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# On the behavior of radial solutions to a parabolic-elliptic system related to biology (走化性方程式の解の挙動について)

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## §0 Introduction.

In the present paper, we consider the behavior of radial solutions to the following problem.

$$\text{(PE)}\qquad \begin{cases} U_t = \nabla\cdot(\nabla U - U\nabla V) & \text{in } \mathbf{R}^n\times(0,\infty),\\ 0=\Delta V + U \ \text{ in } \mathbf{R}^n\times(0,\infty), & V(0,\cdot)=0 \quad \text{in } (0,\infty),\\ U(\cdot,0)=U^{\mathcal{I}}\ge 0 & \text{in } \mathbf{R}^n. \end{cases}$$

Here, $n=1,2,3,\cdots$.

In two dimensional case, the system (PE) is a simplified version of so called Keller-Segel system, and is also a model of self-interacting particles. In the Keller-Segel model, $U$ represents density of cells, and $V$ represents the concentration of a chemoattractant secreted by themselves. In the physical model, $U$ represents the density of particles, and $V$ represents the potential.

We consider the behavior of radial solutions to (PE).

## §1 Time local existence and uniqueness of radial solutions

In this paper, we consider radial solutions. The radial solutions exists uniquely under some conditions.

If $U^{\mathcal{I}}$ is radial, positive and

$$U^{\mathcal{I}}(x)=\begin{cases} O(1)/|x|^2 \ (n\ge 3),\\ O(1)/|x|^4 \ (n=2),\end{cases}$$

as $|x|\to\infty$, there exists a unique solution $(U,V)$ as follows.

$$\begin{aligned} U(x,t) &= \int_{\mathbf{R}^n}\mathcal{G}(x-\tilde{x},t)U^{\mathcal{I}}(\tilde{x})d\tilde{x}\\ &\quad -\int_0^t\int_{\mathbf{R}^n}\left\{\nabla_{\tilde{x}}\mathcal{G}(x-\tilde{x},t-\tilde{t})\cdot\frac{\tilde{x}}{\omega_n|\tilde{x}|^n}\int_{|\hat{x}|<|\tilde{x}|}U(\hat{x},\tilde{t})d\hat{x}\right\}U(\tilde{x},\tilde{t})d\tilde{x}d\tilde{t} \end{aligned}$$

in $\mathbf{R}^n\times[0,T)$ with a constant $T\in(0,\infty]$. And we defined the function $V$ as

$$V(x,t)=-\int_0^{|x|}\frac{1}{\omega_n r^{n-1}}\int_{|\tilde{x}|<r}U(\tilde{x},t)d\tilde{x}dr \quad \text{in } \mathbf{R}^n\times[0,T),$$

since we define $V = 0$ at the origin.

Here, $\mathcal{G}$ is the Gauss kernel of $\partial_t - \Delta$ in $\mathbf{R}^n$ and $\omega_n = |S^{n-1}|$.

## §2 Fundamental properties of solutions

In this section, we explain some fundamental properties of solutions.

**Lemma 1** *The following hold.*
*(i) $U$ is non-negative in $\mathbf{R}^n \times (0,T)$.*
*(ii) In the case where $n \geq 2$, for any $\alpha > 0$ there exists a unique radial stationary solutions $(U_\alpha, V_\alpha)$ satisfying $U_\alpha(0) = \alpha$,*

$$(SPE)\begin{cases} 0 = \Delta V_\alpha + \alpha e^{V_\alpha} & \text{in } \mathbf{R}^n, \\ V_\alpha(0) = 0, \quad U_\alpha = \alpha e^{V_\alpha} & \text{in } \mathbf{R}^n. \end{cases}$$

*(iii) In the case where $n = 2$, for $\alpha > 0$ the function $U_\alpha$ satisfies*

$$U_\alpha(x) = \frac{\alpha}{(1 + (\alpha/8)|x|^2)^2} \quad \text{and} \quad \int_{\mathbf{R}^2} U_\alpha(x)dx = 8\pi.$$

*(iv) In the case where $n \geq 10$, the function $U_\alpha$ satisfies*

$$U_\alpha(x) = \frac{O(1)}{|x|^2} \quad \text{as } |x| \to \infty.$$

*(v) In the case where $n \geq 10$ and $n = 2$, the function $U_\alpha$ is continuous with respect to $\alpha$ and satisfies*

$$\lim_{\alpha \to 0} U_\alpha = 0 \quad \text{and} \quad \lim_{\alpha \to \infty} U_\alpha = U_\infty.$$

