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Fokas-Lenellsの微分型非線形シュレーディンガー方程式の直接解法

Direct method of solution for the Fokas-Lenells derivative nonlinear Schrödinger equation

山口大学大学院理工学研究科　松野　好雅 (Yoshimasa Matsuno)
Division of Applied Mathematical Science
Graduate School of Science and Engineering
Yamaguchi University
E-mail address: matsuno@yamaguchi-u.ac.jp

Abstract

We developed a systematic method for obtaining soliton solutions of the Fokas-Lenells derivative nonlinear Schrödinger equation (FL equation shortly) under non-vanishing boundary condition. In particular, we deal with dark soliton solutions with a plane wave boundary condition. We first derive the novel system of bilinear equations which is reduced from the FL equation through a dependent variable transformation and then construct the general dark N-soliton solution of the system, where N is an arbitrary positive integer. We then investigate the properties of the one-soliton solutions in detail, showing that both the dark and bright solitons appear on the nonzero background which reduce to algebraic solitons in specific limits. Last, the interaction process of two solitons is described.

1. Introduction

We consider the following Fokas-Lenells (FL) equation which can be derived from its original version by a simple change of variables combined with a gauge transformation:

$$u_{xt} = u - 2\mathrm{i}|u|^2 u_x. \tag{1.1}$$

Here, $u = u(x,t)$ is a complex-valued function of x and t, and subscripts x and t appended to u denote partial differentiations. The known results about the FL equation are:

• An integrable generalization of the nonlinear Schrödinger equation, Fokas [1].

• Inverse scattering transform method under the vanishing boundary condition, Lenells and Fokas [2].

• A model equation for the propagation of nonlinear light pulses in monomode optical fibers, Lenells [3].

• The first negative member of the integrable hierarchy of the derivative nonlinear Schrödinger equation, Lenells [3].

• Derivation of the bright soliton solutions, Lenells [4], Matsuno [5].

The purposes of the present report are:

• To construct the dark N-soliton solution of the FL equation on the background of a plane wave. Explicitly, we consider the boundary condition

$$u \to \rho \exp\left\{\mathrm{i}\left(\kappa x - \omega t + \phi^{(\pm)}\right)\right\}, \quad x \to \pm\infty, \tag{1.2}$$

where $\rho(>0)$ and κ are real constants representing the amplitude and wavenumber, respectively, $\phi^{(\pm)}$ are real phase constants and the angular frequency $\omega = \omega(\kappa)$ obeys the dispersion relation $\omega = 1/\kappa + 2\rho^2$.

• To investigate the properties of dark soliton solutions.

This report is the summary of the paper by Matsuno [6].

2. Exact method of solution

2.1. Bilinearization

Proposition 2.1. *By means of the dependent variable transformation*

$$u = \rho\,\mathrm{e}^{\mathrm{i}(\kappa x - \omega t)}\,\frac{g}{f}, \tag{2.1}$$

with $\omega = 1/\kappa + 2\rho^2$, equation (1.1) can be decoupled into the following system of bilinear equations for the tau functions f and g

$$D_t f \cdot f^* - \mathrm{i}\rho^2(gg^* - ff^*) = 0, \tag{2.2}$$

$$D_x D_t f \cdot f^* - \mathrm{i}\rho^2 D_x g \cdot g^* + \mathrm{i}\rho^2 D_x f \cdot f^* + 2\kappa\rho^2(gg^* - ff^*) = 0, \tag{2.3}$$

$$D_x D_t g \cdot f + \mathrm{i}\kappa D_t g \cdot f - \mathrm{i}\omega D_x g \cdot f = 0. \tag{2.4}$$

Here, $f = f(x,t)$ and $g = g(x,t)$ are complex-valued functions of x and t, and the asterisk appended to f and g denotes complex conjugate and the bilinear operators D_x and D_t are defined by

$$D_x^m D_t^n f \cdot g = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^m \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^n f(x,t)g(x',t')\Big|_{x'=x,\, t'=t}, \tag{2.5}$$

where m and n are nonnegative integers.

Proof. Substituting (2.1) into (1.1) and rewriting the resultant equation in terms of the bilinear operators, equation (1.1) can be rewritten as

$$\frac{1}{f^2}(D_x D_t g \cdot f + \mathrm{i}\kappa D_t g \cdot f - \mathrm{i}\omega D_x g \cdot f)$$

$$-\frac{g}{f^3f^*}\{f^*D_xD_tf\cdot f-2\kappa\rho^2f^2f^*-2\mathrm{i}\rho^2g^*(g_xf-gf_x+\mathrm{i}\kappa fg)\}=0. \qquad (2.6)$$

Inserting the identity

$$f^*D_xD_tf\cdot f=fD_xD_tf\cdot f^*-2f_xD_tf\cdot f^*+f(D_tf\cdot f^*)_x, \qquad (2.7)$$

which can be verified by direct calculation, into the second term on the left-hand side of (2.6), one modifies it in the form

$$\frac{1}{f^2}(D_xD_tg\cdot f+\mathrm{i}\kappa D_tg\cdot f-\mathrm{i}\omega D_xg\cdot f)$$

$$-\frac{g}{f^3f^*}\Big[f\{D_xD_tf\cdot f^*-\mathrm{i}\rho^2D_xg\cdot g^*+\mathrm{i}\rho^2D_xf\cdot f^*+2\kappa\rho^2(gg^*-ff^*)\}$$

$$-2f_x\{D_tf\cdot f^*-\mathrm{i}\rho^2(gg^*-ff^*)\}+f\{D_tf\cdot f^*-\mathrm{i}\rho^2(gg^*-ff^*)\}_x\Big]=0. \qquad (2.8)$$

By virtue of equations (2.2)-(2.4), the left-hand side of (2.8) vanishes identically. □

It follows from (2.1) and (2.2) that

$$|u|^2=\rho^2+\mathrm{i}\frac{\partial}{\partial t}\ln\frac{f^*}{f}. \qquad (2.9)$$

2.2. Trilinear equation

Proposition 2.2. *The trilinear equation for f and g*

$$f^*\left\{g_{xt}f-(f_x-\mathrm{i}\kappa f)g_t-\mathrm{i}\left(\frac{1}{\kappa}+\rho^2\right)(g_xf-gf_x)\right\}=f_t^*(g_xf-gf_x+\mathrm{i}\kappa fg), \qquad (2.10)$$

is a consequence of the bilinear equations (2.2)-(2.4).

