4.4.   *For any two HBCs α and β in X encircling ∂_2 and ∂_3, there exists a path in $\mathscr{D}$ connecting α and β.*

PROOF.   Let U be the set of curves in X described in Figure 5. As already mentioned in the proof of Lemma 4.1, $\mathrm{PMod}(X)$ is generated by Dehn twists about curves in U. After labeling $\partial_1, \ldots, \partial_p$ appropriately in Figure 5, one can find an HBC γ in X encircling ∂_2 and ∂_3 such that

- there exists a unique curve δ in U intersecting γ; and
- there exists an HBC γ' in X which separates ∂_1 and ∂_2 and is disjoint from γ and δ.

The vertices $t_\delta \gamma$, γ' and γ of $\mathscr{D}$ form a path in $\mathscr{D}$. Note that $\mathrm{PMod}(X)$ sends γ to any HBC in X encircling ∂_2 and ∂_3. The claim is now obtained in the same way as in the construction of a sequence of pentagons in the proof of Lemma 4.1. $\qquad\square$

The proposition follows from Claims 4.3 and 4.4. $\qquad\square$

LEMMA 4.5.   *Let $Y = S_{g,p}$ be a surface with $g \geq 2$ and $p \geq 2$, and pick a non-separating curve α in Y. Then the full subcomplex of $\mathscr{CP}(Y)$ spanned by all vertices that correspond to HBPs in Y containing α is connected.*

PROOF.   We set $X = Y_\alpha$, which is homeomorphic to $S_{g-1,p+2}$. Using the natural one-to-one correspondence between HBPs in Y containing α and HBCs in X which separates the two boundary components of X corresponding to α, one can deduce the lemma from Proposition 4.2. $\qquad\square$

The following lemma gives information on the image of the pentagon in Figure 4 via an automorphism ϕ of $\mathscr{CP}(S)$.

LEMMA 4.6.   *Let $S = S_{g,2}$ be a surface with $g \geq 2$. Let (a, b_2, b_1, c_1, c_2) be a 5-tuple defining a pentagon Π in $\mathscr{CP}(S)$ such that*

- *b_j and c_j are non-separating j-HBPs in S for each $j = 1, 2$; and*
- *each of the three edges $\{b_2, b_1\}$, $\{b_1, c_1\}$ and $\{c_1, c_2\}$ is rooted.*

Then the root curves of the three edges in the second condition are equal.

PROOF.   Let α, β and γ be the root curves of the edges $\{b_2, b_1\}$, $\{b_1, c_1\}$ and $\{c_1, c_2\}$, respectively. Because of $i(b_1, c_1) = 0$, the equality $i(\alpha, \beta) = i(\beta, \gamma) =$

$i(\gamma, \alpha) = 0$ holds. We deduce a contradiction by assuming that either (a) α, β and γ are mutually distinct; (b) $\alpha = \beta \neq \gamma$ or $\beta = \gamma \neq \alpha$ holds; or (c) $\alpha = \gamma \neq \beta$ holds.

(a) We assume that α, β and γ are mutually distinct. The equalities $b_1 = \{\alpha, \beta\}$ and $c_1 = \{\beta, \gamma\}$ then holds. The HBP $\{\alpha, \gamma\}$ is a 2-HBP because b_1 and c_1 are both 1-HBPs. Since b_1 intersects a and since b_2 is disjoint from a and contains α, we see that β intersects a. Put $b_2 = \{\alpha, \alpha_2\}$ and denote by R the component of S_{b_2} of genus zero, which is homeomorphic to $S_{0,4}$. Note that a (or a curve in a if a is an HBP in S) and β lie in R and fill R. Since γ is disjoint from a and β, the equality $\alpha_2 = \gamma$ holds. This is a contradiction because the equality implies that b_2 and c_1 are disjoint.

(b) We suppose $\alpha = \beta \neq \gamma$. The equality $c_1 = \{\beta, \gamma\}$ then holds. It follows from $i(b_2, a) = i(c_2, a) = 0$ that we have $i(\alpha, a) = i(\gamma, a) = 0$ and thus $i(c_1, a) = 0$. This is a contradiction. We can also deduce a contradiction in the same manner if $\beta = \gamma \neq \alpha$ holds.

(c) If $\alpha = \gamma \neq \beta$, then we have $b_1 = \{\alpha, \beta\} = c_1$. This is a contradiction. $\square$

LEMMA 4.7.   *Let $S = S_{g,2}$ be a surface with $g \geq 2$, and let ϕ be an automorphism of $\mathscr{CP}(S)$. For each $k = 1, 2, 3, 4$, let $a_k = \{\alpha, \alpha_k\}$ be a non-separating HBP in S such that $\{a_1, a_2\}$ and $\{a_3, a_4\}$ are edges of $\mathscr{CP}(S)$. Then the root curves of the two edges $\{\phi(a_1), \phi(a_2)\}$ and $\{\phi(a_3), \phi(a_4)\}$ of $\mathscr{CP}(S)$ are equal.*

PROOF.   By Lemma 4.5, there exists a sequence of non-separating HBPs in S, $a_2 = b_2, b_3, \ldots, b_{m-1} = a_3$, such that any two successive HBPs are disjoint and each b_k contains α. We set $b_1 = a_1$ and $b_m = a_4$. For each $k = 1, \ldots, m-2$, if we have $i(b_k, b_{k+2}) \neq 0$, then there exists a sequence of pentagons in $\mathscr{CP}(S)$ connecting the two edges $\{b_k, b_{k+1}\}$, $\{b_{k+1}, b_{k+2}\}$ and satisfying conditions in Lemma 4.1. Otherwise at least two of b_k, b_{k+1} and b_{k+2} are equal. We then obtain a sequence of pentagons in $\mathscr{CP}(S)$, $\Pi_1, \Pi_2, \ldots, \Pi_n$, with $a_1, a_2 \in \Pi_1$ and $a_3, a_4 \in \Pi_n$. By Lemma 4.6, $\phi(\Pi_k)$ is a pentagon consisting of one HBC and four HBPs sharing a single non-separating curve in S. Since $\phi(\Pi_k)$ and $\phi(\Pi_{k+1})$ share at least two HBPs for each k, the non-separating curve shared by all HBPs of $\phi(\Pi_k)$ is equal to that of $\phi(\Pi_{k+1})$. The lemma follows. $\square$

4.2.   Hexagons and squares in $\mathscr{CP}_s(S)$.

We mean by a *hexagon* in $\mathscr{CP}_s(S)$ the full subgraph of $\mathscr{CP}_s(S)$ spanned by six vertices $v_1, \ldots, v_6$ with $i(v_k, v_{k+1}) = 0$, $i(v_k, v_{k+2}) \neq 0$ and $i(v_k, v_{k+3}) \neq 0$ for each $k \mod 6$ (see Figure 7). In this case, let us say that the hexagon is defined by the 6-tuple $(v_1, \ldots, v_6)$.

Similarly, we mean by a *square* in $\mathscr{CP}_s(S)$ the full subgraph of $\mathscr{CP}_s(S)$ spanned by four vertices $v_1, \ldots, v_4$ with $i(v_k, v_{k+1}) = 0$ and $i(v_k, v_{k+2}) \neq 0$ for
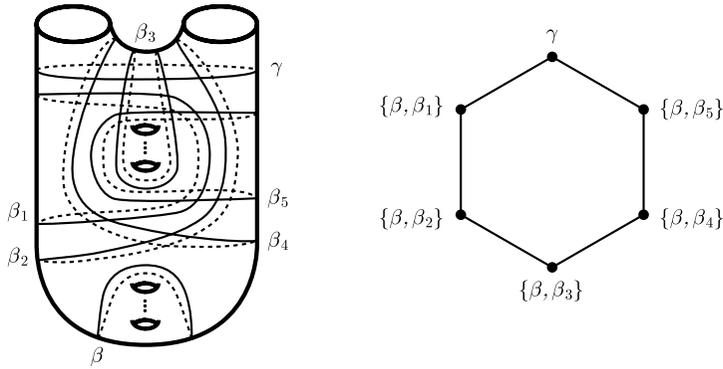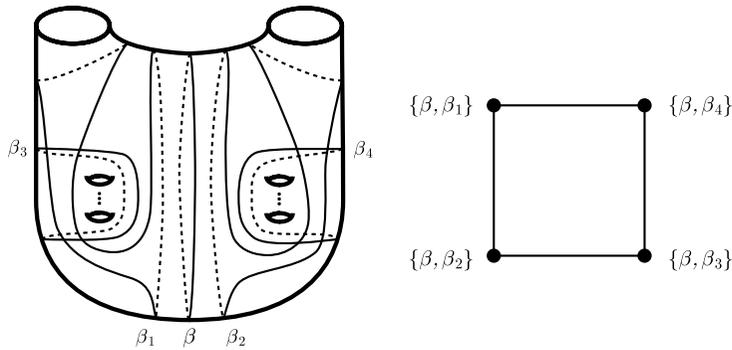
Figure 7.   A hexagon in $\mathscr{CP}_s(S)$.

Figure 8.   A square in $\mathscr{CP}_s(S)$.

each k mod 4 (see Figure 8). In this case, we say that the square is defined by the 4-tuple $(v_1, \ldots, v_4)$.

Hexagons and squares in $\mathscr{CP}_s(S)$ are used to show that $\Phi(\alpha)$ is well-defined for each separating curve α in S which is not an HBC in S. A conclusion similar to Lemma 4.7 for such an α is established in Lemma 4.14. To prove it, we connect any two edges of $\mathscr{CP}_s(S)$ consisting of two HBPs in S containing α by a sequence of hexagons and squares in $\mathscr{CP}_s(S)$.

