PAPER

Energy Conversion and Phase Regulation in Transient States of Frequency Entrainment Described by van der Pol and Phase-Locked Loop Equations*

Yuichi YOKOI[†a)] ***and*** **Yoshihiko SUSUKI**[††b)], ***Members***

SUMMARY We study the role of energy conversion in phase regulation of frequency entrainment. For an open dynamical system that interacts with its environment, energy conversion in the system is the key to a wide variety of nonlinear phenomena including frequency entrainment. In this paper, using the standard notion of energy, we study the phenomena of frequency entrainment by periodic forces in two different types of oscillations: libration and rotation. Theoretical analysis shows a relationship between phase regulation and energy conversion in the entrainment phenomena. Both of them are explained as a common phase regulation. On the other hand, no common relationship between transient behaviors and energy conversion is identified for the two different types of oscillations. For libration, the development of frequency entrainment does not depend on the energy conversion. The energy input to the oscillator affects the amplitude of libration. For the rotation, the development of frequency entrainment is governed by the amount of energy conversion. The energy input to the system directly regulates the phase of rotation, in other words, controls the entrainment phenomenon. These results suggest a different dynamical and control origin behind the two types of entrainment phenomena as the energy conversion in the systems.

key words: *synchronization, energy conversion, phase regulation, libration, rotation*

1. Introduction

Frequency entrainment is a subject of research on interacting oscillators in which they behave at a common frequency [1]. In particular, frequency entrainment by a periodic force has been extensively studied as one of the simplest synchronization phenomena in nonlinear oscillations [2], [3]. The underlying mechanism of the entrainment provides a concept of nonlinear control to establish an entrained oscillation. The periodic force is regarded as a control input to a self-sustained oscillator to be regulated. The control concept is based on the regulation of phase in synchronization [4]. It has been widely recognized that in a self-oscillatory system with a periodic force, phase is regulated by energy input from the force. The energy input can be carried out directly or indirectly. Profound insight into the energy aspect of entrainment is essential for better understanding of the control concept and its technological applications. In this paper, we study the relationship between phase regulation and energy conversion in frequency entrainment by a periodic force.

Several groups of researchers have focused on the energy aspect of synchronization. In [5], Cartwright examined the time average of energy stored in the van der Pol oscillator for various steady states and showed that the entrained state appears with higher energy than the drift and asynchronous states. In [6], [7], Kuramitsu et al. use the Brayton-Moser formulation [8] of nonlinear electrical circuits and define the averaged potential as the time average of power dissipated from a system of coupled self-sustained oscillators. They showed that the frequency entrainment develops with the decrease of the averaged potential. In [9], [10], Sarasola et al. analyzed synchronization in coupled chaotic systems by using the notion of energy stored in and dissipated from the systems. In [11], Paley et al. employed the time derivative of kinetic energy to control collective motions of coupled phase oscillators.

Here we introduce suggestive articles that associate the energy aspect of synchronization with the *type* of self-sustained oscillations. Takagi mentioned the difference between synchronization in oscillators and rotators from the viewpoint of energy [12]. In the context, he uses the term *oscillators* to represent vibrating objects, in comparison with rotating objects for which he uses the term *rotators*. For the synchronization in oscillators and rotators, Blekhman made the following statement in his book [13]: "*The most substantial difference is that usually only clocks with sufficiently close partial frequencies can self-synchronize, while in the case of rotating rotors the effect can occur also at very different partial angular velocities. This is partially related to another practical and important difference: In the case of the self-synchronization of oscillating objects, like pendulum clocks, much less power can be transmitted between*

Manuscript received February 1, 2012.
Manuscript revised September 4, 2012.
[†]The author was with the Department of Electrical Engineering, Kyoto University, Japan, where most parts of this work were carried out. He is currently with the Division of Electrical Engineering and Computer Science, Nagasaki University, Nagasaki-shi, 852-8521 Japan.
[††]The author is with the Department of Electrical Engineering, Kyoto University, Kyoto-shi, 615-8510 Japan.
[*]This work was partially supported by the Global COE Program of Kyoto University to YY when he was with Kyoto University and partially by the Grant-in-Aid for Research Activity Start-up (23860039) to YY with Nagasaki University. The work was also partially supported by the Grant-in-Aid for Young Researcher (B) (23760390) to YS. These programs are funded from the Ministry of Education, Culture, Sports, Science and Technology, Japan. In addition to them, this work was partially supported by the Itoh Chubei Foundation to YS.
a) E-mail: yyokoi@nagasaki-u.ac.jp
b) E-mail: susuki.yoshihiko.5c@kyoto-u.ac.jp
DOI: 10.1587/transfun.E96.A.591

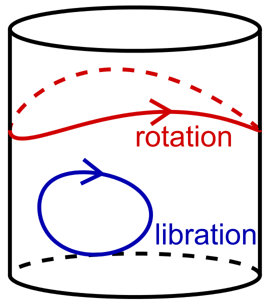


Fig. 1 Cylindrical phase space with libration and rotation.

the objects than in the case of rotors (vibro-exciters)." Note that he employs clocks as an example of oscillators. Their statements suggest that the two different types of oscillations offer important information for understanding the energy aspect of synchronization. The different types of oscillations can be regarded as topologically different kinds of periodic orbits in a dynamical system defined in the cylindrical phase space: see Fig. 1. The oscillation of vibrating object is called *libration*, and the oscillation of rotator is *rotation* [14]. As shown in Fig. 1, rotation is defined as a closed orbit that encircles the cylindrical phase space. On the other hand, libration can be defined either in the cylindrical phase space or on the phase plane.

Based on the above statements, the previous paper [15] analyzed the entrained steady states of libration and rotation by quantifying energy supplied by periodic forces. Two dynamical systems with the oscillations were considered in this paper: the van der Pol oscillator for libration [16] and the phase-locked system for rotation [17], [18]. Also, the paper [19] examined the stability of libration and rotation with respect to dissipated power in the systems, which was motivated by [7].

In this paper, we analyze energy stored in the dynamical systems with libration and rotation during transient states of the entrainment phenomena. For libration and rotation, we adopt the above dynamical systems as examples showing frequency entrainment of the oscillations. By focusing on the phase space close to the entrained oscillation, it is expected that one gains general features for the frequency entrainment. In fact, we derive phase equations that describe the dynamics of phase regulation in the phenomena and have the same structure as the standard phase equation or Kuramoto model [20], [21]. The derivation of phase equation for the frequency entrainment of rotation is an auxiliary result of this analysis. The phase equation for the entrainment of libration is well-known in literature [22]. Thus, by using the standard notion of energy, we formulate the dynamics of phase regulation in terms of energy conversion in the systems. The formulation enables us to describe the physics behind the entrainment phenomena of libration and rotation, in particular, the energy conversion during transient behaviors toward entrained states. In this way, we clarify a relationship between phase regulation and energy conversion in the entrainment phenomena.

This paper is organized as below. In Sect. 2, we describe an energy aspect of the frequency entrainment of libration in the van der Pol oscillator. In Sect. 3, we investigate the frequency entrainment of rotation in the phase-locked system and associate the regulation of phase with the conversion of energy in the system. In Sect. 4, we conclude this paper by comparing the relationship between phase regulation and energy conversion for the entrainment phenomena of libration and rotation. Several remarks and future work related to this analysis are also presented.

