Title	N-Fractional Calculus of the Function $f(z)=log((z-b)^3-c)$ and Identities (Some inequalities concerned with the geometric function theory)
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N- Fractional Calculus of the Function $f(z)=\log((z-b)^{3}-c)$ and Identities

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Abstract

In this article, N- fractional calculus of the logarithmic function

$$f(z)=\log((z-b)^{3}-c)\ ,\qquad (z-b)^{3}-c\neq 0,1\ ,$$

is discussed and some identities derived from them are reported.

That is, it is discussed in the manner below, for example.

$$(\log((z-b)^{3}-c))_{\gamma}=\left((\log((z-b)^{3}-c))_{1}\right)_{\gamma-1}$$

$$=-3e^{-i\pi\gamma}(z-b)^{-\gamma}\sum_{k=0}^{\infty}\frac{1}{k!\,\Gamma(3k+3)}T^{k}$$

$$\times\{\Gamma(3k+2+\gamma)-2(\gamma-1)\Gamma(3k+1+\gamma)+(\gamma-1)(\gamma-2)\Gamma(3k+\gamma)\}$$

$$(|\Gamma(\gamma)|<\infty)$$

where

$$T=\frac{c}{(z-b)^{3}},\qquad |T|<1,\qquad \Gamma(\cdots)\text{ ; Gamma Function}$$

and

$$[\lambda]_{k}=\lambda(\lambda+1)\cdots(\lambda+k-1)=\Gamma(\lambda+k)/\Gamma(\lambda)\ with\ [\lambda]_{0}=1.$$

(Notation of Pochhammer)

§ 0. Introduction (Definition of Fractional Calculus)

(I) Definition. (by K. Nishimoto) ([1] Vol. 1)

Let $D = \{D_-, D_+\}$, $C = \{C_-, C_+\}$,

C_- be a curve along the cut joining two points z and $-\infty + i\,\mathrm{Im}(z)$,

C_+ be a curve along the cut joining two points z and $\infty + i\,\mathrm{Im}(z)$,

D_- be a domain surrounded by C_-; D_+ be a domain surrounded by C_+.

(Here D contains the points over the curve C.)

Moreover, let $f = f(z)$ be a regular function in D ($z \in D$),

$$f_\nu = (f)_\nu = {}_C(f)_\nu = \frac{\Gamma(\nu+1)}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{\nu+1}}\, d\zeta \qquad (\nu \notin Z^-), \tag{1}$$

$$(f)_{-m} = \lim_{\nu \to -m} (f)_\nu \qquad (m \in Z^+), \tag{2}$$

where $-\pi \le \arg(\zeta - z) \le \pi$ for C_-, $\quad 0 \le \arg(\zeta - z) \le 2\pi$ for C_+,

$\zeta \ne z$, $z \in C$, $\nu \in R$, Γ; Gamma function,

then $(f)_\nu$ is the fractional differintegration of arbitrary order ν (derivatives of order ν for $\nu > 0$, and integrals of order $-\nu$ for $\nu < 0$), with respect to z, of the function f, if $|(f)_\nu| < \infty$.

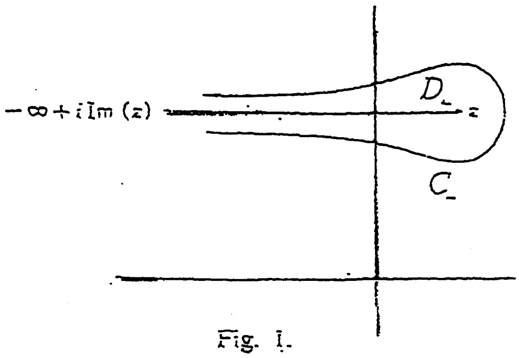

Fig. 1.

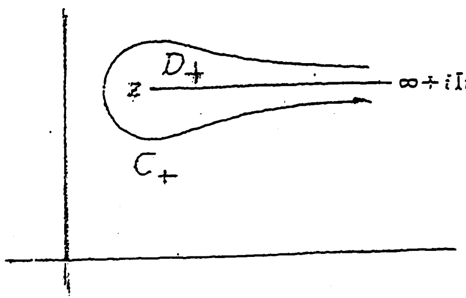

Fig. 2.

Notice that (1) is reduced to Goursat's integral for $\nu = n$ ($\in Z^+$) and is reduced to the famous Cauchy's integral for $\nu = 0$. That is, (1) is an extention of Cauchy's integral and of Goursat's one, conversely Cauchy's and Goursat's ones are special cases of (1).

Moreover, notice that (1) is the representation which unifies the derivatives and integrations.