LEMMA 4.8.   *Let $S = S_{g,2}$ be a surface with $g \geq 2$. For each $k = 1, 2, 3$, let $a_k = \{\alpha, \alpha_k\}$ be a separating HBP in S such that $\{a_1, a_2\}$ and $\{a_2, a_3\}$ are edges of $\mathscr{CP}_s(S)$. We assume that ∂S is contained in a single component of S_α. Then there exists a sequence of hexagons in $\mathscr{CP}_s(S)$, $\Pi_1, \Pi_2, \ldots, \Pi_n$, satisfying the following: For each $k = 1, 2, \ldots, n$,*
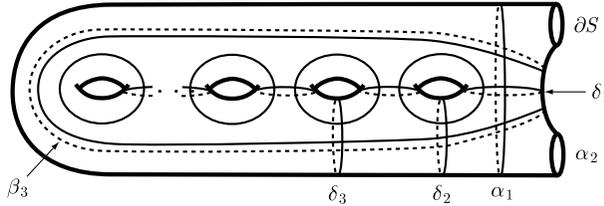
Figure 9.

(i) Π_k *is defined by a 6-tuple consisting of a 2-HBC, a 2-HBP, a 1-HBP, a*
       2-HBP, a 1-HBP and a 2-HBP in this order;
(ii) *any of the five HBPs of Π_k contains α;*
(iii) *$a_1 \in \Pi_1$, $a_3 \in \Pi_n$ and $a_2 \in \Pi_k$;*
(iv) *if $k < n$, then Π_k and Π_{k+1} share at least two HBPs.*

Proof.   We find a desired sequence of hexagons in the following two cases:
(a) a_2 is a 2-HBP; and (b) a_2 is a 1-HBP.

(a) If a_2 is a 2-HBP, then a_1 and a_3 are 1-HBPs since any two distinct separating 2-HBPs in S intersect. We denote by Q the component of S_{a_2} homeomorphic to $S_{0,4}$. Since we have $i(\alpha_1, \alpha_2) = i(\alpha_3, \alpha_2) = 0$, α_1 and α_3 are elements of $V(Q)$. As in Figure 7, we can find a hexagon Π defined by a 6-tuple $(v_1, \ldots, v_6)$ with v_1 a 2-HBC; $v_2 = a_2$; and $v_3 = a_1$. We note that v_4 is a 2-HBP; v_5 is a 1-HBP; and v_6 is a 2-HBP. The vertex v_1 lies in $V(Q)$ since it is disjoint from $v_2 = a_2$. Let $h \in \mathrm{Mod}(S)$ be the half twist about v_1 exchanging the two components of ∂S. Let $x \in \mathrm{Mod}(S)$ be the Dehn twist about α_1. Define Γ to be the subgroup of $\mathrm{Mod}(S)$ generated by h and x. Since a_2 is fixed by Γ, there exists a natural homomorphism $p \colon \Gamma \to \mathrm{Mod}(Q)$. We denote by $\mathrm{Mod}(Q; \alpha, \alpha_2)$ the subgroup of $\mathrm{Mod}(Q)$ consisting of all elements that fix each of the two components of ∂Q corresponding to α and α_2. As in the proof of Lemma 4.1, we obtain the equality $p(\Gamma) = \mathrm{Mod}(Q; \alpha, \alpha_2)$. Since α and α_2 are contained in distinct components of Q_{α_1} (resp. Q_{α_3}), α_3 lies in the orbit of α_1 for the action of $\mathrm{Mod}(Q; \alpha, \alpha_2)$ on $V(Q)$. Setting $H = \{h^{\pm 1}, x^{\pm 1}\}$, we can thus find $h_1, \ldots, h_n \in H$ with $\alpha_3 = p(h_1) \cdots p(h_n)\alpha_1$. Along an argument of the same kind as in Lemma 4.1, we obtain a desired sequence of hexagons.

(b) We next suppose that a_2 is a 1-HBP. Since ∂S is contained in a single component of S_α, both a_1 and a_3 are 2-HBPs. Let R denote the component of S_{a_2} of positive genus and containing a component of ∂S. Note that the number of boundary components of R is equal to two and that α_1 and α_3 are elements of $V(R)$. As in Figure 7, we can find a hexagon Π defined by a 6-tuple $(v_1, \ldots, v_6)$ with $v_1 = a_1$; $v_2 = a_2$; and v_6 a 2-HBC. Note that v_3 is a 2-HBP; v_4 is a 1-HBP; and v_5 is a 2-HBP. We put $v_3 = \{\alpha, \beta_3\}$. It then follows that β_3 lies in $V(R)$.

Let U be the set of the curves in R described in Figure 9 other than α_1 and

β_3, which satisfies the following two conditions:

- The set of Dehn twists about curves in U generates $\mathrm{PMod}(R)$; and
- The curves δ_1, δ_2 and δ_3 in U satisfy $i(\alpha_1, \delta_1) \neq 0$ and $i(\alpha_1, \delta) = 0$ for any $\delta \in U \setminus \{\delta_1\}$; and $i(\beta_3, \delta_2) \neq 0$, $i(\beta_3, \delta_3) \neq 0$ and $i(\beta_3, \delta) = 0$ for any $\delta \in U \setminus \{\delta_2, \delta_3\}$.

Note that α_1 and α_3 lie in the same orbit for the action of $\mathrm{PMod}(R)$ on $V(R)$. Along an argument of the same kind as in Lemma 4.1, we obtain a desired sequence of hexagons. $\square$

LEMMA 4.9.   *Let $S = S_{g,2}$ be a surface with $g \geq 2$. For each $k = 1, 2, 3$, let $a_k = \{\alpha, \alpha_k\}$ be a separating HBP in S such that $\{a_1, a_2\}$ and $\{a_2, a_3\}$ are distinct edges of $\mathscr{CP}_s(S)$. We assume that each component of S_α contains a component of ∂S. Then there exists a square Π in $\mathscr{CP}_s(S)$ such that*

- *a_1, a_2 and a_3 are vertices of Π; and*
- *the other vertex of Π is a 1-HBP in S containing α.*

PROOF.   We first note that for each $k = 1, 2, 3$, a_k is a 1-HBP in S because each component of S_α contains a component of ∂S. It follows that α_1 and α_3 lie in the same component of S_α and that α_2 lies in another component of S_α, denoted by R. Choose a curve α_4 in R with $i(\alpha_4, \alpha_2) \neq 0$ and $a_4 = \{\alpha, \alpha_4\}$ a 1-HBP in S. The 4-tuple (a_1, a_2, a_3, a_4) then defines a square in $\mathscr{CP}_s(S)$. $\square$

The following lemma is a variant of Lemma 4.5 for separating HBPs.

LEMMA 4.10.   *Let $X = S_{g,p}$ be a surface with $g \geq 2$ and $p \geq 2$, and pick a separating curve α in X which is not an HBC in X. Then the full subcomplex $\mathscr{E}$ of $\mathscr{CP}_s(X)$ spanned by all vertices that correspond to separating HBPs in X containing α is connected.*

To prove this lemma, we need the following:

PROPOSITION 4.11.   *Let $Y = S_{g,p}$ be a surface with $g \geq 1$ and $p \geq 3$, and choose a component ∂ of ∂Y. Then the full subcomplex $\mathscr{F}$ of $\mathscr{C}(Y)$ spanned by all vertices that correspond to HBCs in Y cutting off a holed sphere containing ∂ from Y is connected.*

PROOF.   The proof is based on Lemma 2.1 in [20] as in the proof of Proposition 4.2. Label components of ∂Y as $\partial_1, \ldots, \partial_p$ with $\partial = \partial_1$. We first claim that for any curve β in Y corresponding to a vertex of $\mathscr{F}$, there exists a path in $\mathscr{F}$ connecting β to an HBC in Y encircling ∂_1 and ∂_2. If the holed sphere cut off by β from Y contains ∂_2, then one can find an HBC in Y disjoint from β and encircling
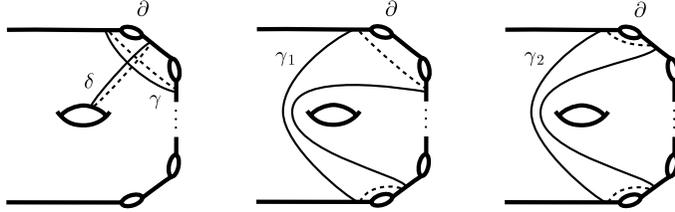
Figure 10.

∂_1 and ∂_2. Otherwise, choose $j \in \{3, \ldots, p\}$ so that ∂_j is contained in the holed sphere cut off by β from Y. One can then find a path in $\mathscr{F}$, β, β_1, β_2, β_3, such that β_1 encircles ∂_1 and ∂_j; β_2 encircles ∂_1, ∂_2 and ∂_j; and β_3 encircles ∂_1 and ∂_2. The claim follows.

To prove the proposition, it suffices to show that if γ and δ are the curves described in Figure 10, then $t_\delta\gamma$ and γ are connected by a path in $\mathscr{F}$. Choose the two HBCs γ_1, γ_2 in Y described in Figure 10. The sequence γ, γ_1, $\gamma_2 = t_\delta\gamma_2$, $t_\delta\gamma_1$, $t_\delta\gamma$ is then a desired path in $\mathscr{F}$. □

PROOF OF LEMMA 4.10. Let $a_1 = \{\alpha, \alpha_1\}$ and $a_2 = \{\alpha, \alpha_2\}$ be vertices of $\mathscr{E}$. To find a path in $\mathscr{E}$ connecting a_1 and a_2, we may assume $i(a_1, a_2) \neq 0$. Note that α_1 and α_2 lie in the same component of X_α. Let R denote that component of X_α and R' denote another component of X_α.

