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Univalence and starlikeness of a function defined by convolution of analytic function and hypergeometric function ${}_3F_2$

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Abstract

We consider functions defined by a condition of functions in the subclass $\mathcal{U}(\lambda)$ of analytic functions with generalized Gauss hypergeometric functions. In this paper, we give a condition of the parameter λ for which the function to be univalent and starlike.

1 Introduction

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \tag{1.1}$$

that are analytic in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}\ :\ |z|<1\}$, and let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of $f(z)$ that are univalent in $\mathbb{U}$.

Obradović and Ponnusamy define in [4] the class $\mathcal{U}(\lambda)$ of $f(z)\in\mathcal{A}$ satisfing the condition

$$\left|\left(\frac{z}{f(z)}\right)^2 f'(z)-1\right|\leq\lambda \quad (z\in\mathbb{U}) \tag{1.2}$$

for some real $\lambda>0$, where f' denotes the derivative of f with respect to the variable z. We set $\mathcal{U}(1)=\mathcal{U}$. It is easy to see that the the condition (1.2) is equivalent to

$$\left|z^2\left(\frac{1}{f(z)}-\frac{1}{z}\right)'\right|\leq\lambda \quad (z\in\mathbb{U}).$$

If $f(z)\in\mathcal{S}$ maps $\mathbb{U}$ onto a starlike domain (with respect to the origin), i.e. if $tw\in f(\mathbb{U})$ whenever $t\in[0,1]$ and $w\in f(\mathbb{U})$, then we say that f is a starlike function. The class of all starlike functions is denoted by $\mathcal{S}^*$. A necessary and sufficient condition for $f(z)\in\mathcal{A}$ to be starlike is that the inequality

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$$\mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0 \qquad (z \in \mathbf{U})$$

holds.

For these facts, the following lemmas hold.

Lemma 1 ([3]) *If* $f(z) \in \mathcal{U}(\lambda)$, $a := \dfrac{|f''(0)|}{2} \leq 1$ *and* $0 \leq \lambda \leq \dfrac{\sqrt{2-a^2}-a}{2}$, *then* $f(z) \in \mathcal{S}^*$.

Lemma 2 ([7]) *If* $f(z) = z + a_{n+1}z^{n+1} + \cdots$ $(n \geq 2)$ *belongs to* $\mathcal{U}(\lambda)$ *and*

$$0 \leq \lambda \leq \frac{n-1}{\sqrt{(n-1)^2+1}},$$

then $f(z) \in \mathcal{S}^*$.

For analytic functions $f(z)$ and $g(z)$ on $\mathbf{U}$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, the power series $\sum_{n=0}^{\infty} a_n b_n z^n$ is said the convolution of $f(z)$ and $g(z)$, denoted by $f * g$ (cf ([5])]).

For $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ in $\mathcal{A}$, we have a natural convolution operator defined by

$$zF(a,b;c;z) * f(z) := \sum_{n=1}^{\infty} \frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}} a_n z^n, \qquad c \in \{-1,\ -2,\ -3,\ \cdots\}, z \in \mathbf{U},$$

where $(a)_n$ denotes the Pochhammer symbol $(a)_0 = 1$, $(a)_n = a(a+1)\cdots(a+n-1)$ for $n \in \mathbb{N}$. Here $F(a,b;c;z)$ denotes the Gauss hypergeometric function which is analytic in $\mathbf{U}$. As a special case of the Euler integral representation for the hypergeometric function, one has

$$F(1,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{1}{1-tz} t^{b-1}(1-t)^{c-b-1} dt, \quad z \in \mathbf{U},\ \mathrm{Re}\, c > \mathrm{Re}\, b > 0.$$

Using this representation, we have, for $f(z) \in \mathcal{A}$,

$$zF(1,c;c+1;z) * f(z) = z\left(F(1,c;c+1;z) * \frac{f(z)}{z}\right) = zc\int_0^1 \frac{f(tz)}{tz} t^{c-1} dt, z \in \mathbf{U},\ \mathrm{Re}\, c > 0.$$

Obradović and Ponnusamy have shown the following result.