2. Van der Pol Oscillator

As a dynamical system describing the frequency entrainment of libration, we focus on the forced van der Pol oscillator [2]. The dynamics of the forced oscillator are represented by the non-autonomous system on the plane as follows:

$$\begin{cases} \dfrac{\mathrm{d}u}{\mathrm{d}t} = v, \\ \dfrac{\mathrm{d}v}{\mathrm{d}t} = \mu(1 - \beta u - \gamma u^2)v - u + B \sin \nu t, \end{cases} \tag{1}$$

where $\mu > 0$ determines the magnitude of the nonlinear damping. The parameters β and γ characterize the damping effect. In this section, the three parameters of the damping term are fixed at $\mu = 0.15$, $\beta = 4/3$, and $\gamma = 4/3$, which are originated by [2]. Under the parameters, the oscillator (1) has a stable limit cycle that we call a stable *librational* limit cycle. The term $B \sin \nu t$ represents the periodic force to the oscillator with amplitude B and angular frequency ν, denoted by $f_{\mathrm{L}}(t)$ with period $T_{\mathrm{L}} := 2\pi/\nu$.

Figure 2 shows an example of the frequency entrainment caused by the periodic force $f_{\mathrm{L}}(t)$ in the van der Pol oscillator (1). The forcing parameters are fixed at $B = 0.05$ and $\nu = 0.99$ to induce the frequency entrainment. The upper figure shows the waveform of the periodic force $f_{\mathrm{L}}(t)$, and the lower figure does the waveform of $u(t)$. In the lower figure, the librational limit cycle is observed before the periodic force is applied at $t = t_0$, where $u(t_0) = 0$ and $v(t_0) > 0$. Here the circles on the waveforms indicate the stroboscopic points sampled every period T_{L} of the forcing term. The time evolution of the stroboscopic points implies that the application of periodic force induces the frequency entrainment of the librational limit cycle.

2.1 Review of Phase Equation

Now we review the phase equation that describes transient behaviors to the entrained libration. For this purpose, we make the following assumptions for the forced van der Pol oscillator (1).

(A-L1) The two constants μ and B are sufficiently small and the same order of magnitude.
(A-L2) The forcing frequency ν is close to the frequency of the librational limit cycle, denoted by ν_0, in this

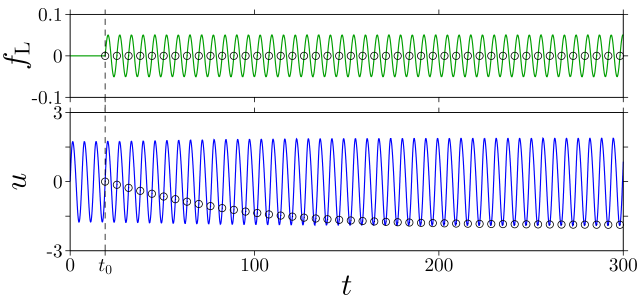


Fig. 2 Frequency entrainment of libration in the van der Pol oscillator (1). The parameters of the system (1) are fixed at $\mu = 0.15$, $\beta = \gamma = 4/3$, $B = 0.05$, and $\nu = 0.99$. The periodic force $f_L(t - t_0) = B \sin\{\nu(t - t_0)\}$ is applied at $t = t_0$. The time evolution of the stroboscopic points, denoted by the circles ⊙, shows that the librational limit cycle is entrained by the periodic force.

case unity. More precisely, the two numbers $1 - \nu^2$ and μ are the same order of magnitude.

(A-L3) The transient behaviors which we analyze here appear in the sub-domain of phase space close to a stable periodic orbit that corresponds to the entrained libration.

These assumptions enable us to approximate the transient behaviors by using a perturbation of the librational limit cycle:

$$\begin{cases} u(t) = b_0 \sin(\nu t + \theta_L(t)), \\ v(t) = \nu b_0 \cos(\nu t + \theta_L(t)), \end{cases} \tag{2}$$

where b_0 is the amplitude of the librational limit cycle, uniquely determined with the averaging method (see e.g. [23]), in this case, $b_0 = 2/\sqrt{\gamma}$. Under the above assumptions, Eq. (2) implies that any librational solution can be described by the single variable $\theta_L(t)$. The variable θ_L stands for the initial phase of libration and corresponds to the phase variable for the frequency entrainment. The choice of the phase variable is the same as in [22], [24].

By substituting (2) into (1) and applying the averaging method to (1), we have the following phase equation that is the same one as in [22], [24]:

$$\frac{d\theta_L}{dt} = \frac{1 - \nu^2}{2\nu} - \frac{B}{2\nu b_0} \cos\theta_L. \tag{3}$$

On the right-hand side, the first term represents the difference between the frequencies of the limit cycle and the periodic force. The second term represents the effect of periodic force on the phase dynamics.

2.2 Entrainment Time

For transient behaviors toward the entrained libration in the system (1), we evaluate the time required for achieving the frequency entrainment. The transient time is defined as the convergence time from a state in the sub-domain of phase space, which is stated in Assumption (A-L3), to the entrained state. We call the convergence time *the entrainment time*.

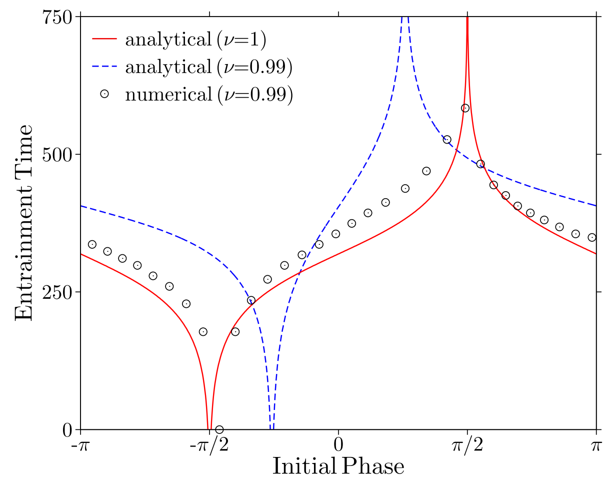


Fig. 3 Entrainment time in the van der Pol oscillator (1). The parameters of the system (1) are fixed at $\mu = 0.15$, $\beta = \gamma = 4/3$, and $B = 0.05$. The convergence condition is set at $\epsilon = 0.02$. The solid and dashed curves depict the analytical results obtained from (4) at $\nu = 1$ and 0.99, respectively. The circles correspond to the numerical result from (1) at $\nu = 0.99$.

The entrainment time is evaluated in the following manner. The analytical evaluation is obtained from the following solution of (3):

$$t = t_0 + D \ln \left| \frac{x(t) - \xi}{x(t) + \xi} \cdot \frac{x(t_0) + \xi}{x(t_0) - \xi} \right|, \tag{4}$$

where t_0 denotes the initial time. The new variable $x(t)$ is implicitly defined as $x(t) = \tan(\theta_L(t)/2)$, and the new parameters D and ξ are

$$\begin{cases} D = \dfrac{2\nu b_0}{\sqrt{B^2 - b_0^2(1 - \nu^2)^2}}, \\ \xi = \sqrt{\dfrac{B - b_0(1 - \nu^2)}{B + b_0(1 - \nu^2)}}. \end{cases} \tag{5}$$

On the other hand, the entrainment time is computed with the direct integration of (1). In both the analytical and numerical evaluations, the convergence condition with a bounded error ϵ is used. The detail of the evaluations is explained in Appendix A.