If R' contains a component of ∂X, then there exists a curve β in R' with $\{\alpha, \beta\}$ an HBP in X. Since $\{\alpha, \beta\}$ is then disjoint from a_1 and a_2, we obtain the path a_1, $\{\alpha, \beta\}$, a_2 in $\mathscr{E}$.

Suppose that R' contains no component of ∂X. It then follows that R contains at least two components of ∂X. By Proposition 4.11, there exists a sequence $\alpha_1 = \beta_1, \beta_2, \ldots, \beta_n = \alpha_2$ of HBCs in R cutting off from R a holed sphere containing the component of ∂R corresponding to α. The pair $\{\alpha, \beta_k\}$, denoted by b_k, is an HBP in X, and the sequence $a_1 = b_1, b_2, \ldots, b_n = a_2$ is thus a path in $\mathscr{E}$. □

The following two lemmas give us information on the images of the hexagons and squares constructed in Lemmas 4.8 and 4.9, respectively, via an automorphism ϕ of $\mathscr{CP}_s(S)$.

LEMMA 4.12. *Let $S = S_{g,2}$ be a surface with $g \geq 2$. Let $(v_1, \ldots, v_6)$ be a 6-tuple defining a hexagon in $\mathscr{CP}_s(S)$ such that*

- *v_1, v_3 and v_5 are 2-HBPs; v_2 and v_4 are 1-HBPs; and*
- *each of the four edges $\{v_1, v_2\}$, $\{v_2, v_3\}$, $\{v_3, v_4\}$ and $\{v_4, v_5\}$ is rooted.*

Then the root curves of the four edges in the second condition are equal.

PROOF.   We note that in general, if $\{u_1, u_2\}$ and $\{u_2, u_3\}$ are rooted edges of $\mathscr{CP}_s(S)$ such that u_1 and u_3 are 2-HBPs and u_2 is a 1-HBP, then the root curves of $\{u_1, u_2\}$ and $\{u_2, u_3\}$ are equal. This fact implies that the root curves of $\{v_1, v_2\}$ and $\{v_2, v_3\}$ are equal and denoted by α. Similarly, the root curves of $\{v_3, v_4\}$ and $\{v_4, v_5\}$ are equal and denoted by β. The equality $i(v_6, \alpha) = i(v_6, \beta) = 0$ then holds because we have $\alpha \in v_1$ and $\beta \in v_5$. If $\alpha \neq \beta$, then the equality $v_3 = \{\alpha, \beta\}$ would hold. This contradicts $i(v_3, v_6) \neq 0$. We thus have $\alpha = \beta$. $\square$

LEMMA 4.13.   *Let $S = S_{g,2}$ be a surface with $g \geq 2$. Let $(v_1, \ldots, v_4)$ be a 4-tuple defining a square in $\mathscr{CP}_s(S)$ such that*

- *each of $v_1, \ldots, v_4$ is an HBP; and*
- *each of the four edges $\{v_1, v_2\}$, $\{v_2, v_3\}$, $\{v_3, v_4\}$ and $\{v_4, v_1\}$ is rooted.*

Then the root curves of the four edges in the second condition are equal.

PROOF.   Let α and β denote the root curves of $\{v_1, v_2\}$ and $\{v_4, v_1\}$, respectively. If $\alpha \neq \beta$, then we would have $v_1 = \{\alpha, \beta\}$. Since v_2 and v_4 are disjoint from v_3, so is v_1. This is a contradiction. The lemma is obtained by repeating this argument. $\square$

LEMMA 4.14.   *Let $S = S_{g,2}$ be a surface with $g \geq 2$, and let ϕ be an automorphism of $\mathscr{CP}_s(S)$. For each $k = 1, 2, 3, 4$, let $a_k = \{\alpha, \alpha_k\}$ be a separating HBP in S such that $\{a_1, a_2\}$ and $\{a_3, a_4\}$ are edges of $\mathscr{CP}_s(S)$. Then the root curves of the two edges $\{\phi(a_1), \phi(a_2)\}$ and $\{\phi(a_3), \phi(a_4)\}$ of $\mathscr{CP}_s(S)$ are equal.*

PROOF.   Using Lemma 4.10, one can find a path in $\mathscr{CP}_s(S)$, $a_2 = b_2, b_3, \ldots, b_{m-1} = a_3$, such that each b_k is an HBP containing α. We put $b_1 = a_1$ and $b_m = a_4$. For each $k = 1, \ldots, m-2$, if $i(b_k, b_{k+2}) = 0$, then at least two of b_k, b_{k+1} and b_{k+2} are equal. If $i(b_k, b_{k+2}) \neq 0$, we apply either Lemma 4.8 or Lemma 4.9 to b_k, b_{k+1} and b_{k+2}. We then obtain a sequence $\Pi_1, \Pi_2, \ldots, \Pi_n$ of hexagons or squares in $\mathscr{CP}_s(S)$ such that we have $a_1, a_2 \in \Pi_1$ and $a_3, a_4 \in \Pi_n$; and for each k, Π_k and Π_{k+1} share at least two HBPs. It follows from Lemmas 4.12 and 4.13 that for each k, there exists a curve shared by all HBPs in $\phi(\Pi_k)$. Since $\phi(\Pi_k)$ and $\phi(\Pi_{k+1})$ share at least two HBPs, the curve shared by all HBPs of $\phi(\Pi_k)$ is equal to that of $\phi(\Pi_{k+1})$. The root curve of $\{\phi(a_1), \phi(a_2)\}$ is therefore equal to that of $\{\phi(a_3), \phi(a_4)\}$. $\square$

4.3.   Definition of Φ.

Let $S = S_{g,2}$ be a surface with $g \geq 2$, and let ϕ be an automorphism of $\mathscr{CP}(S)$. We define a bijection Φ from $V(S)$ onto itself as follows. If α is an HBC in S, then we set $\Phi(\alpha) = \phi(\alpha)$.

If β is a non-separating curve in S, then choose disjoint and distinct curves

β_1, β_2 in S such that $\{\beta, \beta_1\}$ and $\{\beta, \beta_2\}$ are both HBPs in S, and define $\Phi(\beta)$ to be the root curve of the edge in $\mathscr{CP}(S)$ consisting of $\phi(\{\beta, \beta_1\})$ and $\phi(\{\beta, \beta_2\})$. This is well-defined by Lemma 4.7.

In a similar way, if γ is a separating curve in S which is not an HBC in S, then choose disjoint and distinct curves γ_1, γ_2 in S such that $\{\gamma, \gamma_1\}$ and $\{\gamma, \gamma_2\}$ are both HBPs in S, and define $\Phi(\gamma)$ to be the root curve of the edge in $\mathscr{CP}(S)$ consisting of $\phi(\{\gamma, \gamma_1\})$ and $\phi(\{\gamma, \gamma_2\})$. Lemma 4.14 shows that this is well-defined since ϕ induces an automorphism of $\mathscr{CP}_s(S)$ by Lemmas 3.9 and 3.14.

We thus obtain a map $\Phi\colon V(S) \to V(S)$. Considering ϕ^{-1}, we see that Φ is a bijection. Note that if ψ is an automorphism of $\mathscr{CP}_s(S)$, then we obtain a bijection $\Psi\colon V_s(S) \to V_s(S)$ by applying the argument in the previous paragraph.

5. Construction of Φ in the case $p \geq 3$.

Let $S = S_{g,p}$ be a surface with $g \geq 2$ and $p \geq 3$. For an automorphism ϕ of $\mathscr{CP}(S)$, we define a map $\Phi\colon V(S) \to V(S)$ as in Section 2.4. Most of this section is devoted to showing that Φ is well-defined as in the previous section. To define $\Phi(\alpha) \in V(S)$ for each curve α in S which is not an HBC in S, choosing any two edges e_1, e_2 of $\mathscr{CP}(S)$ consisting of two HBPs containing α, we have to show that the root curves of $\phi(e_1)$ and $\phi(e_2)$ are equal. To show it, we connect e_1 and e_2 by a sequence of rooted 2-simplices of $\mathscr{CP}(S)$ whose root curve is equal to α. Note that rooted 2-simplices of $\mathscr{CP}(S)$ exist thanks to the assumption $p \geq 3$.

LEMMA 5.1. *Let $S = S_{g,p}$ be a surface with $g \geq 2$ and $p \geq 3$. For each $k = 1, 2, 3$, let $a_k = \{\alpha, \alpha_k\}$ be a non-separating HBP in S such that $\{a_1, a_2\}$ and $\{a_2, a_3\}$ are edges of $\mathscr{CP}(S)$. Then there exists a sequence of non-separating HBPs in S, $a_1 = b_1, b_2, \ldots, b_n = a_3$, such that for each $k = 1, 2, \ldots, n-1$, the set $\{a_2, b_k, b_{k+1}\}$ is a rooted 2-simplex of $\mathscr{CP}(S)$ whose root curve is equal to α.*

To prove this lemma, we use the following:

PROPOSITION 5.2 ([**15**, Proposition 4.4]). *Let $X = S_{0,p}$ be a surface with $p \geq 5$, and choose two distinct components ∂_1, ∂_2 of ∂X. Then the full subcomplex of $\mathscr{C}(X)$ spanned by all vertices that correspond to curves in X separating ∂_1 and ∂_2 is connected.*

PROOF OF LEMMA 5.1. If $i(a_1, a_3) = 0$, then the lemma obviously holds. We assume $i(a_1, a_3) \neq 0$. Let R denote the component of S_{a_2} of genus zero, and let R' denote another component of S_{a_2}. Because of $i(a_1, a_2) = i(a_2, a_3) = 0$ and $i(a_1, a_3) \neq 0$, either (a) $\alpha_1, \alpha_3 \in V(R)$; or (b) $\alpha_1, \alpha_3 \in V(R')$ occurs.