(II) On the fractional calculus operator N^{ν} [3]

Theorem A. *Let fractional calculus operator (Nishimoto's Operator) N^{ν} be*

$$N^{\nu}=\left(\frac{\Gamma(\nu+1)}{2\pi i}\int_{C}\frac{d\zeta}{(\zeta-z)^{\nu+1}}\right)\qquad(\nu\notin Z^{-}),\quad [\text{Refer to}(1)]\qquad(3)$$

with

$$N^{-m}=\lim_{\nu\to -m}N^{\nu}\qquad(m\in Z^{+}),\qquad(4)$$

and define the binary operation $\circ$ *as*

$$N^{\beta}\circ N^{\alpha}f=N^{\beta}N^{\alpha}f=N^{\beta}(N^{\alpha}f)\qquad(\alpha,\beta\in \boldsymbol{R}),\qquad(5)$$

then the set

$$\{N^{\nu}\}=\{N^{\nu}\mid \nu\in \boldsymbol{R}\}\qquad(6)$$

is an Abelian product group (having continuous index ν) which has the inverse transform operator $(N^{\nu})^{-1}=N^{-\nu}$ to the fractional calculus operator N^{ν} , for the function f such that $f\in F=\left\{f_{\nu};\ 0\neq |f_{\nu}|<\infty,\ \nu\in \boldsymbol{R}\right\}$, where $f=f(z)$ and $z\in \boldsymbol{C}$.
(vis. $-\infty<\nu<\infty$).

(For our convenience, we call $N^{\beta}\circ N^{\alpha}$ as product of N^{β} and N^{α} .)

Theorem B. " F.O.G. $\{N^{\nu}\}$ " *is an " Action product group which has continuous index ν " for the set of F .* (*F.O.G.* ; Fractional calculus operator group) [3]

Theorem C. *Let*

$$S:=\{\pm N^{\nu}\}\cup\{0\}=\{N^{\nu}\}\cup\{-N^{\nu}\}\cup\{0\}\quad(\nu\in \boldsymbol{R}).\qquad(7)$$

Then the set S is a commutative ring for the function $f\in F$, when the identity

$$N^{\alpha}+N^{\beta}=N^{\gamma}\qquad(N^{\alpha},N^{\beta},N^{\gamma}\in S)\qquad(8)$$

holds. [5]

(III) **Lemma.** We have [1]

(i) $$((z-c)^{b})_{\alpha}=e^{-i\pi\alpha}\frac{\Gamma(\alpha-b)}{\Gamma(-b)}(z-c)^{b-\alpha}\qquad\left(\left|\frac{\Gamma(\alpha-b)}{\Gamma(-b)}\right|<\infty\right),$$

(ii) $$(\log(z-c))_{\alpha}=-e^{-i\pi\alpha}\Gamma(\alpha)(z-c)^{-\alpha}\qquad(|\Gamma(\alpha)|<\infty),$$

(iii) $$((z-c)^{-\alpha})_{-\alpha}=-e^{i\pi\alpha}\frac{1}{\Gamma(\alpha)}\log(z-c)\qquad(|\Gamma(\alpha)|<\infty),$$

where $z-c\neq 0$ for (i) and $z-c\neq 0,1$ for (ii), (iii) ,

(iv) $$(u\cdot v)_{\alpha}:=\sum_{k=0}^{\infty}\frac{\Gamma(\alpha+1)}{k!\,\Gamma(\alpha+1-k)}u_{\alpha-k}v_{k}\qquad\left(\begin{array}{l}u=u(z),\\ v=v(z)\end{array}\right).$$

§ 1. Preliminary

[I] The theorem D below is reported by K. Nishimoto already (cf. J. Frac. Calc.Vol.29, May (2006), p.37). [12]

Theorem D. We have

(i) $$(((z-b)^\beta - c)^\alpha)_\gamma = e^{-i\pi\gamma}(z-b)^{\alpha\beta-\gamma}$$

$$\times \sum_{k=0}^{\infty} \frac{[-\alpha]_k \Gamma(\beta k - \alpha\beta + \gamma)}{k!\,\Gamma(\beta k - \alpha\beta)} T^k \qquad \left(\left| \frac{\Gamma(\beta k - \alpha\beta + \gamma)}{\Gamma(\beta k - \alpha\beta)} \right| < \infty \right) \tag{1}$$

and

(ii) $$(((z-b)^\beta - c)^\alpha)_n = (-1)^n (z-b)^{\alpha\beta - n}$$

$$\times \sum_{k=0}^{\infty} \frac{[-\alpha]_k [\beta k - \alpha\beta]_n}{k!} T^k \qquad (n \in Z_0^+) . \tag{2}$$

where

$$T = \frac{c}{(z-b)^\beta}, \qquad |T| < 1, \tag{3}$$

and

$$[\lambda]_k = \lambda(\lambda+1)\cdots(\lambda+k-1) = \Gamma(\lambda-k)/\Gamma(\lambda) \ with \ [\lambda]_0 = 1 . \tag{4}$$

(Notation of Pochhammer

[II] The theorem E below for the fractional calculus of a logarithmic function is reported by K. Nishimoto already (cf. J. Frac. Calc.Vol.29, May (2006), p.40).[12]

Theorem E. We have

(i) $$(\log((z-b)^\beta - c))_\gamma = -e^{-i\pi\gamma}\beta(z-b)^{-\gamma}$$

$$\times \sum_{k=0}^{\infty} \frac{\Gamma(\beta k + \gamma)}{\Gamma(\beta k + 1)} T^k \qquad \left(\left| \frac{\Gamma(\beta k + \gamma)}{\Gamma(\beta k + 1)} \right| < \infty \right) \tag{5}$$

and.

