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Report on Bounded Insurance Contracts

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Abstract: Insurance contract can be considered from a view point of two person game between two participants, *i.e.*, the buyer and the seller. This report is the republication of the results on optimal insurance contracts from the viewpoint of each of the two participants, under some plausible conditions, presented by Teraoka about forty years ago. These were the advancement of the ideas suggested from Arrow and Miller.

1. Introduction

In this report, we republish to make instruction on the problem of choosing the "optimal bounded insurance contracts" from the view point of each of the two participants, i.e., the buyer and the seller, under some plausible conditions. Arrow [1] and Miller [2] have already described case where the monetary refund does not have an upper bound. In the real world, however, the insurance company does not pay more than certain amount of money to the beneficiary, so we shall call such a contract "bounded". We gave four theorems which state that the "optimal bounded contract" for the buyer is "bounded stop-loss" and one for the seller is "bounded proportional". These were evidently generalization of the results from Arrow and Miller in [1] and [2]. Those were appeared by Teraoka[3], however, the journal which printed it has ceased to published more than thirty years ago, consequently it is very difficult to read the paper now.

We shall also instruct two optimal insurance contracts from the view point of each of the two participants, under a kind of duality conditions for the above two results. It was shown that the optimal insurance contracts are "minimum truncated" for the seller and "bounden proportional" for the buyer, and the latter contract is the common one for the two under disadvantage conditions for each other. There were also given by Teraoka[4] 1977, however, the paper which printed them contains the printer's error in the main result. Thus we shall instruct our results obtained about forty years ago in this report...

2. Assumptions

Suppose that buyer faces a positive-valued monetary risk with a cumulative distribution $F(x)$ and has utility-of-money function $u(\cdot)$. Thus his expected utility of facing the risk is $\int_0^\infty u(-x)dF(x)$. We also assume that the seller of the insurance sells a contract $T(\cdot)$ in such a way that if the loss x is incurred by the buyer, the seller will pay the buyer an amount $T(x)$ which satisfies $0 \le T(x) \le \min(x, K)$, where K is a pre-assigned positive constant. Let π be the premium which is usually equal to $\pi = \int_0^\infty T(x)dF(x)$. We also assume that the seller has utility-of-money function $v(\cdot)$, and that both of $u(\cdot)$ and $v(\cdot)$ are twice differentiable and concave, that is, $u'(\cdot) \ge 0$, $u''(\cdot) \le 0$, $v'(x) \ge 0$, and $v''(x) \le 0$ for all x. Then the expected utilities for each are

$$\int_0^\infty u[-\pi - x + T(x)]dF(x) \quad \text{and} \quad \int_0^\infty v[\pi - T(x)]dF(x)$$

by making the contract.

Let $S_F(z) = \int_z^\infty (x-z)dF(x) = \int_z^\infty (1-F(x))dx$, where the expected value $E(X) = \int_0^\infty xdF(x)$ is assumed to exist. For any cumulative distribution function F with finite mean $E(X)$ $S_F(z)$ is non-negative, convex, and strictly decreasing on the set where it is positive. Furthermore, $S_F(z) \ge E(X) - z$, $(0 \le z \le \infty)$, and $S_F(0) = E(X)$, $\lim_{z\to\infty} S_F(z) = 0$. We denote the inverse function of $S_F(z)$ by $S_F^{-1}(c)$ for $0 < c \le E(X)$..

3. The Optimal Bounded Insurance Contracts

Result.1 presents an optimal bounded insurance contract from the viewpoint of the buyer, and Result.2 offers it for the seller under some condition. We find that the optimal contract for the buyer is "bounded stop-loss "and one for the seller is "bounded proportional".

Result.1. *Let π be a positive number, and let τ_K be the set of all insurance agreement of $T(\cdot)$ such that $\int_0^\infty T(x)dF(x) = \pi$ and $0 \le T(x) \le \min(x, K)$, where $K \ge S_F^{-1}(E(X) - \pi)$.*

Then for any utility function $u(\cdot)$

(1) $$\max_{T \in \tau_K} \int_0^\infty u[-\pi - x + T(x)]dF(x) = \int_0^\infty u[-\pi - x + T_K^*(x)]dF(x),$$

where

(2) $$T_K^*(x) = \begin{cases} 0, & 0 \le x < a_K \\ \min(x - a_K, K), & x \ge a_K \end{cases}$$

and a_K *is the unique root of equation*

(3) $$S_F(a_K) - S(a_K + K) = \pi$$

and satisfies

(4) $$0 \le a_K \le S_F^{-1}(\pi).$$

Proof. Since $u''(x) \le 0$, we have

(5) $$u[-\pi - x + T(x)] - u[-\pi - x + T_K^*(x)] \le \{T(x) - T_K^*(x)\} u'(-\pi - x + T_K^*(x)).$$

We find that

(6) $$u'(-\pi - x + T_K^*(x)) = \begin{cases} u'(-\pi - x) \le u'(-\pi - a_K), & 0 \le x \le a_K \\ u'(-\pi - a_K), & a_K < x \le a_K + K \\ u'(-\pi - x + K) \ge u'(-\pi - a_K), & x \ge a_K + K \end{cases}.$$