(a) Suppose $\alpha_1, \alpha_3 \in V(R)$. We note that R contains at least two components of ∂S. If R contains at least three components of ∂S, then we can find a desired

sequence of HBPs in S by using Proposition 5.2.

We now suppose that R contains exactly two components of ∂S. It then follows that R' contains at least one component of ∂S. There exists a curve β_2 in R' with $\{\alpha, \beta_2\}$ an HBP in S. The sequence $a_1, \{\alpha, \beta_2\}, a_3$ is a desired one.

(b) Suppose $\alpha_1, \alpha_3 \in V(R')$. If a_2 is not a 1-HBP in S, then we can find a curve β_2 in R with $\{\alpha, \beta_2\}$ an HBP in S. The sequence $a_1, \{\alpha, \beta_2\}, a_3$ is a desired one.

If a_2 is a 1-HBP in S, then R' contains at least two components of ∂S and thus has at least four boundary components. Note that both α_1 and α_3 are HBCs in R' which separate α and α_2 as curves in R'. By Proposition 4.2, there exists a sequence $\alpha_1 = \gamma_1, \gamma_2, \ldots, \gamma_n = \alpha_3$ of HBCs in R' such that each γ_j separates α and α_2; and any two successive HBCs in that sequence are disjoint and distinct. For each $k = 1, 2, \ldots, n$, the pair $\{\alpha, \gamma_k\}$, denoted by b_k, is an HBP in S. The sequence $a_1 = b_1, b_2, \ldots, b_n = a_3$ then satisfies the condition in the lemma. $\qquad\square$

The following lemma is an analogue of Lemma 5.1, dealing with separating HBPs in S in place of non-separating ones.

LEMMA 5.3. *Let $S = S_{g,p}$ be a surface with $g \geq 2$ and $p \geq 3$. For each $k = 1, 2, 3$, let $a_k = \{\alpha, \alpha_k\}$ be a separating HBP in S such that $\{a_1, a_2\}$ and $\{a_2, a_3\}$ are edges of $\mathscr{CP}_s(S)$. Then there exists a sequence of separating HBPs in S, $a_1 = b_1, b_2, \ldots, b_n = a_3$, such that for each $k = 1, 2, \ldots, n-1$, the set $\{a_2, b_k, b_{k+1}\}$ is a rooted 2-simplex of $\mathscr{CP}_s(S)$ whose root curve is equal to α.*

PROOF. If $i(a_1, a_3) = 0$, then the lemma obviously holds. We assume $i(a_1, a_3) \neq 0$. We denote by R, R' and R'' the three components of S_{a_2} so that R is of genus zero, R' and R'' are of positive genus; and R' (resp. R'') has the boundary component corresponding to α (resp. α_2). Note that either (a) $\alpha_1, \alpha_3 \in V(R)$; (b) $\alpha_1, \alpha_3 \in V(R')$; or (c) $\alpha_1, \alpha_3 \in V(R'')$ occurs.

(a) Suppose $\alpha_1, \alpha_3 \in V(R)$. We note that R contains at least two components of ∂S. If R contains at least three components of ∂S, then a desired sequence of HBPs can be obtained as an application of Proposition 5.2 since as a curve in R, each of α_1 and α_3 separates the two components of ∂R corresponding to α and α_2.

If R contains exactly two components of ∂S, then R' or R'' contains at least one component of ∂S. Using this component of ∂S, we can find a curve β_2 in R' or R'' with $\{\alpha, \beta_2\}$ an HBP in S. The sequence $a_1, \{\alpha, \beta_2\}, a_3$ is then a desired one.

(b) Suppose $\alpha_1, \alpha_3 \in V(R')$. If a_2 is not a 1-HBP in S, then pick a curve β_2 in R with $\{\alpha, \beta_2\}$ an HBP in S. The sequence $a_1, \{\alpha, \beta_2\}, a_3$ is a desired one.

Assume that a_2 is a 1-HBP in S. If R'' contains at least one component of

∂S, then there exists a curve γ_2 in R'' with $\{\alpha, \gamma_2\}$ an HBP in S. The sequence a_1, $\{\alpha, \gamma_2\}$, a_3 is then a desired one. If R'' contains no component of ∂S, then R' contains at least two components of ∂S and thus has at least three boundary components. Note that α_1 and α_3 are HBCs in R' cutting off from R' a holed sphere containing α as a boundary component. It follows from Proposition 4.11 that there exists a sequence $\alpha_1 = \delta_1, \delta_2, \ldots, \delta_n = \alpha_3$ of HBCs in R' such that each δ_k cuts off from R' a holed sphere containing α as a boundary component; and any two successive HBCs in that sequence are disjoint and distinct. Since for each $k = 1, 2, \ldots, n$, the pair $\{\alpha, \delta_k\}$, denoted by b_k, is an HBP in S, the sequence $a_1 = b_1, b_2, \ldots, b_n = a_3$ satisfies the condition of the lemma.

(c) If $\alpha_1, \alpha_3 \in V(R'')$, then we can find a desired sequence of HBPs in essentially the same way as in case (b). $\qquad\square$

Let $S = S_{g,p}$ be a surface with $g \geq 2$ and $p \geq 3$, and let ϕ be an automorphism of $\mathscr{CP}(S)$. We define a map $\Phi\colon V(S) \to V(S)$ in the same way as in Section 4.3. This is well-defined thanks to the following:

LEMMA 5.4.   *Let $S = S_{g,p}$ be a surface with $g \geq 2$ and $p \geq 3$. Let ϕ be an automorphism of $\mathscr{CP}(S)$, and for each $k = 1, 2, 3, 4$, let $a_k = \{\alpha, \alpha_k\}$ be an HBP in S such that $\{a_1, a_2\}$ and $\{a_3, a_4\}$ are edges of $\mathscr{CP}(S)$. Then the root curves of the two edges $\{\phi(a_1), \phi(a_2)\}$ and $\{\phi(a_3), \phi(a_4)\}$ of $\mathscr{CP}(S)$ are equal.*

PROOF.   By Lemmas 4.5 and 4.10, there exists a sequence of HBPs in S, $a_1 = b_1, a_2 = b_2, b_3, \ldots, b_{n-1} = a_3, b_n = a_4$, such that for each k, we have $i(b_k, b_{k+1}) = 0$, $b_k \neq b_{k+1}$ and $\alpha \in b_k$. Applying Lemmas 5.1 and 5.3 to the three HBPs b_{k-1}, b_k and b_{k+1} for each $k = 2, \ldots, n-1$, we can find a sequence of HBPs in S, $b_{k-1} = b_k^1, b_k^2, \ldots, b_k^{m_k} = b_{k+1}$, such that for each $l = 1, \ldots, m_k - 1$, the set $\{b_k, b_k^l, b_k^{l+1}\}$, denoted by σ_k^l, is a rooted 2-simplex of $\mathscr{CP}(S)$ whose root curve is equal to α. For any k and l, each of $\sigma_k^l \cap \sigma_k^{l+1}$ and $\sigma_k^{m_k-1} \cap \sigma_{k+1}^1$ contains at least two HBPs. It turns out that the root curve of the edge $\{\phi(a_1), \phi(a_2)\}$ is equal to that of the 2-simplex $\phi(\sigma_k^l)$ for any k and l and is thus equal to that of the edge $\{\phi(a_3), \phi(a_4)\}$. $\qquad\square$

Exchanging symbols appropriately in the above proof, we obtain the following lemma. For any automorphism ψ of $\mathscr{CP}_s(S)$, if we define a map $\Psi\colon V_s(S) \to V_s(S)$ in the same way as in Section 4.3, then the lemma shows that Ψ is well-defined.

LEMMA 5.5.   *Let $S = S_{g,p}$ be a surface with $g \geq 2$ and $p \geq 3$. Let ψ be an automorphism of $\mathscr{CP}_s(S)$, and for each $k = 1, 2, 3, 4$, let $b_k = \{\beta, \beta_k\}$ be a separating HBP in S such that $\{b_1, b_2\}$ and $\{b_3, b_4\}$ are edges of $\mathscr{CP}_s(S)$. Then the root curves of the two edges $\{\psi(b_1), \psi(b_2)\}$ and $\{\psi(b_3), \psi(b_4)\}$ of $\mathscr{CP}_s(S)$ are equal.*

6. Simpliciality of Φ.

Let $S = S_{g,p}$ be a surface with $g \geq 2$ and $p \geq 2$. We show that the bijection Φ from $V(S)$ onto itself associated to an automorphism of $\mathscr{CP}(S)$, defined in Sections 4 and 5, induces an automorphism of $\mathscr{C}(S)$.