Theorem A ([5])
Let $f \in \mathcal{U}(\lambda)$ *and* $c \in \mathbb{C}$ *with* $\mathrm{Re}\, c > 0$ *such that*

$$\left(\frac{z}{f(z)}\right) * F(1,c;c+1;z) \neq 0 \quad \text{in} \quad z \in \mathbf{U},$$

and $G(z)=G_f^c(z)$ *be the transformed function defined by*

$$G(z)=\frac{z}{\left(\frac{z}{f(z)}\right)*F(1,c;c+1;z)}\qquad (z\in\mathbf{U}).$$

Then we have the following;

(1) $G\in\mathcal{U}\left(\frac{\lambda|c|}{|c+2|}\right)$. *The result is sharp especially when* $\left|\frac{f''(0)}{2}\right|\le 1-\lambda$. *In particular,* $G\in\mathcal{U}$ *whenever* $0<\lambda\le\left|\frac{c+2}{c}\right|$.

(2) $G\in\mathcal{S}^*$ *whenever* $0<\lambda\le\frac{|c+2|}{2|c|}(\sqrt{2-A^2}-A)$ *with* $A=\left|\frac{c}{c+1}\frac{f''(0)}{2}\right|\le 1$.

2 Main Result

For the generalized hypergeometric function ${}_3F_2(1,\alpha,\beta;\alpha+1,\beta+1;z)$, we obtain

Theorem1

Let $f(z)\in\mathcal{U}(\lambda)$. *Let* $\alpha,\ \beta\in\mathbb{C}$ *satisfying*

$$\text{Re }\alpha\ge 0,\ \text{Re }\beta\ge 0,\frac{1}{|\alpha+\beta|}\left(\frac{|\alpha||\beta|}{|\beta+2|}+\frac{|\beta||\alpha|}{|\alpha+2|}\right)<1 \text{ and } |\alpha+\beta|>|\alpha\beta|$$

and

$$\frac{z}{f(z)}*{}_3F_2(1,\alpha,\beta;\alpha+1,\beta+1;z)\neq 0,\quad z\in\mathbf{U}.$$

Denote by $G(z)=G_f^{\alpha,\beta}(z)$ *the function defined by*

$$G(z)=\frac{z}{\frac{z}{f(z)}*{}_3F_2(1,\alpha,\beta;\alpha+1,\beta+1;z)},\quad z\in\mathbf{U},\tag{2.1}$$

where ${}_3F_2(1,\alpha,\beta;\alpha+1,\beta+1;z)$ *is the generalized hypergeometric function. Then we have the following:*

(1) $G(z)\in\mathcal{U}\left(\frac{\lambda|\alpha+\beta|}{|\alpha+\beta+4|}\right)$. *The result is sharp especially when* $\left|\frac{f''(0)}{2}\right|\le 1-\lambda$.

In particular, $G(z)\in\mathcal{U}$ *whenever* $0<\lambda\le\frac{|\alpha+\beta+4|}{|\alpha+\beta|}$.

(2) $G(z)\in\mathcal{S}^*$

whenever $0<\lambda\le\frac{|\alpha+\beta+4|}{2|\alpha+\beta|}(\sqrt{2-A^2}-A)$ *with* $A=\left|\frac{\alpha\beta}{(\alpha+1)(\beta+1)}\frac{f''(0)}{2}\right|\le 1$.

Proof.

Since

$${}_3F_2(1,\alpha,\beta;\alpha+1,\beta+1;z)=\sum_{n=0}^{\infty}\frac{\alpha\beta}{(\alpha+n)(\beta+n)}z^n=1+\sum_{n=1}^{\infty}\frac{\alpha\beta}{(\alpha+n)(\beta+n)}z^n,\tag{2.2}$$

we have

$$\frac{z}{f(z)} * {}_3F_2(1,\alpha,\beta;\alpha+1,\beta+1;z) = 1 - \frac{\alpha\beta a_2}{(\alpha+1)(\beta+1)}z + \frac{\alpha\beta(a_2^2-a_3)}{(\alpha+2)(\beta+2)}z^2 + \cdots$$

$$= \left\{1 - \frac{\alpha a_2}{\alpha+1}z + \frac{\alpha(a_2^2-a_3)}{\alpha+2}z^2 + \cdots\right\} * \left\{1 - \frac{\beta a_2}{\beta+1}z + \frac{\beta(a_2^2-a_3)}{\beta+2}z^2 + \cdots\right\}$$

$$= \left\{\frac{z}{f(z)} * F(1,\alpha;\alpha+1;z)\right\} * F(1,\beta;\beta+1;z).$$