*Here,*

$$U_\infty(x) = \begin{cases} \dfrac{2(n-2)}{|x|^2} & \text{if } n \geq 3, \\ 8\pi\delta_0 & \text{if } n = 2. \end{cases}$$

**Sketch of proof.** (i) comes from the comparison theorem, since we assume that $U^{\mathcal{I}} \geq 0$ in $\mathbf{R}^n$.
(ii) radial stationary solutions satisfies $\nabla U_\alpha - U_\alpha \nabla V_\alpha = 0$, $V_\alpha(0) = 0$ and $U_\alpha(0) = \alpha$. These ensure $U_\alpha = \alpha e^{V_\alpha}$, which together with the second equation of (PE) implies (SPE).
(iii) The straightforward calculation gives us this property.
(iv) This property is shown in [8, Lemma 2.1].
(v) This property is shown in the proof of [8, Theorem 3.1]. □

## §3 Known results ∼ radial case ∼

• Finite time blowup solutions.

There exist radial solutions to (PE) satisfying

$$\limsup_{t\to T} \|U(\cdot,t)\|_{L^\infty(\mathbf{R}^n)} = \infty.$$

Many persons contribute to this problem (see [3]).

• Time-global solutions.

If the initial function $U^{\mathcal{I}}$ is radial and satisfies

$$\begin{cases} 0 \le U^{\mathcal{I}} \le U_\infty, \quad U^{\mathcal{I}} \not\equiv U_\infty \quad (n \ge 3), \\ U^{\mathcal{I}} \ge 0, \quad \Lambda = \displaystyle\int_{\mathbf{R}^2} U^{\mathcal{I}}(x)dx \le 8\pi \quad (n=2), \end{cases}$$

the radial solution exists globally in time.

In the case where $n \ge 3$, the property is shown by the comparison theorem for the mass function $M(r,t) = \int_{|x|,r} U(x,t)dx$. In the case where $n=2$, the property is shown in [1]. In the non-radial case, there exists many open problems.

• Infinite time blowup solution.

There exist solutions satisfying

$$\limsup_{t\to\infty} \|U(\cdot,t)\|_{L^\infty(\mathbf{R}^n)} = \infty.$$

In two dimensional case, these solutions are found in [2, 4]. In [2], non-radial solutions are treated. In [4], radial solutions in a disk are treated and investigated blowup rate. Moreover, such radial solutions are found also in the case where $n \ge 11$ (see [7]).

## §4 Oscillating solutions in two dimensional case

Although system (PE) has several solutions, the behavior of each solution is not so complicated. However, there exists solutions having complicate behavior. We define $\omega$-limit set as

$$\begin{aligned} \omega(U^{\mathcal{I}} : C(\mathbf{R}^2)) &= \Big\{ F \in C(\mathbf{R}^2) \cap L^\infty(\mathbf{R}^2) \ : \ \lim_{n\to\infty} t_n = \infty, \\ &\qquad \lim_{n\to\infty} \|U(\cdot,t_n) - F\|_{L^\infty(\mathbf{R}^2)} = 0 \quad \text{for some } \{t_n\} \subset (0,\infty) \Big\}. \end{aligned}$$

**Theorem 1** *[5]*

*(i) For $a$ and $d$ with $0 < a < d$ there exists a radial solution $(U,V)$ with $U(\cdot,0) = U^{\mathcal{I}}$ satisfying*

$$\{U_b\}_{b\in[a,d]} \subset \omega(U^{\mathcal{I}} : C(\mathbf{R}^2)), \quad \int_{\mathbf{R}^2} U(x,t)dx = 8\pi.$$

*(ii) For* $\{b_j\}_{j=1}^{\infty} \subset (0,\infty)$ *with* $\lim_{j\to\infty} b_j = \infty$ *there exists a radial solution* $(U,V)$ *with* $U(\cdot,0) = U^{\mathcal{I}}$ *satisfying*

$$\{U_{b_j}\}_{j=1}^{\infty} \subset \omega(U^{\mathcal{I}} : C(\mathbf{R}^2)), \qquad \int_{\mathbf{R}^2} U(x,t)dx = 8\pi$$

According to the definition of $\omega$-limit set, these solutions satisfies the following.

Concerning the solution in (i), for each $b \in [a,d]$ there exists a sequence $\{t_k\}_{k\geq 1} \subset (0,\infty)$ satisfying

$$\lim_{k\to\infty} \|U(\cdot,t_k) - U_b\|_{L^\infty(\mathbf{R}^2)} = 0, \quad \lim_{k\to\infty} t_k = \infty.$$

Then, the solution oscillates among any stationary solutions between $U_a$ and $U_d$.

