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# On Algebraic Structures of Petri Net Morphisms based on Place Connectivity

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## 1 Introduction

A Petri net is a useful mathematical model applied to descriptions of various parallel processing systems. So far, some types of morphisms related to Petri nets (or condition/event net) have been studied in terms of the category theory, in order to investigate the relationship between different Petri nets and understand the concurrency in other computation models [4][10].

Usually such a morphism is defined based on connection of transitions and thier nearby places. It is one of necessary conditions that such morphisms commute with the transition function of a Petri net.

Studying how the structure of Petri nets have an effect on Petri net languages and codes, we often realize that the ratio between the number of tokens in a place and the weights of edges connected to the place is important. We give our definition of morphims between Petri nets focusing on the connection state/level of edges which come in or go out a place. This is an extension of an automorphism which we used to introduce to a net in [5][6].

After summarising the monoid of all surjective morphisms of a Petri net and ideals in the monoid, we state the decomposition of automorphism group $G = \mathbf{Aut}(\mathcal{P})$ of a Petri net $\mathcal{P}$ into $G = KN = NK$, where $N$ is a kind of normal subgroup of $G$.

## 2 Preliminaries

Here we give our definition of morphisms of a Petri net and state the properties of some monoids composed of these morphisms.

### 2.1 Petri Nets and Morphisms

In this section, we give definitions and fundamental properties related to Petri nets. We denote the set of all nonnegative integers by $\boldsymbol{N}_0$, that is, $\boldsymbol{N}_0 = \{0, 1, 2, \ldots\}$.

First of all, a Petri net is viewed as a particular kind of directed graph, together with an initial state $\mu_0$, called the *initial marking*. The underlying graph $N$ of a Petri net is a directed, weighted, bipartite graph consisting of two kinds of nodes, called *places* and *transitions*, where arcs are either from a place to a transition or from a transition to a place.

**DEFINITION 2.1 (Petri net)** A *Petri net* is a 4-tuple $(P, T, W, \mu_0)$ where
(1) $P = \{p_1, p_2, \ldots, p_m\}$ is a finite set of places,
(2) $T = \{t_1, t_2, \ldots, t_n\}$ is a finite set of transitions,
(3) $W : \boldsymbol{E}(P,T) \to \{0, 1, 2, 3, \ldots\}$, i.e.,$W \in {\boldsymbol{N}_0}^{\boldsymbol{E}(P,T)}$, is a *weight function*, where $\boldsymbol{E}(P,T) = (P \times T) \cup (T \times P)$,
(4) $\mu_0 : P \to \{0, 1, 2, 3, \ldots\}$, i.e., $\mu_0 \in {\boldsymbol{N}_0}^P$, is the initial marking,
(5) $P \cap T = \emptyset$ and $P \cup T \neq \emptyset$.

A Petri net structure (net, for short) $N = (P, T, W)$ without any specific initial marking is denoted by $N$, a Petri net with a given initial marking $\mu_0$ is denoted by $(N, \mu_0)$. □

In the graphical representation, the places are drawn as circles and the transitions are drawn as bars or boxes. Arcs are labeled with their weights(positive integers), where a $k$-weighted arc can be interpreted as the set of $k$ parallel arcs. Labels for unity weights are usually omitted. A marking (state) assigns a nonnegative integer $k$ to each place. If a marking assigns a nonnegative integer $k$ to a place $p$, we say that $p$ is *marked with* $k$ tokens. Pictorially, we put $k$ black dots (tokens) in place $p$. A marking is denoted by $\mu$, an $n$-dimensional row vector, where $n$ is the total number of places. The $i$-th component of $\mu$, denoted by $\mu(p_i)$, is the number of tokens in the $i$-th place $p_i$.

**EXAMPLE 2.1** Fig. 1 shows a graphical representation of a Petri net $\mathcal{P} = (P, T, W, \mu_0)$. $P = \{\mathbf{a}, \mathbf{b}\}$ and $T = \{\mathbf{t}\}$. $(\mathbf{a}, \mathbf{t})$ and $(\mathbf{t}, \mathbf{b})$ are arcs of weights 2 and 1 respectively. $(\mathbf{t}, \mathbf{a})$ and $(\mathbf{b}, \mathbf{t})$ are arcs of weight 0, which are not usually drawn in the picture. Note that the weight of $(\mathbf{t}, \mathbf{b})$ is omitted since it is unity. That is, $W(\mathbf{a}, \mathbf{t}) = 2, W(\mathbf{b}, \mathbf{t}) = 1, W(\mathbf{t}, \mathbf{a}) = W(\mathbf{b}, \mathbf{t}) = 0$. The initial marking $\mu_0$ with $\mu_0(\mathbf{a}) = 3, \mu_0(\mathbf{b}) = 0$ is often written like a row vector $\mu_0 = (3, 0)$. □

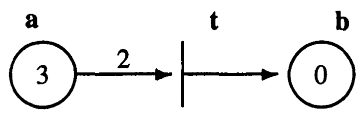


**Figure 1. Graphical representation of a Petri net**

Now we introduce a Petri net morphism based on place connectivity. We denote the set of all positive rational numbers by $\boldsymbol{Q}_+$.