Figure 3 shows the entrainment time in the oscillator (1) at $\mu = 0.15$, $\beta = \gamma = 4/3$, $B = 0.05$, $\nu = 0.99$, and $\epsilon = 0.02$. The dashed curve depicts the analytical result, and the circles correspond to the numerical result. The values of initial phase with short entrainment time are observed in the neighborhood of the entrained state. In contrast, long entrainment time is observed in the neighborhood of an unstable state. Here, the results in Fig. 3 indicate a discrepancy between phases obtained by the analytical and numerical evaluations, in particular, the values of phase close to the entrained and unstable states. The discrepancy is due to the difference in the values of $\nu_0^2 - \nu^2$ for the analytical and numerical evaluations. For the analytical evaluation, the frequency ν_0 is fixed at unity from Assumption (A-L2). On the other hand, the frequency is numerically obtained at $\nu_0 = 0.9937$ for the parameter setting. In fact, the analytical result at $\nu = 1$ denoted by the solid curve in Fig. 3 provides better approximation of the numerical result than that at $\nu = 0.99$. Thus, by adjusting the parameter, the analytical and numerical results of entrainment time become consistent.

2.3 Energy-Based Analysis of Transient Behaviors

Next, we introduce the definition of energy to investigate the energy conversion during the transient behaviors. The energy $S_{\rm L}$ stored in the van der Pol oscillator (1) is defined as

$$S_{\rm L}(u, v) := \frac{1}{2}v^2 + \frac{1}{2}u^2. \tag{6}$$

If the van der Pol oscillator is realized in an electrical circuit [2], [16], then the energy corresponds to the sum of energy stored in an oscillation circuit with a negative resistance. Because the definition of the stored energy is also used for linear harmonic oscillators, the energy is relevant to general analysis of such oscillators. Since the phase equation (3) is derived with the averaging method, $\theta_{\rm L}$ is subject to slow dynamics in the sense that it slowly changes compared with the forcing frequency ν. In comparison with the phase $\theta_{\rm L}$, the stored energy $S_{\rm L}$, which is a function of u and v, changes fast. Thus, $\theta_{\rm L}$ and $S_{\rm L}$ behave in different time scales. By averaging the stored energy $S_{\rm L}$ over the forcing period $T_{\rm L}$, namely the fast time scale, the slowly changing component of the stored energy with the same time scale as $\theta_{\rm L}$ is extracted as follows:

$$\begin{aligned}\langle S_{\rm L}\rangle(\theta_{\rm L}) &:= \frac{1}{T_{\rm L}}\int_{-T_{\rm L}/2}^{T_{\rm L}/2} S_{\rm L}(u(t,\theta_{\rm L}), v(t,\theta_{\rm L}))\,{\rm d}t \\ &= \frac{1}{4}(1+\nu^2)b_0^2 =: \langle S_{\rm L}\rangle^*,\end{aligned} \tag{7}$$

where $\langle\cdot\rangle$ denotes the time average during $T_{\rm L}$, and $\langle S_{\rm L}\rangle^*$ does the value of $\langle S_{\rm L}\rangle$ at the entrained libration. In (7), $\langle S_{\rm L}\rangle$ is not a function of $\theta_{\rm L}$ but the constant $\langle S_{\rm L}\rangle^*$ with the single parameter b_0. Since b_0 is constant, no correlation between (3) and (7) is identified. That is, any change of the stored energy does not affect the phase dynamics or the transient behaviors in the frequency entrainment.

Figure 4 shows a numerical estimation of the transient behavior predicted by the theoretical result. The numerical result is obtained from (1) under the parameters $B = 0.05$ and $\nu = 0.99$. In the figure, $b(t)$ denotes the envelop of $u(t)$ which is calculated by averaging the amplitude $\sqrt{u(t)^2 + (v(t)/\nu)^2}$ over $T_{\rm L}$. The envelop corresponds to the amplitude of libration. The solution $\phi_{\rm L}(t) := \tan^{-1}(\nu u(t)/v(t))$ is displayed so that $\theta_{\rm L}(t)$ can be numerically estimated through stroboscopic sampling. At $t \le t_0$, the entrained libration appears. At $t = t_0 = (3 + 1/2)T_{\rm L}$, the phase of the periodic force $f_{\rm L}(t)$ is initialized to zero

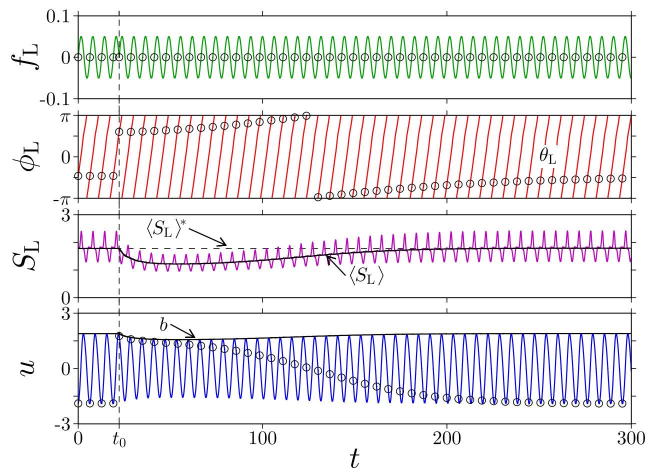


Fig. 4 Transient behavior in the van der Pol oscillator (1). The parameters of the system (1) are fixed at $\mu = 0.15$, $\beta = \gamma = 4/3$, $B = 0.05$, and $\nu = 0.99$. The solution $b(t)$ denotes the envelop of the waveform of $u(t)$ and corresponds to the amplitude of libration. The solution $\phi_{\rm L}(t)$ is plotted for estimating $\theta_{\rm L}(t)$ through the stroboscopic observation, denoted by $\odot$. The system exhibits the entrained libration as the initial state. In order to induce the transient behavior, the phase of the periodic force $f_{\rm L}(t)$ is initialized to zero at $t = t_0 = (3 + 1/2)T_{\rm L}$ by replacing the function $f_{\rm L}(t) = B\sin\nu t$ with $f_{\rm L}(t - t_0)$.

for inducing the transient behavior. The initialization is carried out by replacing the function $f_{\rm L}(t)$ with $f_{\rm L}(t - t_0)$ at $t = t_0$. At the moment, $\theta_{\rm L}(t_0)$ shifts by around π, namely $\theta_{\rm L}(t_0) \approx \theta_{\rm L}^* + \pi$, where $\theta_{\rm L}^*$ corresponds to the entrained state. The assumption $\nu \approx 1$ implies $\langle S_{\rm L}\rangle \approx b^2/2$. In fact, a correlation between the amplitude $b(t)$ and the energy $\langle S_{\rm L}\rangle(t)$ is confirmed. Here the theoretical result does not imply any correlation between the phase $\theta_{\rm L}(t)$ and the energy $S_{\rm L}(t)$. Indeed, no dominant correlation is observed in the numerical result. These numerical results verify the theoretical prediction that the energy conversion does not affect the frequency entrainment of libration in the van der Pol oscillator.

