# TWO-SIDED BGG RESOLUTIONS OF ADMISSIBLE REPRESENTATIONS

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ABSTRACT. We prove the conjecture of Frenkel, Kac and Wakimoto [FKW] on the existence of two-sided BGG resolutions of $G$-integrable admissible representations of affine Kac-Moody algebras at fractional levels. As an application we establish the semi-infinite analogue of the generalized Borel-Weil theorem [Kos] for minimal parabolic subalgebras which enables an inductive study of admissible representations.

## 1. INTRODUCTION

Wakimoto modules are representations of non-twisted affine Kac-Moody algebras introduced by Wakimoto [Wak] in the case of $\widehat{\mathfrak{sl}}_2$ and by Feigin and Frenkel [FF1] in the general case. Wakimoto modules have useful applications in representation theory and conformal field theory. In these applications it is important to have a resolution of an irreducible highest weight representation $L(\lambda)$ of an affine Kac-Moody algebra $\mathfrak{g}$ in terms of Wakimoto modules, that is, a complex

$$C^\bullet(\lambda) :\to C^{i-1}(\lambda) \xrightarrow{d_{i-1}} C^i(\lambda) \xrightarrow{d_i} C^{i+1}(\lambda) \to \dots$$

with a differential $d_i$ which is a $\mathfrak{g}$-module homomorphism such that $C^i(\lambda)$ is a direct sum of Wakimoto modules and

$$H^i(C^\bullet(\lambda)) = \begin{cases} L(\lambda) & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The existence of such a resolution has been proved by Feigin and Frenkel [FF2] for any integrable representations over arbitrary $\mathfrak{g}$ and by Bernard and Felder [BF] and Feigin and Frenkel [FF2] for any admissible representation [KW2] over $\widehat{\mathfrak{sl}}_2$. In their study of $W$-algebras Frenkel, Kac and Wakimoto [FKW, Conjecture 3.5.1] conjectured the existence of such a resolution for any principle admissible representations over arbitrary $\mathfrak{g}$. In this paper we prove the existence of a two-sided resolution in terms of Wakimoto modules for any $\mathring{\mathfrak{g}}$-integrable admissible representations over arbitrary $\mathfrak{g}$ (Theorem 6.11), where $\mathring{\mathfrak{g}}$ is the classical part of $\mathfrak{g}$. For a general principal admissible representation of $\mathfrak{g}$ we obtain the two-sided resolution in terms of twisted Wakimoto modules (Theorem 6.15).

Let us sketch the proof of our result briefly. By Fiebig's equivalence [Fie] the block of the category $\mathcal{O}$ of $\mathfrak{g}$ containing an admissible representation $L(\lambda)$ is equivalent to the block containing an integrable representation[1]. Therefore an admissible

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[1]In the case $L(\lambda)$ is a non-principal $G$-integrable admissible representation this is a block of another Kac-Moody algebra.

representation admits a usual BGG type resolution in terms of Verma modules by the result of [GL, RCW]. Hence the idea of Arkhipov [Ark1] is applicable in our situation: One can obtain a twisted BGG resolution of $L(\lambda)$ in terms of twisted Verma modules by applying the twisting functor $T_w$ [Ark1] to the BGG resolution of $L(\lambda)$ as we have the "Borel-Weil-Bott" vanishing property [AS]

$$\mathcal{L}_i T_w L(\lambda) \cong \begin{cases} L(\lambda) & \text{if } i = \ell(w), \\ 0 & \text{otherwise} \end{cases}$$

for $w \in \mathcal{W}(\lambda)$, where $\mathcal{W}(\lambda)$ is the integral Weyl group of $\lambda$ and $\ell : \mathcal{W}(\lambda) \to \mathbb{Z}_{\geq 0}$ is the length function, see Theorem 5.12. It remains to show that one can construct an inductive system of twisted BGG resolutions $\{B_w^\bullet(\lambda)\}$ of $L(\lambda)$ such that the complex $\varinjlim_{w} B_w^\bullet(\lambda)$ gives the required two-sided resolution of $L(\lambda)$, see §6 for the details.

We note that by applying the (generalized) quantum Drinfeld-Sokolov reduction functor [FKW, KRW] to the (duals of the) two-sided BGG resolutions of admissible representations we obtain resolutions of some of simple modules over $W$-algebras in terms of free field realizations due to the vanishing of the associated BRST cohomology [A1, A2, A3, A4, A5]. In particular we obtain two-sided resolutions of all the minimal series representations [FKW, A7] of the $W$-algebras associated with principal nilpotent elements in terms of free bosonic realizations.

As an application of the existence of two-sided BGG resolution for admissible representations we prove a semi-infinite analogue of the generalized Borel-Weil theorem [Kos] for minimal parabolic subalgebras (Theorem 7.7). This result is important since it enable an inductive study of admissible representations, see our subsequent paper [A6].

This paper is organized as follows. In §2 we collect and prove some basic results about semi-infinite cohomology [Feĭ] and semi-regular bimodules [Vor1] which are needed for later use. In particular we establish an important property of semi-regular bimodules in Proposition 2.1. In §2 we collect basic results on the semi-infinite Bruhat ordering (or the generic Bruhat ordering) of an affine Weyl group defined by Lusztig [Lus] and study the semi-infinite analogue of parabolic subgroups. Semi-infinite Bruhat ordering is important for us since it (conjecturally) describes the space of homomorphisms between Wakimoto modules, see Proposition 4.10 and Conjecture 4.11. The semi-infinite analogue of the minimal (or maximal) length representatives (Theorem 3.3) is important for describing the semi-infinite restriction functors studied in §7. In §4 we define Wakimoto modules and twisted Verma modules following [Vor2] and study some of their basic properties. In particular we prove the uniqueness of Wakimoto modules which was stated in [FF2] without a proof (Theorem 4.7). In §5 we generalize the Borel-Weil-Bott vanishing property of the twisting functor established in [AS] to the affine Kac-Moody algebra cases. In §6 we state and prove the main results of this paper. In §7 we study the semi-infinite restriction functor and establish the semi-infinite analogue of the generalized Borel-Weil theorem [Kos] for minimal parabolic subalgebras. This is a non-trivial fact since admissible representations are not unitarizable unless they are integrable.

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## 2. Semi-regular bimodules and semi-infinite cohomology

**2.1. Some notation.** For $\mathbb{Z}$-graded vector spaces $M=\bigoplus_{n\in\mathbb{Z}} M_n, N=\bigoplus_{n\in\mathbb{Z}} N_n$ with finite-dimensional homogeneous components let

$$\begin{aligned}\mathcal{H}om_{\mathbb{C}}(M,N)=&\bigoplus_{n\in\mathbb{Z}}\mathcal{H}om_{\mathbb{C}}(M,N)_n,\\ &\mathcal{H}om_{\mathbb{C}}(M,N)_n=\{f\in \operatorname{Hom}_{\mathbb{C}}(M,N); f(M_i)\subset N_{i+n}\},\\ \mathcal{E}nd_{\mathbb{C}}(M)=&\mathcal{H}om_{\mathbb{C}}(M,M).\end{aligned}$$

We denote by $M^*=\bigoplus_{n\in\mathbb{Z}}(M^*)_n$ the space $\mathcal{H}om_{\mathbb{C}}(M,\mathbb{C})$, where $\mathbb{C}$ is considered as a graded vector space concentrated in the degree 0 component. If $M$, $N$ are module over an algebra $A$ we denote by $\mathcal{H}om_A(M,N)$ the space of all $A$-homomorphisms in $\mathcal{H}om_{\mathbb{C}}(M,N)$.

**2.2. Semi-infinite structure.** Let $\mathfrak{g}$ be a complex Lie algebra. A *semi-infinite structure* [Vor1] of $\mathfrak{g}$ is is the following data:

(i) a $\mathbb{Z}$-grading $\mathfrak{g}=\bigoplus_{n\in\mathbb{Z}}\mathfrak{g}_n$ of $\mathfrak{g}$ with finite-dimensional homogeneous components, $\dim_{\mathbb{C}}\mathfrak{g}_n<\infty$ for all $n$,

(ii) a *semi-infinite* 1*-cochain* $\gamma:\mathfrak{g}\to\mathbb{C}$.