Proof. By direct calculation, one can show the following trilinear identity among the tau functions f and g:

$$f^*\left\{g_{xt}f-(f_x-\mathrm{i}\kappa f)g_t-\mathrm{i}\left(\frac{1}{\kappa}+\rho^2\right)(g_xf-gf_x)\right\}-f_t^*(g_xf-gf_x+\mathrm{i}\kappa fg)$$

$$=f^*(D_xD_tg\cdot f+\mathrm{i}\kappa D_tg\cdot f-\mathrm{i}\omega D_xg\cdot f)$$

$$-\frac{g}{2}\Big[\{D_tf\cdot f^*-\mathrm{i}\rho^2(gg^*-ff^*)\}_x+(D_xD_tf\cdot f^*-\mathrm{i}\rho^2D_xg\cdot g^*+\mathrm{i}\rho^2D_xf\cdot f^*-2\mathrm{i}\kappa D_tf\cdot f^*)\Big]$$

$$+g_x\{D_tf\cdot f^*-\mathrm{i}\rho^2(gg^*-ff^*)\}. \qquad (2.11)$$

Replacing a term $2\mathrm{i}\kappa D_tf\cdot f^*$ on the right-hand side of (2.11) by (2.2), the right-hand side becomes zero by (2.2)-(2.4). This yields (2.10). □

3. Dark N-soliton solution

3.1. Main result

Theorem 3.1. *The dark N-soliton solution of the system of bilinear equations (2.2)-(2.4) is expressed by the following determinants*

$$f = |D|, \tag{3.1a}$$

$$g = \begin{vmatrix} D & \mathbf{z}^T \\ \frac{1}{\rho^2}\mathbf{z}_t^* & 1 \end{vmatrix} = |D| + \frac{1}{\rho^2}\begin{vmatrix} D & \mathbf{z}^T \\ \mathbf{z}_t^* & 0 \end{vmatrix}. \tag{3.1b}$$

Here, D is an $N \times N$ matrix and $\mathbf{z}$ and $\mathbf{z}_t$ are N-component row vectors defined below and the symbol T denotes the transpose:

$$D = (d_{jk})_{1 \le j,k \le N}, \quad d_{jk} = \delta_{jk} + \frac{\kappa - \mathrm{i}p_j}{p_j + p_k^*} z_j z_k^*,$$

$$z_j = \exp\left(p_j x + \frac{\kappa\rho^2}{p_j} t + \frac{1}{p_j + \mathrm{i}\kappa}\tau + \zeta_{j0}\right), \tag{3.2a}$$

$$\mathbf{z} = (z_1, z_2, ..., z_N), \quad \mathbf{z}_t = \left(\frac{\kappa\rho^2 z_1}{p_1}, \frac{\kappa\rho^2 z_2}{p_2}, ..., \frac{\kappa\rho^2 z_N}{p_N}\right), \tag{3.2b}$$

where p_j are complex parameters satisfying the constraints

$$(p_j + \mathrm{i}\kappa)(p_j^* - \mathrm{i}\kappa) = \frac{1 + \kappa\rho^2}{\kappa\rho^2} p_j p_j^*, \quad j = 1, 2, ..., N, \tag{3.2c}$$

ζ_{j0} $(j = 1, 2, ..., N)$ are arbitrary complex parameters, δ_{jk} is kronecker's delta and τ is an auxiliary variable.

3.2. Remarks

- The dark N-soliton solution is parameterized by $2N$ complex parameters p_j and ζ_{j0} $(j = 1, 2, ..., N)$. The parameters p_j determine the amplitude and velocity of the solitons whereas the parameters ζ_{j0} determine the phase of the solitons. As opposed to the bright soliton case, however, the real and imaginary parts of p_j are not independent because of the constraints (3.2c).
- The dark N-soliton solution (3.1) solves the bilinear equations (2.2) and (2.3) without the constraints (3.2c).
- The trilinear equation (2.10) will be proved in place of the bilinear equation (2.4) where we use the relations

$$f_t = (1 + \kappa\rho^2) f_\tau, \quad g_t = (1 + \kappa\rho^2) g_\tau.$$

as well as the constraints (3.2c).

4. Stability of the plane wave

We have considered the dark solitons on the background of a plane wave $\rho\, \mathrm{e}^{\mathrm{i}(\kappa x-\omega t)}$ with $\omega = 1/\kappa + 2\rho^2$. It is important to see whether the background field is stable or not against perturbations. If unstable, then dark solitons would not exist. Here, we perform the linear stability analysis of the plane wave.

