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Author(s)	DINH, TRUNG HOA; DU, THI HOA BINH; HO, MINH TOAN
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Kyoto University

ON SOME INEQUALITIES WITH MATRIX MEANS

DINH TRUNG HOA, DU THI HOA BINH, AND HO MINH TOAN

ABSTRACT. Let $0 < m \leq A, B \leq M$ and σ, τ two arbitrary means between harmonic and arithmetic means. Then for every positive unital linear map Φ,

$$\Phi(A\sigma B) \leq K(h)\Phi(A\tau B),$$

$$\Phi(A\sigma B) \leq K(h)\left(\Phi(A)\tau\Phi(B)\right),$$

$$\Phi(A)\sigma\Phi(B) \leq K(h)\Phi(A\tau B),$$

and

$$\Phi(A)\sigma\Phi(B) \leq K(h)\Phi(A)\tau\Phi(B),$$

where $K(h) = \dfrac{(h+1)^2}{4h}$ with $h = \dfrac{M}{m}$ is the Kantorovich constant.

1. INTRODUCTION

The axiomatic theory for connections and means for pairs of positive matrices have been studied by Kybo and Ando [4]. A binary operation σ define on the set of positive definite matrices is called a connection if

(i) $A \leq C, B \leq D$ implies $A\sigma B \leq B\sigma D$;
(ii) $C(A\sigma B)C \leq (CAC)\sigma(CBC)$;
(iii) $A_n \downarrow A$ and $B_n \downarrow B$ imply $A_n\sigma B_n \downarrow A\sigma B$.

If $I\sigma I = I$, then σ is called a mean.

Many authors study matrix inequalities containing means and linear unital positive maps on matrix algebras. Such inequalities are interesting by themselves and have many applications in quantum information theory.

In [2], Lin proved the following Theorem.

Theorem 1.1 ([2]). *Let $0 < m \leq A, B \leq M$. Then for every positive unital linear map Φ,*

$$(1) \qquad \Phi^2(A\nabla B) \leq K^2(h)\Phi^2(A\sharp B),$$

and

$$(2) \qquad \Phi^2(A\nabla B) \leq K^2(h)(\Phi(A)\sharp\Phi(B))^2,$$

where $K(h) = \frac{(h+1)^2}{4h}$ with $h = \frac{M}{m}$ is the Kantorovich constant.

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It is well-known that the arithmetic mean ∇ is the biggest among symmetric means (see [4]). A natural question is that *is the theorem above still true if we replace the biggest means by a smaller one?* In this note, we consider such inequalities for two different means with Kantorovich constant. In applications, we give an analogous result of Uchiyama and Yamazaki in [7].

This note is based on preprint [1].

2. MAIN RESULTS

Lemma 2.1. *Let* $0 < m \leq A, B \leq M$ *and* σ, τ *two arbitrary means between harmonic and arithmetic means. Then for every positive unital linear map* Φ,

$$(3) \qquad \Phi(A\sigma B) + Mm\Phi^{-1}(A\tau B) \leq M + m,$$

and

$$(4) \qquad \Phi(A)\sigma\Phi(B) + Mm\Phi^{-1}(A\tau B) \leq M + m.$$

Proof. It is easy to see that

$$(M - A)(m - A)A^{-1} \leq 0,$$

or

$$mMA^{-1} + A \leq M + m.$$

Consequently,

$$\Phi(A) + mM\Phi(A^{-1}) \leq M + m.$$

Similarly,

$$\Phi(B) + mM\Phi(B^{-1}) \leq M + m.$$

Summing up two above inequalities, we get

$$\Phi(A \nabla B) + mM\Phi((A!B)^{-1}) \leq M + m.$$

Besides, from the general theory of matrix means we know that $\nabla \geq \sigma$ and $\tau \geq !$. Hence,

$$\Phi(A\sigma B) + mM\Phi^{-1}(A\tau B) \leq \Phi(A\sigma B) + mM\Phi((A\tau B)^{-1})$$
$$\leq \Phi(A \nabla B) + mM\Phi((A!B)^{-1})$$
$$\leq M + m.$$

By a similar argument, we can get inequality (4) with using the fact that

$$\Phi(A)\sigma\Phi(B) \leq \Phi(A) \nabla \Phi(B) = \Phi(A \nabla B).$$

$\square$

The following Proposition is a generalization of Lin's result mentioned in Introduction.