3. Phase-Locked System

For the analysis of frequency entrainment of rotation, in this paper we focus on the forced phase-locked system described by

$$\begin{cases} \dfrac{{\rm d}\phi}{{\rm d}t} = y, \\ \dfrac{{\rm d}y}{{\rm d}t} = -ky - \sin\phi + N + A\sin\Omega t, \end{cases} \tag{8}$$

where $k > 0$ is the damping coefficient and N the constant torque. Here we express the periodic force by $f_{\rm R}(t) := A\sin\Omega t$ and its period by $T_{\rm R} := 2\pi/\Omega$. Equation (8) represents the dynamics of a driven pendulum [25], the Josephson junction circuit [26], the phase-locked loop circuit [17], and a simple power system [27]. The damping coefficient k and the constant torque N are fixed at $k = 0.1$ and $N = 0.2$, respectively. Under the parameters, there exists a stable limit cycle in the system that we call a stable *rotational* limit

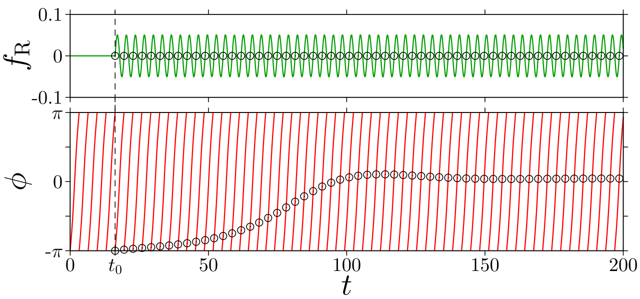


Fig. 5 Frequency entrainment of rotation in the phase-locked system (8). The parameters of the system (8) are fixed at $k = 0.1$, $N = 0.2$, $A = 0.05$, and $\Omega = 1.93$. The periodic force $f_R(t - t_0) = A \sin\{\Omega(t - t_0)\}$ is applied at $t = t_0$. The time evolution of the stroboscopic points, denoted by the circles $\odot$, shows that the rotational limit cycle is entrained by the periodic force.

cycle. Note that the parameter setting satisfies the condition $N > 4k/\pi$ for the existence of a rotational limit cycle. The condition is derived with the Melnikov's method [28].

Figure 5 shows an example of the frequency entrainment by the periodic force $f_R(t)$ in the phase-locked system (8). The forcing parameters are fixed at $A = 0.05$ and $\Omega = 1.93$ in order to induce the frequency entrainment. The upper figure shows the waveform of the periodic force $f_R(t)$, and the lower figure does the waveform of $\phi(t)$. In the lower figure, the rotational limit cycle appears in the situation of no periodic force before $t = t_0$. The force is applied to the system at $t = t_0$ when $\phi(t_0) = -\pi$. The circles on the lines denote the stroboscopic points that are sampled every period T_R of the forcing term. After $t = t_0$, the stroboscopic points converge to a constant value as time goes on. Because the convergent stroboscopic points imply the appearance of a periodic rotation, the frequency of the rotational limit cycle is entrained to the forcing frequency, namely Ω.

3.1 Derivation of Phase Equation

Now we analytically derive the phase equation that describes transient behaviors toward the entrained rotation. To do so, we make the following assumptions for the phase-locked system (8).

(A-R1) The parameters k, N, and A are sufficiently small and the same order of magnitude. Also, the magnitude of N is greater than that of k. Hence, a stable periodic rotation in the system (8) has an almost constant speed.

(A-R2) The forcing frequency Ω is close to the frequency of the rotational limit cycle under $A = 0$, denoted by Ω_0.

(A-R3) The transient behaviors which we analyze here appear in the sub-domain of phase space close to a stable periodic orbit which corresponds to the entrained rotation.

To begin with, we identify the phase variable for the frequency entrainment of rotation. For this, it is necessary to find a perturbative form of the solution $(\phi(t), y(t))$ to the rotational limit cycle. Under (A-R1)–(A-R3), a rotational behavior can be expressed as a small perturbation of the rotational limit cycle. The rotational limit cycle is denoted by $\phi_0(t)$ and can be approximated as the sum of rotatory and fundamental oscillatory components. The approximation is the following:

$$\phi_0(t) = \Omega_0 t + \theta_{R0} + a_0 \sin(\Omega_0 t + \delta_0). \tag{9}$$

On the right-hand side, the first two terms are the rotatory component, and the last term is the oscillatory component with amplitude a_0 and initial phase δ_0. The second term θ_{R0} is constant and depends on the initial condition of (8). Note that under (A-R1) and (A-R2), it is enough to consider the fundamental harmonic term for the current analysis. Without loss of generality, we rewrite (9) as

$$\phi_0(t) = \Omega_0 t + \theta_{R0} + a_0 \sin(\Omega_0 t + \theta_{R0} + \sigma_0), \tag{10}$$

where the new parameter $\sigma_0 := \delta_0 - \theta_{R0}$ is constant. The constants a_0 and σ_0 are uniquely determined for the parameters k and N as shown in Appendix B. Here we use the perturbation of (10) and its time derivative to represent the rotational solution $(\phi(t), y(t))$. Under the perturbation, the constant θ_{R0} becomes a variable subject to slow dynamics, denoted by $\theta_R(t)$. Hence, the variable $\theta_R(t)$ is chosen as the phase variable for the frequency entrainment of rotation. This choice of the phase variable is consistent with that of the determining variable for the frequency-locked rotation [15] (see Sects. IV-C and V in it). In this way, the time derivative of the phase variable $\theta_R(t)$ is important and not negligible, because it contains information on transient behaviors toward the entrained rotation that we now focus on. Now let us denote the time derivative of θ_R by $\omega_R := \mathrm{d}\theta_R/\mathrm{d}t$. As a result, the rotational solution $(\phi(t), y(t))$ is expressed as

$$\begin{cases} \phi(t) = \Omega t + \theta_R(t) + a_0 \sin(\Omega t + \theta_R(t) + \sigma_0), \\ y(t) = \Omega + \omega_R(t) + \Omega a_0 \cos(\Omega t + \theta_R(t) + \sigma_0). \end{cases} \tag{11}$$

These expressions imply that under (A-R1)–(A-R3), the transient behaviors are represented by the two slow variables θ_R and ω_R, that is, the phase variable and its time derivative.

By substituting (11) into (8) and using the technique similar to the averaging method to (8), we obtain the following phase equation:

$$\frac{\mathrm{d}\theta_R}{\mathrm{d}t} = \frac{N}{k} \cdot \frac{\Omega_0 - \Omega}{\Omega_0} - \frac{A a_0}{2k} \sin(\theta_R + \sigma_0). \tag{12}$$

The detailed derivation is given in Appendix B. On the right-hand side of (12), the first term represents the difference between the frequencies of the rotational limit cycle and the periodic force. The second term corresponds to the effect of periodic force on the phase dynamics. Equation (12) has the same structure as (3) in the forced van der Pol oscillator and the well-known phase equations [20], [21].