THEOREM 6.1.   *Let $S = S_{g,p}$ be a surface with $g \geq 2$ and $p \geq 2$, and let ϕ be an automorphism of $\mathscr{CP}(S)$. Then the bijection Φ from $V(S)$ onto itself associated to ϕ induces an automorphism of $\mathscr{C}(S)$.*

PROOF.   It suffices to show that Φ is simplicial. Note that by the definition of Φ and Lemma 3.13, we have $\phi(\{\alpha, \beta\}) = \{\Phi(\alpha), \Phi(\beta)\}$ for each HBP $\{\alpha, \beta\}$ in S.

Let α and β be distinct curves in S with $i(\alpha, \beta) = 0$. We prove the equality $i(\Phi(\alpha), \Phi(\beta)) = 0$ in the following three cases: (a) both α and β are non-separating in S; (b) α is non-separating in S and β is separating in S; and (c) both α and β are separating in S.

(a) Suppose that both α and β are non-separating in S. If α and β are HBP-equivalent, then $\phi(\{\alpha, \beta\}) = \{\Phi(\alpha), \Phi(\beta)\}$ is an HBP in S, and thus $i(\Phi(\alpha), \Phi(\beta)) = 0$. Suppose that α and β are not HBP-equivalent. We choose non-separating curves α', β' in S such that $a = \{\alpha, \alpha'\}$ and $b = \{\beta, \beta'\}$ are disjoint 1-HBPs in S. We then have $i(\phi(a), \phi(b)) = 0$ and thus $i(\Phi(\alpha), \Phi(\beta)) = 0$.

(b) Suppose that α is non-separating in S and that β is separating in S. If β is an HBC in S, then we can choose a non-separating curve α' in S disjoint from β such that $a = \{\alpha, \alpha'\}$ is an HBP in S. We then obtain the equality $i(\phi(a), \phi(\beta)) = 0$ and thus $i(\Phi(\alpha), \Phi(\beta)) = 0$.

We assume that β is not an HBC in S. Let R denote the component of S_β containing α, and let R' denote another component of S_β. If R contains at least one component of ∂S, then we choose a curve α' in R and a curve β' in S such that $a = \{\alpha, \alpha'\}$ and $b = \{\beta, \beta'\}$ are disjoint HBPs in S. We then obtain the equality $i(\phi(a), \phi(b)) = 0$ and thus $i(\Phi(\alpha), \Phi(\beta)) = 0$.

If R contains no component of ∂S, then we choose a curve β' in R' with $b = \{\beta, \beta'\}$ a 2-HBP in S. As in Figure 11 (i), we can find curves α_1, α_2 in S and curves β_1, β_2 in R' such that for each $k = 1, 2$,

- each of $a_k = \{\alpha, \alpha_k\}$, $b_k = \{\beta, \beta_k\}$ and $b'_k = \{\beta', \beta_k\}$ is a 1-HBP in S; and
- $i(a_k, b'_k) = 0$ and $i(\beta_1, \beta_2) \neq 0$.

Let Q denote the component of $S_{\phi(b)}$ homeomorphic to $S_{0,4}$. Since $\phi(b)$ is a 2-HBP in S and since for each $k = 1, 2$, both $\phi(b_k) = \{\Phi(\beta), \Phi(\beta_k)\}$ and $\phi(b'_k) = \{\Phi(\beta'), \Phi(\beta_k)\}$ are 1-HBPs in S, the curve $\Phi(\beta_k)$ lies in $V(Q)$. The two curves $\Phi(\beta_1)$ and $\Phi(\beta_2)$ intersect because b_1 and b_2 intersect. Hence, $\Phi(\beta_1)$ and $\Phi(\beta_2)$ fill
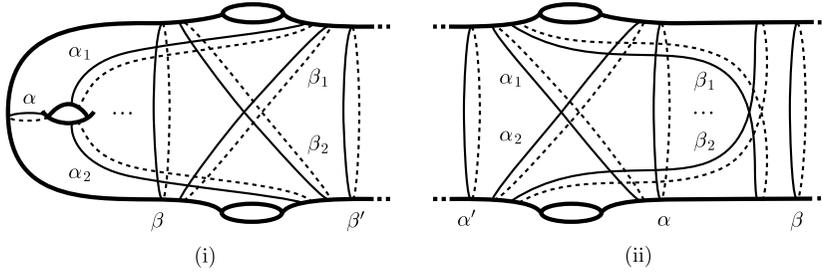
Figure 11.

Q. Since $\phi(a_k)$ and $\phi(b'_k)$ are disjoint for each $k = 1, 2$, the curve $\Phi(\alpha)$ is disjoint from both $\Phi(\beta_1)$ and $\Phi(\beta_2)$. It follows that $\Phi(\alpha)$ is disjoint from $\Phi(\beta)$.

(c) We assume that both α and β are separating in S. Either if both α and β are HBCs in S or if neither α nor β is an HBC in S and they are HBP-equivalent, then we have $i(\Phi(\alpha), \Phi(\beta)) = 0$.

Suppose that α is an HBC in S and that β is not an HBC in S. We choose a curve β' disjoint from α such that $\{\beta, \beta'\}$ is an HBP in S. Since $\phi(\alpha)$ and $\phi(\{\beta, \beta'\})$ are then disjoint, $\Phi(\alpha)$ and $\Phi(\beta)$ are disjoint.

Finally, we suppose that neither α nor β is an HBC in S and they are not HBP-equivalent. We denote by R, R' and R'' the three components of $S_{\{\alpha,\beta\}}$ so that R contains α as a boundary component, but does not contain β as a boundary component; and R'' contains α and β as boundary components. Unless ∂S is contained in either R or R', then there exist two curves α', β' in S such that $\{\alpha, \alpha'\}$ and $\{\beta, \beta'\}$ are disjoint HBPs in S. We thus have $i(\Phi(\alpha), \Phi(\beta)) = 0$.

Suppose that ∂S is contained in either R or R'. Without loss of generality, we may assume that ∂S is contained in R. Let α' be a curve in R with $a = \{\alpha, \alpha'\}$ a 2-HBP in S. As in Figure 11 (ii), we can find curves β_1, β_2 in S and curves α_1, α_2 in R such that for each $k = 1, 2$,

- each of $a_k = \{\alpha, \alpha_k\}$, $a'_k = \{\alpha', \alpha_k\}$ and $b_k = \{\beta, \beta_k\}$ is a 1-HBP in S; and
- $i(a'_k, b_k) = 0$ and $i(\alpha_1, \alpha_2) \neq 0$.

We denote by Q the component of $S_{\phi(a)}$ homeomorphic to $S_{0,4}$, which contains $\Phi(\alpha_1)$ and $\Phi(\alpha_2)$ because both $\phi(a_k) = \{\Phi(\alpha), \Phi(\alpha_k)\}$ and $\phi(a'_k) = \{\Phi(\alpha'), \Phi(\alpha_k)\}$ are 1-HBPs in S for each $k = 1, 2$. The curves $\Phi(\alpha_1)$ and $\Phi(\alpha_2)$ fill Q because they intersect. Since $\Phi(\beta)$ is disjoint from $\Phi(\alpha_1)$ and $\Phi(\alpha_2)$, it is disjoint from $\Phi(\alpha)$. $\square$

The argument in case (c) in the above proof shows the following:

THEOREM 6.2.   *Let $S = S_{g,p}$ be a surface with $g \geq 2$ and $p \geq 2$, and let*

ψ be an automorphism of $\mathscr{CP}_s(S)$. Then the bijection Ψ from $V_s(S)$ onto itself associated to ψ, defined in Sections 4 and 5, induces an automorphism of $\mathscr{C}_s(S)$.

Theorem 1.2 in [**15**] shows that if $S = S_{g,p}$ is a surface with $g \geq 2$ and $p \geq 2$, then any automorphism of $\mathscr{C}_s(S)$ is induced by an element of $\mathrm{Mod}^*(S)$. Combining Theorem 2.1, we obtain the following:

COROLLARY 6.3. *Let $S = S_{g,p}$ be a surface with $g \geq 2$ and $p \geq 2$. Then any automorphism of $\mathscr{CP}(S)$ is induced by an element of $\mathrm{Mod}^*(S)$. Moreover, the same conclusion holds for any automorphism of $\mathscr{CP}_s(S)$.*

7. Characterization of twisting elements.

The aim of this section is to show that any injective homomorphism from a finite index subgroup of $P_s(S)$ into $P(S)$ induces a superinjective map from $\mathscr{CP}_s(S)$ into $\mathscr{CP}(S)$. This map is obtained by characterizing HBC and HBP twists algebraically. Our argument is based on the characterization of such twists due to Irmak-Ivanov-McCarthy [**11**].

7.1. Preliminaries.
Let $S = S_{g,p}$ be a surface with $g \geq 1$ and assume that the Euler characteristic of S, denoted by $\chi(S) = -2g - p + 2$, is negative. Theorem 3.1 in [**11**] asserts that any element x of $P(S)$ is *pure* in the following sense (see Theorem A.1 in [**15**] for a proof): There exists $\sigma \in \Sigma(S) \cup \{\emptyset\}$ such that

(I) x fixes each curve of σ and each component of S_σ and of ∂S; and
(II) x acts on each component of S_σ as either the identity or a pseudo-Anosov element.