**DEFINITION 2.2** Let $\mathcal{P}_1 = (P_1, T_1, W_1, \mu_1)$ and $\mathcal{P}_2 = (P_2, T_2, W_2, \mu_2)$ be Petri nets. Then a triple $(f, (\alpha, \beta))$ of maps is called a *morphism* from $\mathcal{P}_1$ to $\mathcal{P}_2$ if the maps $f : P_1 \to \boldsymbol{Q}_+$, $\alpha : P_1 \to P_2$ and $\beta : T_1 \to T_2$ satisfy the condition that for any $p \in P_1$ and $t \in T_1$,

$$\begin{aligned} W_2(\alpha(p), \beta(t)) &= f(p)W_1(p,t), \\ W_2(\beta(t), \alpha(p)) &= f(p)W_1(t,p), \\ \mu_2(\alpha(p)) &= f(p)\mu_1(p). \end{aligned} \tag{2.1}$$

In this case we write $(f, (\alpha, \beta)) : \mathcal{P}_1 \to \mathcal{P}_2$. □

The morphism $(f, (\alpha, \beta)) : \mathcal{P}_1 \to \mathcal{P}_2$ is called *injective* (resp. *surjective*) if both $\alpha$ and $\beta$ are injective (resp. surjective). In particular, it is called an *isomorphism* from $\mathcal{P}_1$ to $\mathcal{P}_2$ if it is injective and surjective. Then $\mathcal{P}_1$ is said to be *isomorphic* to $\mathcal{P}_2$ and we write $\mathcal{P}_1 \simeq \mathcal{P}_2$. Moreover, in case of $\mathcal{P}_1 = \mathcal{P}_2$, an isomorphism is called an *automorphism* of $\mathcal{P}_1$. By $\mathbf{Aut}(\mathcal{P})$ we denote the set of all the automorphisms of $\mathcal{P}$.

For Petri nets $\mathcal{P}_1$ and $\mathcal{P}_2$, we write $\mathcal{P}_1 \sqsupseteq \mathcal{P}_2$ if there exists a surjective morphism from $\mathcal{P}_1$ to $\mathcal{P}_2$. The relation $\sqsupseteq$ forms a pre-order (a relation satisfying the reflexive law and the transitive law) as shown below. Of course, the pre-order is regarded as an order by identifying isomorphisms.

**PROPOSITION 2.1** *Let $\mathcal{P}_1, \mathcal{P}_2, \mathcal{P}_3$ be Petri nets. Then,*
*(1)* $\mathcal{P}_1 \sqsupseteq \mathcal{P}_1$.
*(2)* $\mathcal{P}_1 \sqsupseteq \mathcal{P}_2$ *and* $\mathcal{P}_2 \sqsupseteq \mathcal{P}_1 \iff \mathcal{P}_1 \simeq \mathcal{P}_2$.
*(3)* $\mathcal{P}_1 \sqsupseteq \mathcal{P}_2$ *and* $\mathcal{P}_2 \sqsupseteq \mathcal{P}_3$ *imply* $\mathcal{P}_1 \sqsupseteq \mathcal{P}_3$. □

**DEFINITION 2.3 (Similar)** *Let $\mathcal{P} = (P, T, W, \mu)$ be a Petri net. Two places $p, q \in P$ are said to be similar if there exists some positive rational number $r$ such that $\mu(p) = r\mu(q)$, $W(q,t) = rW(p,t)$ and $W(t,q) = rW(t,p)$ for all $t \in T$. Two transitions $s, t \in T$ are said to be similar if $W(p,s) = W(p,t)$ and $W(s,p) = W(t,p)$ for all $p \in P$.* □

The similarity defined above is obviously an equivalence relation on $P \cup T$. We denote this relation by $\sim_{\mathcal{P}}$ or simply $\sim$ and the $\sim_{\mathcal{P}}$-class of a place or a transition $u$ by $C(u)$. A place (resp. a transition) is said to be *isolated* if it has no connection to any transitions (resp. any places). Especially, a place $p$ is 0-isolated if it is isolated and $\mu(p) = 0$. Note that two 0-isolated places $p$ and $q$ are similar because for any positive rational number $r$ $\mu(p) = 0 = r\mu(q)$, $W(q,t) = 0 = rW(p,t)$ and $W(t,q) = 0 = rW(t,p)$ for all $t \in T$.

## 2.2 Monoids $\mathcal{S}$ of Surjective Morphisms of Petri Nets

We introduce a composition of morphisms; all the morphisms between Petri nets form a monoid under this composition.

Let $\mathcal{P}_i = (P_i, T_i, W_i, \mu_i)$ $(i = 1,2,3)$ be Petri nets, $(f,\ (\alpha,\beta)) : \mathcal{P}_1 \to \mathcal{P}_2$ and $(g,\ (\gamma,\delta)) : \mathcal{P}_2 \to \mathcal{P}_3$ be morphisms. Then,

$$\begin{aligned} W_3(\gamma(\alpha(p)), \delta(\beta(t))) &= g(\alpha(p))W_2(\alpha(p), \beta(t)) \\ &= g(\alpha(p))f(p)W_1(p,t), \\ W_3(\delta(\beta(t)), \gamma(\alpha(p))) &= g(\alpha(p))W_2(\beta(t), \alpha(p)) \\ &= g(\alpha(p))f(p)W_1(t,p), \\ \mu_3(\gamma(\alpha(p))) = g(\alpha(p))\mu_2(\alpha(p)) &= g(\alpha(p))f(p)\mu_1(p) \end{aligned}$$

hold.

