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The Valuation of Callable Financial Options with Regime Switches: A Discrete-time Model*

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1 Introduction

The purpose of this paper is to develop a dynamic valuation framework for callable financial securities with general payoff function by explicitly incorporating the use of regime switches. Such examples of the callable financial security may include game options (Kifer 2000, Kyprianou 2004), convertible bond (Yagi and Sawaki 2005, 2007), callable put and call options (Black and Scholes 1973, Brennan and Schwartz 1976, Geske and Johnson 1984, McKean 1965). Most studies on these securities have focused on the pricing of the derivatives when the underlying asset price processes follow a Brownian motion defined on a single probability space. In other words the realizations of the price process come from the same source of the uncertainty over the planning horizon.

The Markov regime switching model make it possible to capture the structural changes of the underlying asset prices based on the macro-economic environment, fundamentals of the real economy and financial policies including international monetary cooperation. Such regime switching can be presented by the transition of the states of the economy, which follows a Markov chain. Recently, there is a growing interest in the regime switching model. Naik (1993), Guo (2001), Elliott et al. (2005) address the European call option price formula. Guo and Zhang (2004) presents a valuation model for perpetual American put options. Le and Wang (2010) study the optimal stopping time for the finite time horizon, and derive the optimal stopping strategy and properties of the solution. They also derive the technique for computing the solution and show some numerical examples for the American put option.

In this paper we show that there exists a pair of optimal stopping rules for the issuer and of the investor and derive the value of the coupled game. Should the payoff functions be specified like options, some analytical properties of the optimal stopping rules and their values can be explored under the several assumptions. In particular, we are interested in the cases of callable American put and call options in which we may derive the optimal stopping boundaries of the both of the issuer and the investor, depending on the state of the economy. Numerical examples are also presented to illustrate these properties.

The organization of our paper is as follows: In section 2, we formulate a discrete time valuation model for a callable contingent claim whose payoff functions are in general form. And then we derive optimal policies and investigate their analytical properties by using contraction mappings. Section 3 discusses two special cases of the payoff functions to derive the specific stop

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and continue regions for callable put and call, respectively. Finally, last section concludes the paper with further comments. It summarize results of this paper and raises further directions for future research.

2 A Genetic Model of Callable-Putable Financial Commodities

In this section we formulate the valuation of callable securities as an optimal stopping problem in discrete time. Let $\mathcal{T}$ be the time index set $\{0, 1, \cdots\}$. We consider a complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$, where $\mathcal{P}$ is a real-world probability. We suppose that the uncertainties of an asset price depend on its fluctuation and the economic states which are described by the probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Let $\{1, 2, \cdots, N\}$ be the set of states of the economy and i or j denote one of these states. We denote $Z := \{Z_t\}_{t\in\mathcal{T}}$ be the finite Markov chain with transition probability $P_{ij} = \Pr\{Z_{t+1} = j \mid Z_t = i\}$. A transition from i to j means a regime switch. Let r be the market interest rate of the bank account. We suppose that the price dynamics $B := \{B_t\}_{t\in\mathcal{T}}$ of the bank account is given by

$$B_t = B_{t-1}e^r, \ B_0 = 1.$$

Let $S := \{S_t\}_{t\in\mathcal{T}}$ be the asset price at time t. We suppose that $\{X_t^i\}$ be a sequence of i.i.d. random variable having mean μ_i with the probability distribution $F_i(\cdot)$ and its parameters depend on the state of the economy modeled by Z. Here, the sequence $\{X_t^i\}$ and $\{Z_t\}$ are assumed to be independent. Then, the asset price is defined as

$$S_{t+1} = S_t X_t^i. \tag{2.1}$$

The Esscher transform is well-known tool to determine an equivalent martingale measure for the valuation of options in an incomplete market (Elliott et al. 2005 and Ching et al. 2007). Ching et al. (2007) define the regime-swiching Esscher transform in discrete time and apply it to determine an equivalent martingale measure when the price dynamics is modeled by high-order Markov chain.

We define $Y_t^i = \log X_t^i$ and $Y := \{Y_t\}_{t\in\mathcal{T}}$. Let $\mathcal{F}_t^Z$ and $\mathcal{F}_t^Y$ denote the σ-algebras generated by the values of Z and Y, respectively. We set $\mathcal{G} = \mathcal{F}_t^Z \vee \mathcal{F}_t^Y$ for $t \in \mathcal{T}$. We assume that θ_t be a $\mathcal{F}_T^Z$-measurable random variable for each $t = 1, 2, \cdots$. It is interpreted as the regime-switching Esscher parameter at time t conditional on $\mathcal{F}_T^Z$. Let $M_Y(t, \theta_t)$ denote the moment generating function of Y_t^i given $\mathcal{F}_T^Z$ under $\mathcal{P}$, that is, $M_Y(t, \theta_t) := E[e^{\theta_t Y_t^i} \mid \mathcal{F}_T^Z]$. We define $\mathcal{P}^\theta$ as a equivalent martingale measure for $\mathcal{P}$ on $\mathcal{G}_T$ associated with $(\theta_1, \theta_2, \cdots, \theta_T)$.