Thus $G(z)$ can be written as

$$G(z) = \frac{z}{\left\{\frac{z}{f(z)} * F(1,\alpha;\alpha+1;z)\right\} * F(1,\beta;\beta+1;z)}.$$

In the same manner, $G(z)$ can be also written as

$$G(z) = \frac{z}{\left\{\frac{z}{f(z)} * F(1,\beta;\beta+1;z)\right\} * F(1,\alpha;\alpha+1;z)}.$$

Put

$$h_1(z) = \frac{z}{\frac{z}{f(z)} * F(1,\alpha;\alpha+1;z)}, \qquad h_2(z) = \frac{z}{\frac{z}{f(z)} * F(1,\beta;\beta+1;z)}.$$

then

$$\frac{z}{f(z)} * F(1,\alpha;\alpha+1;z) = \frac{z}{h_1(z)}, \qquad \frac{z}{f(z)} * F(1,\beta;\beta+1;z) = \frac{z}{h_2(z)}$$

By the Theorem A in the introduction, we have

$$h_1(z) \in \mathcal{U}\left(\frac{\lambda|\alpha|}{|\alpha+2|}\right) \quad i.e. \quad \left|\left(\frac{z}{h_1(z)}\right)^2 h_1'(z) - 1\right| < \frac{\lambda|\alpha|}{|\alpha+2|}$$

and

$$h_2(z) \in \mathcal{U}\left(\frac{\lambda|\beta|}{|\beta+2|}\right) \quad i.e. \quad \left|\left(\frac{z}{h_2(z)}\right)^2 h_2'(z) - 1\right| < \frac{\lambda|\beta|}{|\beta+2|}.$$

Since

$$\frac{z}{G(z)} = \frac{z}{h_1(z)} * F(1,\beta;\beta+1;z) \quad (z \in \mathbf{U}),$$

we have

$$(\beta+1)\frac{z}{G(z)} - \left(\frac{z}{G(z)}\right)^2 G'(z) = \beta\frac{z}{G(z)} + z\left(\frac{z}{G(z)}\right)'. \tag{2.3}$$

On the other hand, $\frac{z}{G(z)}$ can be also written as

$$\frac{z}{G(z)} = \frac{z}{h_2(z)} * F(1,\alpha;\alpha+1;z) \quad (z \in \mathbf{U}),$$

we have

$$(2.4)\qquad (\beta+1)\frac{z}{G(z)}-\left(\frac{z}{G(z)}\right)^2 G'(z)=\beta\frac{z}{G(z)}+z\left(\frac{z}{G(z)}\right)'.$$

Then we have

$$(2.5)\qquad (\alpha+1)\frac{z}{G(z)}-\left(\frac{z}{G(z)}\right)^2 G'(z)=\alpha\frac{z}{h_2(z)}\quad (z\in \mathbb{U})$$

and

$$(2.6)\qquad (\beta+1)\frac{z}{G(z)}-\left(\frac{z}{G(z)}\right)^2 G'(z)=\beta\frac{z}{h_1(z)}\quad (z\in \mathbb{U}).$$