Concerning the solution in (ii), for each $j = 1,2,3,\cdots$ there exists a sequence $\{t_k\}_{k=1}^{\infty} \subset (0,\infty)$ satisfying

$$\lim_{k\to\infty} \|U(\cdot,t_k) - U_{b_j}\|_{L^\infty(\mathbf{R}^2)} = 0, \quad \lim_{k\to\infty} t_k = \infty.$$

Since $\lim_{b\to\infty} U_b = 8\pi\delta_0$, there exists a sequence $\{t_k\}_{k=1}^{\infty} \subset (0,\infty)$ satisfying

$$\lim_{k\to\infty} \|U(\cdot,t_k)\|_{L^\infty(\mathbf{R}^2)} = \infty, \quad \lim_{k\to\infty} t_k = \infty.$$

## §5 Idea of proof of Theorem 1

Essentially, using stability of stationary solutions, layer of stationary solutions and Poláčik and Yangida's argument in [9], we construct oscillating solutions.

The stability of radial stationary solutions are shown in [1]. The following proposition is a modified version of the result.

**Proposition 1** *Let* $U^{\mathcal{I}}$ *be nonnegative and radial,* $\|U^{\mathcal{I}}\|_{L^1(\mathbf{R}^2)} = 8\pi$ *and*

$$\sup_{x\in\mathbf{R}^2} (1+|x|)^5 |U^{\mathcal{I}}(x) - U_b(x)| < \infty$$

*with some* $b > 0$. *Then,* $\lim_{t\to\infty} \|U(\cdot,t) - U_b\|_{L^\infty(\mathbf{R}^2)} = 0$.

In two dimensional case, radial stationary solutions layer in the following sense.

$$\begin{cases} \lim_{a\to b} \|U_a - U_b\|_{L^\infty(\mathbf{R}^2)} = 0 \quad (b>0) \\ \displaystyle\int_{|x|<r} U_a(x)dx \leq \int_{|x|<r} U_b(x)dx \quad (r>0), \qquad \text{if } a \leq b. \end{cases}$$

Poláčik and Yanagida [9] show stability of radial stationary solutions to the problem

$$\begin{cases} U_t = \Delta U + U^p & \text{in } \mathbf{R}^n \times (0,\infty), \\ U(\cdot,0) = U^{\mathcal{I}} & \text{in } \mathbf{R}^n \end{cases}$$

with $n \geq 11$ and $p \geq p_{JL} = \{(n-2)^2 - 4n + 8\sqrt{n-1}\}/\{(n-2)(n-10)\}$. Moreover, radial stationary solutions to this problem layer in the case where $n \geq 11$ and $p \geq p_{JL}$. Using the stability and the layer, they construct oscillating solutions to this problem.

In order to describe the idea of proof of Theorem 1, we consider a special case.

**Theorem 2 (Special case of our problem)** *There exists a radial solution $(U,V)$ to (PE) with $U(\cdot,0) = U^{\mathcal{I}}$ such that $\{U_a, U_d\} \subset \omega(U^{\mathcal{I}} : C(\mathbf{R}^2))$ with $0 < a < d < \infty$.*

Let $(U,V)$ be a solutions to (PE). Put

$$u(r,t) = \frac{1}{2\pi r^2}\int_{|x|<r} U(x,t)dx.$$

The function $u$ satisfies

$$(IPE)\begin{cases} \mathcal{L}(u) = u_t - u_{rr} - \frac{3}{r}u_r - u\{ru_r + 2u\} = 0 \quad (0 < r < \infty,\ t > 0), \\ u_r(0,t) = 0 \quad (t > 0), \\ u(x,0) = u^{\mathcal{I}} \quad (0 \leq r < \infty). \end{cases}$$

Put

$$u_\alpha(r) = \frac{1}{2\pi r^2}\int_{|x|<r} U_\alpha(x)dx.$$

The function $u_\alpha$ is a stationary solution to (IPE).