Following the standard procedure, we seek a solution of the form

$$u = (\rho + \Delta\rho)\, \mathrm{e}^{\mathrm{i}(\kappa x-\omega t+\Delta\phi)}, \tag{4.1}$$

where $\Delta\rho = \Delta\rho(x,t)$ and $\Delta\phi = \Delta\phi(x,t)$ are small perturbations. Substituting (4.1) into the FL equation (1.1) and linearizing about the plane wave, we obtain the system of linear PDEs for $\Delta\rho$ and $\Delta\phi$

$$\Delta\rho_{xt} + \rho(\omega - 2\rho^2)\Delta\phi_x - \kappa\rho\Delta\phi_t - 4\kappa\rho^2\Delta\rho = 0, \tag{4.2a}$$

$$\rho\Delta\phi_{xt} - (\omega - 2\rho^2)\Delta\rho_x + \kappa\Delta\rho_t = 0. \tag{4.2b}$$

Assume the perturbations of the form $\mathrm{e}^{\mathrm{i}(\lambda x-\nu t)}$ with λ real and ν possibly complex and substitute them into (4.2) to obtain a homogeneous linear system for $\Delta\rho$ and $\Delta\phi$

$$(\lambda\nu - 4\kappa\rho^2)\Delta\rho + \mathrm{i}\{\rho\lambda(\omega - 2\rho^2) + \kappa\rho\nu\}\Delta\phi = 0, \tag{4.3a}$$

$$-\mathrm{i}\{(\omega - 2\rho^2)\lambda + \kappa\nu\}\Delta\rho + \rho\lambda\nu\Delta\phi = 0. \tag{4.3b}$$

The nontrivial solution exists if ν satisfies the quadratic equation

$$(\lambda^2 - \kappa^2)\nu^2 - 2(2\kappa\rho^2 + 1)\lambda\nu - \frac{\lambda^2}{\kappa^2} = 0. \tag{4.4}$$

Solving this equation, we obtain

$$\nu = \frac{\lambda}{\lambda^2 - \kappa^2}\left[2\kappa\rho^2 + 1 \pm \frac{1}{\kappa}\sqrt{\lambda^2 + 4\kappa^3(\kappa\rho^2 + 1)\rho^2}\right]. \tag{4.5}$$

Thus, if the condition

$$\kappa(\kappa\rho^2 + 1) > 0, \tag{4.6}$$

is satisfied, then ν becomes real for all values of real λ, implying that the plane wave is neutrally stable. It is evident that this condition always holds for $\kappa > 0$. For negative κ, on the other hand, we put $\kappa = -K$ with $K > 0$ and see that the stability criterion turns out to be as $K\rho^2 > 1$.

5. Properties of dark soliton solutions

We first parametrize the complex parameters p_j and ζ_{j0} by the real quantities a_j, b_j, θ_{j0} and χ_{j0} as

$$p_j = a_j + \mathrm{i}b_j, \qquad \zeta_{j0} = \theta_{j0} + \mathrm{i}\chi_{j0}, \qquad j = 1, 2, \ldots, N, \tag{5.1}$$

and introduce the new independent variables θ_j and χ_j according to the relations

$$\theta_j = a_j(x + c_j t) + \theta_{j0}, \qquad c_j = \frac{\kappa\rho^2}{a_j^2 + b_j^2}, \qquad j = 1, 2, ..., N. \tag{5.2a}$$

$$\chi_j = b_j(x - c_j t) + \chi_{j0}, \qquad j = 1, 2, ..., N. \tag{5.2b}$$

In terms of these variables, the variables z_j defined by (3.2a) are put into the form

$$z_j = \mathrm{e}^{\theta_j + \mathrm{i}\chi_j}, \qquad j = 1, 2, ..., N, \tag{5.2c}$$

after setting $\tau = 0$. Substituting (5.1) into (3.2c), the constraints for p_j can be rewritten as a quadratic equation for b_j

$$b_j^2 - 2\kappa^2\rho^2 b_j + a_j^2 - \kappa^3\rho^2 = 0, \qquad j = 1, 2, ..., N. \tag{5.3}$$

The solution to this equation is found to be as follows:

$$b_j = (\kappa\rho)^2 \pm \sqrt{\kappa^3\rho^2(1 + \kappa\rho^2) - a_j^2}, \qquad j = 1, 2, ..., N. \tag{5.4}$$

We can see from the above expression that the real b_j $(j = 1, 2, ..., N)$ exist only when the condition $\kappa^3\rho^2(1 + \kappa\rho^2) > 0$ is satisfied. This coincides with the criterion (4.6) for the stability of the plane wave. Throughout the analysis, we assume this condition to assure the existence of soliton solutions. It is to be noted from (5.2) and (5.3) that the parameters a_j and b_j are expressed in terms of c_j as

$$a_j^2 = \frac{\kappa^2}{4c_j^2}\left(c_{\max} - c_j\right)\left(c_j - c_{\min}\right), \qquad b_j = \frac{1}{2\kappa c_j}(1 - \kappa^2 c_j), \qquad c_{\min} < c_j < c_{\max}, \tag{5.5a}$$

where

$$c_{\max} = \frac{1}{\kappa^2}\left\{1 + 2\kappa\rho^2 + 2\sqrt{\kappa\rho^2(1 + \kappa\rho^2)}\right\},$$

$$c_{\min} = \frac{1}{\kappa^2}\left\{1 + 2\kappa\rho^2 - 2\sqrt{\kappa\rho^2(1 + \kappa\rho^2)}\right\}. \tag{5.5b}$$

Thus, the dark N-soliton solution is characterized by the N velocities $c_j\,(j = 1, 2, ..., N)$ and the $2N$ real phase constants θ_{j0} and $\chi_{j0}\,(j = 1, 2, ..., N)$, the total number of which is $3N$.