(ii) $$(\log((z-b)^\beta - c))_n = (-1)^{n+1}\beta(z-b)^{-n}$$

$$\times \sum_{k=0}^{\infty} [\beta k + 1]_{n-1} T^k \qquad (n \in Z^+) \tag{6}$$

with (3), where

$$(z-b)^\beta - c \neq 0, 1 . \tag{7}$$

§ 2. N-Fractional Calculus of The Function in Title

Theorem 1. *Let be*

$$(z-b)^3 - c \neq 0,1 \qquad (1)$$

we have then

(i) $$(\log((z-b)^3-c))_\gamma = -e^{-i\pi\gamma}3(z-b)^{-\gamma}$$

$$\times \sum_{k=0}^{\infty} \frac{\Gamma(3k+\gamma)}{\Gamma(3k+1)} T^k \qquad (|\Gamma(3k+\gamma)|<\infty) \qquad (2)$$

and.

(ii) $$(\log((z-b)^3-c))_n = (-1)^{n+1}3(z-b)^{-n}$$

$$\times \sum_{k=0}^{\infty} [3k+1]_{n-1} T^k \qquad (n \in Z^+) \qquad (3)$$

(n-th derivatives)

where

$$T = \frac{c}{(z-b)^3}, \qquad |T|<1 \qquad (4)$$

Proof. Set $\beta = 3$ in Theorem E.

Corollary 1. *Let be*

$$z-b \neq 0,1 \qquad (5)$$

we have then

(i) $$(\log(z-b)^3)_\gamma = -e^{-i\pi\gamma}3\Gamma(\gamma)(z-b)^{-\gamma} \qquad (|\Gamma(\gamma)|<\infty) \qquad (6)$$

and

(ii) $$(\log(z-b)^3)_n = (-1)^{n+1}3\Gamma(n)(z-b)^{-n} \qquad (n \in Z^+) \qquad (7)$$

(n-th derivatives)

Proof. Set $c = 0$ in Theorem 1.

Note. We can obtain (6) and (7) from Lemma (ii) directly.

Theorem 2. *Let be*

$$(z-b)^{3/2} \pm \sqrt{c} \neq 0,1, \qquad (8)$$

we have then

(i) $\quad (\log((z-b)^3-c))_\gamma = -e^{-i\pi\gamma}\frac{3}{2}(z-b)^{-\gamma}$

$$\times\sum_{k=0}^{\infty}\frac{\Gamma(\frac{3}{2}k+\gamma)}{\Gamma(\frac{3}{2}k+1)}\{S^k+(-S)^k\} \qquad \left(|\Gamma(\tfrac{3}{2}k+\gamma)|<\infty\right) \tag{9}$$

and

(i i) $\quad (\log((z-b)^3-c))_n = (-1)^{n+1}\frac{3}{2}(z-b)^{-n}$

$$\times\sum_{k=0}^{\infty}\frac{\Gamma(\frac{3}{2}k+n)}{\Gamma(\frac{3}{2}k+1)}\{S^k+(-S)^k\} \qquad (n\in Z^+) \tag{10}$$

(n-th derivatives)

where

$$S=\frac{\sqrt{c}}{(z-b)^{3/2}}, \qquad |S|<1. \tag{11}$$

Proof of (i). We have

$$(\log((z-b)^3-c))_\gamma = (\log((z-b)^{3/2}-\sqrt{c}))_\gamma + (\log((z-b)^{3/2}+\sqrt{c}))_\gamma \,. \tag{12}$$

Now we have

$$(\log((z-b)^{3/2}-\sqrt{c}))_\gamma = -e^{-i\pi\gamma}\frac{3}{2}(z-b)^{-\gamma}$$

$$\times\sum_{k=0}^{\infty}\frac{\Gamma(\frac{3}{2}k+\gamma)}{\Gamma(\frac{3}{2}k+1)}S^k \qquad \left(|\Gamma(\tfrac{3}{2}k+\gamma)|<\infty\right) \tag{13}$$

from Theorem E. (i), setting $\beta=3/2$.

Therefore, we obtain (9) from (12), applying (13).

Proof of (ii). Set $\gamma=n$ in (9).

Note. We can obtain (6) and (7) from (9) and (10), setting $c=0$, clearly.