The definition of $T_K^*(x)$ gives

(7) $$T(x) - T_K^*(x) = \begin{cases} T(x) \ge 0, & 0 \le x \le a_K \\ T(x) - K \le 0, & x \ge a_K \end{cases}.$$

Considering (4), (5) and (6), it follows that for any $T(\cdot) \in \tau_K$

$$\int_0^\infty u[-\pi - x + T(x)]dF(x) - \int_0^\infty u[-\pi - x + T_K^*(x)]dF(x)$$

$$\le \int_0^{a_K} \{T(x) - T_K^*(x)\} u'(-\pi - a_K)dF(x) + \int_{a_K}^{a+K} \{T(x) - T_K^*(x)\} u'(-\pi - a_K)dF(x)$$

$$+ \int_{a_K+K}^\infty \{T(x) - T_K^*(x)\} u'(-\pi - a_K)dF(x)$$

$$= u'(-\pi - a_K) \int_0^\infty \{T(x) - T_K^*(x)\} dF(x) = 0,$$

yielding Equation (1).

Since $S_F(x) - S_F(x+K)$, for $x \ge 0$, is decreasing from $E(X) - S_F(K)$ to zero, the condition $E(X) - S_F(K) \ge \pi$, i.e., $K \ge S_F^{-1}(E(X) - \pi)$ assures the unique existence of a_K satisfying Equation (3) and Inequality (4). This completes the proof of Result 1.

Result 2. *Let* π *be a positive number, and let* $\tau_K{}^0$ *be the set of all insurance agreement of* $T(\cdot)$ *such that* $\int_0^\infty T(x)dF(x) = \pi$ *and* $0 \le T(x) \le \min(x, K)$ *, where* $K \ge S_F{}^{-1}(E(X) - \pi)$*, and* $T(x)/x$ *is a non-decreasing function of* x *if* $T(x) < K$.

Then for any utility function $v(\cdot)$

(8) $$\max_{T \in \tau_K{}^0} \int_0^\infty u[\pi - T(x)]dF(x) = \int_0^\infty u[\pi - T_K{}^0(x)]dF(x),$$

where

(9) $$T_K{}^0(x) = \begin{cases} q_K x, & 0 \le x < K/q_K \\ K, & x \ge K/q_K \end{cases},$$

and q_K *is the unique root of equation*

(10) $$S_F(K/q_K) = E(x) - \pi/q_K$$

and satisfies

(11) $$\pi/E(X) \le q_K \le \min(1, K/E(X))..$$

Proof. First we shall prove Equation (11). Putting $t = K/q$, Equation (10) is rewritten by

(12) $$S_F(t) = E(X) - (\pi/K)t.$$

From the assumption of K, we obtain

$$S_F(K) \le E(X) - \pi \quad \text{and} \quad 0 < \pi < K.$$

Hence the root t^0 of (12) exists uniquely and

$$\max(E(X), K) \le t^0 (= K/q^0, say) \le (K/\pi)E(X)$$

giving

$$\pi/E(X) \le q^0 \le \min(K/E(X), 1).$$

Next we shall prove that $T_K{}^0(x)$ is an optimal contract for the seller. We clearly have

(13) $$v[\pi - T(x)] - v[\pi - T_K{}^0(x)] \le \{T_K{}^0(x) - T(x)\}v'[\pi - T_K{}^0(x)].$$

From the definition of $T_K{}^0(x)$ and $0 < q < 1$, we obtain

$$(14) \qquad T_K{}^0(x) - T(x) = \begin{cases} x[q - \{T(x)/x\}], & 0 < x \le K/q \\ K - T(x), & x \ge K/q \end{cases}.$$

If $F(K/q) = \Pr\{X \le K/q\} = 0$, then since

$$\int_0^\infty \{T_K{}^0(x) - T(x)\} v'[\pi - T_K{}^0(x)] dF(x) = v'[\pi - T_K{}^0(x)\} \int_0^\infty \{T_K{}^0(x) - T(x)\} dF(x) = 0,$$

(8) is derived from (13). Therefore, we shall prove the case where $F(K/q) = \Pr\{X \le K/q\} > 0$. Suppose that $T(K/q) < K$, then

$$(15) \qquad T_K{}^0(x) - T(x) = x[q - \{T(x)/x\}], \quad \text{for} \quad 0 < x < K/q,$$

since $T(x)/x$ is a non-decreasing function so far as $T(x) < K$. Therefore we obtain from (14) and (15)

$$\int_0^\infty \{T_K{}^0(x) - T(x)\} dF(x) > 0,$$

contradicting to $T(x) \in \tau_K{}^0$. Hence we find that for any $T(x) \in \tau_K{}^0$,

$$(16) \qquad T(\cdot) = K, \quad \text{for} \quad x \ge K/q$$

and

$$\int_0^\infty \{T_K{}^0(x) - T(x)\} dF(x) = \int_0^{K/q} \{T_K{}^0(x) - T(x)\} dF(x)$$

$$= \int_0^{K/q} x\{q - T(x)/x\} dF(x) = 0.$$

From the above results, if $F(K/q) > 0$ then there exists a $\gamma \in (0, K/q]$ such that

$$(17) \qquad T_K{}^0(x) = qx \begin{Bmatrix} > \\ < \end{Bmatrix} T(x) \quad \text{if} \quad \begin{Bmatrix} 0 < x < \gamma \\ \gamma < x < K/q \end{Bmatrix}.$$