In this manuscript, by writing compositions of maps like $g \circ \alpha$, $\gamma \circ \alpha$ and $\delta \circ \beta$ in the form of multiplications like $\alpha g$, $\alpha\gamma$ and $\beta\delta$ respectively, the *composition* of morphisms is written as $(f \otimes_{P_1} (\alpha g), (\alpha\gamma, \beta\delta))$, where $\otimes_{P_1}$ is the operation in the following fundamental commutative group $(\boldsymbol{Q_+}^{P_1}, \otimes_{P_1})$.

The set $(\boldsymbol{Q_+}^{P}, \otimes_P)$ of all maps from a set $P$ to $\boldsymbol{Q}_+$ forms a commutative group under the operation $\otimes_P$ defined by $f \otimes_P g : p \mapsto f(p)g(p)$. $1_{\otimes_P} : P \to \boldsymbol{Q}_+ : p \mapsto 1$ is the identity and $f^{-1} : P \to \boldsymbol{Q}_+ : p \mapsto 1/f(p)$ is the inverse of a $f \in \boldsymbol{Q_+}^{P}$. Whenever it does not cause confusion, we write $\otimes$ instead of $\otimes_P$. Immediately we obtain the following lemma.

**LEMMA 2.1** Let $\alpha$ and $\beta$ be arbitrary maps on $P$ and $f, g : P \to \boldsymbol{Q}_+$. Then the following equations are true.
(1) $(\alpha\beta)f = \alpha(\beta f)$.
(2) $\alpha(f \otimes g) = (\alpha f) \otimes (\alpha g)$.
(3) $\alpha \mathbf{1}_\otimes = \mathbf{1}_\otimes$.
(4) $(\alpha f) \otimes (\alpha f^{-1}) = 1_\otimes$.
(5) $(\alpha f)^{-1} = \alpha f^{-1}$. □

For a surjective morphim $x : \mathcal{P}_1 \to \mathcal{P}_2$, $\mathcal{P}_1$ is called the domain of $x$, denoted by $Dom(x)$, and $\mathcal{P}_2$ is called the image(or range) of $x$, denoted by $Im(x)$.

We denote the set of all surjective morphisms between two Petri nets and a zero element 0, by $\mathcal{S}_0$. Especially $Dom(0) = Im(0) = \emptyset$. $\mathcal{S}_0$ forms a semigroup, equipped with the multiplication of $x = (f,\ (\alpha,\beta))$ and $y = (g,\ (\gamma,\delta))$:

$$x \cdot y \stackrel{\text{def}}{=} \begin{cases} (f \otimes_{\mathcal{P}} \alpha g, (\alpha\gamma, \beta\delta)) & \text{if} \quad Im(x) = Dom(y). \\ 0 & \text{otherwise.} \end{cases}$$

$\mathcal{S} = \mathcal{S}_0 \cup \{1\}$ is the monoid obtained from $\mathcal{S}_0$ by adjoining an (extra) identity 1, that is, $1 \cdot s = s \cdot 1 = s$ for all $s \in \mathcal{S}_0$ and $1 \cdot 1 = 1$.

# 3 Ideals in the monoid $\mathcal{S}$

In this section we consider ideals and Green's relations on the monoid $\mathcal{S}$. At first, we consider some properties of the structure of the automorphism group of a Petri net $\mathcal{P}$.

## 3.1 Green's equivalences on the monoid $\mathcal{S}$

In general, Green's equivalences $\mathcal{L}, \mathcal{R}, \mathcal{J}, \mathcal{H}, \mathcal{D}$ on a monoid $M$, which are well-known and important equivalence relations in the development of semigroup theory, are defined as follows:

$$\begin{aligned} &x\mathcal{L}y \iff Mx = My, \\ &x\mathcal{R}y \iff xM = yM, \\ &x\mathcal{J}y \iff MxM = MyM, \\ &\mathcal{H} = \mathcal{L} \cap \mathcal{R}, \\ &\mathcal{D} = (\mathcal{L} \cup \mathcal{R})^*, \end{aligned}$$

where $(\mathcal{L} \cup \mathcal{R})^*$ means the reflexive and transitive closure of $\mathcal{L} \cup \mathcal{R}$. $Mx$ (resp. $xM$) is called the *principal left* (resp. *right* ) *ideal generated by* $x$ and $MxM$ the it principal (two-sided) ideal generated by $x$. Then, the following facts are generally true[2, 1].

**FACT 1** *The following relations are true.*

$$\begin{aligned} &(1)\, \mathcal{D} = \mathcal{L}\mathcal{R} = \mathcal{R}\mathcal{L} \\ &(2)\, \mathcal{H} \subset \mathcal{L} \ (\textit{resp.}\mathcal{R}) \ \subset \mathcal{D} \subset \mathcal{J} \end{aligned}$$

**FACT 2** *An $\mathcal{H}$-class of a monoid $M$ is a group if and only if it contains an idempotent.*

Now we consider the case of $M = \mathcal{S}$ in the rest of the maniscript. The following lemma is obviously true.