It follows that $P(S)$ is torsion-free. For $\tau \in \Sigma(S)$, we denote by $P(S)_\tau$ the stabilizer of τ in $P(S)$. Theorem A.1 of [**15**] shows that for any $x \in P(S)$ and $\tau \in \Sigma(S)$, if x fixes τ, then x fixes each curve of τ and each component of S_τ. Moreover, x preserves an orientation of each curve of τ. For each $\tau \in \Sigma(S)$, we thus have the natural homomorphism

$$\theta_\tau \colon P(S)_\tau \to \prod_Q \mathrm{PMod}(Q),$$

where Q runs through all components of S_τ. We define $\theta_Q \colon P(S)_\tau \to \mathrm{PMod}(Q)$ as the composition of θ_τ with the projection onto $\mathrm{PMod}(Q)$.

It is known that for each $x \in P(S)$, there exists the minimal element among all $\sigma \in \Sigma(S) \cup \{\emptyset\}$ satisfying the above conditions (I) and (II). The minimal element is called the *canonical reduction system (CRS)* for x. We note that

- the CRS's for x and its non-zero power are equal; and
- if $y \in P(S)$ lies in the centralizer of x, then y fixes the CRS for x.

Indeed, one can also define the CRS for any element and any subgroup of $\mathrm{Mod}(S)$. If $x \in \mathrm{Mod}(S)$ fixes an element of $\Sigma(S)$, then x is said to be *reducible*. It is known that for each element $x \in \mathrm{Mod}(S)$ of infinite order, x is reducible if and only if the CRS for x is non-empty. We recommend the reader to consult Section 7 of [**12**] for details on CRS's.

Pick $x \in P(S)$ and let $\sigma \in \Sigma(S) \cup \{\emptyset\}$ be the CRS for x. We mean by a *pA component* for x a component Q of S_σ on which x acts as a pseudo-Anosov element.

LEMMA 7.1.   *Let $S = S_{g,p}$ be a surface with $g \geq 1$ and $\chi(S) < 0$. Then the maximal rank of finitely generated abelian subgroups of $P(S)$ (resp. $P_s(S)$) is equal to p if $g \geq 2$, and equal to $p - 1$ if $g = 1$.*

PROOF.   If $g \geq 2$, then for each simplex σ of $\mathscr{CP}_s(S)$ of maximal dimension, HBP-twists about HBPs in σ generate a subgroup of $P_s(S)$ isomorphic to $\mathbf{Z}^p$. If $g = 1$, then there exists a simplex τ of $\mathscr{C}_s(S)$ with $|\tau| = p - 1$, and Dehn twists about curves in τ generate a subgroup of $P_s(S)$ isomorphic to $\mathbf{Z}^{p-1}$. It is thus enough to show that the rank of any finitely generated abelian subgroup of $P(S)$ is at most p if $g \geq 2$, and at most $p - 1$ if $g = 1$. We prove it by induction of p.

If either $g \geq 2$ and $p = 0$ or $g = 1$ and $p = 1$, then $P(S)$ is trivial, and the lemma is obvious. We assume either $g \geq 2$ and $p \geq 1$ or $g = 1$ and $p \geq 2$. In general, for any short exact sequence of groups, $1 \to A \to B \to C \to 1$, the inequality $\mathrm{rk}B \leq \mathrm{rk}A + \mathrm{rk}C$ holds, where for a group D, we denote by $\mathrm{rk}D$ the supremum of the ranks of finitely generated abelian subgroups in D. Fix a component ∂ of ∂S, and let R be the surface obtained from S by attaching a disk to ∂. Restricting the associated Birman exact sequence

$$1 \to \pi_1(R) \xrightarrow{\imath} \mathrm{PMod}(S) \to \mathrm{PMod}(R) \to 1$$

to $P(S)$ and to $P_s(S)$, we obtain the exact sequences

$$1 \to \pi_1(R) \to P(S) \to P(R) \to 1, \quad 1 \to N \to P_s(S) \to P_s(R) \to 1,$$

where we put $N = \imath^{-1}(P_s(S))$. Since we have $\mathrm{rk}\pi_1(R) = \mathrm{rk}N = 1$, the induction is completed.   □

7.2.   Characterization.
We start with the following observation.

LEMMA 7.2. *Let S be a surface of genus at least one with $\chi(S) < 0$. Pick $x \in P(S)$ and $\sigma \in \Sigma(S)$ such that x fixes σ. Let Q be a component of S_σ, and let τ be the set of all curves of σ corresponding to a component of ∂Q. Then the following assertions hold:*

(ⅰ) *If we have $Q \cap \partial S = \emptyset$ and no curve of τ is an HBC in S, then $\theta_Q(x)$ is neutral.*

(ⅱ) *If σ is the CRS for x and Q is a pA component for x, then there exists $y \in P(S)$ such that τ is the CRS for y, Q is a unique pA component for y, and the equality $\theta_Q(x) = \theta_Q(y)$ holds.*

PROOF. Let $\bar{S}$ denote the closed surface obtained from S by attaching disks to all components of ∂S. Let $\iota \colon \mathrm{PMod}(S) \to \mathrm{Mod}(\bar{S})$ be the homomorphism associated with the inclusion of S into $\bar{S}$. On the assumption in assertion (i), any curve of τ is essential in $\bar{S}$ (although some of them may be isotopic in $\bar{S}$). Assertion (i) follows because $\iota(x)$ is neutral.

We prove assertion (ii). Let F and C be representatives of x and τ, respectively, such that $F(C) = C$ and F is the identity on $C \cup \partial S$. Let Q° denote the component of $S \setminus C$ corresponding to Q. Let R denote the surface obtained from Q by attaching disks to all components of ∂Q corresponding to either a component of ∂S or an HBC in S. The element of $\mathrm{PMod}(R)$ induced by $\theta_Q(x)$ is neutral because $\iota(x)$ is neutral.

We define G as the homeomorphism of S obtained by extending the restriction of F to Q° so that G is the identity on $S \setminus Q^\circ$. Let $y \in \mathrm{PMod}(S)$ be the isotopy class of G. The equality $\theta_Q(x) = \theta_Q(y)$ then holds. Let τ' be the set of all curves of τ that are not HBCs in S. Let $\pi \colon \mathscr{C}(S) \to \mathscr{C}^*(\bar{S})$ be the simplicial map associated with the inclusion of S into $\bar{S}$ (see Section 2.2). We put $\bar{\tau} = \pi(\tau')$, which is an element of $\Sigma(\bar{S}) \cup \{\emptyset\}$. The element $\iota(y)$ fixes $\bar{\tau}$ and acts on each component of $\bar{S}_{\bar{\tau}}$ as the identity. Multiplying y with appropriate powers of Dehn twists about curves in τ', we may assume that $\iota(y)$ is neutral. We then have $y \in P(S)$ and obtain assertion (ii). $\qquad\square$

Following terminology in [**11**], we say that an element x of $P(S)$ is *basic* if the center of the centralizer of x in $P(S)$ is isomorphic to $\mathbf{Z}$. The following two propositions characterize basic elements and are stated in Proposition 5.1 of [**11**].

PROPOSITION 7.3. *Let S be a surface of genus at least one with $\chi(S) < 0$. Let Γ be a finite index subgroup of $P(S)$. Pick $x \in \Gamma$ and let $\sigma \in \Sigma(S) \cup \{\emptyset\}$ be the CRS for x. We assume that x satisfies one of the following three conditions:*

(a) *There exists a unique pA component Q for x, and any curve of σ corresponds to a component of ∂Q. Moreover, no curve of σ is an HBC in S, and no two*

curves of σ form an HBP in S.

(b) *x is a non-zero power of an HBC twist.*

(c) *x is a non-zero power of an HBP twist.*

Then the center of the centralizer of x in Γ is isomorphic to $\mathbf{Z}$. In particular, x is basic.

Proof.   We denote by $Z(x)$ the centralizer of x in Γ and denote by Z the center of $Z(x)$. We assume condition (a). By Lemma 2.3, it is enough to show that for each component R of S_σ other than Q, the group $\theta_R(Z(x))$ either is trivial or contains a pair of independent pseudo-Anosov elements of $\mathrm{PMod}(R)$, where two pseudo-Anosov elements are said to be *independent* if they do not generate a virtually cyclic group. If R contains a component of ∂S and is not a pair of pants, then $\theta_R(Z(x))$ contains independent pseudo-Anosov elements. If R contains no component of ∂S, then $Z(x)$ acts on R trivially by Lemma 7.2 (i).

We assume condition (b). In this case, σ consists of a single HBC γ in S and x is a non-zero power of t_γ. Let R be a component of S_γ. If R contains a component of ∂S and is not a pair of pants, then $\theta_R(Z(x))$ contains independent pseudo-Anosov elements. If R contains no component of ∂S, then γ is a p-HBC in S, and R is the component of S_γ of positive genus. When the genus of S is at least two, the group $\theta_R(Z(x))$ contains independent pseudo-Anosov elements because R contains an HBP in S. When the genus of S is equal to one, R is homeomorphic to $S_{1,1}$. The group $\theta_R(Z(x))$ is then trivial because $Z(x)$ acts trivially on the torus obtained by attaching a disk to ∂R. It thus turns out that Z is contained in the cyclic group generated by t_γ.