5.1. One-soliton solution

The tau functions $f = f_1$ and $g = g_1$ for the one-soliton solution are given by

$$f_1 = 1 + \frac{\kappa - \mathrm{i}p_1}{p_1 + p_1^*} z_1 z_1^*, \qquad g_1 = 1 - \frac{\kappa + \mathrm{i}p_1^*}{p_1 + p_1^*}\frac{p_1}{p_1^*} z_1 z_1^*. \tag{5.6}$$

The one-soliton solution u_1 follows from (2.1) with (5.6), yielding

$$u_1 = \rho\, \mathrm{e}^{\mathrm{i}(\kappa x - \omega t)} \frac{1 - \frac{\kappa + b_1 + \mathrm{i}a_1}{2a_1} \frac{a_1 + \mathrm{i}b_1}{a_1 - \mathrm{i}b_1} \mathrm{e}^{2\theta_1}}{1 + \frac{\kappa + b_1 - \mathrm{i}a_1}{2a_1} \mathrm{e}^{2\theta_1}}. \tag{5.7}$$

The above expression can be put into the form

$$u_1 = |u_1|\, \mathrm{e}^{\mathrm{i}(\kappa x - \omega t)} \exp\left\{\mathrm{i}\left(\phi + \phi^{(+)}\right)\right\}, \tag{5.8}$$

where the square of the modulus of u_1 is represented by

$$|u_1|^2 = \rho^2 - \frac{2a_1^2 c\, \mathrm{sgn}(\kappa a_1)}{\sqrt{a_1^2 + (\kappa + b_1)^2}} \frac{1}{\cosh\, 2(\theta_1 + \delta_1) + \frac{(\kappa + b_1)\, \mathrm{sgn}\, a_1}{\sqrt{a_1^2 + (\kappa + b_1)^2}}}, \qquad c = |c_1|, \tag{5.9a}$$

with

$$\theta_1 = a_1(x + c_1 t) + \theta_{10}, \qquad c_1 = \frac{\kappa \rho^2}{a_1^2 + b_1^2}, \qquad \mathrm{e}^{4\delta_1} = \frac{a_1^2 + (\kappa + b_1)^2}{4a_1^2}, \tag{5.9b}$$

and the tangent of the phase ϕ and $\phi^{(+)}$ being given respectively by

$$\tan\, \phi = \frac{\{a_1^2 + b_1(\kappa + b_1)\} \cosh\, 2(\theta_1 + \delta_1) + b_1\, \mathrm{sgn}\, a_1 \sqrt{a_1^2 + (\kappa + b_1)^2}}{\kappa a_1\, \sinh\, 2(\theta_1 + \delta_1)}, \tag{5.10a}$$

$$\tan\, \phi^{(+)} = \frac{a_1^2 + b_1(\kappa + b_1)}{\kappa a_1}. \tag{5.10b}$$

Let us classify the one-soliton solutions in accordance with the sign of κ. We consider the two cases, i.e., case 1 ($\kappa > 0, a_1 \lessgtr 0$) and case 2 ($\kappa < 0, a_1 \lessgtr 0$) separately. For each sign of κ, both dark and bright solitons arise, as we shall show now.

5.1.1. Case 1: $\kappa > 0$

In this case, the velocity $c_1 (= \kappa\rho^2/(a_1^2 + b_1^2))$ of the soliton is positive. We then find from (5.5) and (5.9) that

$$\begin{aligned} A_d &= \rho - \sqrt{\rho^2 - 2c_1\left\{\sqrt{a_1^2 + (\kappa + b_1)^2} - (\kappa + b_1)\right\}} \\ &= \rho - \frac{1}{\sqrt{\kappa}}|\kappa\sqrt{c} - \sqrt{1 + \kappa\rho^2}|, \qquad a_1 > 0, \quad c_1 = c > 0, \end{aligned} \tag{5.11}$$

$$\begin{aligned} A_b &= \sqrt{\rho^2 + 2c_1\left\{\sqrt{a_1^2 + (\kappa + b_1)^2} + (\kappa + b_1)\right\}} - \rho \\ &= \frac{1}{\sqrt{\kappa}}(\kappa\sqrt{c} + \sqrt{1 + \kappa\rho^2}) - \rho \qquad a_1 < 0, \quad c_1 = c > 0, \end{aligned} \tag{5.12}$$

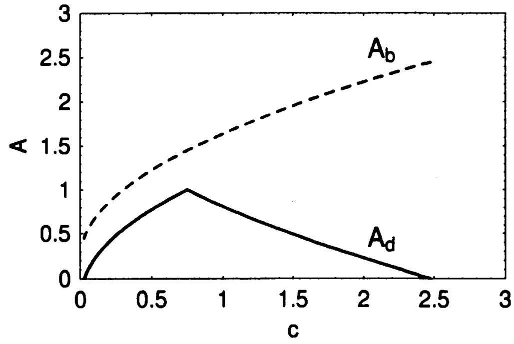

Figure 1. Amplitude-velocity relation for the dark soliton A_d (solid line) and bright soliton A_b (broken line) for $\rho = 1$ and $\kappa = 2$.

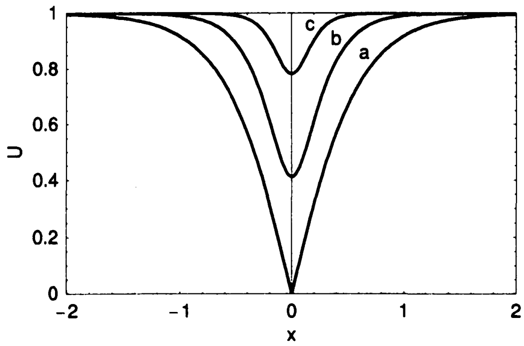

Figure 2. Profile of the amplitude of the dark soliton $U = |u_1|$ at $t = 0$. a: $c = c_0 = 0.75$, b: $c = 0.33$, c: $c = 0.098$. The profile a is a black soliton.

where $c \equiv |c_1|$ lies in the interval $c_{\min} < c < c_{\max}$ with $c_{\max}$ and $c_{\min}$ being given by (5.5b). Note from (5.5a) that $\kappa + b_1 = (1 + \kappa^2 c_1)/(2\kappa c_1) > 0$ for $\kappa > 0$ and $c_1 > 0$.

Figure 1 plots the dependence of the amplitudes $A = A_d$ and $A = A_b$ on the velocity $c = |c_1|$ for $\rho = 1$ and $\kappa = 2$.