3.2 Entrainment Time

In the same manner as Sect. 2.2, we evaluate the time required for achieving the frequency entrainment, based on the notion of entrainment time. The entrainment time is analytically estimated by using the following solution of (12):

$$t = t_0 + E \ln \left| \frac{z(t) - \lambda_+}{z(t) - \lambda_-} \cdot \frac{z(t_0) - \lambda_-}{z(t_0) - \lambda_+} \right|, \tag{13}$$

where t_0 denotes the initial time, the new variable $z(t)$ is defined as $z(t) = \tan\{(\theta_R(t)+\sigma_0)/2\}$, and the new coefficients are

$$\begin{cases} E = \dfrac{2k\Omega_0}{\sqrt{\Omega_0^2 A^2 a_0^2 - 4N^2(\Omega_0 - \Omega)^2}}, \\ \lambda_\pm = \dfrac{\Omega_0 A a_0 \pm \sqrt{\Omega_0^2 A^2 a_0^2 - 4N^2(\Omega_0 - \Omega)^2}}{2N(\Omega_0 - \Omega)}. \end{cases} \tag{14}$$

On the other hand, the numerical estimation of the entrainment time is obtained from the direct integration of (8). For both the analytical and numerical evaluations, the convergence condition with a bounded error ϵ is introduced in the same manner as the system (1). The detail of the procedure is shown in Appendix A.

Figure 6 shows the entrainment time in the system (8) at $k = 0.1$, $N = 0.2$, $A = 0.05$, $\Omega = 1.93$, and $\epsilon = 0.02$. The solid curve depicts the analytical result and the circles correspond to the numerical result. The analytical calculation predicts the characteristics of the entrainment time obtained via numerical simulations. The phases at the entrained and unstable states shift by about $\pi/2$ from those in Fig. 3 because of the difference in the sinusoidal functions in the phase equations. It is confirmed that the result in Fig. 6 shows the same dependence of entrainment time on phase as Fig. 3.

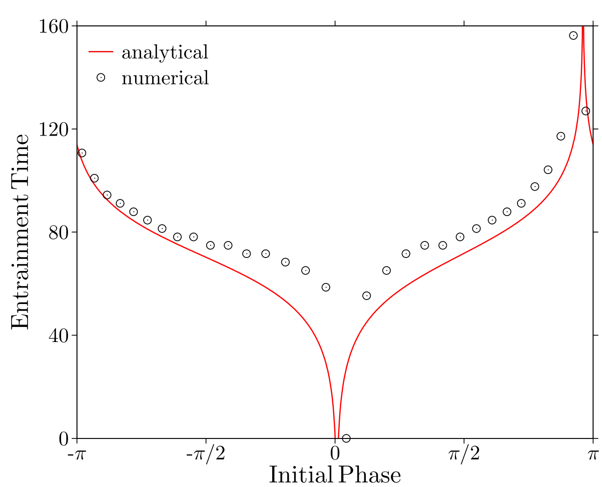


Fig. 6 Entrainment time in the phase-locked system (8). The parameters of the system (8) are fixed at $k = 0.1$, $N = 0.2$, $A = 0.05$, and $\Omega = 1.93$. The convergence condition is set at $\epsilon = 0.02$. The solid curve depicts the analytical result obtained from (13) and the circles correspond to the numerical result of (8).

3.3 Energy-Based Analysis of Transient Behaviors

Next, we introduce the notion of energy to investigate the energy conversion during the transient behaviors. The energy S_R stored in the system (8) is defined as

$$S_R(\phi, y) := \frac{1}{2}y^2 - \cos\phi. \tag{15}$$

The stored energy implies the mechanical energy of a driven pendulum and the sum of energy stored in the Josephson junction circuit. The definition of stored energy is applicable to wide range of dynamical systems. Here S_R is a function of the fast variables ϕ and y. Since θ_R governs the slow dynamics, the two variables S_R and θ_R change in different time scales. By averaging the stored energy S_R over the forcing period T_R, the slowly changing component of the stored energy with the same time scale as θ_R is extracted. That is, we have

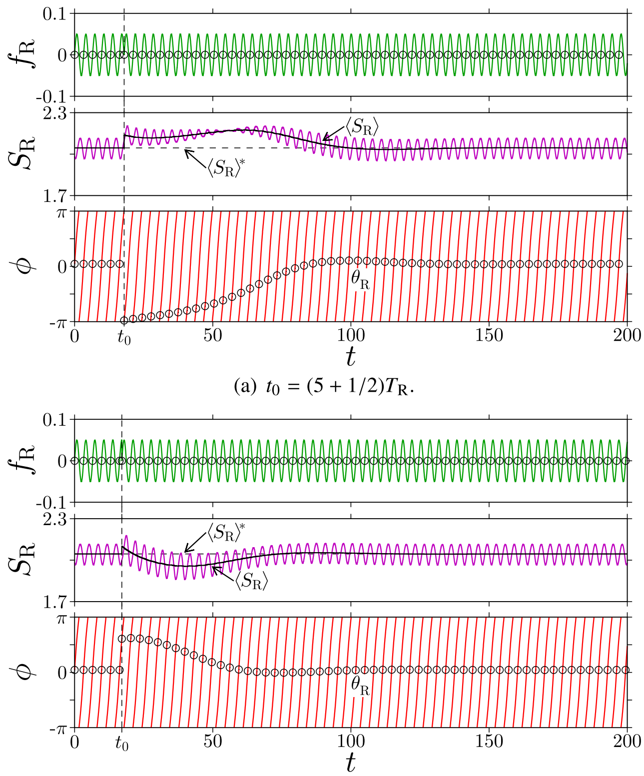
$$\begin{aligned} \langle S_R \rangle(\theta_R) &:= \frac{1}{T_R}\int_{-T_R/2}^{T_R/2} S_R(\phi(t,\theta_R), y(t,\theta_R,\omega_R))\mathrm{d}t \\ &= \frac{1}{2}\Omega^2 + \frac{1}{4}\Omega^2 a_0^2 + J_1(a_0)\cos\sigma_0 + \Omega\omega_R \\ &= \langle S_R\rangle^* + \Omega\frac{\mathrm{d}\theta_R}{\mathrm{d}t}. \end{aligned} \tag{16}$$

The constant $\langle S_R\rangle^*$ denotes the value of $\langle S_R\rangle$ at the entrained rotation. From (16) we finally obtain the following equation:

$$\frac{\mathrm{d}\theta_R}{\mathrm{d}t} = \frac{1}{\Omega}(\langle S_R\rangle(\theta_R) - \langle S_R\rangle^*). \tag{17}$$

This equation clearly shows that the time change of phase is the function of the stored energy $\langle S_R\rangle$. Based on the conservation law of energy, the change of stored energy is identical to the sum of energy supplied by the periodic force and energy dissipated by the damping. Therefore, the amount of energy conversion regulates phase and governs the frequency entrainment of rotation in the phase-locked system (8).