Here by a semi-infinite 1-cochain we mean the following: Decompose $\mathfrak{g}$ into the direct sum of two subalgebras

$$\mathfrak{g}=\mathfrak{g}_+\oplus\mathfrak{g}_-, \tag{1}$$

$$\mathfrak{g}_+=\bigoplus_{i\geq 0}\mathfrak{g}_i,\quad \mathfrak{g}_-=\bigoplus_{i<0}\mathfrak{g}_i. \tag{2}$$

A linear map $\gamma:\mathfrak{g}\to\mathbb{C}$ is called a semi-infinite 1-cochain if $\gamma$ satisfies

$$\gamma([x,y])=\operatorname{tr}((\operatorname{ad}x)_{+-}(\operatorname{ad}y)_{-+}-(\operatorname{ad}y)_{+-}(\operatorname{ad}x)_{-+})\quad\text{for } x,y\in\mathfrak{g},$$

where $(\operatorname{ad}x)_{\pm\mp}$ denotes the composition $\mathfrak{g}_{\mp}\overset{\operatorname{ad}x}{\to}\mathfrak{g}\overset{\text{projection}}{\longrightarrow}\mathfrak{g}_{\pm}$.

In the rest of this section we assume that $\mathfrak{g}$ is equipped with a semi-infinite structure such that $\gamma(\sum_{i\neq 0}\mathfrak{g})=0$.

We denote by $U$, $U_-$, $U_+$, the enveloping algebras of $\mathfrak{g}$, $\mathfrak{g}_+$, $\mathfrak{g}_-$ by respectively. These algebras inherit a $\mathbb{Z}$-grading from the corresponding Lie algebras.

Let $\tilde{\mathcal{O}}^{\mathfrak{g}}$ be the category of $\mathbb{Z}$-graded $\mathfrak{g}$-modules $M=\bigoplus_{n\in\mathbb{Z}}M_n$ with $\dim M_n<\infty$ for all $m$ on which $\bigoplus_{j>0}\mathfrak{g}_+$ acts locally nilpotently and $\mathfrak{g}_0$ acts locally finitely.

**2.3. Semi-infinite cohomology.** Choose a basis $\{x_i; i\in\mathbb{Z}\}$ of $\mathfrak{g}$ such that $\{x_i; i\geq 0\}$ and $\{x_i; i<0\}$ are bases of $\mathfrak{g}_+$ and $\mathfrak{g}_-$, respectively, and let $\{c_{ij}^k\}$ be the structure constant: $[x_i,x_j]=\sum_k c_{ij}^k x_k$.

Denote by $\mathcal{C}l(\mathfrak{g})$ the Clifford algebra associated with $\mathfrak{g}\oplus\mathfrak{g}^*$, which has the following generators and relations:

$$\text{generators: } \psi_i, \psi_i^* \quad \text{for } i\in\mathbb{Z},$$
$$\text{relations: } \{\psi_i,\psi_j^*\}=\delta_{i,j},\ \{\psi_i,\psi_j\}=\{\psi_i^*,\psi_j^*\}=0.$$

Here $\{X,Y\}=XY+YX$. The *space of the semi-infinite forms* $\bigwedge^{\frac{\infty}{2}+\bullet}(\mathfrak{g})$ of $\mathfrak{g}$ is by definition the irreducible representation of $\mathcal{C}l(\mathfrak{g})$ generated by the vector $\mathbf{1}$ satisfying

$$\psi_i\mathbf{1}=0 \quad \text{for } i\geq 0, \quad \psi_i^*\mathbf{1}=0 \quad \text{for } i>0.$$

It is graded by $\deg\mathbf{1}=0$, $\deg\psi_i^*=1$ and $\deg\psi_i=-1$: $\bigwedge^{\frac{\infty}{2}+\bullet}(\mathfrak{g})=\bigoplus_{p\in\mathbb{Z}}\bigwedge^{\frac{\infty}{2}+p}(\mathfrak{g})$.

For $A\in\mathcal{E}nd_{\mathbb{C}}(\bigwedge^{\frac{\infty}{2}+\bullet}(\mathfrak{g}))$ of degree $n$ set

$$:\psi_kA:=\begin{cases}\psi_kA & \text{if } k<0,\\ (-1)^nA\psi_k & \text{if } k\geq 0,\end{cases}\qquad :\psi_k^*A:=\begin{cases}\psi_k^*A & \text{if } k\leq 0,\\ (-1)^nA\psi_k^* & \text{if } k>0.\end{cases} \tag{3}$$

The following defines a $\mathfrak{g}$-module structure on $\bigwedge^{\frac{\infty}{2}+\bullet}(\mathfrak{g})$:

$$x_i\mapsto :\mathrm{ad}(x_i):+\gamma(x_i), \tag{4}$$

where

$$:\mathrm{ad}\,x_i:=\sum_{j,k}c_{ij}^k:\psi_k\psi_j^*:.$$

For $M\in\tilde{\mathcal{O}}^{\mathfrak{g}}$, define $d\in\mathrm{End}(M\otimes\bigwedge^{\frac{\infty}{2}+\bullet}(\mathfrak{g}))$ by

$$d=\sum_{i\in\mathbb{Z}}x_i\otimes\psi_i^*-1\otimes\frac{1}{2}\sum_{i,j,k\in\mathbb{Z}}c_{ij}^k:\psi_i^*(:\psi_j^*\psi_k:):+1\otimes\sum_{i\in\mathbb{Z}}\gamma(x_i)\psi_i^*$$

Then

$$d^2=0,\quad d(M\otimes\bigwedge^{\frac{\infty}{2}+p}(\mathfrak{g}))\subset M\otimes\bigwedge^{\frac{\infty}{2}+p+1}(\mathfrak{g}).$$

The cohomology of the complex $(M\otimes\bigwedge^{\frac{\infty}{2}+\bullet}(\mathfrak{g}),d)$ is called the *semi-infinite* $\mathfrak{g}$-*cohomology* with coefficients in $M$ and denoted by $H^{\frac{\infty}{2}+\bullet}(\mathfrak{g},M)$ ([Feĭ, Vor1]).

2.4. **Semi-regular bimodules.** We consider the full dual space $\mathrm{Hom}_{\mathbb{C}}(U,\mathbb{C})$ of $U$ as a $U$-bimodule by $(Xf)(u)=f(uX)$, $(fX)(u)=f(Xu)$ for $X\in\mathfrak{g}$, $f\in\mathrm{Hom}_{\mathbb{C}}(M,\mathbb{C})$, $u\in U$. The graded duals $U_\pm^*$ of $U_\pm$ are $\mathfrak{g}_\pm$-submodule of $\mathrm{Hom}_{\mathbb{C}}(U,\mathbb{C})$. By abuse of notation we denote by $U^*$ the image of the embedding $U_+^*\otimes_{\mathbb{C}}U_-^*\hookrightarrow\mathrm{Hom}_{\mathbb{C}}(U,\mathbb{C})$, $f_+\otimes f_-\mapsto(u_-u_+\mapsto f_+(u_+)f_-(u_-))$, $f_\pm\in U_\pm^*$, $u_\pm\in U$. Then $U^*$ is a $U$-bisubmodule of $\mathrm{Hom}_{\mathbb{C}}(U,\mathbb{C})$ and coincides with the image of the embedding $U_-^*\otimes_{\mathbb{C}}U_+^*\hookrightarrow\mathrm{Hom}_{\mathbb{C}}(U,\mathbb{C})$, $f_-\otimes f_+\mapsto(u_+u_-\mapsto f_+(u_+)f_-(u_-))$.

Following [Vor2] define

$$US(\mathfrak{g})=H^{\frac{\infty}{2}+0}(\mathfrak{g},U^*\otimes_{\mathbb{C}}U),$$

where $\mathfrak{g}$ is given the opposite semi-infinite structure and the semi-infinite $\mathfrak{g}$-cohomology is taken with respect to the diagonal left action on $U^*\otimes_{\mathbb{C}}U$. Here by the opposite semi-infinite structure we mean the one obtained by replacing $\mathfrak{g}_\pm$ with $\mathfrak{g}_\mp$ and $\gamma$

with $-\gamma$. The space $US(\mathfrak{g})$ inherits the $U$-bimodule structure from $U^*\otimes U$ defined by

$$X(f\otimes u) = -(fX)\otimes u, \quad (f\otimes u)X = f\otimes(uX)$$

for $X \in \mathfrak{g}$, $\in U^*$, $u \in U$. The $U$-bimodule $US(\mathfrak{g})$ is called the *semi-regular bimodule* of $\mathfrak{g}$. One has

$$US(\mathfrak{g}) \cong U_+^*\otimes_{U_+}U \tag{5}$$

as left $\mathfrak{g}_+$-modules and right $\mathfrak{g}$-modules, and the left $\mathfrak{g}$-module structure of $US(\mathfrak{g})$ is defined through the isomorphism

$$U_+\otimes_{U_-}U \cong \mathcal{H}om_{\mathbb{C}}(U_+, U) \cong \mathcal{H}om_{U_-}(U, U_-\otimes_{\mathbb{C}}\mathbb{C}_{-\gamma}) \tag{6}$$

([Vor1, Soe2, Vor2]).