Theorem 3. *Let be* (1), *we have then*

(i) $\quad (\log((z-b)^3-c))_\gamma = -3e^{-i\pi\gamma}(z-b)^{-\gamma}\sum_{k=0}^{\infty}\frac{1}{\Gamma(3k+3)}T^k$

$$\times\ \{\Gamma(3k+2+\gamma)-2(\gamma-1)\Gamma(3k+1+\gamma)+(\gamma-1)(\gamma-2)\Gamma(3k+\gamma)\}$$

$$(|\Gamma(\gamma)|<\infty) \qquad (14)$$

and

(ii) $$(\log((z-b)^3-c))_n=(-1)^{n+1}3(z-b)^{-n}\sum_{k=0}^{\infty}\frac{1}{\Gamma(3k+3)}T^k$$

$$\times\ \{\Gamma(3k+2+n)-2(n-1)\Gamma(3k+1+n)+(n-1)(n-2)\Gamma(3k+n)\}.$$

$$(n\text{-th derivarives})\quad (n\in Z^+) \qquad (15)$$

where T is the one given by (4).

Proof of (i). We have

$$(\log((z-b)^3-c))_\gamma=((\log((z-b)^3-c))_1)_{\gamma-1} \qquad (16)$$

$$=3\,(((z-b)^3-c)^{-1}\cdot(z-b)^2)_{\gamma-1} \qquad (17)$$

$$=3\sum_{s=0}^{\infty}\frac{\Gamma(\gamma)}{m!\Gamma(\gamma-m)}(((z-b)^3-c)^{-1})_{\gamma-1-m}((z-b)^2)_m \qquad (18)$$

(by Lemma (iv))

$$=3\sum_{s=0}^{2}\frac{\Gamma(\gamma)}{m!\Gamma(\gamma-m)}\{e^{-i\pi(\gamma-1-m)}(z-b)^{-3-\gamma+1+m}\sum_{k=0}^{\infty}\frac{[1]_k\Gamma(3k+3+\gamma-1-m)}{k!\Gamma(3k+3)}T^k\}$$

$$\times\{e^{-i\pi m}\frac{\Gamma(m-2)}{\Gamma(-2)}(z-b)^{2-m}\} \qquad (19)$$

$$=-3\,e^{-i\pi\gamma}(z-b)^{-\gamma}\sum_{m=0}^{2}\frac{\Gamma(\gamma)\Gamma(m-2)}{m!\Gamma(\gamma-m)\Gamma(-2)}\sum_{k=0}^{\infty}\frac{\Gamma(3k+2+\gamma-m)}{\Gamma(3k+3)}T^k, \qquad (20)$$

$$=-3\,e^{-i\pi\gamma}(z-b)^{-\gamma}\{\sum_{k=0}^{\infty}\frac{\Gamma(3k+2+\gamma)}{\Gamma(3k+3)}T^k-2(\gamma-1)\sum_{k=0}^{\infty}\frac{\Gamma(3k+1+\gamma)}{\Gamma(3k+3)}T^k$$

$$+(\gamma-1)(\gamma-2)\sum_{k=0}^{\infty}\frac{\Gamma(3k+\gamma)}{\Gamma(3k+3)}T^k\}, \qquad (21)$$

applying

Theorem D. (i) to $(((z-b)^3-c)^{-1})_{\gamma-1-m}$

and Lemma (i) to $((z-b)^2)_m$.

We have then (14) from (21), clearly.

Proof of (ii). Set $\gamma = n$ in (i).

Theorem 4. *Let be (1), we have then*

$$\text{(i)}\quad (\log((z-b)^3-c))_\gamma = -3\,e^{-i\pi\gamma}(z-b)^{-\gamma}\sum_{k=0}^{\infty}\frac{[2]_k}{k!\,\Gamma(3k+6)}T^k$$
$$\times\{(1+2T)\Gamma(3k+4+\gamma)-(\gamma-2)(4+2T)\Gamma(3k+3+\gamma)$$
$$+6(\gamma-2)(\gamma-3)\Gamma(3k+2+\gamma)-4(\gamma-2)(\gamma-3)(\gamma-4)\Gamma(3k+1+\gamma)$$
$$+(\gamma-2)(\gamma-3)(\gamma-4)(\gamma-5)\Gamma(3k+\gamma)\}\qquad (|\Gamma(\gamma)|<\infty) \tag{22}$$

and

$$\text{(ii)}\quad (\log((z-b)^3-c))_n = (-1)^{n+1}\,3(z-b)^{-n}\sum_{k=0}^{\infty}\frac{[2]_k}{k!\,\Gamma(3k+6)}T^k$$
$$\times\{(1+2T)\Gamma(3k+4+n)-(n-2)(4+2T)\Gamma(3k+3+n)$$
$$+6(n-2)(n-3)\Gamma(3k+2+n)-4(n-2)(n-3)(n-4)\Gamma(3k+1+n)$$
$$+(n-2)(n-3)(n-4)(n-5)\Gamma(3k+n)\}\qquad (n\in Z^+\geq 2) \tag{23}$$

(n-th derivatives)

where T *is the one given by* (4).