Finally, we assume condition (c). In this case, if x is a non-zero power of the element $t_\alpha t_\beta^{-1}$ with $\{\alpha, \beta\}$ an HBP in S, then we have the equality $\sigma = \{\alpha, \beta\}$. We can then apply the same argument as in the case where we assumed condition (a), and we conclude that Z is contained in the cyclic group generated by $t_\alpha t_\beta^{-1}$.   □

Proposition 7.4.   *Let S be a surface of genus at least one with $\chi(S) < 0$. Pick $x \in P(S)$ and let $\sigma \in \Sigma(S) \cup \{\emptyset\}$ be the CRS for x. If x is basic, then x satisfies one of conditions (a), (b) and (c) in Proposition 7.3.*

Proof.   We first claim that the center of $P(S)$ is trivial. When the genus of S is at least two, the claim follows from Lemma 2.2 (ii). When the genus of S is equal to one, any separating curve in S is an HBC in S. Along argument similar to the proof of Lemma 2.2, we can show that any element of $\mathrm{Mod}^*(S)$ fixing any element of $V_s(S)$ fixes any element of $V(S)$ and that any element of $P(S)$ that commutes any element of $P(S)$ is neutral. The claim thus follows.

We denote by $Z(x)$ the centralizer of x in $P(S)$ and denote by Z the center

of $Z(x)$. The claim shown in the last paragraph implies that x is not neutral. If x is pseudo-Anosov, then condition (a) holds. We now suppose that x is reducible, and let $\sigma \in \Sigma(S)$ denote the CRS for x. If σ contains an HBC γ in S, then t_γ lies in Z because $Z(x)$ fixes σ. Since Z is isomorphic to $\boldsymbol{Z}$, x and t_γ generate a cyclic group. Pureness of x then shows that condition (b) holds. We can apply the same argument in the case where σ contains an HBP in S.

We assume that no curve of σ is an HBC in S and no two curves of σ form an HBP in S. If there were no pA component for x, then x would be neutral by Lemma 2.3. This is a contradiction. If there were two pA components R_1, R_2 for x, then by Lemma 7.2 (ii), we would have $x_1, x_2 \in P(S)$ such that

- both x_1 and x_2 fix σ; and
- for each $j = 1, 2$, R_j is a unique pA component for x_j and we have the equality $\theta_{R_j}(x_j) = \theta_{R_j}(x)$.

The two elements x_1 and x_2 lie in Z, and they generate $\boldsymbol{Z}^2$. This contradicts the assumption that Z is isomorphic to $\boldsymbol{Z}$. It follows that x has a single pA component. Condition (a) therefore holds. $\square$

REMARK 7.5. Let S be a surface of genus at least one with $\chi(S) < 0$. Pick $x \in P(S)$ and denote by Z the center of the centralizer of x in $P(S)$. We note that Z is finitely generated because any abelian subgroup of $\mathrm{Mod}^*(S)$ is finitely generated by Theorem A in [**3**]. Following the proof of Proposition 7.3, we can compute the rank of Z as follows. Let $\sigma \in \Sigma(S) \cup \{\emptyset\}$ be the CRS for x. We define n_1 to be the number of pA components for x and define n_2 to be the number of curves of σ which are HBCs in S. Let $\{\tau_1, \ldots, \tau_m\}$ be the collection of HBP-equivalence classes in σ. The rank of Z is then equal to the sum $n_1 + n_2 + \sum_{i=1}^m (|\tau_i| - 1)$. This result is stated in Proposition 4.1 of [**11**].

We define $\mathfrak{C} = \mathfrak{C}(S)$ as the set of all non-zero powers of HBC twists in $P(S)$ and define $\mathfrak{P} = \mathfrak{P}(S)$ as the set of all non-zero powers of HBP twists in $P(S)$.

LEMMA 7.6. *Let $S = S_{g,p}$ be a surface with $g \geq 2$. Let Γ be a finite index subgroup of $P_s(S)$, and let $f \colon \Gamma \to P(S)$ be an injective homomorphism. If an element x of Γ is basic and lies in a finitely generated abelian subgroup of Γ of rank p, then $f(x)$ is also basic. In particular, $f(y)$ is basic for each $y \in \mathfrak{C} \cup \mathfrak{P}$.*

To prove this lemma, we use the following general fact, which is essentially verified in Lemma 5.2 in [**10**] (see Lemma 6.8 in [**15**] for a proof). For a group A, we denote by $\mathrm{rk}A$ the supremum of the ranks of finitely generated abelian subgroups in A, and denote by $Z(A)$ the center of A. For each $a \in A$, let $Z_A(a)$ denote the centralizer of a in A.

Lemma 7.7.   *Let A and B be groups with $\mathrm{rk}A = \mathrm{rk}B < \infty$ and assume that any abelian subgroup of B is finitely generated. Let $\eta\colon A \to B$ be an injective homomorphism. If a is an element of A lying in a finitely generated, free abelian subgroup of A with its rank equal to $\mathrm{rk}A$, then we have the inequality*

$$\mathrm{rk}Z(Z_B(\eta(a))) \leq \mathrm{rk}Z(Z_A(a)).$$

Proof of Lemma 7.6.   By Propositions 7.3 and 7.4, $Z(Z_\Gamma(x))$ is isomorphic to $\boldsymbol{Z}$. By Lemmas 7.1 and 7.7, $Z(Z_{P(S)}(f(x)))$ is of rank one. The lemma thus follows.   □

The following lemma characterizes HBP twists among basic elements and is a slight variant of Proposition 6.1 in [**11**]. Let us say that an element x of $P(S)$ is *single-pA* if it satisfies condition (a) in Proposition 7.3.

Lemma 7.8.   *Let $S = S_{g,p}$ be a surface with $g \geq 2$ and $p \geq 2$. Then the following assertions hold*:

(i) *For each $x \in \mathfrak{P} \cap P_s(S)$, there exists $y \in \mathfrak{P} \cap P_s(S)$ such that the group generated by x and y is isomorphic to $\boldsymbol{Z}^2$ and the product xy belongs to $\mathfrak{P}$.*
(ii) *Let $z \in P(S)$ be a basic element. If we have a basic element $w \in P(S)$ such that the group generated by z and w is isomorphic to $\boldsymbol{Z}^2$ and the product zw is basic, then z belongs to $\mathfrak{P}$.*

Proof.   We first prove assertion (i). Let $\{\alpha, \beta\}$ be the HBP in S and k the non-zero integer with $x = t_\alpha^k t_\beta^{-k}$. Since we have $g \geq 2$ and $p \geq 2$, there exists a curve γ in S which is disjoint and distinct from α and β and forms an HBP in S with β (and thus with α). The element $y = t_\beta^k t_\gamma^{-k}$ is then a desired one.

We next prove assertion (ii). Note that z and w fix the CRS's for them because z and w commute. Since zw is basic, if z were a single-pA element with Q the pA component for z, then w would also be a single-pA element with Q the pA component for w, and $\theta_Q(z)$ and $\theta_Q(w)$ would generate a virtually cyclic group. It then follows that z and w generate a virtually cyclic group. This is a contradiction. If z were a non-zero power of an HBC twist, then w would be a non-zero power of the same HBC twist since zw is basic. This also contradicts the assumption that z and w generate $\boldsymbol{Z}^2$. Proposition 7.4 then shows $z \in \mathfrak{P}$.   □

For each integer k with $1 \leq k \leq p$, we denote by $\mathfrak{P}_k = \mathfrak{P}_k(S)$ the subset of $\mathfrak{P}$ consisting of all non-zero powers of HBP twists about k-HBPs in S. If $x \in \mathfrak{P}$ is a non-zero power of the HBP twist about an HBP b in S, we call b the *support* of x. Let us say that two elements $x, y \in \mathfrak{P}$ are *equivalent* if the supports of x and y are disjoint and equivalent.

LEMMA 7.9.   *Let $S = S_{g,p}$ be a surface with $g \geq 2$ and $p \geq 2$. Let Γ be a finite index subgroup of $P_s(S)$, and let $f \colon \Gamma \to P(S)$ be an injective homomorphism. Then the following assertions hold*:

(i) *For each $x \in \mathfrak{P} \cap \Gamma$, we have $f(x) \in \mathfrak{P}$.*

(ii) *If the supports of two elements $x, y \in \mathfrak{P} \cap \Gamma$ are disjoint and contain a common curve, then the same holds for the supports of $f(x)$ and $f(y)$.*

(iii) *If two elements $x, y \in \mathfrak{P} \cap \Gamma$ are equivalent, then $f(x)$ and $f(y)$ are also equivalent.*

(iv) *For each $x \in \mathfrak{P}_p \cap \Gamma$, we have $f(x) \in \mathfrak{P}_p$.*

PROOF.   Assertion (i) follows from Lemmas 7.6 and 7.8. Pick $x, y \in \mathfrak{P} \cap \Gamma$ so that the supports of x and y, denoted by $\{\alpha_1, \alpha_2\}$ and $\{\beta_1, \beta_2\}$, respectively, are disjoint. To prove assertion (ii), we first assume $\alpha_1 = \beta_1$ and $\alpha_2 \neq \beta_2$. There then exist non-zero integers j, k with $x^j y^k \in \mathfrak{P}$. Since all of $f(x^j)$, $f(y^k)$ and $f(x^j y^k)$ belong to $\mathfrak{P}$ by assertion (i), the supports of $f(x^j)$ and $f(y^k)$ contain a common curve. Assertion (ii) is proved.

To prove assertion (iii), we next assume that α_1, α_2, β_1 and β_2 are mutually distinct. Let z be an HBP twist about the HBP $\{\alpha_1, \beta_1\}$ and choose non-zero integers a_1, a_2 and b so that $x^{a_1} z^b$ and $y^{a_2} z^b$ belong to $\mathfrak{P}$ and we have $z^b \in \Gamma$. Assertion (ii) implies that the supports of $f(x^{a_1})$ and $f(z^b)$ (resp. $f(y^{a_2})$ and $f(z^b)$) contain a common curve, and thus $f(x)$ and $f(y)$ are equivalent.