**Sketch of Theorem 2.** For two positive constants $\tilde{L}_1 \gg L_1 \gg 1$ put

$$u_1^{\mathcal{I}}(r) = u_d(r)\ (\ r \leq L_1), \quad u_1^{\mathcal{I}}(r) = u_a(r)\ (\ \tilde{L}_1 < r).$$

Let $u_1$ be the solution to (IPE) with $u_1(\cdot,0) = u_1^{\mathcal{I}}$. By the continuity with respect to initial data, there exists $C(T_1) > 0$ such that

$$\begin{aligned} \|u_1(\cdot,t) - u_d\|_{L^\infty((0,\infty))} &\leq C(T_1)\|u_1(\cdot,0) - u_d\|_{L^\infty((0,\infty))} \\ &\leq C(T_1)L_1^{-2}\sup_{L_1<r} r^2|u_a(r) - u_d(r)| \\ &\leq C(T_1,a,d)L_1^{-2} \quad \text{for } t \in [0,T_1]. \end{aligned}$$

Therefore, for $0 < \varepsilon \ll 1$ and $T_1 > 0$ the solution $u_1$ satisfies

$$\|u_1(\cdot,t) - u_d\|_{L^\infty((0,\infty))} < \varepsilon \quad \text{for } t \in [0, T_1],$$

if $1 \ll L_1 \ll \tilde{L}_1$. On the other hand, Proposition 1 guarantees

$$\lim_{t\to\infty} \|u_1(\cdot,t) - u_a\|_{L^\infty((0,\infty))} = 0,$$

since $u_1(\cdot,0) - u_a$ has a compact support.

For $\tilde{L}_2 > L_2 \gg \tilde{L}_1$, putting initial data

$$u_2^{\mathcal{I}}(r) = u_1(r,0) \ ( \ r \le L_2), \quad u_2^{\mathcal{I}}(r) = u_d(r) \ ( \ \tilde{L}_2 < r).$$

Let $u_2$ be a solution to (IPE) with $u_2(\cdot,0) = u_2^{\mathcal{I}}$. Taking $T_2 \gg T_1$ and $\tilde{L}_2 \gg L_2 \gg \tilde{L}_1$ such that

$$\|u_1(\cdot,t) - u_a\|_{\beta,(0,\infty)} < \varepsilon/2 \quad \text{for } t \in [T_2 - 1, \infty)$$

and

$$\|u_2(\cdot,t) - u_1(\cdot,t)\|_{\beta,(0,\infty)} < \varepsilon/2 \quad \text{for } t \in [0, T_2 + 1],$$

we get

$$\|u_2(\cdot,T_2) - u_a\|_{\beta,(0,\infty)} < \varepsilon \quad \text{for } t \in [T_2 - 1, T_2 + 1],$$
$$\lim_{t\to\infty} \|u_2(\cdot,t) - u_d\|_{\beta,(0,\infty)} = 0.$$

Since the initial function $u_2^{\mathcal{I}}$ satisfies the property having the function $u_1^{\mathcal{I}}$, then the solution $u_2$ satisfies

$$\|u_2(\cdot,t) - u_d\|_{\beta,(0,\infty)} < \varepsilon \quad \text{for } t \in [T_1 - 1, T_1 + 1].$$

Repeating this argument, we find a solution $u$ with $u(\cdot,0) = u^0$ such that

$$\{u_a, u_d\} \subset \omega(u^{\mathcal{I}} : C([0,\infty))). \tag{1}$$

Moreover, the parabolic regularity method guarantees the following. There exists a constant $C$ such that

$$\|U(\cdot,t) - U_\alpha\|_{\beta,\mathbf{R}^n} \le C \max_{t-\frac{1}{2}\le s\le t+\frac{1}{2}} \|u(\cdot,s) - u_\alpha\|_{\beta,[0,\infty)} \quad \text{for } t \ge 1.$$

Then, for solution $u$ satisfying (1) we obtain that

$$(U(x,t), V(x,t)) = \left(\frac{1}{|x|}u_r(|x|,t), -\int_0^{|x|} u(r,t)dr\right)$$

is the desired solution to (PE). □

## §6 High dimensional case

As mentioned in the previous section, stability of stationary solutions and layer of stationary solutions guarantee the existence of oscillating solutions.

In the case where $n \geq 11$, stationary solutions are stable in the following sence.

**Theorem 3** *[6] Let* $n \geq 11$. $\beta_- = \{n+2-\sqrt{(n-2)(n-10)}\}/2 \in (2, n)$. *Suppose* $0 \leq U^{\mathcal{I}} \leq U_\infty$ *in* $\mathbf{R}^n$ *and*

$$\lim_{|x|\to\infty} (1+|x|)^{\beta_-} |U^{\mathcal{I}}(x) - U_\alpha(x)| = 0$$

*with some* $\alpha > 0$. *Then, the solution* $(U, V)$ *to (PE) satisfies*

$$\lim_{t\to\infty} \|U(\cdot, t) - U_\alpha\|_{\beta_-, \mathbf{R}^n} = 0,$$

*where* $\|F\|_{\beta,\mathbf{R}^n} = \sup_{x\in\mathbf{R}^n}(1+|x|)^\beta |F(x)|$.

