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Radii of Starlikeness and Convexity for analytic functions with bounded derivative

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Abstract

We consider two families of analytic functions $|f'(z)-1|<r$ $(|z|<1)$, *and* $|f'(z)-1|<1$ $(|z|<R)$, *then investigate radii of starlikeness and convexity for these families*

1 Introduction

Let $\mathbb{D}_R=\{|z|<R\}$ $(0<R\leq 1)$, and for brevity we write $\mathbb{D}_1=\mathbb{D}$.
Let $\mathcal{A}$ be the class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_nz^n \tag{1.1}$$

that are analytic in $\mathbb{D}$. And let $\mathcal{S}^*$ and $\mathcal{C}$ denote the subclass of $\mathcal{A}$ consisting of functions which are starlike and convex, respectively. i.e.

$$f(z)\in\mathcal{S}^* \text{ in } \mathbb{D}_{\mathrm{R}} \Leftrightarrow \mathrm{Re}\frac{zf'(z)}{f(z)}>0\ (z\in\mathbb{D}_{\mathrm{R}}), \tag{1.2}$$

$$f(z)\in\mathcal{C} \text{ in } \mathbb{D}_{\mathrm{R}} \Leftrightarrow \mathrm{Re}\left\{1+\frac{\mathrm{zf''(z)}}{\mathrm{f'(z)}}\right\}>0\ (\mathrm{z}\in\mathbb{D}_{\mathrm{R}}). \tag{1.3}$$

A function $f(z)\in\mathcal{A}$ is said to be in the class $\mathcal{B}_r$ if it satisfies

$$|f'(z)-1|<r\ (z\in\mathbb{D}). \tag{1.4}$$

We investigate the following three problems.

Problem 1. Find the maximum value R_1 of R s.t.

$$f\in\mathcal{B}_1\Rightarrow f(z)\in\mathcal{S}^* \text{ in } \mathbb{D}_R$$

Problem 2. Find the maximum value R_2 of r s.t.

$$f\in\mathcal{B}_r\Rightarrow f(z)\in\mathcal{S}^* \text{ in } \mathbb{D}$$

Problem 3. Find the maximum value $\mathrm{R}_3(r)$ of R s.t.

$$f\in\mathcal{B}_r\Rightarrow f(z)\in\mathcal{C} \text{ in } \mathbb{D}_{\mathrm{R}}$$

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2 Known results

For R_1, first T.MacGregor showed in [1] that

$$R_1 \geq \frac{2}{\sqrt{5}} = 0.894\cdots$$

Lator M.Nunokawa showed in [3]

$$R_1 > 0.926\cdots$$

And Nunokawa, Fukui, Owa, Saitoh and Sekine improved in [4]

$$R_1 > 0.933\cdots$$

On the other hand P.Mocanu showed in [2] that

$$R_1 < 1$$

Using the Mocanu's method, the present author[5] have showed that

$$R_1 < 0.9982$$

For R_2, P. Mocanu[2] also proved

$$R_2 \geq 0.894\cdots$$

R.Yamakawa[5] have showed

$$R_2 < 0.9962$$

For R_3, T.MacGregor showed that

$$R_3(1) = \frac{1}{2}$$

Now, for R_1 and R_2, we improve little. And for R_3 we investigate $R_3(r)$.

3 Results

Theorem 1. $R_1 < 0.99815$ *and* $R_2 < 0.9961$. i.e.