The next proposition follows from Ching et al. (2007).

Proposition 1 *The discounted price process $\{S_t/B_t\}_{t\in\mathcal{T}}$ is a $(\mathcal{G}, \mathcal{P}^\theta)$-martingale if and only if θ_t satisfies*

$$\frac{M_Y(t+1, \theta_{t+1}+1)}{M_Y(t+1, \theta_{t+1})} = e^r. \tag{2.2}$$

A callable contingent claim is a contract between an issuer I and an investor II addressing the asset with a maturity T. The issuer can choose a stopping time σ to call back the claim with the payoff function f_σ and the investor can also choose a stopping time τ to exercise his/her right with the payoff function g_τ at any time before the maturity. Should neither of them stop before the maturity, the payoff is h_T. The payoff always goes from the issuer to the investor. Here, we assume

$$0 \le g_t \le h_t \le f_t, \ 0 \le t < T$$

and

$$g_T = h_T. \tag{2.3}$$

The investor wishes to exercise the right to maximize the expected payoff. On the other hand, the issuer wants to call the contract to minimize the payment to the investor. Then, for any pair of the stopping times (σ, τ), define the payoff function by

$$R(\sigma, \tau) = f_\sigma 1_{\{\sigma<\tau\le T\}} + g_\tau 1_{\{\tau<\sigma\le T\}} + h_T 1_{\{\sigma\wedge\tau=T\}}. \tag{2.4}$$

When the initial asset price $S_0 = s$, our stopping problem becomes the valuation of

$$v_0(s,i) = \min_{\sigma\in\mathcal{J}_{0,T}} \max_{\tau\in\mathcal{J}_{0,T}} E^\theta_{s,i}[\beta^{\sigma\wedge\tau} R(\sigma,\tau)], \tag{2.5}$$

where $\beta \equiv e^{-r}$, $0 < \beta < 1$ is the discount factor, $\mathcal{J}$ is the finite set of stopping times taking values in $\{0, 1, \cdots, T\}$, and $E^\theta[\cdot]$ is an expectation under $\mathcal{P}^\theta$. Since the asset price process follows a random walk, the payoff processes of g_t and f_t are both Markov types. We consider this optimal stopping problem as a Markov decision process. Let $v_n(s,i)$ be the price of the callable contingent claim when the asset price is s and the state is i. Here, the trading period moves backward in time indexed by $n = 0, 1, 2, \cdots, T$. It is easy to see that $v_n(s,i)$ satisfies

$$\begin{aligned} v_{n+1}(s,i) &\equiv (\mathcal{U}v_n)(s,i) \\ &\equiv \min\left\{ f_{n+1}(s,i), \max\left(g_{n+1}(s,i), \beta\sum_{j=1}^{N} P_{ij}\int_0^\infty v_n(sx,j)dF_i(x)\right)\right\} \end{aligned} \tag{2.6}$$

with the boundary conditions are $v_0(s,i) = h_0(s,i)$ for any s, i and $v_n(s,0) \equiv 0$ for any n and s. Define the operator $\mathcal{A}$ as follows:

$$(\mathcal{A}v_n)(s,i) \equiv \beta\sum_{j=1}^{N} P_{ij}\int_0^\infty v_n(sx,j)dF_i(x). \tag{2.7}$$

Remark 1 *The equation (2.6) can be reduced to the non-switching model when we set $P_{ii} = 1$ for all i, or $f_n(s,i) = f_n(s)$, $g_n(s,i) = g_n(s)$, $h_0(s,i) = h_0(s)$ and $\mu_i = \mu$ for all i, n and s.*

Let V be the set of all bounded measurable functions with the norm $\|v\| = \sup_{s\in(0,\infty)} |v(s,i)|$ for any i. For $u, v \in V$, we write $u \le v$ if $u(s,i) \le v(s,i)$ for all $s \in (0,\infty)$. A mapping $\mathcal{U}$ is called a contraction mapping if

$$\|\mathcal{U}u - \mathcal{U}v\| \le \beta\|u - v\|$$

for some $\beta < 1$ and for all $u, v \in V$.

Lemma 1 *The mapping $\mathcal{U}$ as defined by equation (2.6) is a contraction mapping.*

Proof. For any $u_n,\ v_n \in V$, we have

$$\begin{aligned}
(\mathcal{U}u_n)(s,i) - (\mathcal{U}v_n)(s,i) &= \min\{f_{n+1}(s,i), \max(g_{n+1}(s,i), \mathcal{A}u_n)\} \\
&\quad - \min\{f_{n+1}(s,i), \max(g_{n+1}(s,i), \mathcal{A}v_n)\} \\
&\leq \min(f_{n+1}(s,i), \mathcal{A}u_n) - \max(g_{n+1}(s,i), \mathcal{A}v_n) \\
&\leq \mathcal{A}u_n - \mathcal{A}v_n \\
&\leq \beta \sum_{j=1}^{N} P_{ij} \int_0^{\infty} \sup(u_n(sx,j) - v_n(sx,j)) dF_i(x) \\
&\leq \beta \|u_n - v_n\|.
\end{aligned}$$