(i) Dark soliton: $a_1 > 0$

As seen from figure 1, the amplitude A_d of the dark soliton becomes an increasing function of the velocity c in the interval $c_{\min} < c \leq c_0$ and a decreasing function in the interval $c_0 < c < c_{\max}$, where $c_{\max}$ and $c_{\min}$ are given by (5.5b) and a critical velocity c_0 by

$$c_0 = \frac{1 + \kappa\rho^2}{\kappa^2}. \tag{5.13}$$

In the present numerical example ($\rho = 1, \kappa = 2$), $c_{\min} = 0.025, c_0 = 0.75, c_{\max} = 2.47$. Figure 2 depicts the profile of $U = |u_1|$ at $t = 0$ for three different values of c, i.e., a: $c = c_0 = 0.75$, b: $c = 0.33$, c: $c = 0.098$ with the parameters $\rho = 1, \kappa = 2, \theta_{10} = -\delta_1$ and $\chi_{10} = 0$. When $c = c_0$, the amplitude of the dark

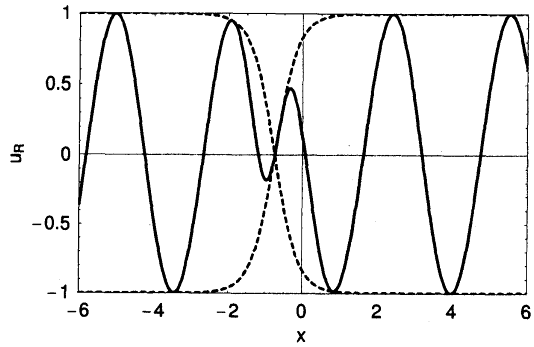

Figure 3. Profile of a black soliton $u_{\rm R} = {\rm Re}\, u_1$ at $t = 1$.

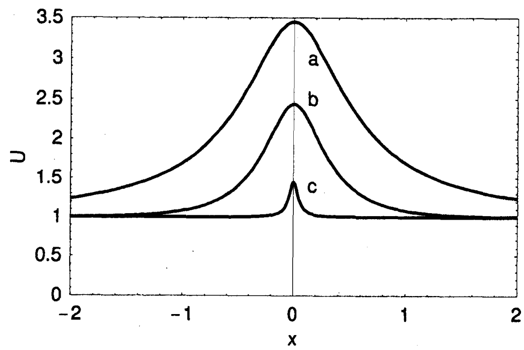

Figure 4. Profile of the amplitude of the bright soliton $U = |u_1|$ at $t = 0$. a: $c = 2.47$, b: $c = 0.73$, c: $c = 0.025$. The profiles a and c are algebraic solitons.

soliton attains the maximum value $A_d = \rho$. See figure 2 a. It then turns out that the intensity of the soliton center falls to zero. Such a soliton is well-known in the field of nonlinear optics. It is sometimes called a *black* soliton.

Figure 3 shows the profile of $u_{\rm R} = {\rm Re}[u_1]$ at $t = 1$ for the black soliton. The broken line indicates $\pm|u_1|$ (see figure 2 a).

(ii) Bright soliton: $a_1 < 0$

Figure 4 depicts the profile of the bright soliton $U = |u_1|$ at $t = 0$ for three different values of c, i.e., a: $c = 2.47$, b: $c = 0.73$, c: $c = 0.025$ with $\rho = 1$ and $\kappa = 2$. The feature of the bright soliton differs substantially from that of the dark soliton. To be specific, the amplitude of the bright soliton always becomes an increasing function of the velocity (see figure 1). It takes the maximum value at $c = c_{\rm max}$ and the minimum value at $c = c_{\rm min}$. At these limiting values of the velocity, the algebraic soliton is produced from the soliton of hyperbolic type.

Indeed, if we put $\theta_{10} = a_1 x_0 - \delta_1$ in (5.7) and (5.9) with x_0 being a real constant and then take the limit $a_1 \to -0$, we find

$$u_1 = \rho\, {\rm e}^{{\rm i}(\kappa x - \omega t)} \frac{x + ct + x_0 - {\rm i}\, \frac{2\kappa + b_1}{2b_1(\kappa + b_1)}}{x + ct + x_0 - {\rm i}\, \frac{1}{2(\kappa + b_1)}}, \tag{5.14a}$$

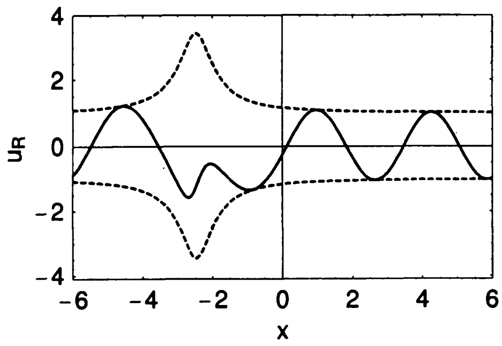

Figure 5. Profile of an algebraic bright soliton $u_R = \mathrm{Re}\, u_1$ at $t = 1$.

$$|u_1|^2 = \rho^2 + \frac{2\kappa c^2}{1+\kappa^2 c} \frac{1}{(x+ct+x_0)^2 + \left(\frac{\kappa c}{1+\kappa^2 c}\right)^2}, \tag{5.14b}$$

where $b_1 = (1-\kappa^2 c)/2\kappa c$ by (4.5a) and $c = c_{\max}$ or $c_{\min}$. Note from (5.9b) that $b_1^2 = \kappa\rho^2/c$ when $a_1 \to -0$.

A representative profile of the algebraic bright soliton $U = |u_1|$ at $t = 0$ and the corresponding profile of $u_R = \mathrm{Re}\, u_1$ at $t = 1$ are shown in figure 4 a and figure 5, respectively.