Let $M$ be a $\mathfrak{g}$-module and consider the following four left $\mathfrak{g}$-module structures on $US(\mathfrak{g})\otimes_{\mathbb{C}}M$:

$$\pi_1(X)(s\otimes m) = -(sX)\otimes m + s\otimes Xm, , \quad \pi_2(X)(s\otimes m) = (Xs)\otimes m, \tag{7}$$

$$\pi_1'(X)(s\otimes m) = -(sX)\otimes m, \quad \pi_2'(X)(s\otimes m) = (Xs)\otimes m + s\otimes(Xm), \tag{8}$$

for $X \in \mathfrak{g}$, $s \in US(\mathfrak{g})$, $m \in M$. Clearly, the two actions $\pi_1$ and $\pi_2$ (resp. $\pi_1'$ and $\pi_2'$) commute.

**Proposition 2.1** (cf. [AG, 6.4])**.** *For $M \in \tilde{\mathcal{O}}^{\mathfrak{g}}$ the two $U$-bimodule structures $(\pi_1, \pi_2)$ and $(\pi_1', \pi_2')$ on $US(\mathfrak{g})\otimes_{\mathbb{C}}M$ are equivalent. Namely there exists a linear isomorphism $\Phi : US(\mathfrak{g})\otimes_{\mathbb{C}}M \xrightarrow{\sim} US(\mathfrak{g})\otimes_{\mathbb{C}}M$ such that $\Phi \circ \pi_i'(X) = \pi_i(X) \circ \Phi$ for $i = 1, 2$, $X \in \mathfrak{g}$.*

*Proof.* Define the linear isomorphism

$$\tilde{\Phi}_1 : U^*\otimes_{\mathbb{C}}U\otimes_{\mathbb{C}}M \xrightarrow{\sim} U^*\otimes_{\mathbb{C}}U\otimes_{\mathbb{C}}M$$

by $\tilde{\Phi}_1(f\otimes u\otimes m) = f\otimes(\Delta(u)(1\otimes m))$ for $f \in U^*$, $u \in U$, $m \in M$, where $\Delta : U \to U\otimes_{\mathbb{C}}U$ is the coproduct. We have

$$\tilde{\Phi}_1 \circ \pi_{2,L}(X) = (\pi_{2,L}(X) + \pi_{3,L}(X)) \circ \tilde{\Phi}_1$$
$$\tilde{\Phi}_1 \circ (\pi_{2,R}(X) + \pi_{3,R}(X)) = \pi_{2,R}(X) \circ \tilde{\Phi}_1,$$

where $\pi_{i,L}$ (resp. $\pi_{i,R}$) denotes the left action (resp. the right action) of $\mathfrak{g}$ on the $i$-th factor of $U^*\otimes U\otimes M$, and $M$ is considered as a right $\mathfrak{g}$-module by the action $mx = -xm$ for $m \in M$, $x \in \mathfrak{g}$.

Next consider the graded dual $M^* = \bigoplus_n (M^*)_n$ as a right module by the action $(fX)(m) = f(Xm)$. Let

$$\Psi : U^*\otimes_{\mathbb{C}}M \xrightarrow{\sim} U^*\otimes_{\mathbb{C}}M$$

be the linear isomorphism defined by $\Psi(f\otimes m)(u\otimes g) = (f\otimes m)((1\otimes g)\Delta(u))$ for $f \in U^*$, $m \in M$, $u \in U$, $g \in M^*$, where $M$ is identified with $(M^*)^*$. Extend this to the linear isomorphism

$$\tilde{\Phi}_2 : U^*\otimes_{\mathbb{C}}U\otimes_{\mathbb{C}}M \xrightarrow{\sim} U^*\otimes_{\mathbb{C}}U\otimes_{\mathbb{C}}M$$

by setting $\tilde{\Phi}_2(f\otimes u\otimes m)=\sum_i f_i\otimes u\otimes m_i$ if $\Psi(f\otimes m)=\sum_i f_i\otimes m_i$ with $f_i\in U^*$, $m_i\in M$. Then

$$\tilde{\Phi}_2\circ\pi_{1,R}(X)=(\pi_{1,R}(X)+\pi_{3,R}(X))\circ\tilde{\Phi}_2,$$
$$\tilde{\Phi}_2\circ(\pi_{1,L}(X)+\pi_{3,L}(X))=\pi_{1,L}(X)\circ\tilde{\Phi}_2.$$

Set

$$\tilde{\Phi}=\tilde{\Phi}_2\circ\tilde{\Phi}_1:U^*\otimes_{\mathbb{C}}U\otimes_{\mathbb{C}}M\xrightarrow{\sim}U^*\otimes_{\mathbb{C}}U\otimes_{\mathbb{C}}M.$$

Then

$$\begin{aligned}\tilde{\Phi}\circ(\pi_{1,L}(X)+\pi_{2,L}(X))&=\tilde{\Phi}_2\circ(\pi_{1,L}(X)+\pi_{2,L}(X)+\pi_{3,L}(X))\circ\tilde{\Phi}_1\\&=(\pi_{1,L}(X)+\pi_{2,L}(X))\circ\tilde{\Phi},\end{aligned}\tag{9}$$

$$\tilde{\Phi}\circ(\pi_{2,R}(X)+\pi_{3,R}(X))=\tilde{\Phi}_2\circ\pi_{2,R}(X)\circ\tilde{\Phi}_1=\pi_{2,R}(X)\circ\tilde{\Phi},\tag{10}$$

$$\tilde{\Phi}\circ\pi_{1,R}(X)=\tilde{\Phi}_2\circ\pi_{1,R}(X)\circ\tilde{\Phi}_1=(\pi_{1,R}(X)+\pi_{3,R}(X))\circ\tilde{\Phi}.\tag{11}$$

By (9) and the definition of $US(\mathfrak{g})$, $\tilde{\Phi}$ gives rise to a linear isomorphism

$$\Phi:US(\mathfrak{g})\otimes_{\mathbb{C}}M\xrightarrow{\sim}US(\mathfrak{g})\otimes_{\mathbb{C}}M.$$

Moreover $\Phi$ satisfies the required properties by (10) and (11). □

2.5. **Semi-infinite induction.** Let $\mathfrak{h}=\bigoplus_{n\in\mathbb{Z}}\mathfrak{h}_n$ be a graded Lie subalgebra of $\mathfrak{g}$ such that $\gamma|_{\mathfrak{h}}$ is a semi-infinite 1-cochain of $\mathfrak{h}$. Following [Vor2] we define the *semi-induced* $\mathfrak{g}$*-module* $\text{S-ind}_{\mathfrak{h}}^{\mathfrak{g}}M$ as

$$\text{S-ind}_{\mathfrak{h}}^{\mathfrak{g}}M:=H^{\frac{\infty}{2}+0}(\mathfrak{h},US(\mathfrak{g})\otimes_{\mathbb{C}}M),$$

where $US(\mathfrak{g})\otimes_{\mathbb{C}}M$ is considered as an $\mathfrak{h}$-module by the action $\pi_1$ defined in (7). The space $\text{S-ind}_{\mathfrak{h}}^{\mathfrak{g}}M$ inherits the structure of a $\mathfrak{g}$-module from the action $\pi_2$ defined in (7).

**Lemma 2.2.** *The assignment* $\text{S-ind}_{\mathfrak{h}}^{\mathfrak{g}}:M\mapsto\text{S-ind}_{\mathfrak{h}}^{\mathfrak{g}}M$ *defines an exact functor from* $\tilde{\mathcal{O}}^{\mathfrak{h}}$ *to* $\tilde{\mathcal{O}}^{\mathfrak{g}}$.

*Proof.* Clearly $\text{S-ind}\,M$ is an object of $\tilde{\mathcal{O}}^{\mathfrak{g}}$ since $US(\mathfrak{g})\otimes_{\mathbb{C}}M$ is. By Proposition 2.1 we may replace the actions of $\pi_1$ and $\pi_2$ on $US(\mathfrak{g})\otimes_{\mathbb{C}}M$ with $\pi_1'$ and $\pi_2'$, simultaneously. It follows that

$$H^{\frac{\infty}{2}+\bullet}(\mathfrak{h},US(\mathfrak{g})\otimes_{\mathbb{C}}M)\cong H^{\frac{\infty}{2}+\bullet}(\mathfrak{h},US(\mathfrak{g}))\otimes_{\mathbb{C}}M.\tag{12}$$

Since $US(\mathfrak{g})$ is free over $\mathfrak{h}_-$ and cofree over $\mathfrak{h}_+$, $H^{\frac{\infty}{2}+i}(\mathfrak{h},US(\mathfrak{g}))=0$ for $i\neq 0$ by [Vor1, Theorem 2.1]. (Note that the spectral sequence on [Vor1] converges since the complex $US(\mathfrak{g})\otimes\bigwedge^{\frac{\infty}{2}+\bullet}(\mathfrak{h})$ is a direct sum of finite-dimensional subcomplexes consisting of homogeneous vectors.) We have shown that the functor $\text{S-ind}_{\mathfrak{h}}^{\mathfrak{g}}$ is exact. □

In the case that $\mathfrak{h}=\mathfrak{g}$ and $\gamma_0=\gamma$, we have the following assertion.