Hence, we obtain

$$\sup_{s\in(0,\infty)} \{(\mathcal{U}u)(s,i) - (\mathcal{U}v)(s,i)\} \leq \beta\|u - v\|. \tag{2.8}$$

By taking the roles of u and v reversely, we have

$$\sup_{s\in(0,\infty)} \{(\mathcal{U}v)(s,i) - (\mathcal{U}u)(s,i)\} \leq \beta\|v - u\|. \tag{2.9}$$

Putting equations (2.8) and (2.9) together, we obtain

$$\|\mathcal{U}u - \mathcal{U}v\| \leq \beta\|u - v\|.$$

□

Corollary 1 *There exists a unique function $v \in V$ such that*

$$(\mathcal{U}v)(s,i) = v(s,i) \quad \textit{for all } s,i. \tag{2.10}$$

Furthermore, for all $u \in V$,

$$(\mathcal{U}^T u)(s,i) \to v(s,i) \textit{ as } T \to \infty,$$

where $v(s,i)$ is equal to the fixed point defined by equation (2.10), that is, $v(s,i)$ is a unique solution to

$$v(s,i) = \min\{f(s,i), \max(g(s,i), \mathcal{A}v)\}.$$

Since $\mathcal{U}$ is a contraction mapping from Corollary 1, the optimal value function v for the perpetual contingent claim can be obtained as the limit by successively applying an operator $\mathcal{U}$ to any initial value function v for a finite lived contingent claim.

To establish an optimal policy, we make some assumptions;

Assumption 1

(i) $F_1(x) \geq F_2(x) \geq \cdots \geq F_N(x)$ *for all* x.

(ii) $f_n(s,i) \geq f_n(s,j)$, $g_n(s,i) \geq g_n(s,j)$ *and* $h_n(s,i) \geq h_n(s,j)$ *for each* n *and* s, *and states* i, j, $1 \leq j < i \leq N$.

(iii) $f_n(s,i)$, $g_n(s,i)$ *and* $h_n(s,i)$ *are monotone in* s *for each* i *and* n, *and are non-decreasing in* n *for each* s *and* i.

(iv) For each $k \leq N$, $\sum_{j=k}^{N} P_{ij}$ *is non-decreasing in* i.

Lemma 2 *Suppose Assumption 1 holds.*

(i) For each i, $(\mathcal{U}^n v)(s,i)$ *is monotone in* s *for* $v \in V$.

(ii) v *satisfying* $v = \mathcal{U}v$ *is monotone in* s.

(iii) Suppose $v_n(s,i)$ *is monotone non-decreasing in* s, *then* $v_n(s,i)$ *is non-decreasing in* i.

(iv) $v_n(s,i)$ *is non-decreasing in* n *for each* s *and* i.

(v) For each i, *there exists a pair* $(s_n^*(i), s_n^{**}(i))$, $s_n^{**}(i) < s_n^*(i)$, *of the optimal boundaries such that*

$$v_n(s,i) \equiv (\mathcal{U}v_{n-1})(s) = \begin{cases} f_n(s,i), & \text{if } s_n^*(i) \leq s, \\ \mathcal{A}v_{n-1}, & \text{if } s_n^{**}(i) < s < s_n^*(i), n = 1, 2, \cdots, T, \\ g_n(s,i), & \text{if } s \leq s_n^{**}(i), \end{cases}$$

with $v_0(s,i) = h_0(s,i)$.

Proof.

(i) The proof follows by induction on n. For $n = 1$, we have

$$(\mathcal{U}^1 v)(s,i) = \min\left\{ f_1(s,i), \max\left(g_1(s,i), \beta \sum_{j=1}^{N} P_{ij} \int_0^{\infty} h_0(sx,j) dF_i(x) \right) \right\} \tag{2.11}$$

which, since Assumption 1 (iii), implies that $(\mathcal{U}^1 v)(s,i)$ is monotone in s. Suppose that $(\mathcal{U}^n v)(s,i)$ is monotone for $n > 1$. Then, we have

$$(\mathcal{U}^{n+1} v)(s,i) = \min\left\{ f_{n+1}(s,i), \max\left(g_{n+1}(s,i), \beta \sum_{i=1}^{n} P_{ij} \int_0^{\infty} (\mathcal{U}^n v)(sx,j) dF_i(x) \right) \right\} \tag{2.12}$$

which is again monotone in s.

(ii) Since $\lim_{n\to\infty}(\mathcal{U}^n v)(s,i)$ point-wisely converges to the limit $v(s,i)$ from Corollary 1, the limit function $v(s,i)$ is also monotone in s.