5.1.2. Case 2: $\kappa < 0$

For negative κ, the expressions of the amplitude for the dark and bright solitons are given respectively by

$$\begin{aligned} A_d &= \rho - \sqrt{\rho^2 + 2c_1\left\{\sqrt{a_1^2 + (\kappa+b_1)^2} + (\kappa+b_1)\right\}} \\ &= \rho - \frac{1}{\sqrt{K}}|K\sqrt{c} - \sqrt{K\rho^2-1}|, \qquad a_1 < 0, \quad c_1 = -c < 0, \end{aligned} \tag{5.15}$$

$$\begin{aligned} A_b &= \sqrt{\rho^2 - 2c_1\left\{\sqrt{a_1^2 + (\kappa+b_1)^2} - (\kappa+b_1)\right\}} - \rho \\ &= \frac{1}{\sqrt{K}}(K\sqrt{c} + \sqrt{K\rho^2-1}) - \rho, \qquad a_1 > 0, \quad c_1 = -c < 0, \end{aligned} \tag{5.16}$$

where $K = -\kappa$ is a positive wavenumber and the velocity c lies in the interval $c'_{\min} < c < c'_{\max}$ with

$$\begin{aligned} c'_{\max} &= \frac{1}{K^2}\left\{2K\rho^2 - 1 + 2\sqrt{K\rho^2(K\rho^2-1)}\right\}, \\ c'_{\min} &= \frac{1}{K^2}\left\{2K\rho^2 - 1 - 2\sqrt{K\rho^2(K\rho^2-1)}\right\}. \end{aligned} \tag{5.17}$$

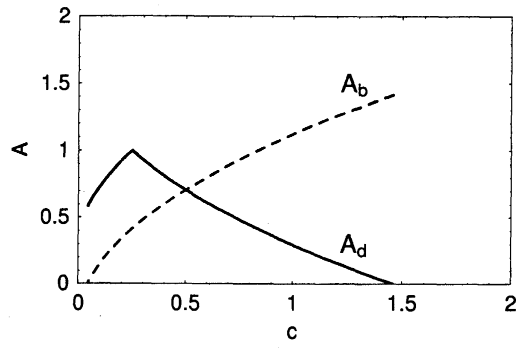

Figure 6. Amplitude-velocity relation for the dark soliton A_d (solid line) and bright soliton A_b (broken line) for $\rho = 1$ and $\kappa = -2$.

Recall that the condition $K\rho^2 - 1 > 0$ must be imposed to assure the existence of the soliton solutions.

Figure 6 plots the dependence of the amplitudes $A = A_d$ and $A = A_b$ on the velocity $c = |c_1|$ for $\rho = 1$ and $\kappa = -2$. When compared with figure 1 for $\kappa > 0$, there appear several different features for $\kappa < 0$. In particular, the algebraic *dark* soliton would arise in the limit $c \to c'_{\text{min}}$ since in this limit, the amplitude A_d tends to a finite value. In addition, the algebraic bright soliton exists only in the limit $c \to c'_{\text{max}}$. We now proceed to the detailed description of the soliton solutions.

(i) Dark soliton: $a_1 < 0$

It follows from (5.5) with $\kappa = -K, c_1 = -c$ that $\kappa + b_1 = 1/2Kc - K/2$. Since $c'_{\text{min}} < c < c'_{\text{max}}$, the possible value of $\kappa + b_1$ is restricted by the inequality

$$K\left[K\rho^2 - 1 - \sqrt{K\rho^2(K\rho^2-1)}\right] < \kappa + b_1 < K\left[K\rho^2 - 1 + \sqrt{K\rho^2(K\rho^2-1)}\right]. \tag{5.18}$$

One can see that the upper limit of $\kappa + b_1$ is attained when $c = c'_{\text{min}}$ and its limiting value is positive by the condition $K\rho^2 > 1$ whereas the lower limit is attained when $c = c'_{\text{max}}$ and is negative. In view of this fact, the algebraic dark soliton would be produced in the limit $c \to c'_{\text{min}}$ for which $\text{sgn}(\kappa + b_1) > 0$. Actually, taking the limit $a_1 \to -0$ for the solutions (5.7) and (5.9), we find that the hyperbolic soliton reduces to the limiting form

$$u_1 = \rho\, \mathrm{e}^{\mathrm{i}(-Kx-\omega t)} \frac{x - ct + x_0 - \mathrm{i}\,\frac{-2K+b_1}{2b_1(-K+b_1)}}{x - ct + x_0 - \mathrm{i}\,\frac{1}{2(-K+b_1)}}, \tag{5.19a}$$

$$|u_1|^2 = \rho^2 - \frac{2Kc^2}{1-K^2c}\frac{1}{(x-ct+x_0)^2 + \left(\frac{Kc}{1-K^2c}\right)^2}, \tag{5.19b}$$

where $b_1 = (1 + K^2c)/2Kc$ and $c = c'_{\text{min}}$. Since $1 - Kc'_{\text{min}} > 0$ by virtue of the

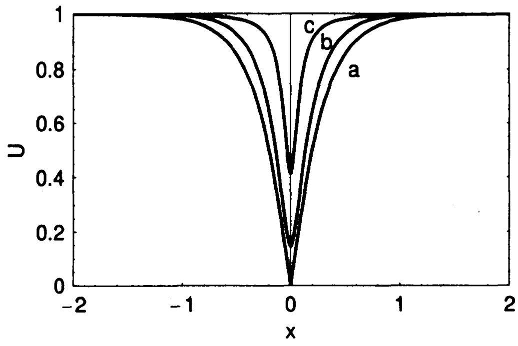

Figure 7. Profile of the amplitude of the dark soliton $U = |u_1|$ at $t = 0$. a: $c = c_0 = 0.25$, b: $c = 0.16$, c: $c = 0.043$. The profile a is a black soliton and the profile c is an algebraic soliton.