**Proposition 2.3** ([Vor2, (1.9)])**.** *The functor* $\text{S-ind}_{\mathfrak{g}}^{\mathfrak{g}}:\tilde{\mathcal{O}}^{\mathfrak{g}}\to\tilde{\mathcal{O}}^{\mathfrak{g}}$ *is isomorphic to the identify functor.*

*Proof.* As $H^{\frac{\infty}{2}+0}(\mathfrak{g},US(\mathfrak{g}))$ is isomorphic to the trivial representation $\mathbb{C}$ of $\mathfrak{g}$ ([Vor1, Theorem 2.1]), (12) gives the $\mathfrak{g}$-module isomorphism $\text{S-ind}_{\mathfrak{g}}^{\mathfrak{g}}M\cong M$ as required. □

2.6. Suppose that $\mathfrak{g}$ admits a decomposition

$$\mathfrak{g} = \mathfrak{a} \oplus \bar{\mathfrak{a}}$$

with graded subalgebras $\mathfrak{a}$ and $\bar{\mathfrak{a}}$ such that the restrictions $\gamma|_{\mathfrak{a}}$ and $\gamma|_{\bar{\mathfrak{a}}}$ of $\gamma$ are semi-infinite 1-cochains of $\mathfrak{a}$ and $\bar{\mathfrak{a}}$, respectively.

**Lemma 2.4.** *$US(\mathfrak{g}) \cong US(\mathfrak{a}) \otimes_{\mathbb{C}} US(\bar{\mathfrak{a}})$ as left $\mathfrak{a}$-modules and right $\bar{\mathfrak{a}}$-modules.*

*Proof.* We have $U_+^* \cong U(\mathfrak{a}_+)^* \otimes_{\mathbb{C}} U(\bar{\mathfrak{a}}_+)^*$ as left $\mathfrak{a}_+$-modules and right $\bar{\mathfrak{a}}_+$-modules. Consider the composition

$$\begin{aligned} US(\mathfrak{a}) \otimes_{\mathbb{C}} US(\bar{\mathfrak{a}}) &\xrightarrow{\sim} (U(\mathfrak{a}_-) \otimes_{\mathbb{C}} U(\mathfrak{a}_+)^*) \otimes_{\mathbb{C}} (U(\bar{\mathfrak{a}}_+)^* \otimes_{\mathbb{C}} U(\bar{\mathfrak{a}}_-)) \\ &\xrightarrow{\sim} U(\mathfrak{a}_+) \otimes_{\mathbb{C}} U_+^* \otimes_{\mathbb{C}} U(\bar{\mathfrak{a}}_+) \to US(\mathfrak{g}), \end{aligned}$$

where the last map is the multiplication map. From the description (5), (6) of the $\mathfrak{g}$-bimodule structure of semi-regular bimodules one sees that the image of the above map is stable under the left and the right action of $\mathfrak{g}$ on $US(\mathfrak{g})$. Hence the image must coincides with $US(\mathfrak{g})$ since it contains $U_+^*$. By the equality of the graded dimensions it follows that above map is an isomorphism. □

**Lemma 2.5.** *For $M \in \tilde{\mathcal{O}}^{\bar{\mathfrak{a}}}$, $\text{S-ind}_{\bar{\mathfrak{a}}}^{\mathfrak{g}} M \cong US(\mathfrak{a}) \otimes_{\mathbb{C}} M$ as $\mathfrak{a}$-modules, where $\mathfrak{a}$ acts only on the first factor $US(\mathfrak{a})$ of $US(\mathfrak{a}) \otimes_{\mathbb{C}} M$.*

*Proof.* We have

$$\begin{aligned} \text{S-ind}_{\bar{\mathfrak{a}}}^{\mathfrak{g}}(M) &\cong H^{\frac{\infty}{2}+0}(\bar{\mathfrak{a}}, US(\mathfrak{a}) \otimes_{\mathbb{C}} US(\bar{\mathfrak{a}}) \otimes_{\mathbb{C}} M) \\ &\cong US(\mathfrak{a}) \otimes_{\mathbb{C}} \text{S-ind}_{\bar{\mathfrak{a}}}^{\bar{\mathfrak{a}}}(M) \cong US(\mathfrak{a}) \otimes_{\mathbb{C}} M \end{aligned}$$

by Lemmas 2.3 and 2.4. □

## 3. Semi-infinite Bruhat ordering

3.1. **Affine Kac-Moody algebras and affine Weyl groups.** We first fix some notation which are used for the rest of the paper.

Let $\mathring{\mathfrak{g}}$ be a finite-dimensional complex simple Lie algebra, and fix a Cartan subalgebra $\mathring{\mathfrak{h}}$ of $\mathring{\mathfrak{g}}$. Let $\mathring{\Delta} \subset \mathring{\mathfrak{h}}^*$ be the set of roots of $\mathring{\mathfrak{g}}$. Choose a subset $\Delta_+ \subset \mathring{\mathfrak{h}}^*$ of positive roots and the set $\mathring{\Pi} = \{\alpha_i; i \in \mathring{I}\} \subset \Delta_+$, $\mathring{I} = \{1, 2, \dots l\}$, of simple roots. Let $\theta$ be the highest root, $\theta_s$ the highest short root, $\Delta_- = -\Delta_+$, $\mathring{\rho} = \frac{1}{2} \sum_{\alpha \in \mathring{\Delta}_+} \alpha$. Let $\mathring{Q} = \sum_{\alpha \in \mathring{\Delta}} \mathbb{Z}\alpha \subset \mathring{\mathfrak{h}}^*$, the root lattice of $\mathring{\mathfrak{g}}$, Set $\mathring{\mathfrak{n}} = \bigoplus_{\alpha \in \mathring{\Delta}_+} \mathring{\mathfrak{g}}_\alpha$, $\mathring{\mathfrak{n}}_- = \bigoplus_{\alpha \in \mathring{\Delta}_-} \mathring{\mathfrak{g}}_\alpha$, where $\mathring{\mathfrak{g}}_\alpha$ is the root space of $\mathring{\mathfrak{g}}$ with root $\alpha$. We have the triangular decomposition

$$\mathring{\mathfrak{g}} = \mathring{\mathfrak{n}}_- \oplus \mathring{\mathfrak{h}} \oplus \mathring{\mathfrak{n}}.$$

Let $( \,|\, )$ be the normalized invariant bilinear form of $\mathring{\mathfrak{g}}$. We identify $\mathring{\mathfrak{h}}$ with $\mathring{\mathfrak{h}}^*$ using $( \,|\, )$. Let $\mathring{\Delta}^\vee = \{\alpha^\vee; \alpha \in \mathring{\Delta}\}$, the set of coroots , $\mathring{Q}^\vee = \sum_{\alpha \in \mathring{\Delta}} \mathbb{Z}\alpha^\vee \subset \mathring{\mathfrak{h}} = \mathring{\mathfrak{h}}^*$, the coroot lattice of $\mathring{\mathfrak{g}}$, $\mathring{\rho}^\vee = \frac{1}{2} \sum_{\alpha \in \mathring{\Delta}_+} \alpha^\vee$, where $\alpha^\vee = 2\alpha/(\alpha|\alpha)$.

Let $\overset{\circ}{\mathcal{W}} \subset GL(\overset{\circ}{\mathfrak{h}}^*)$ be the Weyl group of $\overset{\circ}{\mathfrak{g}}$, $s_\alpha \in \overset{\circ}{\mathcal{W}}$ be the reflection corresponding to $\alpha \in \Delta$: $s_\alpha(\lambda) = \lambda - \lambda(\alpha^\vee)\alpha$.