Set

$$p(z)=\left(\frac{z}{G(z)}\right)^2 G'(z).$$

Then $p(z)$ is analytic on $\mathbb{U}$ with $p(0)=1$ and $p'(0)=0$, and

$$(2.7)\qquad p(z)=(\alpha+1)\frac{z}{G(z)}-\alpha\frac{z}{h_2(z)}$$

and

$$(2.8)\qquad p(z)=(\beta+1)\frac{z}{G(z)}-\beta\frac{z}{h_1(z)}.$$

From (2.3), (2.4), (2.5), (2.6), (2.7) and (2.8) one then obtain that

$$\begin{aligned}\alpha p(z)+zp'(z) &= (\alpha+1)\alpha\frac{z}{G(z)}+(\alpha+1)z\left(\frac{z}{G(z)}\right)'-\alpha^2\frac{z}{h_2(z)}-\alpha z\left(\frac{z}{h_2(z)}\right)'\\ &= \alpha\left[(\alpha+1)\frac{z}{h_2(z)}-\alpha\frac{z}{h_2(z)}-z\left(\frac{z}{h_2(z)}\right)'\right]\\ &= \alpha\left[\frac{z}{h_2(z)}-z\left(\frac{z}{h_2(z)}\right)'\right]\\ &= \alpha\left(\frac{z}{h_2(z)}\right)^2 h_2'(z)\end{aligned}$$

and

$$\begin{aligned}\beta p(z)+zp'(z) &= (\beta+1)\beta\frac{z}{G(z)}+(\beta+1)z\left(\frac{z}{G(z)}\right)'-\beta^2\frac{z}{h_1(z)}-\beta z\left(\frac{z}{h_1(z)}\right)'\\ &= \beta\left[(\beta+1)\frac{z}{h_1(z)}-\beta\frac{z}{h_1(z)}-z\left(\frac{z}{h_1(z)}\right)'\right]\\ &= \beta\left[\frac{z}{h_1(z)}-z\left(\frac{z}{h_1(z)}\right)'\right]\\ &= \beta\left(\frac{z}{h_1(z)}\right)^2 h_1'(z).\end{aligned}$$

Since

$$(\alpha+\beta)p(z)+2zp'(z)=\alpha\left(\frac{z}{h_2(z)}\right)^2 h_2'(z)+\beta\left(\frac{z}{h_1(z)}\right)^2 h_1'(z),$$

we have

$$p(z)+\frac{2}{\alpha+\beta}zp'(z)=\frac{\alpha}{\alpha+\beta}\left(\frac{z}{h_2(z)}\right)^2 h_2'(z)+\frac{\beta}{\alpha+\beta}\left(\frac{z}{h_1(z)}\right)^2 h_1'(z).$$

Now, as $h_1(z)\in\mathcal{U}\left(\frac{\lambda|\alpha|}{|\alpha+2|}\right)$ and $h_2(z)\in\mathcal{U}\left(\frac{\lambda|\beta|}{|\beta+2|}\right)$, it follows that

$$\begin{aligned}\left|p(z)+\frac{2}{\alpha+\beta}zp'(z)-1\right| &= \left|\frac{\alpha}{\alpha+\beta}\left\{\left(\frac{z}{h_2(z)}\right)^2 h_2'(z)-1\right\}+\frac{\beta}{\alpha+\beta}\left\{\left(\frac{z}{h_1(z)}\right)^2 h_1'(z)-1\right\}\right| \\ &\le \left|\frac{\alpha}{\alpha+\beta}\right|\left|\left(\frac{z}{h_2(z)}\right)^2 h_2'(z)-1\right|+\left|\frac{\beta}{\alpha+\beta}\right|\left|\left(\frac{z}{h_1 z)}\right)^2 h_1'(z)-1\right| \\ &< \frac{|\alpha|}{|\alpha+\beta|}\frac{\lambda|\beta|}{|\beta+2|}+\frac{|\beta|}{|\alpha+\beta|}\frac{\lambda|\alpha|}{|\alpha+2|} \\ &= \lambda\left\{\frac{1}{|\alpha+\beta|}\left(\frac{|\alpha||\beta|}{|\beta+2|}+\frac{|\beta||\alpha|}{|\alpha+2|}\right)\right\}.\end{aligned}$$

By the assumption, we have

$$\left|p(z)+\frac{2}{\alpha+\beta}zp'(z)-1\right|<\lambda. \tag{2.9}$$

From the work of Hallenbeck and Rusheweyh ([2],[6]), we deduce that

$$|p(z)-1|\le\frac{\lambda|\alpha+\beta|}{|\alpha+\beta+4|}\quad(z\in\mathbf{U}). \tag{2.10}$$

Thus we have $G(z)\in\mathcal{U}\left(\frac{\lambda|\alpha+\beta|}{|\alpha+\beta+4|}\right)$.