**LEMMA 3.1** *Let $x : \mathcal{P}_1 \to \mathcal{P}_2$, $y : \mathcal{P}_3 \to \mathcal{P}_4 \in \mathcal{S}$. Then,*
(1) $x\mathcal{S} \subset y\mathcal{S} \Longrightarrow \mathcal{P}_1 = \mathcal{P}_3$ *and* $\mathcal{P}_2 \sqsubseteq \mathcal{P}_4$.
(2) $\mathcal{S}x \subset \mathcal{S}y \Longrightarrow \mathcal{P}_1 \sqsubseteq \mathcal{P}_3$ *and* $\mathcal{P}_2 = \mathcal{P}_4$.
(3) $x\mathcal{S} = y\mathcal{S} \Longrightarrow \mathcal{P}_1 = \mathcal{P}_3$ *and* $\mathcal{P}_2 \simeq \mathcal{P}_4$.
(4) $\mathcal{S}x = \mathcal{S}y \Longrightarrow \mathcal{P}_1 \simeq \mathcal{P}_3$ *and* $\mathcal{P}_2 = \mathcal{P}_4$. □

Note that any reverses of the implications above are not necessarily true.

**PROPOSITION 3.1** The following conditions are equivalent.
(1) $H$ is an $\mathcal{H}$-class and a group.
(2) $H = \mathbf{Aut}(\mathcal{P})$ for some Petri net $\mathcal{P}$. □

**PROPOSITION 3.2** *On the monoid $\mathcal{S}$, $\mathcal{J} = \mathcal{D}$ .* □

## 3.2 Intersection of principal ideals

The aim here is that for given $x, y \in \mathcal{S}$ we find a elements $z$ such that $\mathcal{S}x \cap \mathcal{S}y = \mathcal{S}z$ (resp. $x\mathcal{S} \cap y\mathcal{S} = z\mathcal{S}$). $x\mathcal{S} \cap y\mathcal{S} = \{0\}$ (resp. $\mathcal{S}x \cap \mathcal{S}y = \{0\}$) is a trivial case(i.e., $z = 0$). We should only consider the non-trivial case.

**LEMMA 3.2** *Let $\mathcal{P}_i = (P_i, T_i.W_i, \mu_i)(i = 1, 2, 3)$ be Petri nets, $x = (f, (\alpha, \beta)) : \mathcal{P}_1 \to \mathcal{P}_3$, $y = (g, (\gamma, \delta)) : \mathcal{P}_2 \to \mathcal{P}_3$ be elements of $\mathcal{S}$. If $|\alpha^{-1}(p)| \leq |\gamma^{-1}(p)|$ and $|\beta^{-1}(t)| \leq |\delta^{-1}(t)|$ for any $p \in P_3$ and $t \in T_3$, then $\mathcal{S}y \subset \mathcal{S}x$.* □

**LEMMA 3.3** *Let $\mathcal{P}_i = (P_i, T_i.W_i, \mu_i)(i = 0, 1, 2)$ be Petri nets, $x = (f, (\alpha, \beta)) : \mathcal{P}_0 \to \mathcal{P}_1$, $y = (g, (\gamma, \delta)) : \mathcal{P}_0 \to \mathcal{P}_2$ be elements of $\mathcal{S}$. If for any $p \in P_1$ and $t \in T_1$, there exist $q \in P_2$ and $s \in T_2$ such that $\alpha^{-1}(p) \subset \gamma^{-1}(q)$ and $\beta^{-1}(t) \subset \delta^{-1}(s)$, then $y\mathcal{S} \subset x\mathcal{S}$.* □

**PROPOSITION 3.3 (Intersection of Principal Left Ideals)** *Let $\mathcal{P}_i = (P_i, T_i.W_i, \mu_i)(i = 1, 2, 3)$ be Petri nets, $x : \mathcal{P}_1 \to \mathcal{P}_3$ and $y : \mathcal{P}_2 \to \mathcal{P}_3$ be elements of $\mathcal{S}$. Then, there exist a Petri net $P$ and a surjective morphism $z$ such that $\mathcal{S}x \cap \mathcal{S}y = \mathcal{S}z$.* □

**COROLLARY 3.1 (Diamond Property I)** *Let $\mathcal{P}_i = (P_i, T_i, W_i, \mu_i)$ $(i = 1, 2, 3)$ be Petri nets with $\mathcal{P}_1 \sqsupseteq \mathcal{P}_3$ and $\mathcal{P}_2 \sqsupseteq \mathcal{P}_3$. Then there exists a Petri net $\mathcal{P}$ such that $\mathcal{P} \sqsupseteq \mathcal{P}_1$ and $\mathcal{P} \sqsupseteq \mathcal{P}_2$.* □

**PROPOSITION 3.4 (Intersection of Principal Right Ideals)** *Let $\mathcal{P}_i = (P_i, T_i.W_i, \mu_i)(i = 0, 1, 2)$ be Petri nets, $x : \mathcal{P}_1 \longrightarrow \mathcal{P}_3$ and $y : \mathcal{P}_2 \longrightarrow \mathcal{P}_3$ be elements of $S$. Then, there exist a Petri net $\mathcal{P}$ and a surjective morphism $z$ such that $xS \cap yS = zS$.* □