Let $\mathfrak{g}$ be the affine Kac-Moody algebra associated with $\overset{\circ}{\mathfrak{g}}$:

$$\mathfrak{g} = \overset{\circ}{\mathfrak{g}}[t, t^{-1}] \oplus \mathbb{C}K \oplus \mathbb{C}D.$$

The commutation relations of $\mathfrak{g}$ are given by

$$[xt^m, yt^n] = [x, y]t^{m+n} + m\delta_{m+n,0}(x|y)K, \quad [K, \mathfrak{g}] = 0, \quad [D, xt^n] = nxt^n.$$

We consider $\overset{\circ}{\mathfrak{g}}$ as a subalgebra of $\mathfrak{g}$ by the natural embedding $\overset{\circ}{\mathfrak{g}} \hookrightarrow \mathfrak{g}$, $x \mapsto xt^0$. Let

$$\mathfrak{h} = \overset{\circ}{\mathfrak{h}} \oplus \mathbb{C}K \oplus \mathbb{C}D,$$

the Cartan subalgebra of $\mathfrak{g}$. The bilinear form $(\ |\ )$ from $\overset{\circ}{\mathfrak{h}}$ to $\mathfrak{h}$ by letting $(K|\overset{\circ}{\mathfrak{h}}) = (D|\overset{\circ}{\mathfrak{h}}) = (K|K) = (D|D) = 0$ and $(D|K) = 1$. We identify $\overset{\circ}{\mathfrak{h}}^*$ with the subspace of $\mathfrak{h}^*$ consisting of elements which vanishes on $\mathbb{C}K \oplus \mathbb{C}D$. Thus,

$$\mathfrak{h}^* = \overset{\circ}{\mathfrak{h}}^* \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta,$$

where $\Lambda_0$ and $\delta$ are defined by $\Lambda_0(K) = \delta(D) = 1$, $\Lambda_0(\overset{\circ}{\mathfrak{h}} \oplus \mathbb{C}\delta) = \delta(\overset{\circ}{\mathfrak{h}} \oplus \mathbb{C}K) = 0$. The number $\langle \lambda, K \rangle$ is called the *level* of $\lambda$.

Let $\Delta_+^{re} = \overset{\circ}{\Delta}_+ \sqcup \{\alpha + n\delta; \alpha \in \overset{\circ}{\Delta},\ n \in \mathbb{N}\}$, the set of positive real roots of $\mathfrak{g}$, $\Delta_-^{re} = -\Delta_+^{re}$, $\Delta^{re} = \Delta_+^{re} \sqcup \Delta_-^{re}$ the set of real roots, $\Pi = \{\alpha_0 = -\theta + \delta, \alpha_1, \dots, \alpha_\ell\}$ the set of simple roots.

Let $\mathcal{W}$ be the Weyl group of $\mathfrak{g}$, or the *affine Weyl group* of $\overset{\circ}{\mathcal{W}}$. We have

$$\mathcal{W} = \overset{\circ}{\mathcal{W}} \ltimes \overset{\circ}{Q}{}^\vee.$$

The *extended affine Weyl group* $\mathcal{W}^e$ of $\mathfrak{g}$ is the semidirect product

$$\mathcal{W}^e = \overset{\circ}{\mathcal{W}} \ltimes P^\vee$$

where $\overset{\circ}{P}{}^\vee = \{\lambda \in \overset{\circ}{\mathfrak{h}}; \langle \alpha, \lambda \rangle \in \mathbb{Z} \text{ for all } \alpha \in \overset{\circ}{\Delta}\}$, the coweight lattice of $\overset{\circ}{\mathfrak{g}}$. We have

$$\mathcal{W}^e = \mathcal{W}_+^e \ltimes \mathcal{W},$$

where $\mathcal{W}_+^e$ subgroup of $\mathcal{W}^e$ consisting of elements which fix the set $\Pi$.

We denote by $t_\alpha$ or simply by $\alpha$ for the element of $\mathcal{W}^e$ corresponding to $\alpha \in \overset{\circ}{P}{}^\vee$. The reflection $s_\alpha$ corresponding $\alpha = \bar{\alpha} + n\delta \in \Delta^{re}$ is given by $s_\alpha = t_{-n\bar{\alpha}^\vee} s_{\bar{\alpha}}$. We set $s_i = s_{\alpha_i}$ for $i \in I := \{0, 1, \dots, l\}$, so that $\mathcal{W} = \langle s_i; i \in I \rangle$. The action of $\overset{\circ}{\mathcal{W}}$ on $\overset{\circ}{\mathfrak{h}}^*$ is extended to the action of $\mathcal{W}^e$ on $\mathfrak{h}^*$ by

$$w(\Lambda_0) = \Lambda_0,\ w(\delta) = \delta \quad w \in \overset{\circ}{\mathcal{W}},$$
$$t_\alpha(\lambda) = \lambda + \langle \Lambda, K \rangle \alpha - (\langle \lambda, \alpha \rangle + \frac{(\alpha|\alpha)}{2} \langle \lambda, K \rangle)\delta, \quad \lambda \in \mathfrak{h}^*.$$

For $\lambda \in \mathfrak{h}^*$ let $\bar{\lambda} \in \overset{\circ}{\mathfrak{h}}^*$ be its restriction to $\overset{\circ}{\mathfrak{h}}$.

3.2. **Twisted Bruhat ordering.** Let $\ell : \mathcal{W}^e \to \mathbb{Z}_{\geq 0}$ the length function: $\ell(w) = \sharp(\Delta_+^{re} \cap w(\Delta_-^{re}))$. We have

$$\ell(t_\mu y) = \sum_{\alpha \in \Delta_+ \cap y(\Delta_+)} |(\alpha|\mu)| + \sum_{\alpha \in \Delta_+ \cap y(\Delta_-)} |1 - (\alpha|\mu)| \tag{13}$$

for $\mu \in \mathring{P}^\vee$, $y \in \mathring{\mathcal{W}}$.

The *twisted length function* [Ark1] $\ell^y : \mathcal{W}^e \to \mathbb{Z}$ with the twist $y \in \mathcal{W}^e$ is defined by

$$\ell^y(w) = \sharp(\Delta_+^{re} \cap w(\Delta_-^{re}) \cap y(\Delta_+^{re})) - \sharp(\Delta_+^{re} \cap w(\Delta_-^{re}) \cap y(\Delta_-^{re})).$$

**Lemma 3.1.** *Let $w, y \in \mathcal{W}^e$.*

(i) *Suppose that $\ell(ys_i) = \ell(y) + 1$ for $i \in I$. Then*

$$\ell^{ys_i}(w) = \begin{cases} \ell^y(w) & \text{if } w^{-1}y(\alpha_i) \in \Delta_+^{re}, \\ \ell^y(w) - 2 & \text{if } w^{-1}y(\alpha_i) \in \Delta_-^{re}. \end{cases}$$

(ii) $\ell^y(w) = \ell(y^{-1}w) - \ell(y^{-1})$.

*Proof.* (i) The assertion follows from the definition and the fact that

$$\Delta_+^{re} \cap ys_i(\Delta_-^{re}) = \Delta_+^{re} \cap y(\Delta_-^{re}) \sqcup \{y(\alpha_i)\} \quad \text{if } \ell(ys_i) = \ell(y) + 1.$$

(ii) We prove by induction on $\ell(y)$. If $\ell(y) = 0$ then $\ell^y(w) = \ell(w) = \ell(y^{-1}w)$. Suppose that $\ell(ys_i) = \ell(y) + 1$. If $w^{-1}y(\alpha_i) \in \Delta_+^{re}$ then $\ell(s_i y^{-1} w) = \ell(y^{-1}w) + 1$. Hence by (i) and induction hypothesis,

$$\ell^{ys_i}(w) = \ell^y(w) = \ell(y^{-1}w) - \ell(y^{-1}) = \ell(s_i y^{-1} w) - \ell(s_i y^{-1}).$$

If $w^{-1}y(\alpha_i) \in \Delta_-^{re}$ then $\ell(s_i y^{-1} w) = \ell(y^{-1}w) - 1$. Again by (i) and induction hypothesis,

$$\ell^{ys_i}(w) = \ell^y(w) - 2 = \ell(y^{-1}w) - 2 - \ell(y^{-1}) = \ell(s_i y^{-1} w) - \ell(s_i y^{-1}).$$

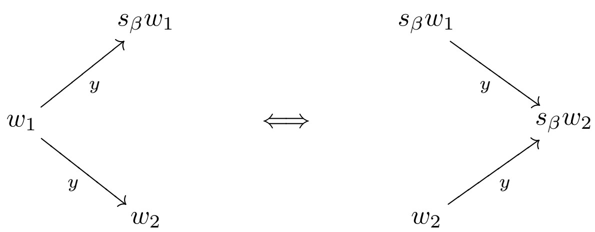
This completes the proof. □

For $w_1, w_2, y \in \mathcal{W}$ and $\gamma \in \Delta^{re}$, write $w_1 \xrightarrow[y]{\gamma} w_2$ if $w_1 = s_\gamma w_2$ and $\ell^y(w_1) > \ell^y(w_2)$. Below, we shall often omit the symbol $\gamma$ above the arrow. Also, we shall omit the symbol $y$ under the arrow if $y = 1$. By Lemma 3.1 (ii) we have

$$w_1 \xrightarrow[y]{y(\gamma)} w_2 \iff y^{-1}w_1 \xrightarrow{\gamma} y^{-1}w_2. \tag{14}$$

In particular for $\beta = y(\alpha_i) \in \Delta_+^{re}$, $\alpha_i \in \Pi$, and $w_1, w_2 \in \mathcal{W}$ such that $\ell^y(w_2) - \ell^y(w_1) = 1$ we have the equivalence

(15)




by [BGG, Lemma 11.3].