Transient behaviors toward the entrained rotation in the system (8) are illustrated in Fig. 7. The parameters are fixed at $k = 0.1$, $N = 0.2$, $A = 0.05$, and $\Omega = 1.93$. The periodic force $f_R(t)$ is initialized at $t = t_0$, and then the transient behavior appears. The stroboscopic points at $t = \tau := t_0 + nT_R\,(n = 0, 1, 2, \cdots)$ extract the time response of the phase variable θ_R, because of $\phi(\tau) \approx \theta_R(\tau) + a_0 \sin(\theta_R(\tau) + \sigma_0) \approx \theta_R(\tau)$. The entrained rotation is achieved at $t \le t_0$. In Fig. 7(a), the phase of the force $f_R(t)$ is initialized to zero at $t = t_0 = (5 + 1/2)T_R$. That is, the function $f_R(t)$ is replaced with $f_R(t - t_0)$ at $t = t_0$. Here let us use θ_R^* to represent the phase at the entrained rotation. At the moment, $\theta_R(t_0) \approx \theta_R^* - \pi$, and then $\langle S_R\rangle(t_0)$ increases. After $t = t_0$, $\theta_R(t)$ increases when $\langle S_R\rangle(t)$ is higher than $\langle S_R\rangle^*$. On the other hand, in Fig. 7(b), the phase of the force $f_R(t)$ is initialized to zero at $t = t_0 = (5 + 1/4)T_R$. At this moment,



(a) $t_0 = (5 + 1/2)T_{\rm R}$.

(b) $t_0 = (5 + 1/4)T_{\rm R}$.

Fig. 7 Transient behaviors of the phase-locked system (8). The parameters of the system (8) are fixed at $k = 0.1$, $N = 0.2$, $A = 0.05$, and $\Omega = 1.93$. The system exhibits the entrained rotation as the initial states. In order to induce the transient behaviors, the phase of the periodic force $f_{\rm R}(t)$ is initialized to zero at $t = t_0$ by replacing the function $f_{\rm R}(t) = A\sin\Omega t$ with $f_{\rm R}(t - t_0)$. The phase is evaluated through the stroboscopic points that are sampled every forcing period, denoted by ⊙.

$\theta_{\rm R}(t_0) \approx \theta_{\rm R}^* + \pi/2$, and then $\langle S_{\rm R}\rangle(t_0)$ increases. If the deviation $\langle S_{\rm R}\rangle(t) - \langle S_{\rm R}\rangle^*$ is negative, then $\theta_{\rm R}(t)$ decreases. These numerical results confirm the validity of the theoretical formulation (17) that predicts the quantitative relationship between the deviation $\langle S_{\rm R}\rangle - \langle S_{\rm R}\rangle^*$ and the time derivative of $\theta_{\rm R}$.

4. Conclusion

In this paper, we identified a different dynamical and control origin behind the two types of entrainment phenomena as the energy conversion in the nonlinear systems. Here we compare the relationships between phase regulation and energy conversion for the frequency entrainment phenomena of libration in the forced van der Pol oscillator (1) and the phenomenon of rotation in the phase-locked system (8). Equations (3) and (12) for both the phenomena have the same structure as the well-known phase equation. That is, the entrainment phenomena of the different types of oscillations are explained with a common phase regulation. However, no common relationship between transient behaviors and energy conversion is identified in the two different systems. For the forced van der Pol oscillator (1), the development of frequency entrainment does not depend on the energy conversion. The energy input to the oscillator affects the amplitude of libration. On the other hand, for the phase-locked system (8), the development of frequency entrainment is governed by the amount of energy conversion. The energy input to the system directly regulates the phase of rotation, in other words, controls the entrainment phenomenon. Hence, the energy conversion plays a different role in the two types of entrainment phenomena.

Finally, several remarks related to this analysis are presented. First, in the current paper, we clarified the role of energy conversion in the *transient* states of frequency entrainments. On the other hand, the previous paper [15] studied the properties of energy conversion in the *steady* states of entrainment. Thus, these results are unified to offer an energy-based approach to characterization of frequency entrainment. One direction of future work is to generalize the unified results into other nonlinear systems exhibiting frequency entrainment and synchronization. Second, also in [15], analytical studies on the dynamics of frequency entrainment of rotation were remaining work. In the current paper, we indeed performed the analytical studies and explained the dynamics of the phase equation in terms of the notion of energy conversion. Last, the control viewpoint in this paper is closely related to energy control [29] and passivity-based control [30]. Another direction of future work is to explore how to use the obtained result for control of nonlinear systems.

Acknowledgments

The authors appreciate Professor Takashi Hikihara of Kyoto University for fruitful discussions. Y. Yokoi would like to acknowledge Professor Tsuyoshi Higuchi of Nagasaki University for his support to this research.

References

[1] R. Brown and L. Kocarev, "A unifying definition of synchronization for dynamical systems," Chaos, vol.10, no.2, pp.344–349, 2000.
[2] C. Hayashi, Nonlinear Oscillations in Physical Systems, McGraw-Hill, New York, 1964.
[3] A.A. Andronov, A.A. Vitt, and S.E. Khaikin, Theory of Oscillations, Pergamon, Oxford, 1966.
[4] R. Adler, "A study of locking phenomena in oscillators," Proc. IRE, vol.34, no.6, pp.351–357, 1946.
[5] M.L. Cartwright, "Forced oscillations in nearly sinusoidal systems," J. Inst. Electr. Eng., vol.95, no.34, pp.88–96, 1948.
[6] M. Kuramitsu and F. Takase, "An analytical method for multimode oscillators using the averaged potential," IEICE Trans. Fundamentals (Japanese Edition), vol.J66-A, no.4, pp.336–343, April 1983.
[7] M. Kuramitsu, F. Takase, and H. Nitta, "A new method for the analysis of forced self-oscillatory systems using the averaged potential," Proc. IEEE International Symposium on Circuits and Systems, pp.863–866, 1985.
[8] R.K. Brayton and J.K. Moser, "A theory of nonlinear networks – I," Q. Appl. Math., vol.22, no.1, pp.1–33, 1964.
[9] C. Sarasola, F.J. Torrealdea, A. d'Anjou, A. Moujahid, and M. Graña, "Energy balance in feedback synchronization of chaotic systems," Phys. Rev. E, vol.69, no.1, 011606, 2004.