**COROLLARY 3.2 (Diamond Property II)** Let $\mathcal{P}_i = (P_i, T_i, W_i, \mu_i)$ $(i = 0, 1, 2)$ be Petri nets with $\mathcal{P}_0 \sqsupseteq \mathcal{P}_1$ and $\mathcal{P}_0 \sqsupseteq \mathcal{P}_2$. Then there exists a Petri net $\mathcal{P}_3$ such that $\mathcal{P}_1 \sqsupseteq \mathcal{P}_3$ and $\mathcal{P}_2 \sqsupseteq \mathcal{P}_3$. □

We define the concept of irreducible forms of a Petri net with respect to $\sqsupseteq$ and show the uniqueness of them up to isomophism.

**DEFINITION 3.1 (Irreducible)** *A Petri net $\mathcal{P}$ is called a $\sqsupseteq$-irreducible if $\mathcal{P} \sqsupseteq \mathcal{P}'$ implies $\mathcal{P} \simeq \mathcal{P}'$ for any Petri net $\mathcal{P}'$. Then $\mathcal{P}$ is called an $\sqsupseteq$-irreducible form.* □

**COROLLARY 3.3** Let $\mathcal{P}$, $\mathcal{P}'$ and $\mathcal{P}''$ be Petri nets with $\mathcal{P} \sqsupseteq \mathcal{P}'$ and $\mathcal{P} \sqsupseteq \mathcal{P}''$. If $\mathcal{P}'$ and $\mathcal{P}''$ are $\sqsupseteq$-irreducible, then $\mathcal{P}' \simeq \mathcal{P}''$. □

# 4 Structure of the automorphism group of a Petri net

Our aim in this section is to decompose the automorphism group $G = \mathbf{Aut}(\mathcal{P})$ of a Petri net $\mathcal{P}$ into $G = KN = NK$, where $N$ is a kind of normal subgroup of $G$.

At first, we consider some properties of the structure of the automorphism group of a fixed (given) Petri net $\mathcal{P} = (\mathcal{P}, \mathcal{T}, \mathcal{W}, \mu)$.

## 4.1 The group of automorphisms of a Petri net

Let $\boldsymbol{Q}_+{}^P \rtimes (P^P \times T^T)$ be the semi-direct product of the group $\boldsymbol{Q}_+{}^P$ and the monoid $P^P \times T^T$, equipped with the multiplication defined by

$$(f, (\alpha, \beta))(g, (\alpha', \beta')) \stackrel{\text{def}}{=} (f \otimes \alpha\, g, (\alpha\alpha', \beta\beta')), \tag{4.1}$$

where $P^P$ is the set of all maps from $P$ to $P$ and $T^T$ is the set of all maps from $T$ to $T$. $\boldsymbol{Q}_+{}^P \rtimes (P^P \times T^T)$ forms a monoid with the identity $(\mathbf{1}_\otimes, (\mathbf{1}_P, \mathbf{1}_T))$, where $\mathbf{1}_\otimes$ is the identity of the group $\boldsymbol{Q}_+{}^P$, $\mathbf{1}_P$ and $\mathbf{1}_T$ are the identity maps on $P$ and $T$ respectively.

Let $\mathcal{P} = (P, T, W, \mu)$ be a Petri net. Now we consider the following set related to the Petri net $\mathcal{P}$.

| | |
|---|---|
| $\mathbf{Mor}(\mathcal{P})$ : | the set of all the morphisms of $\mathcal{P}$. |
| $\mathbf{Aut}(\mathcal{P})$ : | the set of all the automorphisms of $\mathcal{P}$. |

By changing the weight function and the markings of $\mathcal{P}$, we can construct another Petri net $\mathcal{P}_0 = (P, T, 0^{E(P,T)}, 0^P)$ be Petri nets, where $0^P$ denotes the special marking with $0^P : P \longrightarrow \boldsymbol{N}_0, p \mapsto 0$ and $0^{E(P,T)}$ the special weight function with $0^{E(P,T)} : E(P,T) \longrightarrow \boldsymbol{N}_0, e \mapsto 0$. Then the following inclusion relation holds.

**PROPOSITION 4.1** *Let $\mathcal{P} = (P, T, W, \mu)$ and $\mathcal{P}_0 = (P, T, 0^{E(P,T)}, 0^P)$ be Petri nets. And let $S_P$ and $S_T$ be the symmetric groups of $P$ and $T$, respectively.*
*(1) The subset $\boldsymbol{Q}_+{}^P \rtimes (S_P \times S_T)$ of $\boldsymbol{Q}_+{}^P \rtimes (P^P \times T^T)$ forms a group with the identity $(\mathbf{1}_\otimes, (\mathbf{1}_P, \mathbf{1}_T))$.*
*(2) $\mathbf{Mor}(\mathcal{P}_0) = \boldsymbol{Q}_+{}^P \rtimes (P^P \times T^T)$.*
*(3) $\mathbf{Mor}(\mathcal{P})$ is a submonoid of $\mathbf{Mor}(\mathcal{P}_0)$.*
*(4) $\mathbf{Aut}(\mathcal{P}_0) = \boldsymbol{Q}_+{}^P \rtimes (S_P \times S_T)$.*
*(5) $\mathbf{Aut}(\mathcal{P})$ is a subgroup of $\mathbf{Aut}(\mathcal{P}_0)$.* □