(iii) For $n = 0$, it follows from Assumption 1 (ii) that $v_0(s,i) = h_0(s,i)$ is non-decreasing in i. Suppose (iii) holds for n. If $v_n(s,i)$ is monotone non-decreasing in s, then $v_n(sx,i)$ is also

monotone non-decreasing in x for each s. Then, from Assumption 1 (i), we obtain

$$\begin{aligned}
\beta\sum_{j=1}^{N} P_{ij}\int_0^{\infty} v_n(sx,j)dF_i(x) &\leq \beta\sum_{j=1}^{N} P_{ij}\int_0^{\infty} v_n(sx,j)dF_{i+1}(x) \\
&= \beta\int_0^{\infty}\sum_{k=1}^{N}(v_n(sx,k)-v_n(sx,k-1))\sum_{j=k}^{N} P_{ij}dF_{i+1}(x) \\
&\leq \beta\int_0^{\infty}\sum_{k=1}^{N}(v_n(sx,k)-v_n(sx,k-1))\sum_{j=k}^{N} P_{i+1j}dF_{i+1}(x) \\
&= \beta\sum_{j=1}^{N} P_{i+1j}\int_0^{\infty} v_n(sx,j)dF_{i+1}(x),
\end{aligned}$$

where the second inequality follows from Assumption 1 (iv). Hence, we obtain

$$\begin{aligned}
v_{n+1}(s,i) &= \min\{f_{n+1}(s,i), \max(g_{n+1}(s,i), \mathcal{A}v_n(s,i)\} \\
&\leq \min\{f_{n+1}(s,i+1), \max(g_{n+1}(s,i+1), \mathcal{A}v_n(s,i+1)\} \\
&= v_{n+1}(s,i+1). \qquad (2.13)
\end{aligned}$$

(iv) For $n=1$ in equation (2.6), it follows from Assumption 1 (iii) that

$$\begin{aligned}
v_1(s,i) &= \min\{f_1(s,i), \max(g_1(s,i), \mathcal{A}v_0)\} \\
&\geq \min\{f_1(s,i), g_1(s,i)\} = g_1(s,i) \geq g_0(s,i) = v_0(s,i).
\end{aligned}$$

Suppose (iv) holds for n. We obtain

$$\begin{aligned}
v_{n+1}(s,i) &= \min\{f_{n+1}(s,i), \max(g_{n+1}(s,i), \mathcal{A}v_n)\} \\
&\geq \min\{f_n(s,i), \max(g_n(s,i), \mathcal{A}v_{n-1})\} \\
&= v_n(s,i).
\end{aligned}$$

(iv) Should $v_n(s,i) = (\mathcal{U}^{n-1}v)(s,i)$ be monotone in s, then there exists at least one pair of boundary values $s_n^*(i)$ and $s_n^{**}(i)$ such that

$$v_n(s,i) = \begin{cases} f_n(s,i), & \text{if } s \geq s_n^*(i), \\ \max(g_n(s,i), \mathcal{A}v_{n-1}), & \text{otherwise,} \end{cases}$$

and

$$\max(g_n(s,i), \mathcal{A}v_{n-1}) = \begin{cases} g_n(s,i), & \text{for } s \leq s_n^{**}(i), \\ \mathcal{A}v_{n-1}, & \text{otherwise.} \end{cases}$$

□

Corollary 2 *The relationship between g_n, f_n and $v_n(s,i)$ is given by*

$$g_n(s,i) \leq v_n(s,i) \leq f_n(s,i).$$

Proof. The proof directly follows from equation (2.6). □

We define the stopping regions S^I for the issuer and S^{II} for the investor as

$$S_n^I(i) = \{(s,n,i) \mid v_n(s,i) \geq f_n(s,i)\}, \tag{2.14}$$

$$S_n^{II}(i) = \{(s,n,i) \mid v_n(s,i) \leq g_n(s,i)\}. \tag{2.15}$$

Moreover, the optimal exercise boundaries for the issuer and the investor are defined as

$$s_n^*(i) = \inf\{s \in S_n^I(i)\}, \tag{2.16}$$

$$s_n^{**}(i) = \inf\{s \in S_n^{II}(i)\}. \tag{2.17}$$

3 A Simple Callable American Option with Regime Switching

Interesting results can be obtained for the special cases when the payoff functions are specified. In this section we consider callable American options whose payoff functions are specified as a special case of callable contingent claim. If the issuer call back the claim in period n, the issuer must pay to the investor $g_n(s,i) + \delta_n^i$. Note that δ_n^i is the compensate for the contract cancellation, and varies depending on the state and the time period. If the investor exercises his/her right at any time before the maturity, the investor receives the amount $g_n(s,i)$. In the following subsections, we discuss the optimal cancel and exercise policies both for the issuer and investor and show the analytical properties under some conditions.

3.1 Callable Call Option

We consider the case of a callable call option where $g_n(s,i) = (s-K^i)^+$ and $f_n(s,i) = g_n(s,i)+\delta_n^i$, $0 < \delta_n^i < K^i$. Here, K^i is the strike price on the state i. We set out the assumptions to show the analytical properties of the optimal exercise policies.

Assumption 2

(i) $\beta\mu_N \leq 1$

(ii) $K^1 \geq K^2 \geq \cdots \geq K^N \geq 0$.