Define $w \succeq_y w'$ if there exists a sequence $w_1, w_2, \dots, w_k$ of elements of $\mathcal{W}$ such that

$$w \underset{y}{\longrightarrow} w_1 \underset{y}{\longrightarrow} w_2 \underset{y}{\longrightarrow} \dots \underset{y}{\longrightarrow} w_k \underset{y}{\longrightarrow} w'.$$

Note that

$$w \succeq_y w' \iff y^{-1}w \succeq y^{-1}w', \tag{16}$$

by (14), where $\succeq = \succeq_1$, the usual Bruhat ordering of $\mathcal{W}$. It follows that $\succeq_y$ defines a partial ordering of $\mathcal{W}$.

We will use the symbol $w \rhd_y w'$ to denote a covering in the twisted Bruhat order $\succeq_y$. Thus $w \rhd_y w'$ means that $w \succeq_y w'$ and $\ell^y(w) = \ell^y(w') + 1$.

3.3. **Semi-infinite Bruhat ordering.** Define the *semi-infinite length* [FF2] $\ell^{\frac{\infty}{2}}(w)$ of $w \in \mathcal{W}^e$ by

$$\ell^{\frac{\infty}{2}}(w) = \sharp\{\alpha \in \Delta_+^{re} \cap w(\Delta_-^{re}); \bar{\alpha} \in \mathring{\Delta}_+\} - \sharp\{\alpha \in \Delta_+^{re} \cap w(\Delta_-^{re}); \bar{\alpha} \in \mathring{\Delta}_-\}.$$

We have

$$\ell^{\frac{\infty}{2}}(t_\lambda y) = \ell(y) - 2(\mathring{\rho}|\lambda) \tag{17}$$

for $\lambda \in \mathring{P}^\vee$, $w \in \mathring{\mathcal{W}}$.

Set

$$\mathring{P}_+^\vee = \{\lambda \in \mathring{P}^\vee; \alpha(\lambda) \geq 0 \text{ for all } \alpha \in \mathring{\Delta}_+\},$$

We say that $\lambda \in \mathring{P}_+^\vee$ is sufficiently large if $\alpha(\lambda)$ if sufficiently large for all $\alpha \in \mathring{\Delta}_+$.

By (13) and (17) we have the following assertion.

**Lemma 3.2.** *$\ell^{\frac{\infty}{2}}(w) = \ell^\lambda(w) = -\ell^{-\lambda}(w)$ for a sufficiently large $\lambda \in \mathring{P}_+^\vee$ .*

We write

$$w_1 \underset{\frac{\infty}{2}}{\overset{\gamma}{\longrightarrow}} w_2$$

for $w_1, w_2 \in \mathcal{W}$ and $\gamma \in \Delta^{re}$ if $w_1 = w_2 s_\gamma$ and $\ell^{\frac{\infty}{2}}(w_1) < \ell^{\frac{\infty}{2}}(w_2)$. (We shall often omit the symbol $\gamma$ above the arrow.) Define $w \succeq_{\frac{\infty}{2}} w'$ if there exists a sequence $w_1, w_2, \dots, w_k$ of elements of $\mathcal{W}$ such that

$$w \underset{\frac{\infty}{2}}{\longrightarrow} w_1 \underset{\frac{\infty}{2}}{\longrightarrow} w_2 \underset{\frac{\infty}{2}}{\longrightarrow} \dots \underset{\frac{\infty}{2}}{\longrightarrow} w_k \underset{\frac{\infty}{2}}{\longrightarrow} w'.$$

By Lemma 3.2

$$\begin{aligned} w \succeq_{\frac{\infty}{2}} w' &\iff w' \succeq_{t_\lambda} w \quad \text{for a sufficiently large } \lambda \in \mathring{P}_+^\vee, \\ &\iff w \succeq_{t_{-\lambda}} w' \quad \text{for a sufficiently large } \lambda \in \mathring{P}_+^\vee. \end{aligned}$$

It follows that $\succeq_{\frac{\infty}{2}}$ defines a partial ordering of $\mathcal{W}$. Following Arkhipov [Ark1], we call it the *semi-infinite Bruhat ordering* on $\mathcal{W}$. By [Soe1, Claim 4.14] the semi-infinite Bruhat ordering coincides with the *generic Bruhat ordering* defined by Lusztig [Lus].

We will use the symbol $w \rhd_{\frac{\infty}{2}} w'$ to denote a covering in the twisted Bruhat order $\succeq_{\frac{\infty}{2}}$. Thus $w \rhd_{\frac{\infty}{2}} w'$ means that $w \succeq_{\frac{\infty}{2}} w'$ and $\ell^{\frac{\infty}{2}}(w) = \ell^{\frac{\infty}{2}}(w') - 1$.

3.4. **Semi-infinite analogue of parabolic subgroups and minimal (maximal) length representatives.** Let $S$ be a subset of $\mathring{\Pi}$, $\mathring{\Delta}_S$ the subroot system of $\mathring{\Delta}$ generated by $\alpha_i \in S$, $\mathring{\Delta}_S = \bigsqcup_{i=1}^r \mathring{\Delta}_{S,i}$ the decomposition into the simple subroot systems $\mathring{\Delta}_{1,S}, \dots, \mathring{\Delta}_{r,S}$. Let $\theta_i$ be the longest root of $\mathring{\Delta}_{S,i}$.

Set

$$\Delta_S = \{\alpha + n\delta \in \Delta^{re}; \alpha \in \mathring{\Delta}_S, n \in \mathbb{Z}\}, \quad \mathcal{W}_S = \langle s_\alpha; \alpha \in \Delta_S \rangle \subset \mathcal{W}.$$

Then $\Delta_S$ is a subroot system of $\Delta^{re}$ isomorphic to the affine root system associated with $\mathring{\Delta}_S$. Put $\Delta_{S,+} = \Delta_S \cap \Delta^{re}_+$, the set of positive root of $\Delta_S$. Then $\Pi_S = S \sqcup \{-\theta_1 + \delta, \dots, -\theta_s + \delta\}$ is a set of simple roots of $\Delta_S$. We have $\mathcal{W}_S = \mathring{\mathcal{W}}_S \ltimes t_{\mathring{Q}^\vee{}_S}$, where $\mathring{Q}^\vee_S = \sum_{\alpha \in \mathring{\Delta}_S} \mathbb{Z}\alpha^\vee$. By (17), the restriction of the semi-infinite length function to $\mathcal{W}_S$ coincides with the semi-infinite length function of the affine Weyl group $\mathcal{W}_S$.

Define

$$\mathcal{W}^S = \{w \in \mathcal{W}; w^{-1}(\Delta_{S,+}) \subset \Delta^{re}_+\}.$$

**Theorem 3.3** ([Pet])**.** *The multiplication map $\mathcal{W}_S \times \mathcal{W}^S \to \mathcal{W}$, $(u, v) \mapsto uv$, is a bijection. Moreover, we have*

$$\ell^{\frac{\infty}{2}}(uv) = \ell^{\frac{\infty}{2}}(u) + \ell^{\frac{\infty}{2}}(v) \quad \textit{for } u \in \mathcal{W}_S,\ v \in \mathcal{W}^S.$$

*Proof.* First, we show the injectivity of the multiplication map. Suppose that $u_1 v_1 = u_2 v_2$ with $u_i \in \mathcal{W}_S$, $v_i \in \mathcal{W}^S$. Then $v_1 = u v_2$ with $u = u_1^{-1} u_2 \in \mathcal{W}_S$. If $u \neq 1$ then there exists $\alpha \in \Delta_{S,+}$ such that $u^{-1}(\alpha) \in -\Delta_{S,+}$. But then $v_2 \in \mathcal{W}^S$ implies that $v_1^{-1}(\alpha) = v_2^{-1} u^{-1}(\alpha) \in \Delta^{re}_-$, and this contradicts that $v_1 \in \mathcal{W}^S$. Hence $u_1 = u_2$, and so $v_1 = v_2$.