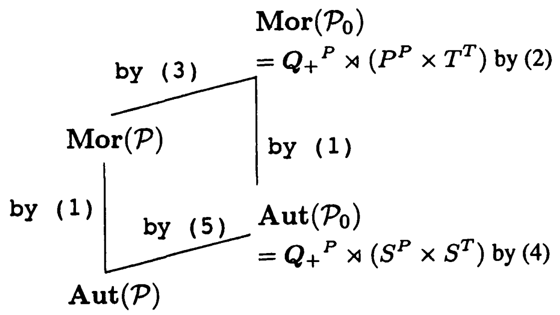


**Figure 2. Inclusion relations among monoids of morphisms and groups of automorphisms related to the Petri nets $\mathcal{P}$ and $\mathcal{P}_0$, as a result of Propositon 4.1.**

## 4.2 Similarity and automorphism

Recall that $(Q_+{}^P, \otimes_P)$ is an abelian group and a 0-isolated place does not have any connection to any transition and is marked with 0 tokens.

**LEMMA 4.1** *Let $P$ be a nonempty set and $P_1, P_2$ be subsets of $P$.*
(1) $Q_+{}^{P_1} = \{f \in Q_+{}^P \mid f(p) = 1, p \in P \setminus P_1\}$ *is a subgroup of* $(Q_+{}^P, \otimes_P)$.
(2) $Q_+{}^{P_1} \otimes_P Q_+{}^{P_2} = Q_+{}^{P_1 \cup P_2}$. □

**LEMMA 4.2 (Transposition-type automorphisms)** *Let $\mathcal{P} = (P, T, W, \mu)$ be a Petri net, $p, q \in P$ be two distinct similar places in $P$ and $s, t \in T$ be two distinct similar transitions in $T$. Then*
(1) *If $p$ is not 0-isolated, $N_{\{p,q\}} = \langle (f_{p,q}, ((p\ q), 1_T)) \rangle$ is a subgroup of $\mathbf{Aut}(\mathcal{P})$ and its order is 2, where $(p\ q)$ is the transposition of $p$ and $q$, $f_{p,q}(p) = r$, $f_{p,q}(q) = 1/r$, $f_{p,q}(x) = 1$ for $x \in P \setminus \{p, q\}$, and $r$ is the rational number such that $\mu(p) = r\mu(q)$, $W(p,t) = rW(q,t)$ and $W(t,p) = rW(t,q)$ for all $t \in T$.*
(2) *If $p$ is 0-isolated, $N_{\{p,q\}} = Q_+{}^{\{p,q\}} \times \langle ((p\ q), 1_T)) \rangle$ is a subgroup of $\mathbf{Aut}_+(\mathcal{P})$.*
(3) *$N_{\{t,s\}} = \langle (1_{\otimes_P}, (1_P, (s\ t))) \rangle$ is a subgroup of $\mathbf{Aut}(\mathcal{P})$ and its order is 2.* □

For a $\sim_{\mathcal{P}}$-class $C(u)$ of $u$, the subgruop $N_{C(u)}$ of $\mathbf{Aut}(\mathcal{P})$ is defined as follows:

$$N_{C(u)} = \begin{cases} \langle S_{\{a,b\}} | a, b \in C(u), a \neq b \rangle & \text{if } |C(u)| \geq 2, \\ \{(1_{\otimes_P}, (1_P, 1_T))\} & \text{if } |C(u)| = 1. \end{cases}$$

If $u$ is a 0-isolated place, the $\sim_{\mathcal{P}}$-class $Z = C(u)$ is the set of all 0-isolated places in $P$ and we can easily verify that $N_Z = Q_+{}^Z \times (S_Z \times \{1_T\})$, where $S_Z$ is the symmetric group of $Z$. The following proposition holds with respect to $N_Z$.

**PROPOSITION 4.2 (Separation of 0-isolated places)** Let $\mathcal{P} = (P, T, W, \mu)$ be a Petri net, $Z \subset P$ be $\sim_{\mathcal{P}}$-class of all the 0-isolated places, $N_Z = Q_+{}^Z \times (S_Z \times \{1_T\})$, $H = \{(f, (\alpha, \beta)) \in (\mathbf{Aut}(\mathcal{P}) | f|_Z = 1_{\otimes_Z}, \alpha|_Z = 1_Z\}$. Then, $\mathbf{Aut}(\mathcal{P}) = N_Z \times H$.