[10] C. Sarasola, A. d'Anjou, F.J. Torrealdea, and M. Graña, "Minimization of the energy flow in the synchronization of nonidentical chaotic systems," Phys. Rev. E, vol.72, no.2, 026223, 2005.
[11] D.A. Paley, N.E. Leonard, R. Sepulchre, D. Grünbaum, and J.K. Parrish, "Oscillator models and collective motion," IEEE Contr. Syst. Mag., vol.27, no.4, pp.89–105, 2007.
[12] J. Takagi, "History and future tasks in research on synchronization phenomena," Proc. Joint Convention Record of Four Institutes of Electrical Engineers, Japan, vol.6, pp.97–100, 1974 (in Japanese).
[13] I.I. Blekhman, Synchronization in Science and Technology, ASME Press, New York, 1988.
[14] S.H. Strogatz, Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry, and Engineering, Addison-Wesley, Reading, MA, 1994; Westview Press, 2000.
[15] Y. Susuki, Y. Yokoi, and T. Hikihara, "Energy-based analysis of frequency entrainment described by van der Pol and phase-locked loop equations," Chaos, vol.17, no.2, 023108, 2007.
[16] B. van der Pol, "Forced oscillations in a circuit with non-linear resistance (reception with reactive triode)," Phil. Mag., vol.3, pp.65–80, 1927.
[17] T. Endo and L.O. Chua, "Chaos from phase-locked loops," IEEE Trans. Circuits & Syst., vol.35, no.8, pp.987–1003, 1988.
[18] Y. Susuki and Y. Ueda, "Amplitude response curves of frequency-locked rotations," IEICE Trans. Fundamentals, vol.E90-A, no.10, pp.2250–2252, Oct. 2007.
[19] Y. Yokoi, Y. Susuki, and T. Hikihara, "A dissipated power-based analysis of frequency entrainment described by van der Pol and PLL equations," Proc. 2007 International Symp. Nonlinear Theory and its Applications, pp.317–320, 2007.
[20] Y. Kuramoto, "Self-entrainment of a population of coupled nonlinear oscillators," Proc. International Symposium on Mathematical Problems in Theoretical Physics, ed. H. Araki, Lecture Notes in Physics, vol.39, pp.420–422, 1975.
[21] Y. Kuramoto, Chemical Oscillations, Waves, and Turbulence, Springer-Verlag, Berlin, 1984; Dover Publications, New York, 2003.
[22] I.I. Blekhman, P.S. Landa, and M.G. Rosenblum, "Synchronization and chaotization in interacting dynamical systems," Appl. Mech. Rev., vol.48, no.11, pp.733–752, 1995.
[23] J.K. Hale, Ordinary Differential Equations, John Wiley & Sons, New York, 1969.
[24] A. Pikovsky, M. Rosenblum, and J. Kurths, Synchronization: A universal concept in nonlinear sciences, Cambridge University Press, Cambridge, 2001.
[25] D. D'Humieres, M.R. Beasley, B.A. Huberman, and A. Libchaber, "Chaotic states and routes to chaos in the forced pendulum," Phys. Rev. A, At. Mol. Opt. Phys., vol.26, no.6, pp.3483–3496, 1982.
[26] R.L. Kautz, "The ac Josephson effect in hysteretic junctions: Range and stability of phase lock," J. Appl. Phys., vol.52, no.5, pp.3528–3541, 1981.
[27] Y. Susuki and T. Hikihara, "Transient dynamics in electric power system with dc transmission: Fractal growth in stability boundary," IEE Proc., Circuits Devices Syst., vol.152, no.2, pp.159–164, 2005.
[28] J. Guckenheimer and P. Holmes, Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields, Springer-Verlag, New York, 1983.
[29] K.J. Åström and K. Furuta, "Swinging up a pendulum by energy control," Automatica, vol.36, no.2, pp.287–295, 2000.
[30] R. Ortega, A.J. van der Schaft, I. Mareels, and B. Maschke, "Putting energy back in control," IEEE Contr. Syst. Mag., vol.21, no.2, pp.18–33, 2001.

Appendix A: Detailed Procedure of Evaluation of Entrainment Time

This appendix describes the detail of the evaluation procedure of entrainment time for the systems (1) and (8).

For the system (1), the convergence condition is mathematically checked whether the value of phase exists in the ϵ-neighborhood of phase at the entrained libration, where ϵ is a small positive constant. In the analytical evaluation, by using θ_{L}^* to represent the phase at the entrained libration, the convergence condition is described as $|\theta_{\mathrm{L}}(t)-\theta_{\mathrm{L}}^*| < \epsilon$. For an arbitrary initial state $\theta_{\mathrm{L}}(t_0)$, the final state along a trajectory of (4) starting from $\theta_{\mathrm{L}}(t_0)$ is selected from $\theta_{\mathrm{L}}(t) = \theta_{\mathrm{L}}^* \pm \epsilon$ so that the trajectory does not go through the unstable state. The entrainment time is then obtained as the time interval $t - t_0$ of (4) with the initial phase $\theta_{\mathrm{L}}(t_0)$ and the corresponding entrained state $\theta_{\mathrm{L}}(t) = \theta_{\mathrm{L}}^* \pm \epsilon$. On the other hand, in the numerical evaluation, the phase $\theta_{\mathrm{L}}(t)$ is computed from the stroboscopic observation of the value $\phi_{\mathrm{L}}(t) := \tan^{-1}(\nu u(t)/v(t))$. The initial state $(u(t_0), v(t_0))$ is set at a point on the entrained orbit. By substituting $t_0 = 0$ or replacing the function $f_{\mathrm{L}}(t) = B\sin\nu t$ with $f_{\mathrm{L}}(t-t_0)$, the transient behavior is calculated with the direct integration of (1). This operation corresponds to the initialization that we use in Sect. 2.3. Therefore, the convergence condition is set at $|\phi_{\mathrm{L}}(t_0 + nT_{\mathrm{L}}) - \phi_{\mathrm{L}}^*| < \epsilon$, through stroboscopic observation for a finite positive integer n, where ϕ_{L}^* represents the phase at the entrained libration. After checking the convergence of phase to the entrained state based on the above condition, the entrainment time is given as the time interval $t - t_0$, where t corresponds to the final time of iteration of stroboscopic observation.

For the system (8), the convergence condition is described as $|\theta_{\mathrm{R}}(t)-\theta_{\mathrm{R}}^*| < \epsilon$ or $|\phi(t_0+nT_{\mathrm{R}})-\phi^*| < \epsilon$, where θ_{R}^* and ϕ^* denote the entrained state. In the analytical evaluation, for an arbitrary initial state $\theta_{\mathrm{R}}(t_0)$, the final state along a trajectory of (12) starting from $\theta_{\mathrm{R}}(t_0)$ corresponds to $\theta_{\mathrm{R}}(t) = \theta_{\mathrm{R}}^* + \epsilon$ or $\theta_{\mathrm{R}}^* - \epsilon$ so that the trajectory does not go through the unstable state. The entrainment time is obtained as the time interval $t-t_0$ of (13) with the initial phase $\theta_{\mathrm{R}}(t_0)$ and the corresponding entrained state $\theta_{\mathrm{R}}(t) = \theta_{\mathrm{R}}^* \pm \epsilon$. In the numerical evaluation, the initial state $(\phi(t_0), v(t_0))$ is set at a point on the entrained orbit. By substituting $t_0 = 0$ or replacing the function $f_{\mathrm{R}}(t) = A\sin\Omega t$ with $f_{\mathrm{R}}(t-t_0)$, the transient behavior is calculated with the numerical integration of (8). The entrainment time is given as $t - t_0$ for the numerical evaluation.