(iii) $0 \leq \delta_n^1 \leq \delta_n^2 \leq \cdots \leq \delta_n^N$ *for each* n.

(iv) $\delta_0^i = 0$ *and* δ_n^i *is non-decreasing and concave in* $n > 0$ *for each* i.

Remark 2 *For example,* $\delta_n^i = \delta^i e^{-r(T-n)} = \frac{\delta^i}{(1+r)^{T-n}}$ *satisfies Assumption 2(iv).*

By the form of payoff function, the value function v_n is not bounded. To apply the result of Corollary 1, we assume that the issuer has to call back the claim when the payoff value exceeds a value $M > K^1$. Define $\tilde{s}_n^i \equiv \inf\{s \mid f_n(s,i) \geq M\}$. Since $f_n(s,i)$ is increasing in s and i for any n, we have $\tilde{s}_n^1 > \tilde{s}_n^i$ for any i, n and $\tilde{s}_n^1 = M + K^1 - \delta_n^1$ for any n.

The stopping regions for the issuer $S_n^I(i)$ and investor $S_n^{II}(i)$ with respect to the callable call option are given by

$$\begin{cases} S_n^I(i) = \{s \mid v_n(s,i) \geq (s-K^i)^+ + \delta_n^i\} \cup \{\tilde{s}_n^1\}, & \text{for } n = 1, \cdots, T, \\ S_n^I(i) = \phi, & \text{for } n = 0, \\ S_n^{II}(i) = \{s \mid v_n(s,i) \leq (s-K^i)^+\}, & \text{for } n = 0, 1, \cdots, T. \end{cases}$$

For each i and n, we define the thresholds for the callable call option as

$$\begin{aligned} s_n^*(i) &= \inf\{s \mid v_n(s,i) = (s-K^i)^+ + \delta_n^i\} \wedge \tilde{s}_n^1, \\ s_n^{**}(i) &= \inf\{s \mid v_n(s,i) = (s-K^i)^+\}. \end{aligned}$$

The following lemma represents the well known result that American call options are identical to the corresponding European call options.

Lemma 3 *Callable call option with the maturity $T < \infty$ can be degenerated into callable European, that is $S_n^{II}(i) = \phi$ for $n > 0$ and $S_0^{II}(i) = \{K^i\}$ for each i.*

Proof. Since the discounted price process $\{S_t/B_t\}_{t\in\mathcal{T}}$ is $(\mathcal{G}, \mathcal{P}^\theta)$-martingale, $\beta^{\sigma\wedge\tau} g_t(S_{\sigma\wedge\tau}, i) = \beta^{\sigma\wedge\tau}\max(S_{\sigma\wedge\tau} - K^i, 0)$ is a $(\mathcal{G}, \mathcal{P}^\theta)$-submartingale. Applying the Optional Sampling Theorem, we obtain that

$$\begin{aligned} v_t(s,i) &= \min_{\sigma\in\mathcal{J}_{t,T}} \max_{\tau\in\mathcal{J}_{t,T}} E_s^\theta[\beta^{\sigma\wedge\tau} R(\sigma,\tau)] \\ &= \min_{\sigma\in\mathcal{J}_{t,T}} \max_{\tau\in\mathcal{J}_{t,T}} E_s^\theta[\beta^{\sigma\wedge\tau}(f_t(S_{\sigma\wedge\tau}, i)1_{\{\sigma<\tau\}} + g_t(S_{\sigma\wedge\tau}, i)1_{\{\tau<\sigma\}} + h_T 1_{\{\sigma\wedge\tau=T\}})] \\ &= \min_{\sigma\in\mathcal{J}_{t,T}} E_s^\theta[\beta^\sigma f_t(S_\sigma, i)1_{\{\sigma<T\}} + \beta^T h_T 1_{\{\sigma=T\}}]. \end{aligned} \tag{3.1}$$

This completes the proof. □

It implies that it is optimal for the investor not to exercise his/her putable right before the maturity. However, the issuer should choose an optimal call stopping time so as to minimize the expected payoff function.

Lemma 4 *If Assumption 2 (i) holds, then $v_n(s,i) - s$ is decreasing in s for $s > K^i$, and $v_n(s,i)$ is non-decreasing in s for $s \leq K^i$ for each n, i.*

Proof. We prove it by induction. For $n = 0$, the claim certainly holds. It is sufficient to prove for the case of $s > K^i$. Suppose the claim holds for n, then we have

$$\begin{aligned} v_{n+1}(s,i) - s &= \min\{s - K^i + \delta_{n+1}^i, \max(s-K^i, \mathcal{A}v_n)\} - s \\ &= \min\left\{-K^i + \delta_{n+1}^i, \max\left(-K^i, \beta\sum_{j=1}^N P_{ij}\int_0^\infty (v_n(sx,j) - sx)dF_i(x) + (\beta\mu_i - 1)s\right)\right\}. \end{aligned}$$