Second, we show that the multiplication map $\mathcal{W}_S \times \mathcal{W}^S \to \mathcal{W}$ is surjective. We will prove by induction on $\sharp(w^{-1}(\Delta_{S,+}) \cap \Delta^{re}_-)$ that there exists $u \in \mathcal{W}_S$ such that $u^{-1} w \in \mathcal{W}^S$. If $\sharp(w^{-1}(\Delta_{S,+}) \cap \Delta^{re}_-) = 0$, $w \in \mathcal{W}^S$ there is nothing to show. Suppose that $\sharp(w^{-1}(\Delta_{S,+}) \cap \Delta^{re}_-) > 0$. Then there exists $\beta \in \Pi_S$ such that $w^{-1}(\beta) \in \Delta^{re}_-$. Indeed, any element $\alpha \in \Delta_{S,+}$ is expressed as $\alpha = \sum_{\beta \in \Pi_S} n_\beta \beta$ with $n_\beta \in \mathbb{Z}_{\geq 0}$. Thus $w^{-1}(\alpha) = \sum_{\beta \in \Pi_S} n_\beta w^{-1}(\beta) \in \Delta^{re}_-$ implies that one of $w^{-1}(\beta)$ must belong to $\Delta^{re}_-$. Now because $(s_\beta w)^{-1}(\Delta_{S,+}) = w^{-1} s_\beta(\Delta_{S,+}) = w^{-1}(\Delta_{S,+} \backslash \{\beta\} \sqcup \{-\beta\}) = w^{-1}(\Delta_{S,+}) \backslash \{w^{-1}(\beta)\} \sqcup \{-w^{-1}(\beta)\}$,

$$(s_\beta w)^{-1}(\Delta_{S,+}) \cap \Delta^{re}_- = w^{-1}(\Delta_{S,+}) \cap \Delta^{re}_- \backslash \{w^{-1}(\beta)\}.$$

Hence by applying the induction hypothesis to $s_\beta w$ we find an element $u \in \mathcal{W}_S$ such that $u^{-1} s_\beta w \in \mathcal{W}^S$.

Finally, we prove the equality of the semi-infinite length. By (17), we have $\ell^{\frac{\infty}{2}}(t_\mu w) = \ell^{\frac{\infty}{2}}(t_\mu) + \ell^{\frac{\infty}{2}}(w)$ for any $\mu \in \mathring{Q}^\vee$. Hence we may assume that $u \in \mathring{\mathcal{W}}_S$. We will prove by induction on the length $\ell(u)$ of $u \in \mathring{\mathcal{W}}_S$ that $\ell^{\frac{\infty}{2}}(uv) = \ell^{\frac{\infty}{2}}(u) + \ell^{\frac{\infty}{2}}(v)$ for any $v \in \mathcal{W}^S$. Suppose that $\ell(u) = 1$, so that $u = s_i$ for some $\alpha_i \in S$. Let

$v = t_\mu y \in \mathcal{W}^S$ with $\mu \in \mathring{Q}^\vee$, $y \in \mathring{\mathcal{W}}$. Note that $v \in \mathcal{W}^S$ is equivalent to that

$$\text{if } \alpha \in \mathring{\Delta}_{S,+} \text{ then} \quad \alpha(\mu) = \begin{cases} 0 & \text{if } y^{-1}(\alpha) \in \mathring{\Delta}_+, \\ 1 & \text{if } y^{-1}(\alpha) \in \mathring{\Delta}_-. \end{cases} \tag{18}$$

Since

$$\ell^{\frac{\infty}{2}}(s_i t_\mu y) = \ell(t_{s_i(\mu)} s_i y) = \ell(s_i y) - 2(\rho|\mu - \alpha_i(\mu)\alpha_i^\vee) = \ell(s_i y) - 2(\rho|\mu) + 2\alpha_i(\mu),$$

(18) implies that $\ell^{\frac{\infty}{2}}(s_i v) = \ell^{\frac{\infty}{2}}(v) + 1$. Next let $u = s_i u_1 \in \mathring{\mathcal{W}}_S$ with $u_1 \in \mathring{\mathcal{W}}_S$, $\alpha_i \in S$, $\ell(u) = \ell(u_1) + 1$, so that $u_1^{-1}(\alpha_i) \in \mathring{\Delta}_+$. Let $v = t_\mu y \in \mathcal{W}^S$ as above. We have

$$\ell^{\frac{\infty}{2}}(uv) = \ell^{\frac{\infty}{2}}(t_{s_i u_1(\mu)} s_i u_1 y) = \ell(s_i u_1 y) - 2(\rho|s_i u_1(\mu)).$$

If $\ell(s_i u_1 y) = \ell(u_1 y) + 1$, then $\mathring{\Delta}_+ \ni (u_1 y)^{-1}(\alpha_i) = y^{-1}(u_1^{-1}(\alpha_i))$. Hence $(\mu|u_1^{-1}(\alpha_i)) = 0$ by (18), which means $s_i u_1(\mu) = u_1(\mu)$. By the induction hypothesis, this proves that $\ell^{\frac{\infty}{2}}(uv) = \ell^{\frac{\infty}{2}}(u) + \ell^{\frac{\infty}{2}}(v)$. If $\ell(s_i u_1 y) = \ell(u_1 y) - 1$, then $\mathring{\Delta}_- \ni (u_1 y)^{-1}(\alpha_i) = y^{-1}(u_1^{-1}(\alpha_i))$. So (18) gives $(\mu|u_1^{-1}(\alpha_i)) = 1$, which means $s_i u_1(\mu) = u_1(\mu) - \alpha_i^\vee$. By the induction hypothesis, this proves that $\ell^{\frac{\infty}{2}}(uv) = \ell^{\frac{\infty}{2}}(u) + \ell^{\frac{\infty}{2}}(v)$ as required. □

## 4. Wakimoto modules and twisted Verma modules

4.1. **The category $\mathcal{O}$ of $\mathfrak{g}$.** For any $\mathfrak{h}$-module $M$ we set $M_\mu = \{m \in M; hm = \mu(h)m \text{ for all } h \in \mathfrak{h}\}$.

Let $\mathcal{O}^{\mathfrak{g}}$ be the full subcategory of $\tilde{\mathcal{O}}^{\mathfrak{g}}$ consisting of modules on which $\mathfrak{h}$ acts semisimply. The formal character of $M \in \mathcal{O}^{\mathfrak{g}}$ is defined by

$$\operatorname{ch} M = \sum_{\mu \in \mathfrak{h}^*} (\dim_{\mathbb{C}} M_\mu) e^\mu.$$

Let $\mathcal{O}_k^{\mathfrak{g}}$ be the full subcategory of $\mathcal{O}^{\mathfrak{g}}$ consisting of objects of level $k$, where a $\mathfrak{g}$-module $M$ is said to be of level $k$ if $K$ acts as the multiplication by $k$.

4.2. **Twisting functors and twisted Verma modules.** By abuse of notation we denote also by $w$ a Tits lifting of $w \in \mathcal{W}^e$ to $\operatorname{Aut}(\mathfrak{g})$.

For each $w \in \mathcal{W}$ the twisting functor $T_w : \mathcal{O}^{\mathfrak{g}} \to \mathcal{O}^{\mathfrak{g}}$ is defined as follows ([Ark1]): Let $\mathfrak{n}_w = \mathfrak{n}_- \cap w^{-1}(\mathfrak{n}_+)$ and set $N_w = U(\mathfrak{n}_w)$. Put

$$S_w = U \otimes_{N_w} N_w^*.$$

The space $S_w$ has a $U$-bimodule structure, which is described as follows: Let $f \in \mathfrak{n}_- \backslash \{0\}$, and set $U_{(f)} = U \otimes_{\mathbb{C}[f]} \mathbb{C}[f, f^{-1}]$. Then $U_{(f)}$ is an associative algebra which contains $U$ as a subalgebra. We set $S_f = U_{(f)}/U$. Choose a filtration $\mathfrak{n}_w = F^0 \supset F^1 \supset \cdots \supset F^r \supset 0$, $r = \ell(w)$, consisting of ideals $F^p \subset \mathfrak{n}_w$ of codimension $p$. If $f_p \in F^{p-1} \backslash F^p$ we have an isomorphism of $U$-bimodules

$$S_w = S_{f_1} \otimes_U S_{f_2} \otimes_U \ldots \otimes_U S_{f_r}. \tag{19}$$

We have

$$S_w \cong N_w^* \otimes_{N_w} U \tag{20}$$

as right $U$-modules and left $N_w$-modules. Put

$$\mathbf{1}_w^* = f_1^{-1} \otimes f_2^{-1} \otimes \ldots \otimes f_r^{-1} \in S_w.$$

For $M \in \mathcal{O}^{\mathfrak{g}}$ define

$$T_w(M) = \phi_w(S_w \otimes_{U(\mathfrak{g})} M),$$

where $\phi_w$ means that the action of $\mathfrak{g}$ is twisted by the automorphism $w$ of $\mathfrak{g}$. This define a right exact functor $T_w : \mathcal{O}^{\mathfrak{g}} \to \mathcal{O}^{\mathfrak{g}}$ such that

$$T_{ws_i} \cong T_w T_i \quad \text{if } \alpha_i \in \Pi \text{ and } \ell(ws_i) = \ell(w) + 1, \tag{21}$$

where $T_i = T_{s_i}$.