Appendix B: Detailed Derivation of the Phase Equation (12)

This appendix provides the detailed derivation of the phase equation (12) that describes the transient dynamics of the frequency entrainment phenomenon in the phase-locked system (8). By substituting (11) into (8) we have

$$\begin{aligned}&\Omega + \frac{\mathrm{d}\theta_{\mathrm{R}}}{\mathrm{d}t} + \left(\Omega + \frac{\mathrm{d}\theta_{\mathrm{R}}}{\mathrm{d}t}\right)a_0\cos(\Omega t + \theta_{\mathrm{R}} + \sigma_0)\\ &\quad = \Omega + \omega_{\mathrm{R}} + \Omega a_0\cos(\Omega t + \theta_{\mathrm{R}} + \sigma_0),\end{aligned} \qquad (\mathrm{A{\cdot}1})$$

$$\begin{aligned}\frac{\mathrm{d}\omega_\mathrm{R}}{\mathrm{d}t} &- \left(\Omega + \frac{\mathrm{d}\theta_\mathrm{R}}{\mathrm{d}t}\right)\Omega a_0 \sin(\Omega t + \theta_\mathrm{R} + \sigma_0)\\ &= -k\left\{\Omega + \omega_\mathrm{R} + \Omega a_0 \cos(\Omega t + \theta_\mathrm{R} + \sigma_0)\right\}\\ &\quad - \sin\left\{\Omega t + \theta_\mathrm{R} + a_0 \sin(\Omega t + \theta_\mathrm{R} + \sigma_0)\right\}\\ &\quad + N + A\sin\Omega t. \qquad \text{(A· 2)}\end{aligned}$$

Here the two variables θ_R and ω_R are slowly variable. By averaging both sides of (A· 2) in time t over one period $2\pi/\Omega$, we have

$$\frac{\mathrm{d}\omega_\mathrm{R}}{\mathrm{d}t} = -k\omega_\mathrm{R} + N - k\Omega - J_1(a_0)\sin\sigma_0, \qquad \text{(A· 3)}$$

where $J_n(\cdot)\,(n = 0, 1, 2, \cdots)$ is the Bessel function of the first kind. In addition to this, by averaging both sides of (A· 1) $\times \sin(\Omega t + \theta_\mathrm{R} + \sigma_0) +$ (A· 2) $\times \cos(\Omega t + \theta_\mathrm{R} + \sigma_0)/\Omega$ in time t over $2\pi/\Omega$, we obtain

$$J_1(a_0)\sin\sigma_0 = \frac{k\Omega a_0^2}{2} + \frac{Aa_0}{2}\sin(\theta_\mathrm{R} + \sigma_0), \qquad \text{(A· 4)}$$

where the relationship $2J_1(a_0)/a_0 = J_0(a_0) + J_2(a_0)$ was implicitly used. By substituting (A· 4) into (A· 3), we derive the following equation:

$$\begin{aligned}\frac{\mathrm{d}\omega_\mathrm{R}}{\mathrm{d}t} &= -k\omega_\mathrm{R} + N - k\Omega\\ &\quad - \frac{k\Omega a_0^2}{2} - \frac{Aa_0}{2}\sin(\theta_\mathrm{R} + \sigma_0). \qquad \text{(A· 5)}\end{aligned}$$

Here, Assumption (A-R1) in Sect. 3 is explicitly formulated as $k = \epsilon' k'$, $N = \epsilon' N'$, and $A = \epsilon' A'$ with the small, positive parameter ϵ'. Equation (A· 5) is thus rewritten as follows:

$$\begin{aligned}\frac{\mathrm{d}\omega_\mathrm{R}}{\mathrm{d}t} &= \epsilon'\Big\{ - k'\omega_\mathrm{R} + N' - k'\Omega\\ &\quad - \frac{k'\Omega a_0^2}{2} - \frac{A' a_0}{2}\sin(\theta_\mathrm{R} + \sigma_0)\Big\}. \qquad \text{(A· 6)}\end{aligned}$$

This equation clearly indicates that the time derivative of ω_R is sufficiently small. In this way, the effect of the time derivative is negligible, and its magnitude is fixed at zero. By using the definition of ω_R, namely, $\omega_\mathrm{R} = \mathrm{d}\theta_\mathrm{R}/\mathrm{d}t$, we obtain the following equation which describes the time derivative of θ_R:

$$\frac{\mathrm{d}\theta_\mathrm{R}}{\mathrm{d}t} = \frac{N}{k} - \Omega - \frac{\Omega a_0^2}{2} - \frac{Aa_0}{2k}\sin(\theta_\mathrm{R} + \sigma_0), \qquad \text{(A· 7)}$$

where the original parameters k, N, and A are again used. Here, under $A = 0$, that is, no external forcing, the above derivation is still valid and gives the following equation for $\theta_{\mathrm{R}0}$:

$$\frac{\mathrm{d}\theta_{\mathrm{R}0}}{\mathrm{d}t} = \frac{N}{k} - \Omega_0 - \frac{\Omega_0 a_0^2}{2} = 0, \qquad \text{(A· 8)}$$

where Ω changes to Ω_0 in this setting. This equation gives the relationship of a_0 and Ω_0 in the rotational limit cycle. By substituting (A· 8) into (A· 7), we derive the phase equation (12).

Moreover, we determine the parameters Ω_0, a_0, and σ_0 in the phase equation (12). These parameters are associated with the rotational limit cycle in the system (8) without the periodic force $A\sin\Omega t$. Equation (A· 3) for the limit cycle is rewritten as

$$0 = N - k\Omega_0 - J_1(a_0)\sin\sigma_0, \qquad \text{(A· 9)}$$

where $\omega_\mathrm{R} = 0$ and $\Omega = \Omega_0$ are used. In the same way, Eq. (A· 4) for the limit cycle is rewritten as

$$J_1(a_0)\sin\sigma_0 = \frac{k\Omega_0 a_0^2}{2}, \qquad \text{(A· 10)}$$

where $A = 0$ and $\Omega = \Omega_0$. By averaging both sides of (A· 1) $\times \cos(\Omega_0 t + \theta_\mathrm{R} + \sigma_0) -$ (A· 2) $\times \sin(\Omega_0 t + \theta_\mathrm{R} + \sigma_0)/\Omega_0$ in time t over $2\pi/\Omega_0$ with $A = 0$ and $\Omega = \Omega_0$, we obtain

$$\left\{J_0(a_0) - J_2(a_0)\right\}\cos\sigma_0 = \Omega_0^2 a_0^2. \qquad \text{(A· 11)}$$

The solution $(\Omega_0, a_0, \sigma_0)$ of the three Eqs. (A· 9)–(A· 11) corresponds to the parameters of the rotational limit cycle.

Yuichi Yokoi received B.E., M.E., and Ph.D. degrees from Kyoto University, Japan in 2006, 2008, and 2011, respectively. He has been with the Division of Electrical Engineering and Computer Science, Nagasaki University, Japan as an assistant professor since 2011. His research interests include the technological application of nonlinear dynamics and the design of electrical machines. He is a member of IEEJ, ISCIE, and IEEE.

Yoshihiko Susuki received the bachelor, master, and Ph.D. degrees, all in engineering from Kyoto University, Kyoto, Japan, in 2000, 2002, and 2005, respectively. In 2005, he joined the Department of Electrical Engineering at Kyoto University, where he is currently a Lecturer. In 2008–2010, he was a Visiting Researcher at the Department of Mechanical Engineering, University of California, Santa Barbara, United States under JSPS Postdoctoral Fellowship for Research Abroad. His research interests are in power engineering, nonlinear dynamical systems, and control systems. He is a member of IEEJ, ISCIE, SICE, IEEE, SIAM, and so on.