Since the statement is true for n, $v_n(sx,j) - sx$ is decreasing in s for $x > K^i$. Assumption 1 (i) implies that $\mu_1 \leq \mu_2 \leq \cdots \leq \mu_N$. If $\mu_N \leq \frac{1}{\beta}$, then $(\beta\mu_i - 1)s$ is non-increasing in s. Hence, $v_{n+1}(s,i) - s$ is decreasing in s for $s > K^i$. □

Lemma 5

(i) Suppose that $n_i^* = \inf\{n \mid \delta_n^i < v_n^a(K^i, i)\}$, *where* $v_n^a(s,i) = \max\{(s-K^i)^+, \mathcal{A}v_{n-1}(s,i)\}$ *and* $v_0^a(s,i) = (s-K^i)^+$. *If* $n_i^* \le n \le T$, *we have* $S_n^I(i) = \{K^i\}$. *If* $0 \le n < n_i^*$, *we have* $S_n^I(i) = \{\tilde{s}_n^1\}$.

(ii) n_i^* *is non-decreasing in* i.

Proof.

(i) Let $\Psi_n^I(s,i) = v_n(s,i) - (s-K^i)^+ - \delta_n^i$. For $s = K^i$, we have

$$\begin{aligned}\Psi_n^I(K^i,i) &= v_n(K^i,i) - \delta_n^i \\ &= \min\{0, \max\{0, \mathcal{A}v_{n-1}(K^i,i)\} - \delta_n^i\} \\ &= \min\{0, v_n^a(K^i,i) - \delta_n^i\}.\end{aligned}$$

If $v_n^a(K^i,i) > \delta_n^i$, then $\Psi_n^I(K^i,i) = 0$ for any i and n. Since δ_n^i is non-decreasing and concave in n by Assumption 1 (iv) and $v_n(s,i)$ is non-decreasing in n by Assumption 2 (iv), there exists at least one value n_i^* such that $n_i^* = \inf\{n \mid \delta_n^i < v_n^a(K^i,i)\}$.

By Lemma 4, the function $\Psi_n^I(s,i)$ is non-decreasing for $s \le K^i$ and is decreasing for $K^i < s$. It implies that it is unimodal function in s, and K^i is a maximizer of $\Psi_n^I(s,i)$. Thus, $v_n(s,i) < (s-K^i)^+ + \delta_n^i$ if $s \ne K^i$. Moreover, $\tilde{s}_n^1 = M + K^1 - \delta_n^1 > K^i$ for any i. Therefore, $S_n^I(i) = \{K^i\}$ for $n_i^* \le n \le T$. For $0 \le n < n_i^*$, since $\delta_n^i > v_n^a(K^i,i)$ for each i and n, we have

$$v_n(K^i,i) = \min\{0, v_n^a(K^i,i) - \delta_n^i\} + \delta_n^i = v_n^a(K^i,i) < \delta_n^i \le (s-K^i)^+ + \delta_n^i.$$

Hence, we have $\Psi_n^I(K^i,i) < 0$, so $S_n^I(i) = \{\tilde{s}_n^1\}$.

(ii) For $n = 0$, $v_0^a(K^i,i) - \delta_0^i$ is non-increasing in i. By induction, we can show that $v_n^a(K^i,i) - \delta_n^i$ is non-increasing in i. Thus, since $v_n^a(K^i,i) - \delta_n^i$ is non-decreasing in n, the value n_i^* is non-decreasing in i.

□

Theorem 1 *Suppose that Assumption 2 (i)-(iv) holds. The stopping regions for the issuer and investor can be obtained as follows;*

(i) The optimal stopping region for the issuer:

$$\begin{cases} S_n^I(i) = \{K^i\}, & \text{if } n_i^* \le n \le T, \\ S_n^I(i) = \{\tilde{s}_n^1\}, & \text{if } 0 \le n < n_i^*, \end{cases} \tag{3.2}$$

where $K^1 \ge K^2 \ge \cdots \ge K^N \ge 0$, *and* $n_i^* \equiv \inf\{n \mid \delta_n^i \le v_n^a(K^i,i)\}$ *which is non-decreasing in* i. *Here,* $v_n^a(s,i) = \max\{(s-K^i)^+, \mathcal{A}v_{n-1}(s,i)\}$.

(ii) The optimal stopping region for the investor:

$$\begin{cases} S_n^{II}(i) = \phi, & \text{if } n > 0, \\ S_0^{II}(i) = \{K^i\}, & \text{if } n = 0. \end{cases} \tag{3.3}$$

Moreover, the thresholds for the issuer and investor are $s_n^*(i) = K^i$ *for* $n_i^* \le n \le T$ *and* $s_0^{**}(i) = K^i$, *respectively.*

Proof. Part (i) follows from Lemma 5. Part (ii) is obtained from Lemma 3. In addition, since $s_n^{**}(i) = \inf\{s \mid (s - K^i)^+ \le s - K^i\} = K^i$ for $n = 0$, we obtain $S_0^{II}(i) = \{K^i\}$. □