To prove the sharpness, we consider functions $f(z)$ in $\mathcal{U}(\lambda)$ of the form

$$f(z)=\frac{z}{1-a_2z+\lambda z^2},$$

where $a_2=\frac{f''(0)}{2}$ and $|a_2|\le 1-\lambda$, so that $1-a_2z+\lambda z^2\neq 0$ for all $z\in\mathbf{U}$. Since Re $\alpha\ge 0$ and Re $\beta\ge 0$, it follows that $|\alpha+2|>|\alpha+1|>|\alpha|$ and $|\beta+2|>|\beta+1|>|\beta|$ and, therefore

$$\left|1-a_2\frac{\alpha\beta}{(\alpha+1)(\beta+1)}z+\lambda\frac{\alpha\beta}{(\alpha+2)(\beta+2)}z^2\right|\neq 0$$

for all $z\in\mathbf{U}$, provided $|a_2|\le 1-\lambda$. By the series expantion (2.2) of ${}_3F_2(1,\alpha,\beta;\alpha+1,\beta+1;z)$, we have

$$G(z)=\frac{z}{1-\frac{a_2\alpha\beta}{(\alpha+1)(\beta+1)}z+\frac{\lambda(\alpha\beta)}{(\alpha+2)(\beta+2)}z^2}.$$

Obviously, $G(z)$ is analytic on $\mathbb{U}$ and $\frac{z}{G(z)} \neq 0$ **on U**. Since

$$\left(\frac{z}{G(z)}\right)^2 G'(z) - 1 = -\frac{\lambda\alpha\beta}{(\alpha+2)(\beta+2)}z^2,$$

we have that

$$\left|\left(\frac{z}{G(z)}\right)^2 G'(z) - 1\right| \leq \frac{\lambda|\alpha\beta|}{|(\alpha+2)(\beta+2)|}. \tag{2.11}$$

Now, let us compare the right hand sides of (2.10) and (2.11). Firstly, since $|\alpha+\beta+4| < |(\alpha+2)(\beta+2)|$, then $\frac{1}{|(\alpha+2)(\beta+2)|} < \frac{1}{|\alpha+\beta+4|}$. From the assumption, we see

$$\frac{|\alpha\beta|}{|(\alpha+2)(\beta+2)|} < \frac{|\alpha+\beta|}{|(\alpha+2)(\beta+2)|} < \frac{|\alpha+\beta|}{|\alpha+\beta+4|}.$$

Then, we have that

$$\left|\left(\frac{z}{G(z)}\right)^2 G'(z) - 1\right| \leq \frac{\lambda|\alpha\beta|}{|(\alpha+2)(\beta+2)|} < \frac{|\alpha+\beta|}{|\alpha+\beta+4|}.$$

Thus, we have that the bound $\frac{|\alpha+\beta|}{|\alpha+\beta+4|}$ is sharp. We conclude that the first assertion of Theorem 1.

The second assertion is a direct consequence of Lemma 1. In fact, obviously

$$A = \frac{G''(0)}{2} = \frac{\alpha\beta}{(\alpha+1)(\beta+1)}\frac{f''(0)}{2}$$

is smaller than or equal to 1.

Theorem 2
For a fixed $n \geq 2$, let $f(z) = z + a_{n+1}z^{n+1} + \cdots$ belong to $\mathcal{U}(\lambda)$. Let α, $\beta \geq 0$ and

$$\text{Re}\ \alpha \geq 0,\ \text{Re}\ \beta \geq 0, \frac{1}{|\alpha+\beta|}\left(\frac{|\alpha||\beta|}{|\beta+n|} + \frac{|\alpha||\beta|}{|\alpha+n|}\right) < 1,$$

and

$$\frac{z}{f(z)} * {}_3F_2(1,\alpha,\beta;\alpha+1,\beta+1;z) \neq 0, \quad z \in \mathbb{U}.$$

and $G(z) = G_f^{\alpha,\beta}(z)$ be the transform function defined by (2.1). Then we have the following:

(1) $G(z) \in \mathcal{U}\left(\frac{\lambda|\alpha+\beta|}{|\alpha+\beta+2n|}\right)$. *In paticular, $G(z) \in \mathcal{U}$ whenever $0 < \lambda \leq \frac{|\alpha+\beta+2n|}{|\alpha+\beta|}$.*

(2) $G(z) \in S^*$ *whenever* $0 < \lambda \leq \frac{(n-1)|\alpha+\beta+2n|}{|\alpha+\beta|\sqrt{(n-1)^2+1}}$.