Proof of (i). We have

$$(\log((z-b)^3-c))_\gamma = ((\log((z-b)^3-c))_2)_{\gamma-2} \tag{24}$$

$$= 3\,(((z-b)^3-c)^{-1}\cdot(z-b)^2)_1)_{\gamma-2} \tag{25}$$

$$= -3\,(((z-b)^3-c)^{-2}\cdot((z-b)^4+2c\,(z-b)))_{\gamma-2} \tag{26}$$

$$= -3\sum_{m=0}^{\infty}\frac{\Gamma(\gamma-1)}{m\,!\,\Gamma(\gamma-1-m)}(((z-b)^3-c)^{-2})_{\gamma-2-m}((z-b)^4+2c(z-b))_m \tag{27}$$

$$= -3e^{-i\pi\gamma}(z-b)^{-4-\gamma}\sum_{m=0}^{4}\frac{\Gamma(\gamma-1)}{m!\,\Gamma(\gamma-1-m)}\{e^{i\pi m}(z-b)^m\sum_{k=0}^{\infty}\frac{[2]_k\Gamma(3k+4+\gamma-m)}{k!\,\Gamma(3k+6)}T^k\}$$

$$\times((z-b)^4+2c(z-b))_m \tag{28}$$

$$= -3e^{-i\pi\gamma}(z-b)^{-4-\gamma}[\sum_{k=0}^{\infty}\frac{[2]_k\Gamma(3k+4+\gamma)}{k!\,\Gamma(3k+6)}T^k\{(z-b)^4+2c(z-b)\}$$

$$-(\gamma-2)(z-b)\sum_{k=0}^{\infty}\frac{[2]_k\Gamma(3k+3+\gamma)}{k!\,\Gamma(3k+6)}T^k\{4(z-b)^3+2c\}$$

$$+\frac{1}{2}(\gamma-2)(\gamma-3)(z-b)^2\sum_{k=0}^{\infty}\frac{[2]_k\Gamma(3k+2+\gamma)}{k!\,\Gamma(3k+6)}T^k\{12(z-b)^2\}$$

$$-\frac{1}{3!}(\gamma-2)(\gamma-3)(\gamma-4)(z-b)^3\sum_{k=0}^{\infty}\frac{[2]_k\Gamma(3k+1+\gamma)}{k!\,\Gamma(3k+6)}T^k\{24(z-b)\}$$

$$+\frac{1}{4!}(\gamma-2)(\gamma-3)(\gamma-4)(\gamma-5)(z-b)^4\sum_{k=0}^{\infty}\frac{[2]_k\Gamma(3k+\gamma)}{k!\,\Gamma(3k+6)}T^k\cdot 24], \tag{29}$$

applying

Theorem D. (i) to $(((z-b)^3-c)^{-2})_{\gamma-2-m}$

and

Lemma (i) to $((z-b)^4+2c(z-b))_m=((z-b)^4)_m+2c(z-b)_m$.

We have then (22) from (29).

Proof of (ii). Set $\gamma=n$ in (i).

§ 3. Identities

Theorem 5. *Let be* § **2**.(2) *and* § **2**.(9) *,we have then the identities as follows.*

(i)
$$\sum_{k=0}^{\infty}\frac{\Gamma(3k+\gamma)}{\Gamma(3k+1)}T^k=\frac{1}{2}\sum_{k=0}^{\infty}\frac{\Gamma(\frac{3}{2}k+\gamma)}{\Gamma(\frac{3}{2}k+1)}\{S^k+(-S)^k\} \tag{1}$$

$$=\sum_{k=0}^{\infty}\frac{1}{\Gamma(3k+3)}T^k\{\Gamma(3k+2+\gamma)-2(\gamma-1)\Gamma(3k+1+\gamma)$$

$$+(\gamma-1)(\gamma-2)\Gamma(3k+\gamma)\}\qquad(|\Gamma(\gamma)|<\infty) \tag{2}$$

$$= \sum_{k=0}^{\infty} \frac{[2]_k}{k!\,\Gamma(3k+6)} T^k \{(1+2T)\Gamma(3k+4+\gamma) - (\gamma-2)(4+2T)\Gamma(3k+3+\gamma)$$

$$+ 6(\gamma-2)(\gamma-3)\Gamma(3k+2+\gamma) - 4(\gamma-2)(\gamma-3)(\gamma-4)\Gamma(3k+1+\gamma)$$

$$+(\gamma-2)(\gamma-3)(\gamma-4)(\gamma-5)\Gamma(3k+\gamma)\} \qquad (|\Gamma(\gamma)|<\infty) \quad (3)$$

and

(ii) $$\sum_{k=0}^{\infty} \frac{\Gamma(3k+n)}{\Gamma(3k+1)} T^k = \frac{1}{2} \sum_{k=0}^{\infty} \frac{\Gamma(\frac{3}{2}k+n)}{\Gamma(\frac{3}{2}k+1)} \{S^k + (-S)^k\} \quad (4)$$