Proof) Here set $G = \mathbf{Aut}(\mathcal{P})$ and $\mathbf{1} = (1_\otimes, (1_P, 1_T))$. What we have to do is to prove that
(a) $G = N_Z H$, (b) $N_Z \cap H = \{\mathbf{1}\}$, and (c) $xy = yx$ for any $x \in N_Z, y \in H$.
(a) Let $(f, (\alpha, \beta))$ be an arbitrary element in $G$. $f = f_0 \otimes f_1 = f_1 \otimes f_0$ for some $f_0 \in Q_+{}^Z, f_1 \in Q_+{}^{P \setminus Z}$. Since $\alpha(Z) = Z$ and $\alpha(P \setminus Z) = P \setminus Z$ hold, $\alpha = \alpha_0 \alpha_1$ for some $\alpha_0 \in S_Z, \alpha_1 \in S_{P \setminus Z}$. Because $\alpha_0$ and $f_1$ are constant on $P \setminus Z$ and $Z$ respectively, we have $\alpha_0 f_1 = f_1$ and $(f_0, (\alpha_0, 1_T))(f_1, (\alpha_1, \beta)) = (f_0 \otimes \alpha_0 f_1, (\alpha_0 \alpha_1, \beta)) = (f, (\alpha, \beta))$. Therefore $G = N_Z H$.

The condition(b) is trivial by the construction of $H$. (c) Let $x = (f, (\alpha, \beta)) \in H, y = (g, (\gamma, 1_T)) \in N_Z$. Since $\alpha$ and $\gamma$ are constant on $Z$ and $P \backslash Z$ respectively, $xy = (f \otimes \alpha g, (\alpha\gamma, \beta)) = (g \otimes \gamma f, (\gamma\alpha, \beta)) = yx$, that is, $x$ and $y$ commute. □

**LEMMA 4.3** Let $\mathcal{P} = (P, T, W, \mu)$,$\{p, q\} \subset P$,$\{s, t\} \subset T$ and $C(u)$ be the $\sim_{\mathcal{P}}$-class of $u \in P \cup T$. If $(f, (\alpha, \beta))$ is an automorphism of $\mathcal{P}$, then
(1) $p \sim_{\mathcal{P}} q \iff \alpha(p) \sim_{\mathcal{P}} \alpha(q)$,
(1') $s \sim_{\mathcal{P}} t \iff \beta(s) \sim_{\mathcal{P}} \beta(t)$,
(2) $\alpha(C(p)) = \{\alpha(q) | q \sim_{\mathcal{P}} p\} = C(\alpha(p))$,
(2') $\beta(C(t)) = \{\beta(s) | s \sim_{\mathcal{P}} t\} = C(\beta(t))$,
(3) $\min\{i | C(\alpha^i(u)) = C(u)\} = \min\{i | C(\beta^i(v)) = C(v)\}$ if $u, v \in P \cup T$ are connected,. □

Note that $|C(\alpha(p))| = |C(p)|$ for all $p \in P$ and $|C(\beta(t))| = |C(t)|$ for all $t \in T$.
Let $C_1, C_2, \ldots, C_k$ be the all $\sim_{\mathcal{P}}$-classes on $P \cup T$ and $\pi = \{C_1, C_2, \ldots, C_k\}$ be the partition of $P \cup T$ determinded by $\sim_{\mathcal{P}}$. Then we introduce the permutation group $S_\pi = \{\sigma \in S_{P \cup T} \,|\, \forall X \in \pi, X^\sigma = X\} = S_{C_1} \times S_{C_2} \times \cdots \times S_{C_k}$, which does not move any elements of $\pi$.

**PROPOSITION 4.3 (Embedding into a symmetric group)** Let $\mathcal{P} = (P, T, W, \mu)$ be a Petri net without 0-isolated places.
(1) $\phi : \mathbf{Aut}(\mathcal{P}) \to S_{P \cup T}, (f, (\alpha, \beta)) \mapsto (\alpha, \beta)$ is a monomorphims, i.e. $\mathbf{Aut}(\mathcal{P}) \simeq \phi(G) \subset S_{P \cup T}$.
(2) $S_\pi \subset \phi(G)$.
(3) $X \in \pi \Longrightarrow g(X) \in \pi$ for any $g \in \phi(G)$.
(4) $S_\pi$ is a normal subgroup of $\phi(G)$, that is, $S_\pi \lhd \phi(G)$.
(5) Let $a_1, a_2, \ldots, a_k$ be a system of representatives for $S_\pi$ of $\phi(G)$ and $A = \langle a_1, a_2, \ldots, a_k \rangle$. Putting $K = \phi^{-1}(A), N = \phi^{-1}(S_\pi), \mathbf{Aut}(\mathcal{P}) = KN = NK$.

Proof) Here set $G = \mathbf{Aut}(\mathcal{P})$ and $\mathbf{1} = (1_\otimes, (1_P, 1_T))$.
(1) $\phi$ is a homomorphim from $G$ to $S_{P \cup T}$. Indeed, for any $x = (f, (\alpha, \beta)), y = (g, (\gamma, \delta)) \in \mathbf{Aut}_+(\mathcal{P})$, Since $xy = (f \otimes \alpha g, (\alpha\gamma, \beta\delta))$ holds, $\phi(xy) = (\alpha\gamma, \beta\delta) = (\alpha, \beta)(\gamma, \delta) = \phi(x)\phi(y)$. Next, suppose $\phi(x) = (\alpha, \beta) = \mathbf{1}_{P \cup T} = (1_P, 1_T)$. $x = (f, (1_P, 1_T))$ must hold. Since $\mathcal{P}$ has no 0-isolated places, $f = \mathbf{1}_\otimes$, that is, $\ker(\phi) = \mathbf{1}$. Therefore $\phi$ is a monomorphism.
(2) $N = N_{C_1} N_{C_2} \ldots N_{C_k}$ is a subgroup of $G$.