Moreover, stationary solutions layer in the case where $n \geq 11$ in the following sense.

**Proposition 2** *[8] Let* $n \geq 11$. *For* $\alpha > 0$, *there exists a unique stationary solutions* $(U_\alpha, V_\alpha)$ *to (PE) satisfying* $U_\alpha(0) = 0$ *and (SPE). Moreover, the set of functions* $\{U_\alpha\}_{\alpha>0}$ *satisfies the following.*

*(i)* $\lim_{a\to b} \|U_a - U_b\|_{\beta_-,\mathbf{R}^n} = 0 \quad (b > 0)$

*(ii)* $U_b(x) = \dfrac{2(n-2)}{|x|^2} - \dfrac{A(b)}{|x|^{\beta_-}}$ *as* $|x| \to \infty$.

*(iii)* $U_a < U_b$ *in* $\mathbf{R}^n$, *if* $a < b$.

*Here,* $A(b)$ *is continuous and strictly decreasing with respect to* $b > 0$.

In order to describe our result, we define some functional spaces and $\omega$-limit sets.

For a non-negative constant $\beta$, put

$$C_\beta(\mathbf{R}^n) = \left\{F \in C(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n) : \sup_{x\in\mathbf{R}^n} (1+|x|)^\beta |F(x)| < \infty\right\}.$$

Let $(U, V)$ be a solution to (PE) with initial data $U^{\mathcal{I}}$ satisfying $U \in C([0,\infty) : C(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n))$. We put

$$\omega(U^{\mathcal{I}} : C_\beta(\mathbf{R}^n)) = \Big\{F \in C(\mathbf{R}^n) \cap L^\infty(\mathbf{R}^n) \ : \ \lim_{n\to\infty} t_n = \infty,$$
$$\lim_{n\to\infty} \|U(\cdot, t_n) - F\|_{\beta,\mathbf{R}^n} = 0 \quad \text{for some } \{t_n\} \subset (0,\infty)\Big\}.$$

Using Theorem 3 and Proposition 2, we construct the following solutions.

**Theorem 4** *[6] Let* $n \geq 11$ *and let* $\Lambda$ *be a set of* $[0, \infty)$. *Then, there exists a radial and continuous function* $U^{\mathcal{I}}$ *such that*

$$0 \leq U^{\mathcal{I}} \leq U_\infty \equiv \frac{2(n-2)}{|x|^2} \quad \text{in } \mathbf{R}^n.$$

*and*

$$\{U_a\}_{a \in \Lambda} \subset \omega(U^{\mathcal{I}} : C_\beta(\mathbf{R}^n)) \quad \text{for any } \beta \in [0, 2).$$

*Moreover, suppose* $\inf \Lambda > 0$. *Then, we can take* $\beta \in [0, \beta_-)$.

# References


[1] P. Biler, G. Kerch, P. Laurençot and T. Nadzieja, *The* $8\pi$ *problem for radially symmetric solutions of a chemotaxis model in the plane*, Math. Meth. Appl. Sci. **29** (2006), 1563-1583.

[2] A. Blanchet, J. A. Carrillo and N. Masmoudi, *Infinite time aggregation for the critical two-dimensional Patlak-Keller-Segel model*, Comm. Pure Appl. Math. **61** (2008), 1449-1481.

[3] D. Horstmann, From 1970 until present: The Keller-Segel model in chemotaxis and its consequences I, Jahresber. Deutsch. Math.-Verein. *105* (2003), 103-165.

[4] N. I. Kavallaris and P. Souplet, *Grow-up rate and refined asymptotics for a two-dimensional Patlak-Keller-Segel model in a disk*, SIAM J. Math. Anal. *40* (2009), 1852-1881.

[5] Y. Naito and T. Senba, *Bounded and unbounded oscillating solutions to a parabolic-elliptic system in two dimensional space*, Commun. Pure Appl. Math., submitted.

[6] T. Senba, *Stability of stationary solutions and existence of oscillating solutions to a chemotaxis system in high dimensional spaces*, Funkcial. Ekvac., accepted.

[7] T. Senba, *Blowup in infinite time of radial solutions for a parabolic-elliptic system in high-dimensional Euclidean spaces*, Nonlinear Analysis TMA, **70** (2009), 2549-2562.

[8] J. I. Tello, *Stability of steady states of the Cauchy problem for the exponential reaction-diffusion equation*, J. Math. Anal. Appl. **324** (2006), 381-396.

[9] Poláčik and E. Yanagida, *On bounded and unbounded global solutions of a supercritical semilinear heat equation*, Math. Ann. **337** (2003), 745-771.