$$= \sum_{k=0}^{\infty} \frac{1}{\Gamma(3k+3)} T^k \{\Gamma(3k+2+n) - 2(n-1)\Gamma(3k+1+n)$$

$$+(n-1)(n-2)\Gamma(3k+n)\} \qquad (n \in Z^+) \quad (5)$$

$$= \sum_{k=0}^{\infty} \frac{[2]_k}{k!\,\Gamma(3k+6)} T^k \{(1+2T)\Gamma(3k+4+n) - (n-2)(4+2T)\Gamma(3k+3+n)$$

$$+ 6(n-2)(n-3)\Gamma(3k+2+n) - 4(n-2)(n-3)(n-4)\Gamma(3k+1+n)$$

$$+(n-2)(n-3)(n-4)(n-5)\Gamma(3k+n)\} \qquad (n \in Z^+ \geq 2) \quad (6)$$

where

$$T = \frac{c}{(z-b)^3}, \quad |T|<1, \quad \textit{and} \quad S = \frac{\sqrt{c}}{(z-b)^{3/2}}, \quad |S|<1,$$

respectively.

Proof I. It is clear from Theorem 1, 2, 3 and 4.

Proof II. We have

$$\text{RHS of } (1) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{\Gamma(\frac{3}{2}\cdot 2k+\gamma)}{\Gamma(\frac{3}{2}\cdot 2k+1)} 2S^{2k} \quad (7)$$

$$= \sum_{k=0}^{\infty} \frac{\Gamma(3k+\gamma)}{\Gamma(3k+1)} T^k = \text{LHS of } (1). \quad (8)$$

Proof III. We have

$$\text{RHS of }(2) = = \sum_{k=0}^{\infty}\frac{\Gamma(3k+2+\gamma)}{\Gamma(3k+3)}T^k - 2(\gamma-1)\sum_{k=0}^{\infty}\frac{\Gamma(3k+1+\gamma)}{\Gamma(3k+3)}T^k$$

$$+(\gamma-1)(\gamma-2)\sum_{k=0}^{\infty}\frac{\Gamma(3k+\gamma)}{\Gamma(3k+3)}T^k \qquad (9)$$

$$= \sum_{k=0}^{\infty}\frac{\Gamma(3k+\gamma)}{\Gamma(3k+1)}T^k\{\frac{(3k+1+\gamma)(3k+\gamma)}{(3k+2)(3k+1)} - 2(\gamma-1)\frac{(3k+\gamma)}{(3k+2)(3k+1)}$$

$$+(\gamma-1)(\gamma-2)\frac{1}{(3k+2)(3k+1)}\} \qquad (10)$$

$$= \sum_{k=0}^{\infty}\frac{\Gamma(3k+\gamma)}{\Gamma(3k+1)}T^k = \text{LH of }(1)\,. \qquad (11)$$

§ 4. Semi Derivatives and Integrals

Set $\gamma = 1/2$ and $-1/2$ in (i) of Theorem1 ~ 4, we have then the semi-derivatives and semi-integrals as follows.

(I) Semi - derivatives ;

1. $$(\log((z-b)^3-c))_{1/2} = 3i(z-b)^{-1/2}\sum_{k=0}^{\infty}\frac{\Gamma(3k+\frac{1}{2})}{\Gamma(3k+1)}\left(\frac{c}{(z-b)^3}\right)^k, \qquad (1)$$

2. $$(\log((z-b)^3-c))_{1/2} = i\frac{3}{2}(z-b)^{-1/2}\sum_{k=0}^{\infty}\frac{\Gamma(\frac{3}{2}k+\frac{1}{2})}{\Gamma(\frac{3}{2}k+1)}$$

$$\times\{\left(\frac{\sqrt{c}}{(z-b)^{3/2}}\right)^k + \left(\frac{-\sqrt{c}}{(z-b)^{3/2}}\right)^k\}\,, \qquad (2)$$

3. $$(\log((z-b)^3-c))_{1/2} = i3(z-b)^{-1/2}\sum_{k=0}^{\infty}\frac{1}{\Gamma(3k+3)}\left(\frac{c}{(z-b)^3}\right)^k$$

$$\times\{\Gamma(3k+\tfrac{5}{2}) + \Gamma(3k+\tfrac{3}{2}) + \tfrac{3}{4}\Gamma(3k+\tfrac{1}{2})\}\,. \qquad (3)$$

4. $$(\log((z-b)^3-c))_{1/2} = i\,3(z-b)^{-1/2}\sum_{k=0}^{\infty}\frac{[2]_k}{k!\,\Gamma(3k+6)}\left(\frac{c}{(z-b)^3}\right)^k$$

$$\times\{(1+2T)\Gamma(3k+\tfrac{9}{2})+\tfrac{3}{2}(4+2T)\Gamma(3k+\tfrac{7}{2})+6(\tfrac{3\cdot 5}{2^2})\Gamma(3k+\tfrac{5}{2})$$

$$+4(\tfrac{3\cdot 5\cdot 7}{2^3})\Gamma(3k+\tfrac{3}{2})+(\tfrac{3\cdot 5\cdot 7\cdot 9}{2^4})\Gamma(3k+\tfrac{1}{2})\} \quad (T=\frac{c}{(z-b)^3}) \ . \qquad (4)$$