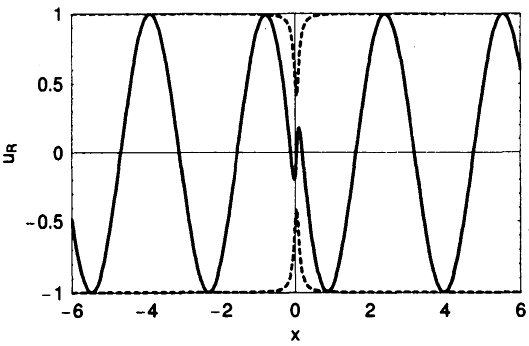

Figure 8. Profile of an algebraic dark soliton $u_{\mathrm{R}} = \mathrm{Re}\, u_1$ at $t = 1$.

condition $K\rho^2 > 1$, the expression (5.19b) actually represents an algebraic dark soliton.

The black soliton appears when the velocity c takes a specific value $c = c_0'$, where

$$c_0' = (K\rho^2 - 1)/K^2. \tag{5.20}$$

Its profile is represented by

$$|u_1|^2 = \rho^2 \left[1 - \frac{3K\rho^2 - 4}{2(K\rho^2 - 1)} \frac{1}{\cosh 2(\theta_1 + \delta_1) + \frac{K\rho^2 - 2}{2(K\rho^2 - 1)}}\right]. \tag{5.21}$$

It is important to notice that the inequality $c'_{\mathrm{min}} < c_0' < c'_{\mathrm{max}}$ requires the condition $K\rho^2 > 4/3$ for the wavenumber K. It then turns out that expression (5.21) takes the form of a black soliton.

Figure 7 depicts the profile of $U = |u_1|$ at $t = 0$ for three different values of c, i.e., a: $c = c_0' = 0.25$, b: $c = 0.16$, c: $c = 0.043$ with the parameters $\rho = 1, \kappa = -2, \theta_{10} = -\delta_1$ and $\chi_{10} = 0$. In this example, $c'_{\mathrm{min}} = 0.043, c_0' = 0.25$ and $c'_{\mathrm{max}} = 1.46$ (see figure 6). An algebraic soliton appears at the lower limit of

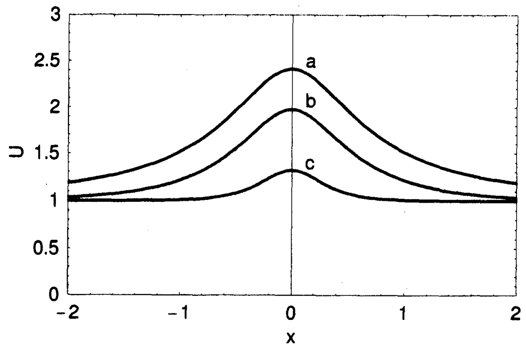

Figure 9. Profile of the amplitude of the bright soliton $U = |u_1|$ at $t = 0$. a: $c = 1.46$, b: $c = 0.81$, c: $c = 0.19$. The profiles a is an algebraic soliton.

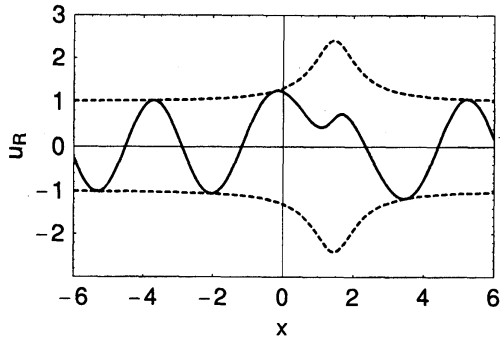

Figure 10. Profile of an algebraic bright soliton $u_R = \mathrm{Re}\, u_1$ at $t = 1$.

the velocity, i.e., $c = c'_{\rm min}$ whereas a black soliton arises at $c = c'_0$. Figure 8 shows the profile of $u_R = \mathrm{Re}\ u_1$ at $t = 1$ for an algebraic dark soliton.

(ii) Bright soliton: $a_1 > 0$

Figure 9 depicts the profile of $U = |u_1|$ at $t = 0$ for three different values of c, i.e., a: $c = 1.46$, b: $c = 0.73$, c: $c = 0.025$ with $\rho = 1$ and $\kappa = -2$. Figure 10 shows the profile $u_{\rm R} = \mathrm{Re}\, u_1$ of an algebraic bright soliton at $t = 1$ which corresponds to the profile a in figure 9.

• **Summary**

i) $\kappa > 0$, $a_1 > 0$: dark soliton (no algebraic soliton)

ii) $\kappa > 0$, $a_1 < 0$: bright soliton (algebraic soliton)

iii) $\kappa < 0$, $a_1 > 0$: bright soliton (algebraic soliton)

iv) $\kappa < 0$, $a_1 < 0$: dark soliton (algebraic soliton)

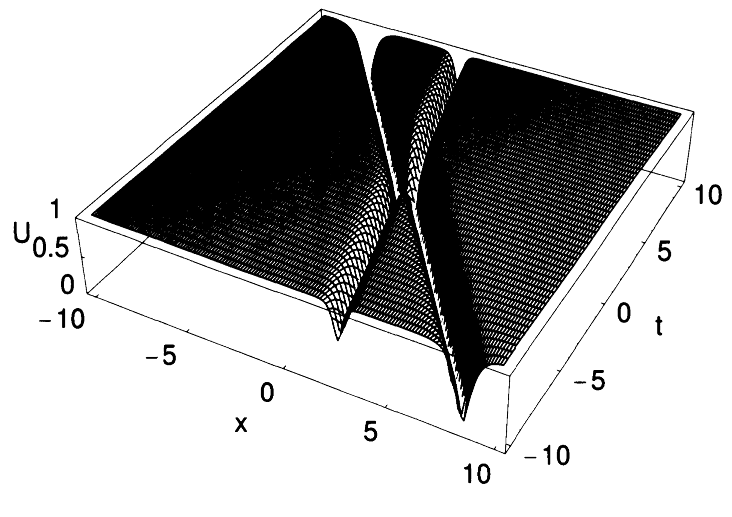

Figure 11. The interaction of two dark solitons.