Proposition 2.1. *Let* $0 < m \le A, B \le M$ *and* σ, τ *two arbitrary means between harmonic and arithmetic means. Then for every positive unital linear map* Φ,

$$(5) \qquad \Phi^2(A\sigma B) \le K^2(h)\Phi^2(A\tau B),$$

$$(6) \qquad \Phi^2(A\sigma B) \le K^2(h)\left(\Phi(A)\tau\Phi(B)\right)^2,$$

$$(7) \qquad (\Phi(A)\sigma\Phi(B))^2 \le K^2(h)\Phi^2(A\tau B),$$

and

$$(8) \qquad (\Phi(A)\sigma\Phi(B))^2 \le K^2(h)(\Phi(A)\tau\Phi(B))^2,$$

where $K(h) = \dfrac{(h+1)^2}{4h}$ *with* $h = \dfrac{M}{m}$ *is the Kantorovich constant.*

Proof. We prove (2.1). The inequality (2.1) is equivalent to the following

$$\Phi^{-1}(A\tau B)\Phi^2(A\sigma B)\Phi^{-1}(A\tau B) \le K^2(h),$$

or

$$\|\Phi(A\sigma B)\Phi^{-1}(A\tau B)\| \le K(h).$$

On the other hand, it is well known that [5, Theorem 1] for $A, B \ge 0$,

$$\|AB\| \le \frac{1}{4}\|A + B\|^2.$$

So, it is necessary to prove that

$$\frac{1}{4mM}\|\Phi(A\sigma B) + mM\Phi^{-1}(A\tau B)\|^2 \le \frac{(M+m)^2}{4Mm},$$

or,

$$\|\Phi(A\sigma B) + mM\Phi^{-1}(A\tau B)\| \le M + m.$$

The last inequality follows from Lemma 2.1.

Remain inequalities in Proposition can be proved analogously. $\square$

Remark 1. As we mentioned in the proof of Proposition 2.1 that for any positive matrices A, B, $\Phi(A\sigma B) \le \Phi(A\nabla B)$. From that, it can rise a wrong intuition that the proof of Proposition 2.1 can be obtained easily from Theorem 1.1. Unfortunately, the last inequality could not be squared as it was shown in [2, Proposition 1.2].

Theorem 2.1. *Let* $0 < m \le A, B \le M$ *and* σ, τ *are two arbitrary symmetric means. Then for every positive unital linear map* Φ,

$$\Phi(A\sigma B) \le K(h)\Phi(A\tau B),$$

$$\Phi(A\sigma B) \le K(h)\left(\Phi(A)\tau\Phi(B)\right),$$

$$\Phi(A)\sigma\Phi(B) \le K(h)\Phi(A\tau B),$$

and

$$\Phi(A)\sigma\Phi(B) \le K(h)\Phi(A)\tau\Phi(B),$$

where $K(h) = \dfrac{(h+1)^2}{4h}$ *with* $h = \dfrac{M}{m}$ *is the Kantorovich constant.*

Proof. The proof follows from Proposition 2.1 and the fact that the function $f(t) = t^{1/2}$ is operator monotone on $[0, \infty)$. $\square$

Corrolary 2.1. *Let f, g be symmetric operator monotone functions on $[0, \infty)$. Then for any pair $0 < m < M$,*

$$(9) \qquad max\{\frac{f(t)}{g(t)}, \frac{f(t)}{g(t)}\} \leq K(h) = \frac{(m+M)^2}{4mM}, \quad t \in [m, M].$$

Proof. It is necessary to apply above Theorem for the symmetric matrix means σ and δ corresponding to the functions f and g, and definition of matrix means via it representation functions. $\square$

Inequality (9) is interesting by itself, and the authors do not know an elementary proof even in the case when $f(t) = \sqrt{t}$.

As an application, now we give a similar result as in [7]. Uchiyama and Yamazaki showed that for an operator monotone function f on $[0, \infty)$ if $f(\lambda B + I)^{-1} \sharp f(\lambda A + I) \leq I$ for all sufficiently small $\lambda > 0$, then $f(\lambda A + I) \leq f(\lambda B + I)$ and $A \leq B$. By applying Theorem 2.1 we get a similar result for any symmetric means.

Corrolary 2.2. *Let f be operator monotone function on $[0, \infty)$ and σ an arbitrary mean between harmonic and arithmetic ones. If for a given pair of positive invertible matrices A, B,*

$$f(\lambda B + I)^{-1} \sigma f(\lambda A + I) \leq K$$

for all sufficiently small $\lambda > 0$ (where K is Kantorovich constant), then $f(\lambda A + I) \leq f(\lambda B + I)$ and $A \leq B$.

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DUY TAN UNIVERSITY, K7/25 QUANG TRUNG, DANANG, VIETNAM
E-mail address: trunghoa.math@gmail.com

HA TAY COLLEGE OF PEDAGOGY, HA NOI, VIETNAM
E-mail address: hoabinhcdsp@gmail.com

INSTITUTE OF MATHEMATICS
E-mail address: hmtoan@math.ac.vn