(I) Semi - integrals;

1. $$(\log((z-b)^3-c))_{-1/2} = -3i(z-b)^{1/2}\sum_{k=0}^{\infty}\frac{\Gamma(3k-\frac{1}{2})}{\Gamma(3k+1)}\left(\frac{c}{(z-b)^3}\right)^k , \qquad (5)$$

2. $$(\log((z-b)^3-c))_{-1/2} = -i\frac{3}{2}(z-b)^{1/2}\sum_{k=0}^{\infty}\frac{\Gamma(\frac{3}{2}k-\frac{1}{2})}{\Gamma(\frac{3}{2}k+1)}$$

$$\times\left\{\left(\frac{\sqrt{c}}{(z-b)^{3/2}}\right)^k+\left(\frac{-\sqrt{c}}{(z-b)^{3/2}}\right)^k\right\} , \qquad (6)$$

3. $$(\log((z-b)^3-c))_{-1/2} = -i\,3(z-b)^{1/2}\sum_{k=0}^{\infty}\frac{1}{\Gamma(3k+3)}\left(\frac{c}{(z-b)^3}\right)^k$$

$$\times\{\Gamma(3k+\tfrac{3}{2})+3\Gamma(3k+\tfrac{1}{2})+\tfrac{15}{4}\Gamma(3k-\tfrac{1}{2})\} , \qquad (7)$$

4. $$(\log((z-b)^3-c))_{-1/2} = -i3(z-b)^{1/2}\sum_{k=0}^{\infty}\frac{[2]_k}{k!\,\Gamma(3k+6)}\left(\frac{c}{(z-b)^3}\right)^k$$

$$\times\{(1+2T)\Gamma(3k+\tfrac{7}{2})+\tfrac{5}{2}(4+2T)\Gamma(3k+\tfrac{5}{2})+6(\tfrac{5\cdot 7}{2^2})\Gamma(3k+\tfrac{3}{2})$$

$$+4(\tfrac{5\cdot 7\cdot 9}{2^3})\Gamma(3k+\tfrac{1}{2})+(\tfrac{5\cdot 7\cdot 9\cdot 11}{2^4})\Gamma(3k-\tfrac{1}{2})\} \quad (T=\frac{c}{(z-b)^3}) \ . \qquad (8)$$

§ 5. Example

1. **Examples for Theorem 1. (ii) and Theorem 2. (ii).**

(I) When $n=1$, we obtain the below from Theorem 1. (ii);

$$(\log((z-b)^3-c))_1 = 3\,(z-b)^{-1}\sum_{k=0}^{\infty} T^k \qquad (1)$$

$$= 3(z-b)^{-1}\sum_{k=0}^{\infty}\frac{[1]_k}{k!}T^k \quad (T=\frac{c}{(z-b)^3}) \qquad (2)$$

$$= 3(z-b)^{-1}(1-T)^{-1} \qquad (3)$$

$$= 3(z-b)^2((z-b)^3-c)^{-1}. \qquad (4)$$

This result coincides with the obtained one by classical calculus

$$\frac{d}{dz}\log((z-b)^3-c)\ .$$

(II) When $n=2$, we obtain the below from Theorem 2. (ii);

$$(\log((z-b)^3-c))_2 = -\frac{3}{2}\,(z-b)^{-2}\sum_{k=0}^{\infty}\frac{\Gamma(\frac{3}{2}k+2)}{\Gamma(\frac{3}{2}k+1)}\{S^k+(-S)^k\} \qquad (5)$$

$$= -\frac{3}{2}\,(z-b)^{-2}\sum_{k=0}^{\infty}\frac{[1]_k(\frac{3}{2}k+1)}{k!}\{S^k+(-S)^k\} \quad (S=\frac{\sqrt{c}}{(z-b)^{3/2}})\ . \qquad (6)$$

Now we have

$$\sum_{k=0}^{\infty}\frac{[1]_k}{k!}\{S^k+(-S)^k\} = (1-S)^{-1}+(1+S)^{-1} \qquad (7)$$

$$= \frac{2}{1-S^2} = 2(z-b)^3((z-b)^3-c)^{-1}\ , \qquad (8)$$

and

$$\sum_{k=0}^{\infty}\frac{[1]_k(\frac{3}{2}k)}{k!}\{S^k+(-S)^k\} = \frac{3}{2}\sum_{k=0}^{\infty}\frac{[1]_{k+1}}{k!}\{S^{k+1}+(-S)^{k+1}\} \qquad (9)$$

$$= \frac{3}{2}\sum_{k=0}^{\infty}\frac{[2]_k}{k!}\{S^{k+1}+(-S)^{k+1}\} \qquad (10)$$

$$=\frac{3}{2}\{S(1-S)^{-2}+(-S)(1+S)^{-2}\} \qquad (11)$$

$$=6c(z-b)^3((z-b)^3-c)^{-2}. \qquad (12)$$

Therefore, we obtain

$$(6)=-\frac{3}{2}(z-b)^{-2}[2(z-b)^3((z-b)^3-c)^{-1}+6c(z-b)^3((z-b)^3-c)^{-2}] \qquad (13)$$

$$=-3(z-b)((z-b)^3-c)^{-2}((z-b)^3+2c) \qquad (14)$$

using (8) and (12).