5.2. Two-soliton solution

5.2.1. Dark-dark solitons

The tau functions f_2 and g_2 representing the dark two-soliton solution are given by (3.1)-(3.3) with $N = 2$ subjected to the conditions $\kappa > 0, a_1 > 0, a_2 > 0$. They read

$$f_2 = 1 + \frac{\kappa - \mathrm{i}p_1}{p_1 + p_1^*} z_1 z_1^* + \frac{\kappa - \mathrm{i}p_2}{p_2 + p_2^*} z_2 z_2^*$$
$$+ \frac{(\kappa - \mathrm{i}p_1)(\kappa - \mathrm{i}p_2)(p_1 - p_2)(p_1^* - p_2^*)}{(p_1 + p_1^*)(p_1 + p_2^*)(p_2 + p_1^*)(p_2 + p_2^*)} z_1 z_2 z_1^* z_2^*, \tag{5.22a}$$

$$g_2 = 1 - \frac{\kappa + \mathrm{i}p_1^*}{p_1 + p_1^*} \frac{p_1}{p_1^*} z_1 z_1^* - \frac{\kappa + \mathrm{i}p_2^*}{p_2 + p_2^*} \frac{p_2}{p_2^*} z_2 z_2^*$$
$$+ \frac{(\kappa + \mathrm{i}p_1^*)(\kappa + \mathrm{i}p_2^*)(p_1 - p_2)(p_1^* - p_2^*)}{(p_1 + p_1^*)(p_1 + p_2^*)(p_2 + p_1^*)(p_2 + p_2^*)} \frac{p_1 p_2}{p_1^* p_2^*} z_1 z_2 z_1^* z_2^*. \tag{5.22b}$$

Figure 11 shows the intercaction of two dark solitons with the parameters $\rho = 1, \kappa = 2, c_1 = 0.75, c_2 = 0.24$ and $\zeta_{10} = \zeta_{20} = 0$ so that from (4.14), $A_{d1} = 1.0$ and $A_{d2} = 0.47$.

5.2.2. Dark-bright solitons

Figure 12 depicts the interaction between a dark soliton and a bright soliton with the parameters $\rho = 1, \kappa = 2, c_1 = 0.75, c_2 = 0.24$ and $\zeta_{10} = \zeta_{20} = 0$, showing that the dark soliton propagates faster than the bright soliton. The asymptotic amplitudes of the dark and bright solitons are given respectively by $A_{d1} = 1.0$ and $A_{b2} = 0.92$ and hence the former is a black soliton.

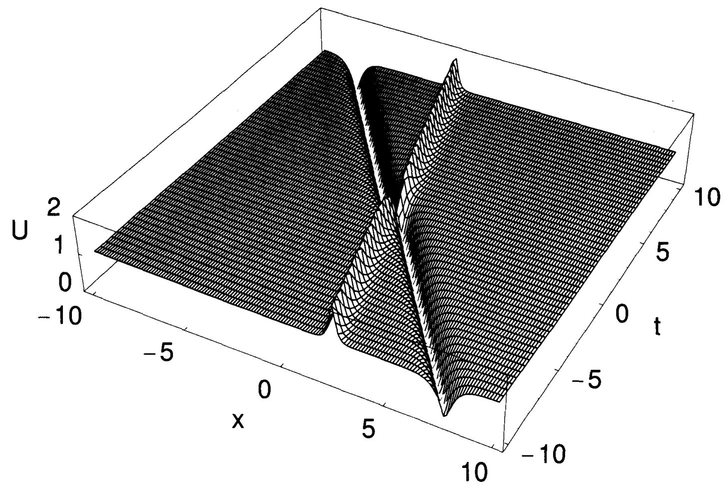

Figure 12. The interaction between a dark soliton and a bright soliton.

• Phase shift

Dark-dark solitons:

$$\Delta x_1 = \frac{1}{a_1}\ln\left|\frac{p_1+p_2^*}{p_1-p_2}\right|, \quad \Delta x_2 = -\frac{1}{a_2}\ln\left|\frac{p_2+p_1^*}{p_2-p_1}\right|, \; a_1>0,\; a_2>0. \tag{5.23}$$

Dark-bright solitons:

$$\Delta x_1 = -\frac{1}{a_1}\ln\left|\frac{p_1+p_2^*}{p_1-p_2}\right|, \quad \Delta x_2 = -\frac{1}{a_2}\ln\left|\frac{p_2+p_1^*}{p_2-p_1}\right|, \; a_1>0,\; a_2<0. \tag{5.24}$$

$$\Delta x_1 > 0,\; \Delta x_2 < 0.$$

• **Summary**

i) $\kappa > 0, \quad a_1 > 0,\; a_2 > 0$: dark-dark solitons

ii) $\kappa > 0, \quad a_1 > 0,\; a_2 < 0$: dark-bright solitons

iii) $\kappa > 0, \quad a_1 < 0,\; a_2 < 0$: bright-bright solitons

6. Conclusion

• The dark soliton solutions of the FL equation have been obtained by means of a direct method.

• The linear stability analysis of the plane wave has been performed to assure the existence of soliton solutions.

• The classification of the one-soliton solutons has been done, showing that both the dark and bright solitons exist on a constant background which reduce to algebraic solitons under certain conditions.
• The two-soliton solutions can be classified into three types, i.e., dark-dark solitons, dark-bright solitons and bright-bright solitons.

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