Since we obtain

$$v'[\pi - T_K{}^0(x)] = v'(\pi - qx) \begin{Bmatrix} \le \\ \ge \end{Bmatrix} v'(\pi - q\gamma), \quad \text{if} \quad \begin{Bmatrix} 0 < x \le \gamma \\ \gamma \le x \le K/q \end{Bmatrix},$$

(13), (16) and (17) give

$$\int_0^\infty v[\pi - T(x)] dF(x) - \int_0^\infty v[\pi - T_K{}^0(x)] dF(x) \le \int_0^\infty \{T_K{}^0(x) - T(x)\} v'[\pi - T_K{}^0(x)] dF(x)$$

$$= \int_0^{K/q} \{T_K{}^0(x) - T(x)\} v'[\pi - T_K{}^0(x)] dF(x)$$

$$\le \int_0^\gamma \{T_K{}^0(x) - T(x)\} v'(\pi - q\gamma) dF(x) + \int_\gamma^{K/q} \{T_K{}^0(x) - T(x)\} v'(\pi - q\gamma) dF(x)$$

$$= v'(\pi - q\gamma) \int_0^{K/q} \{T_K{}^0(x) - T(x)\} dF(x) = 0.$$

This complete the proof.

If $F(x) > 0$ for any finite $x \geq 0$, then letting $K \to \infty$, we have the unbounded cases (Arrow[1] and Miller[2]), in which the optimal contract for the buyer is (from Result 1) of stop-loss type with stop-loss point $a_\infty = S_F^{-1}(\pi)$, and the optimal contract for the seller is (from Result 2) of proportional type with the rate $q_\infty = \pi / E(X)$.

Here we consider the optimal insurance contracts under a kind of duality conditions for the above two resuts. Result 3 shows an optimal bounded insurance contract from the view point of the seller under generous conditions, and Result 4 suggests it for the buyer under disadvantage conditions. It is found that the optimal contract for the seller is "minimum truncated" and one for trhe buyer is "bounded proportional" it is the very same contract as one for the seller under disadvantage conditions given by Result 2.

Result 3. *Let π be a positive number, and let τ_K be the set of all insurance agreement of $T(\cdot)$ such that $\int_0^\infty T(x)dF(x) = \pi$ and $0 \leq T(x) \leq \min(x, K)$, where $K \geq S_F^{-1}(E(X) - \pi)$.*

Then for any utility function $v(\cdot)$

$$\max_{T \in \tau_K} \int_0^\infty v[\pi - T(x)]dF(x) = \int^\infty v[\pi - T_*(x)]dF(x),$$

where

$$T_*(x) = \begin{cases} x, & 0 \leq x < a_K \\ b, & x \geq a_K \end{cases}$$

and b is the unique root of equation $S_F(b) = E(X) - \pi$, that is

$$b = S_F^{-1}(E(X) - \pi)..$$

(We omit the proof since it can be found in Teaoka[4]).

Result 4. *Let π be a positive number, and let $\tau_K{}'$ be the set of all insurance agreement of $T(\cdot)$ such that $\int_0^\infty T(x)dF(x) = \pi$ and $0 \leq T(x) \leq \min(x, K)$, where*

$K \geq S_F^{-1}(E(X)-\pi)$, *and* $T(x)/x$ *is a non-increasing function of* x *if* $T(x) < K$.

Then for any utility function $u(\cdot)$

$$\max_{T\in\tau_K'} \int_0^\infty u[-\pi - x + T(x)]dF(x) = \int_0^\infty u[-\pi - x + T_K^{\,0}(x)]dF(x),$$

where

$$T_K^{\,0}(x) = \begin{cases} q_K x, & 0 \leq x < K/q_K \\ K, & x \geq K/q_K \end{cases},$$

and q_K *is the unique root of equation*

$$S_F(K/q_K) = E(x) - \pi/q_K$$

and satisfies

$$\pi/E(X) \leq q_K \leq \min(1, K/E(X)).$$

(We also omit the proof since we can refer to Teraoka[4].)

Note that $T_K^{\,0}(x)$ is a common contract under disadvantage conditions for the two participants, the buyer and the seller of the insurance. Furthermore, $T_*(x)$ is in contrast with $T_K^{\,+}(x)$ and $T_K^{\,0}(x)$ takes a compromised position between $T_K^{\,+}(x)$ and $T_*(x)$.

As a simple example of our results we examined the case of automobile physical damage insurance for private passenger automobile (small-size) in [4].

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