This result coincides with the obtained one by the Classical calculus

$$\frac{d^2}{dz^2}\log((z-b)^3-c) .$$

2. Examples for the identities.

(I) When $n=2$ we have

$$\sum_{k=0}^{\infty}(3k+1)T^k=\sum_{k=0}^{\infty}\frac{[2]_k}{k!}T^k(1+2T), \qquad (T=\frac{c}{(z-b)^3}) \qquad (15)$$

from §3. (6).

Indeed we have

$$\text{LHS of (15)}=3\sum_{k=0}^{\infty}kT^k+\sum_{k=0}^{\infty}T^k=3\sum_{k=0}^{\infty}\frac{[1]_k k}{k!}T^k+\sum_{k=0}^{\infty}\frac{[1]_k}{k!}T^k \qquad (16)$$

$$=3T(1-T)^{-2}+(1-T)^{-1} \qquad (17)$$

$$=(1-T)^{-2}(2T+1). \qquad (18)$$

And

$$\text{RHS of (15)}=(1+2T)\sum_{k=0}^{\infty}\frac{[2]_k}{k!}T^k=(1+2T)(1-T)^{-2} . \qquad (19)$$

That is, the identity (15) holds true.

(II) When $n=3$ we have

$$\sum_{k=0}^{\infty}(3k+2)(3k+1)T^k=\sum_{k=0}^{\infty}\frac{[2]_k}{k!}T^k\{(1+2T)(3k+6)-(4+2T)\}, \qquad (20)$$

from § 3. (6).

Indeed we have

$$\text{LHS of } (20) = 9\sum_{k=0}^{\infty}\frac{[1]_k k^2}{k!}T^k + 9\sum_{k=0}^{\infty}\frac{[1]_k k}{k!}T^k + 2\sum_{k=0}^{\infty}\frac{[1]_k}{k!}T^k \qquad (21)$$

$$= 9(1-T)^{-3}(T^2+T) + 9T(1-T)^{-2} + 2(1-T)^{-1} \qquad (22)$$

$$= 2(1-T)^{-3}(T^2+7T+1)\ , \qquad (23)$$

and

$$\text{RHS of } (20) = 3\sum_{k=0}^{\infty}\frac{[2]_k k}{k!}T^k + 6T\sum_{k=0}^{\infty}\frac{[2]_k k}{k!}T^k$$

$$+ 2\sum_{k=0}^{\infty}\frac{[2]_k}{k!}T^k + 10T\sum_{k=0}^{\infty}\frac{[2]_k}{k!}T^k \qquad (24)$$

$$= 6T(1-T)^{-3} + 12T^2(1-T)^{-3} + 2(1-T)^{-2} + 10T(1-T)^{-2} \qquad (25)$$

$$= 2(1-T)^{-3}(T^2+7T+1) = (23) = \text{LHS of } (20). \qquad (26)$$

That is, the identity (20) holds true.

Note. We have

1. $$\sum_{k=0}^{\infty}\frac{[\lambda]_k}{k!}T^k = (1-T)^{-\lambda}. \qquad (27)$$

2. $$\sum_{k=0}^{\infty}\frac{[2]_k k}{k!}T^k = \sum_{k=1}^{\infty}\frac{[2]_k}{(k-1)!}T^k = \sum_{k=0}^{\infty}\frac{[2]_{k+1}}{k!}T^{k+1} \qquad (28)$$

$$= T\sum_{k=0}^{\infty}\frac{\frac{2\Gamma(3+K)}{2\Gamma(2)}}{k!}T^k = 2T\sum_{k=0}^{\infty}\frac{[3]_k}{k!}T^k = 2T(1-T)^{-3}. \qquad (29)$$

3. $$\sum_{k=0}^{\infty}\frac{[1]_k k^2}{k!}T^k = \sum_{k=1}^{\infty}\frac{[1]_k k}{(k-1)!}T^k = T\sum_{k=0}^{\infty}\frac{[1]_{k+1}(k+1)}{k!}T^k \qquad (30)$$

$$= T\sum_{k=0}^{\infty}\frac{[2]_k(k+1)}{k!}T^k = T\{\sum_{k=1}^{\infty}\frac{[2]_k k}{k!}T^k + \sum_{k=0}^{\infty}\frac{[2]_k}{k!}T^k\}. \qquad (31)$$

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