3.2 Callable Put Option

We consider the case of a callable put option where $g_n(s,i) = \max\{K^i - s, 0\}$ and $f_n(s,i) = g_n(s,i) + \delta_n^i$. The stopping regions for the issuer $S_n^I(i)$ and the investor $S_n^{II}(i)$ with respect to the callable put option are given by

$$\begin{cases} S_n^I(i) = \{s \mid v_n(s,i) \ge (K^i - s)^+ + \delta_n^i\}, & \text{for } n = 1, \cdots, T, \\ S_n^I(i) = \phi, & \text{for } n = 0, \\ S_n^{II}(i) = \{s \mid v_n(s,i) \le (K^i - s)^+\}, & \text{for } n = 0, 1, \cdots, T. \end{cases}$$

For each i and n, we define the optimal exercise boundaries for the issuer $\tilde{s}_n^*(i)$ and the investor $\tilde{s}_n^{**}(i)$ as

$$\tilde{s}_n^*(i) = \inf\{s \mid v_n(s,i) = (K^i - s)^+ + \delta_n^i\}, \tag{3.4}$$

$$\tilde{s}_n^{**}(i) = \inf\{s \mid v_n(s,i) = (K^i - s)^+\}. \tag{3.5}$$

Assumption 3

(i) $\beta\mu_N \le 1$

(ii) $0 \le K^1 \le K^2 \le \cdots \le K^N$.

(iii) $0 \le \delta_n^1 \le \delta_n^2 \le \cdots \le \delta_n^N$ *for each* n.

(iv) $\delta_0^i = 0$ *and* δ_n^i *is non-decreasing and concave in* $n > 0$ *for each* i.

(v) $\beta \sum_{j=1}^N P_{ij} K^j - K^i$ *is non-decreasing in* i.

Lemma 6 *If Assumption 3 (i) holds, then* $v_n(s,i) + s$ *is increasing in* s *for* $s < K^i$, *and* $v_n(s,i)$ *is non-increasing in* s *for* $K^i \le s$.

Proof. It is sufficient to prove for the case of $s < K^i$. The claim holds for $n = 0$. Suppose the assertion holds for n. Then, we have

$$\begin{aligned} v_{n+1}(s,i) + s &= \min\{K^i - s + \delta_{n+1}^i, \max(K^i - s, \mathcal{A}v_n)\} + s \\ &= \min\left\{K^i + \delta_{n+1}^i, \max\left(K^i, \beta \sum_{j=1}^N P_{ij} \int_0^\infty (v_n(sx, j) + sx) dF_i(x) + (1 - \beta\mu_i)s\right)\right\}. \end{aligned}$$

Hence, from Assumption 3 (i), $v_{n+1}(s,i) + s$ is increasing in s for $s < K^i$. □

Lemma 7 *$v_n(s,i) - K^i$ is non-decreasing in i for each $s < K^i$ and n.*

Proof. When $n = 0$, the claim holds. For $K^i > s$, we set $w_n(s,i) \equiv v_n(s,i) + s$. Suppose (ii) holds for n. Then, we have

$$\begin{aligned} w_{n+1}(s,i) - K^i &= \min\{K^i - s + \delta^i_{n+1}, \max(K^i - s, \mathcal{A}w_n(s,i)\} - K^i \\ &= \min\{-s + \delta^i_{n+1}, \max(-s, \mathcal{A}w_n(s,i) - K^i)\} \end{aligned}$$

By Lemma 6, we have

$$\begin{aligned} \mathcal{A}w_n(s,i) &= \beta \sum_{j=1}^{N} P_{ij} \int_0^\infty w_n(sx,j) dF_i(x) \\ &\le \beta \sum_{j=1}^{N} P_{ij} \int_0^\infty w_n(sx,j) dF_{i+1}(x) \\ &= \beta \sum_{j=1}^{N} P_{ij} \int_0^\infty (w_n(sx,j) - K^j) dF_{i+1}(x) + \beta \sum_{j=1}^{N} P_{ij} K^j \\ &\le \mathcal{A}w_n(s,i+1) - \beta \sum_{j=1}^{N} (P_{i+1j} - P_{ij}) K^j \end{aligned}$$

From Assumption 3 (v), we obtain $\mathcal{A}w_n(s,i) - K^i \le \mathcal{A}w_n(s,i+1) - K^{i+1}$. Hence, $w_{n+1}(s,i) - K^i \le w_{n+1}(s,i+1) - K^{i+1}$, so $v_{n+1}(s,i) - K^i \le v_{n+1}(s,i+1) - K^{i+1}$. □

Lemma 8

(i) There exists a time n^*_i for each i such that $n^*_i \equiv \inf\{n \mid \delta^i_n \le v^a_n(K^i,i)\}$, where $v^a_n(s,i) = \max\{(K^i - s)^+, \mathcal{A}v_{n-1}(s,i)\}$. Moreover, if $n^*_i \le n \le T$, we have $S^I_n(i) = \{K^i\}$. If $0 \le n < n^*_i$, we have $S^I_n(i) = \phi$.

(ii) n^*_i is non-decreasing in i.