$$\begin{aligned} \phi(N) &= \phi(N_{C_1})\phi(N_{C_2}) \ldots \phi(N_{C_k}) \\ &= S_{C_1} S_{C_2} \ldots S_{C_k} \\ &= S_\pi \subset \phi(G). \end{aligned}$$

(3) Let $g \in \phi(G)$. By LEMMA4.3 (2) and (2'), if $X = C_i \in \pi \, (1 \leq i \leq k)$, then $g(X) \in \pi$.
(4) Let $\sigma \in S_\pi$, $g \in \phi(G)$ and $x$ be an arbitrary element of $P \cup T$. Suppose that $x \in X, X \in \pi$. Since $g(x) \in g(X)$ and $g(X) \in \pi$ by (3), $(g\sigma)(X) = g(X)$. $(g\sigma g^{-1})(X) = gg^{-1}(X) = X$ and $g\sigma g^{-1} \in S_{P \cup T}$ imply $g\sigma g^{-1} \in S_\pi$, that is, $gS_\pi g^{-1} \subset S_\pi$. Therefore $S_\pi$ is a normal subgroup of $\phi(G)$.
(5) It is trivial. □

**THEOREM 4.1** *Let $\mathcal{P} = (P, T, W, \mu)$ be a Petri net and $C_1, C_2, \ldots, C_k$ be the all $\sim_{\mathcal{P}}$-classes on $P \cup T$. $N = N_{C_1} \times N_{C_2} \times \cdots \times N_{C_k}$ is a normal subgroup of $G = \mathbf{Aut}(\mathcal{P})$ and $K = \langle \{a_i | i \in \Lambda\} \rangle$ is a subgroup generated by $\{a_i | i \in \Lambda\}$ with $G = \bigcup_{i \in \Lambda} a_i N$.*
*(1) If $P$ has no 0-isolated places, $G = KN = NK$.*
*(2) Otherwise, $G = \mathbf{Q_+}^Z \times (KN) = (KN) \times \mathbf{Q_+}^Z$, where $Z \subset P$ be $\sim_{\mathcal{P}}$-class of a 0-isolated place.*

**LEMMA 4.4 (1-step reduction)** Let $\mathcal{P} = (P, T, W, \mu)$ be a Petri net.
(1) $p, q \in P$ be two distinct similar places in $P$. Then $\mathcal{P} \sqsupseteq \mathcal{P}' = (P', T, W', \mu')$, where $P' = P - \{q\}$, $W' = W|(P' \times T) \cup (T \times P'), \mu' = \mu|P'$.
(2) $s, t \in T$ be two distinct similar transitions in $T$. Then $\mathcal{P} \sqsupseteq \mathcal{P}' = (P, T', W', \mu)$, where $T' = T - \{s\}$, $W' = W|(P \times T') \cup (T' \times P)$. □

In the lemma above, $|P' \cup T| = |P \cup T'| = |P \cup T| - 1$ holds. So we call such a relation *1-step reduction*, denoted by $\sqsupseteq_1$.

**PROPOSITION 4.4** Let $\mathcal{P}_i = (P_i, T_i, W_i, \mu_i)(i = 1, 2)$ be Petri nets with $\mathcal{P}_1 \sqsupseteq \mathcal{P}_2$, $(f, (\alpha, \beta)) : \mathcal{P}_1 \to \mathcal{P}_2$ be a surjective morphism. If $\mathcal{P}_2$ is a normal form, then
(1) For any $p, q \in P, p \sim_{\mathcal{P}} q \iff \alpha(p) = \alpha(q)$,
(2) For any $t, s \in T, t \sim_{\mathcal{P}} s \iff \beta(t) = \beta(s)$. □

Proof) (1)(if part) For an arbitrary transition $t \in T$,

$$f(p)W_1(p,t) = W_2(\alpha(p), \beta(t)) = W_2(\alpha(q), \beta(t)) = f(q)W_1(q,t),$$
$$f(p)W_1(t,p) = W_2(\beta(t), \alpha(p)) = W_2(\beta(t), \alpha(q)) = f(q)W_1(t,q), and$$
$$f(p)\mu_1(p) = \mu_2(\alpha(p)) = \mu_2(\alpha(q)) = f(q)\mu_1(q)$$

hold. So setting $r = f^{-1}(p)f(q)$, we have $\mu_1(p) = r\mu_1(q)$ and $W_1(p,t) = rW_1(q,t)$ and $W_1(t,p) = rW_1(t,q)$ for all $t \in T$. Therefore $p \sim_{\mathcal{P}} q$.

(only if part) Suppose that $\alpha(p) \neq \alpha(q)$. Since $p \neq q$, By lemma 4.4 there exists a Petri net $\mathcal{P}_2'$ such that $\mathcal{P}_2 \sqsupseteq_1 \mathcal{P}_2'$ and thus $\mathcal{P}_2 \not\simeq \mathcal{P}_2'$. This contradicts that $\mathcal{P}_2$ is a normal form.
(2) The claim is proved in a similar way to (1). □

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