We prove assertion (iv). It will be shown that for each $y \in (\mathfrak{P} \backslash \mathfrak{P}_p) \cap \Gamma$, we have $f(y) \in \mathfrak{P} \setminus \mathfrak{P}_p$. Once this is verified, assertion (iv) can be deduced by considering a maximal family of pairwise disjoint and equivalent HBPs in S because such a maximal family contains exactly one p-HBP in S.

Pick $y \in (\mathfrak{P} \backslash \mathfrak{P}_p) \cap \Gamma$ and let $\{\gamma, \gamma'\}$ be the support of y, which is a separating k-HBP in S with $1 \leq k \leq p - 1$. There exists a component Q of $S_{\{\gamma, \gamma'\}}$ such that the genus of Q, denoted by g_1, is positive and Q contains a component of ∂S. We may assume that Q contains γ as a boundary component. Choose a separating curve δ in Q cutting off a surface homeomorphic to $S_{g_1,1}$ from Q. We then have a curve $\delta' \in V(Q) \cup \{\gamma\}$ which is separating in S, is disjoint from δ and cuts off from S a surface R homeomorphic to $S_{g_1,2}$ and containing δ (see Figure 12). Choose a separating curve ϵ in R such that ϵ cuts off a pair of pants from R; and δ and ϵ fill R. Both $\{\gamma, \delta\}$ and $\{\gamma, \epsilon\}$ are then HBPs in S. Assertion (i) implies that f induces a map from the set of separating HBPs in S into the set of HBPs in S, which preserves disjointness and non-disjointness. We denote this map by the same symbol f. We now prove the following:

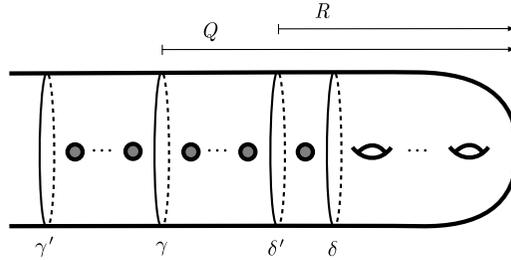CLAIM 7.10.   *The two HBPs $f(\{\gamma, \delta\})$ and $f(\{\gamma, \epsilon\})$ contain a common curve.*

Figure 12.

Proof. Choose a curve γ'' in S such that $\{\gamma, \gamma''\}$ is an HBP in S and we have $i(\gamma', \gamma'') \neq 0$. We set

$$c' = \{\gamma, \gamma'\}, \quad c'' = \{\gamma, \gamma''\}, \quad d = \{\gamma, \delta\}, \quad e = \{\gamma, \epsilon\}.$$

Note that by assertion (ii), the two HBPs $f(c')$ and $f(d)$ (resp. $f(c')$ and $f(e)$) contain a common curve. If the claim were not true, then we would have the inclusion $f(c') \subset f(d) \cup f(e)$. Since c'' is disjoint from d and e, $f(c'')$ is disjoint from $f(c')$. This contradicts $i(\gamma', \gamma'') \neq 0$. $\qquad\square$

We put $f(\{\gamma, \delta\}) = \{\gamma_1, \delta_1\}$ and $f(\{\gamma, \epsilon\}) = \{\gamma_1, \epsilon_1\}$. Since δ and ϵ intersect, so do δ_1 and ϵ_1. Fix a non-zero integer l with $t_\gamma^l t_\delta^{-l}, t_\gamma^l t_\epsilon^{-l} \in \Gamma$, and let m and n be the non-zero integers with

$$f\big(t_\gamma^l t_\delta^{-l}\big) = t_{\gamma_1}^m t_{\delta_1}^{-m}, \quad f\big(t_\gamma^l t_\epsilon^{-l}\big) = t_{\gamma_1}^n t_{\epsilon_1}^{-n}.$$

For each integer q, we then have $t_\delta^{-lq} t_\epsilon^{lq} = (t_\gamma^l t_\delta^{-l})^q (t_\gamma^l t_\epsilon^{-l})^{-q} \in \Gamma$ and

$$f\big(t_\delta^{-lq} t_\epsilon^{lq}\big) = f\big(t_\gamma^{lq} t_\delta^{-lq}\big) f\big(t_\gamma^{-lq} t_\epsilon^{lq}\big) = t_{\gamma_1}^{mq} t_{\delta_1}^{-mq} t_{\gamma_1}^{-nq} t_{\epsilon_1}^{nq} = t_{\gamma_1}^{mq-nq} t_{\delta_1}^{-mq} t_{\epsilon_1}^{nq}.$$

By Theorem 7 in [**21**] or Exposé 13, Section III in [**7**], there exists a non-zero integer r satisfying the following three conditions:

- $t_\gamma^r t_\delta^{-r}$ and $t_\gamma^r t_\epsilon^{-r}$ belong to Γ, and thus so does $t_\delta^{-r} t_\epsilon^r$;
- $t_\delta^{-r} t_\epsilon^r$ acts on R as a pseudo-Anosov element; and
- $f(t_\delta^{-r} t_\epsilon^r)$ acts on the subsurface of S filled by δ_1 and ϵ_1 as a pseudo-Anosov element.

The element $t_\delta^{-r} t_\epsilon^r$ is single-pA and lies in a free abelian subgroup of Γ of rank p. By Lemma 7.6, $f(t_\delta^{-r} t_\epsilon^r)$ is basic and is thus single-pA.

If $f(y)$ were in $\mathfrak{P}_p$, then by Lemma 7.2 (i), there would exist no single-pA element in $P(S)$ commuting $f(y)$. This is a contradiction because $f(t_\delta^{-r} t_\epsilon^r)$ commutes $f(y)$. We thus have $f(y) \in \mathfrak{P} \setminus \mathfrak{P}_p$. $\qquad\square$

LEMMA 7.11.   *Let S, Γ and $f \colon \Gamma \to P(S)$ be the symbols in Lemma 7.9. Then for each $x \in \mathfrak{C} \cap \Gamma$, we have $f(x) \in \mathfrak{C} \cup \mathfrak{P}$.*

PROOF.   Pick $x \in \mathfrak{C}$, and let α be the HBC in S with x a non-zero power of t_α. There then exists a separating p-HBP b in S disjoint from α. Let $y \in \Gamma$ be a non-zero power of the HBP twist about b. Since $f(y)$ lies in $\mathfrak{P}_p$ by Lemma 7.9 (iv) and since $f(y)$ commutes $f(x)$, the element $f(x)$ is not single-pA. $\qquad\square$

Lemmas 7.9 and 7.11 show that any injective homomorphism from a finite index subgroup of $P_s(S)$ into $P(S)$ preserves powers of HBC and HBP twists. The next lemma proves the same conclusion for any injective homomorphism from a finite index subgroup of $P(S)$ into $P(S)$.

LEMMA 7.12.   *Let $S = S_{g,p}$ be a surface with $g \geq 2$ and $p \geq 2$. Let Γ be a finite index subgroup of $P(S)$, and let $f \colon \Gamma \to P(S)$ be an injective homomorphism. Then for each $x \in (\mathfrak{C} \cup \mathfrak{P}) \cap \Gamma$, we have $f(x) \in \mathfrak{C} \cup \mathfrak{P}$.*

PROOF.   It suffices to prove that if $x \in \mathfrak{P} \cap \Gamma$ is a non-zero power of the HBP twist about a non-separating HBP in S, then we have $f(x) \in \mathfrak{P}$. Along an argument of the same kind as in the proof of Lemma 7.8 (i), we can find a non-zero integer k and an element $y \in \mathfrak{P} \cap \Gamma$ such that the group generated by x^k and y is isomorphic to $\boldsymbol{Z}^2$ and the product $x^k y$ belongs to $\mathfrak{P}$. Following argument in the proof of Lemma 7.6, we can show that f preserves basic elements. It then follows from Lemma 7.8 (ii) that $f(x^k)$ belongs to $\mathfrak{P}$, and thus so does $f(x)$. $\qquad\square$

For each HBC α in S, we denote by T_α the cyclic group generated by t_α. For each HBP $b = \{\beta, \gamma\}$ in S, we denote by T_b the cyclic group generated by $t_\beta t_\gamma^{-1}$. As a consequence of Lemmas 7.9, 7.11 and 7.12, we obtain the following:

THEOREM 7.13.   *Let $S = S_{g,p}$ be a surface with $g \geq 2$ and $p \geq 2$. Then the following assertions hold*:

(i) *Let Γ be a finite index subgroup of $P(S)$, and let $f \colon \Gamma \to P(S)$ be an injective homomorphism. Then there exists a superinjective map $\phi \colon \mathscr{CP}(S) \to \mathscr{CP}(S)$ with $f(T_v \cap \Gamma) < T_{\phi(v)}$ for any vertex v of $\mathscr{CP}(S)$.*

(ii) *Let Λ be a finite index subgroup of $P_s(S)$, and let $h \colon \Lambda \to P(S)$ be an injective homomorphism. Then there exists a superinjective map $\psi \colon \mathscr{CP}_s(S) \to \mathscr{CP}(S)$ with $h(T_v \cap \Lambda) < T_{\psi(v)}$ for any vertex v of $\mathscr{CP}_s(S)$.*

Theorem 1.1 can be deduced from Corollary 6.3 and Theorem 7.13 along the argument in Section 3 of [**13**]. We omit details of this part because an argument of the same kind appears in many works [**4**], [**10**], [**15**], [**17**], etc.

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