Proof. The proof can be done similarly as in the case of the call option in Lemma 5. □

Lemma 9 *Suppose Assumption 3 (i) holds. Then, there exists an optimal exercise policy for the both players, and $\tilde{s}^{**}_n(i) < \tilde{s}^*_n(i)$ such that the investor exercises the option if $s \le s^{**}_n(i)$ and the issuer exercises the option if $s^*_n(i) \le s$.*

Proof. We first consider the optimal exercise policy for the investor. Let $\Psi^{II}_n(s,i) \equiv v_n(s,i) - (K^i - s)^+$. The investor does not exercise the option when $s > K^i$ because he/she wishes to exercise the right so as to maximize the expected payoff. For $s \le K^i$, $\Psi^{II}_n(s,i)$ is increasing in s by Lemma 6. Since $v_n(K^i,i) \ge 0$, there exists a value $\tilde{s}^{**}_n(i)$ satisfying (3.5). For $s \le \tilde{s}^{**}_n(i)$, $v_n(s,i) \le (s - K^i)^+$. Hence, it is optimal for the investor to exercise the option when $s \le s^{**}_n(i)$.

It follows from Lemma 8 (i) that the optimal exercise policy for the issuer is $\tilde{s}^*_n(i) = K^i$ for $n^*_i \le n \le T$ and $\tilde{s}^*_n(i) = \infty$ for $0 \le n < n^*_i$. Since $\Psi^{II}_n(s,i)$ is increasing in s for $s \le K^i$, we have $\tilde{s}^{**}_n(i) < \tilde{s}^*_n(i)$ for each i and $n \in [n^*_i, T]$. □

Lemma 10

(i) $\tilde{s}_n^{**}(i)$ *is non-increasing in* i *for each* n.

(ii) $\tilde{s}_n^{**}(i)$ *is non-increasing in* n *for each* i.

Proof. We only consider the case of $K^i > s$.

(i) By Lemma 6, $v_n(s,i)+s$ is increasing in s for $K^i > s$. Therefore, from Lemma 7, we have

$$\begin{aligned} \tilde{s}_n^{**}(i) &= \inf\{s \mid v_n(s,i)+s = K^i\} \\ &\geq \inf\{s \mid v_n(s,i+1)+s = K^{i+1}\} \\ &= \tilde{s}_n^{**}(i+1). \end{aligned}$$

(ii) By Lemma 2 (iv), $v_n(s,i)$ is non-increasing in n, so we have

$$\begin{aligned} \tilde{s}_n^{**}(i) &= \inf\{s \mid v_n(s,i)+s = K^i\} \\ &\geq \inf\{s \mid v_{n+1}(s,i)+s = K^i\} \\ &= \tilde{s}_{n+1}^{**}(i). \end{aligned}$$

□

Theorem 2 *Suppose that Assumption 3 (i)-(v) holds. The stopping regions for the issuer and investor can be obtained as follows;*

(i) The optimal stopping region for the issuer:

$$\begin{cases} S_n^I(i) = \{K^i\}, & \text{if } n_i^* \leq n \leq T, \\ S_n^I(i) = \phi, & \text{if } 0 \leq n < n_i^*, \end{cases} \tag{3.6}$$

where $0 \leq K^1 \leq K^2 \leq \cdots \leq K^N$, *and* $n_i^* \equiv \inf\{n \mid \delta_n^i \leq v_n^a(K^i,i)\}$ *which is non-decreasing in* i. *Here,* $v_n^a(s,i) = \max\{(K^i - s)^+, \mathcal{A}v_{n-1}(s,i)\}$.

(ii) The optimal stopping region for the investor:

$$\begin{cases} S_n^{II}(i) = [0, \tilde{s}_n^{**}(i)], & \text{if } n > 0, \\ S_0^{II}(i) = \{K^i\}, & \text{if } n = 0, \end{cases} \tag{3.7}$$

where $\tilde{s}_n^{**}(i)$ *is non-increasing in* n *and* i. *Moreover,* $\tilde{s}_n^{**}(i) \leq \tilde{s}_n^*(i)$ *for each* i *and* n.

Proof. Part (i) follows from Lemma 8. Part (ii) can be obtained by Lemma 9 and 10. For $n = 0$, since $\tilde{s}_n^{**}(i) = \inf\{s \mid (s-K^i)^+ \leq s - K^i\} = K^i$, we have $S_0^{II}(i) = \{K^i\}$. □

4 Concluding Remarks

In this paper we consider the discrete time valuation model for callable contingent claims in which the asset price depends on a Markov environment process. The model explicitly incorporates the use of the regime switching. It is shown that such valuation model with the Markov regime switches can be formulated as a coupled optimal stopping problem of a two person game between the issuer and the investor. In particular, we show under some assumptions that there exists a simple optimal call policy for the issuer and optimal exercise policy for the investor which can be described by the control limit values. If the distributions of the state of the economy are stochastically ordered, then we investigate analytical properties of such optimal stopping rules for the issuer and the investor, respectively, possessing a monotone property.

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