Proof. Using the Gaussian hypergeometric function, $G(z)$ can be written as

$$G(z) = \frac{z}{\left\{\frac{z}{f(z)} * F(1, \alpha; \alpha+1; z)\right\} * F(1, \beta; \beta+1; z)}$$

and

$$G(z) = \frac{z}{\left\{\frac{z}{f(z)} * F(1, \beta; \beta+1; z)\right\} * F(1, \alpha; \alpha+1; z)}.$$

Put

$$h_3(z) = \frac{z}{\frac{z}{f(z)} * F(1, \alpha; \alpha+1; z)}, \quad h_4(z) = \frac{z}{\frac{z}{f(z)} * F(1, \beta; \beta+1; z)}.$$

Then

$$\frac{z}{f(z)} * F(1, \alpha; \alpha+1; z) = \frac{z}{h_3(z)}, \quad \frac{z}{f(z)} * F(1, \beta; \beta+1; z) = \frac{z}{h_4(z)},$$

We see

$$h_3(z) \in \mathcal{U}\left(\frac{\lambda|\alpha|}{|\alpha+n|}\right) \quad i.e. \quad \left|\left(\frac{z}{h_3(z)}\right)^2 h_3'(z) - 1\right| < \frac{\lambda|\alpha|}{|\alpha+n|}$$

and

$$h_4(z) \in \mathcal{U}\left(\frac{\lambda|\beta|}{|\beta+n|}\right) \quad i.e. \quad \left|\left(\frac{z}{h_4(z)}\right)^2 h_4'(z) - 1\right| < \frac{\lambda|\beta|}{|\beta+n|}.$$

Since

$$\frac{z}{f(z)} = \frac{1}{1 + a_{n+1}z^n + \cdots} = 1 - a_{n+1}z^n + \cdots,$$

so that

$$\frac{z}{f(z)} * {}_3F_2(1, \alpha, \beta; \alpha+1, \beta+1; z) = 1 - a_{n+1}\left\{\frac{\alpha\beta}{(\alpha+n)(\beta+n)}\right\} z^n + \cdots.$$

Thus, $G(z)$ can be written in the form

$$G(z) = z + a_{n+1}\left\{\frac{\alpha\beta}{(\alpha+n)(\beta+n)}\right\} z^{n+1} + \cdots.$$

Therefore, as in the proof of Theoreml, the function $p(z)$ defined by

$$p(z) = \left(\frac{z}{G(z)}\right)^2 G'(z) = 1 + (n-1)a_{n+1}\left\{\frac{\alpha\beta}{(\alpha+n)(\beta+n)}\right\} z^n + \cdots$$

is analytic in U and $p(0) = 1$, $p'(0) = \cdots = p^{(n-1)}(0) = 0$. $p(z)$ can be written as

$$p(z) = (\alpha+1)\frac{z}{G(z)} - \alpha\frac{z}{h_3(z)}$$

and

$$p(z) = (\beta+1)\frac{z}{G(z)} - \beta\frac{z}{h_4(z)}.$$

By the same argument of proof of Therorem 1 using $h_3(z)$ and $h_4(z)$ instead of $h_1(z)$ and $h_2(z)$, $p(z)$ satisfies (2.9). Consequentry, we obtain that

$$|p(z) - 1| \leq \frac{\lambda|\alpha+\beta||z|^n}{|\alpha+\beta+2n|} \quad (z \in \mathbb{U}),$$

and the proof of part(1) is complete. The second part is a direct consequence of Lemma 2.

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