(1) *Suppose* $f(z) \in \mathcal{A}$ *satisfies* $|f'(z) - 1| < 1$, *and suppose* $R < 0.99815$ *then*

$$\mathrm{Re}\,\frac{zf'(z)}{f(z)} > 0 \quad (z \in \mathbb{D}_R).$$

(2) *Suppose* $r < 0.9961$, *and suppose* $f(z) \in \mathcal{A}$ *satisfies* $|f'(z) - 1| < r$, *then*

$$\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \quad (z \in \mathbb{D}).$$

Proof (1) We only have to show that there exist $f(z) \in \mathcal{B}_1$, and $z_0 \in \{|z| = 0.99815\}$ such that $\mathrm{Re}\frac{z_0 f'(z_0)}{f(z_0)} < 0$. Let

$$f'(z) = 1 + z\frac{1 + az}{a + z} \quad (a > 1), \tag{3.1}$$

then from $f(0)=0$, we have

$$f(z)=(2-a^2)z+\frac{a}{2}z^2+a(a^2-1)\log\left(1+\frac{z}{a}\right). \tag{3.2}$$

Putting

$$a=1.06559,\quad z_0=re^{i\theta}$$

where

$$r=0.99815,\quad \theta=\pi+\cos^{-1}0.9479$$

we obtain

$$\mathrm{Re}\frac{z_0f'(z_0)}{f(z_0)}=-1.77497\times10^{-6}<0$$

■

(2) Similarly we set

$$f(z)=(1+(1-a^2)\mathrm{R})\mathrm{z}+\frac{\mathrm{aR}}{2}\mathrm{z}^2+\mathrm{a}(\mathrm{a}^2-1)\mathrm{R}\log\left(1+\frac{\mathrm{z}}{\mathrm{a}}\right). \tag{3.3}$$

Then

$$|f'(z)-1|=\mathrm{R}|\mathrm{z}|\left|\frac{1+\mathrm{az}}{\mathrm{a}+\mathrm{z}}\right|<\mathrm{R}\ \ (\mathrm{z}\in\mathbb{D}).$$

Putting

$$a=1.065,\ \ \mathrm{R}=0.9961,\ \ \mathrm{z}_0=0.949\,\mathrm{e}^{\mathrm{i}\theta}$$

where

$$\theta=\pi+\cos^{-1}0.949.$$

We have

$$\mathrm{Re}\frac{\mathrm{z}_0\mathrm{f}'(\mathrm{z}_0)}{\mathrm{f}(\mathrm{z}_0)}=-0.000159\cdots<0$$

So

$$\mathrm{R}_2<0.9961$$

■

Theorem 2.

$$R_3(r)\begin{cases}=\alpha(r) & \left(0<r\le\dfrac{1+\sqrt5}{4}\right)\\[2mm] =\dfrac{1}{2r} & \left(\dfrac{1+\sqrt5}{4}<r\le1\right),\end{cases}\quad \text{where}\quad \alpha(r)=\left\{\frac{\sqrt{(1-r)(1+3r)}-(1-r)}{2r}\right\}^{\frac12}.$$

Proof From

$$|f'(z)-1|<r\quad(0<r\le1)\quad(z\in\mathbb{D}),$$

we can write

$$f'(z)=1+rzg(z). \tag{3.4}$$

Where $g(z)$ is analytic and

$$|g(z)|\le1, \tag{3.5}$$

$$|g'(z)|\le\frac{1-|g(z)|^2}{1-|z|^2} \tag{3.6}$$

in $\mathbb{D}$. And evidently

$$\left|\frac{zf''(z)}{f'(z)}\right| = \frac{|rz\{g(z)+zg'(z)\}|}{|1+rzg(z)|}.$$

So,from (3.6), setting

$$|z| = s, \quad |g(z)| = t \ (0 \leq s, \ t \leq 1)$$

we have

$$\left|\frac{zf''(z)}{f'(z)}\right| \leq rs\frac{(1-s^2)t + s(1-t^2)}{(1-rst)(1-s^2)}.$$

Let

$$\phi(t) = rs^2t^2 - 2rs(1-s^2)t + 1 - s^2 - rs^2,$$

then

$$\phi(t) > 0 \Rightarrow \left|\frac{zf''(z)}{f'(z)}\right| < 1.$$

Since

$$\left|\frac{zf''(z)}{f'(z)}\right| < 1 \Rightarrow \mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0,$$

$$\phi(t) > 0 \Longrightarrow f(z) \in C. \tag{3.7}$$

■

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