The functor $T_w$ admits a right adjoint functor $G_w$ in the category $\mathcal{O}^{\mathfrak{g}}$ ([AS, §4]):

$$G_w(M) = \mathcal{H}om_U(S_w, \phi_w^{-1}(M)).$$

It is straightforward to extend the definition of $T_w$ and $G_w$ to $w \in \mathcal{W}^e$ ([A1]).

The following assertion follows in the same manner as [Soe2, Theorem 2.1].

**Lemma 4.1.** *Let $M \in \mathcal{O}^{\mathfrak{g}}$, $w \in \mathcal{W}^e$*

(i) *Suppose that $M$ is free over $\mathfrak{n}_w$. Then $M \cong G_w T_w(M)$.*
(ii) *Suppose that $M$ is cofree over $w(\mathfrak{n}_w)$. Then $M \cong T_w G_w(M)$.*

For $\lambda \in \mathfrak{h}^*$, let $M(\lambda)$ be the Verma module of $\mathfrak{g}$ with highest weight $\lambda$. Set

$$M^w(\lambda) = T_w M(w^{-1} \circ \lambda).$$

The $\mathfrak{g}$-module $M^w(\lambda) \in \mathcal{O}^{\mathfrak{g}}$ is called the *twisted Verma module* $M^w(\lambda)$ with highest weight $\lambda$ and twist $w \in \mathcal{W}^e$. Note that by (20) we have

$$M^w(\lambda)_\mu \cong \phi_w(N_w^* \otimes_{N_w} U(\mathfrak{n}_-))_{\mu-\lambda} \cong (U(w(\mathfrak{n}_-) \cap \mathfrak{n}_+)^* \otimes_{\mathbb{C}} U(w(\mathfrak{n}_-) \cap \mathfrak{n}_-))_{\mu-\lambda} \tag{22}$$

as $\mathfrak{h}$-modules. Hence

$$\operatorname{ch} M^w(\lambda) = \operatorname{ch} M(\lambda).$$

In particular $M^w(\lambda)$ is an object of $\mathcal{O}^{\mathfrak{g}}$.

By Lemma 4.1 (1) we have

$$M(\mu) \cong G_w M^w(w \circ \mu).$$

Hence the functor $T_w$ gives the isomorphism

$$\operatorname{Hom}_{\mathfrak{g}}(M(\lambda), M(\mu)) \xrightarrow{\sim} \operatorname{Hom}_{\mathfrak{g}}(M^w(w \circ \lambda), M^w(w \circ \mu)) \tag{23}$$

for $\lambda, \mu \in \mathfrak{h}^*$.

We have [AL, Proposition 6.3]

$$M^w(\lambda) \cong M(\lambda) \quad \text{if } \langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{N} \quad \text{for all } \alpha \in \Delta_+^{re} \cap w(\Delta_-^{re}). \tag{24}$$

4.3. **Hom spaces between twisted Verma modules.** For $\lambda \in \mathfrak{h}^*$ let $\Delta(\lambda)$ and $\mathcal{W}(\lambda)$ be its *integral root system* and *integral Weyl group*, respectively:

$$\Delta(\lambda) = \{\alpha \in \Delta^{re}; \langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}\},$$
$$\mathcal{W}(\lambda) = \langle s_\alpha; \alpha \in \Delta(\lambda) \rangle \subset \mathcal{W}.$$

Let $\Delta(\lambda)_+ = \Delta(\lambda) \cap \Delta_+^{re}$ the set of positive roots of $\Delta(\lambda)$, $\Pi(\lambda) \subset \Delta(\lambda)_+$ the set of simple roots of $\Delta(\lambda)$, $\ell : \mathcal{W}(\lambda) \to \mathbb{Z}_{\geq 0}$ the length function.

For $y \in \mathcal{W}(\lambda)$ the twisted length function $\ell^y$ and the twisted Bruhat ordering $\succeq_{\lambda,y}$ are defined for $\mathcal{W}(\lambda)$. We will use the symbol $w \rhd_{\lambda,y} w'$ to denote a covering in the twisted Bruhat order $\succeq_{\lambda,y}$.

Recall that a weight $\lambda \in \mathfrak{h}^*$ is called *regular dominant* if $\langle \lambda+\rho, \alpha^\vee \rangle \notin \{0, -1, -2, \dots\}$ for all $\alpha \in \Delta_+^{re}$. It is called *regular anti-dominant* if $\langle \lambda + \rho, \alpha^\vee \rangle \notin \{0, 1, 2, \dots\}$ for all $\alpha \in \Delta_+^{re}$.

**Theorem 4.2.** *Let $w, w', y \in \mathcal{W}(\lambda)$.*

(i) *If $\lambda$ is regular dominant then*

$$\dim_{\mathbb{C}} \operatorname{Hom}_{\mathfrak{g}}(M^y(w \circ \lambda), M^y(w' \circ \lambda)) = \begin{cases} 1 & \text{if } w \succeq_{\lambda,y} w', \\ 0 & \text{otherwise.} \end{cases}$$

(ii) *If $\lambda$ is regular anti-dominant then*

$$\dim_{\mathbb{C}} \operatorname{Hom}_{\mathfrak{g}}(M^y(w \circ \lambda), M^y(w' \circ \lambda)) = \begin{cases} 1 & \text{if } w \preceq_{\lambda,y} w', \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* (i) By (23) the assertion follows from (16) and [KT, Proposition 2.5.5 (ii)]. Proof of (ii) is similar. □

4.4. **Wakimoto modules.** Let $\mathfrak{g}$, $\mathfrak{h}$ be as in §3.1, and let us consider the $\mathbb{Z}$-grading of $\mathfrak{g}$ with $\mathfrak{g}_0 = \mathfrak{h}$, $\mathfrak{g}_1 = \bigoplus_{\alpha \in \Pi} \mathfrak{g}_\alpha$, where $\mathfrak{g}_\alpha$ is the root space of $\mathfrak{g}$ of root $\alpha$. Let $\rho = \mathring{\rho} + h^\vee \Lambda_0 \in \mathfrak{h}^*$, where $h^\vee$ is the dual Coxeter number of $\mathring{\mathfrak{g}}$. Then $\langle \rho, \alpha^\vee \rangle = 1$ for all $\alpha \in \Pi$ and $2\rho$ define a semi-infinite 1-cochain of $\mathfrak{g}$ [Ark2].

Let $L\mathring{\mathfrak{n}}$, $L\mathring{\mathfrak{n}}_-$, $\mathfrak{a}$ and $\bar{\mathfrak{a}}$ be graded subalgebras of $\mathfrak{g}$ defined by

$$L\mathring{\mathfrak{n}} = \mathring{\mathfrak{n}}[t, t^{-1}], \quad L\mathring{\mathfrak{n}}_- = \mathring{\mathfrak{n}}_-[t, t^{-1}],$$
$$\mathfrak{a} = L\mathring{\mathfrak{n}} \oplus \mathring{\mathfrak{h}}[t^{-1}]t^{-1}, \quad \bar{\mathfrak{a}} = L\mathring{\mathfrak{n}}_- \oplus \mathring{\mathfrak{h}}[t] \oplus \mathbb{C}K \oplus \mathbb{C}D.$$

Then $0 = 2\rho|_{L\mathring{\mathfrak{n}}} = 2\rho|_{L\mathring{\mathfrak{n}}_-} = 2\rho|_{\mathfrak{a}}$ gives semi-infinite 1-cochains of $L\mathring{\mathfrak{n}}$, $L\mathring{\mathfrak{n}}_-$, $\mathfrak{a}$, and $2\rho|_{\bar{\mathfrak{a}}}$ gives a semi-infinite 1-cochain of $\bar{\mathfrak{a}}$.

Following [Vor2] we define the *Wakimoto module* $W(\lambda)$ of $\mathfrak{g}$ with highest weight $\lambda \in \mathfrak{h}^*$ by

$$W(\lambda) = \text{S-ind}_{\bar{\mathfrak{a}}}^{\mathfrak{g}} \mathbb{C}_\lambda,$$

where $\mathbb{C}_\lambda$ is the one-dimensional representation of $\mathfrak{h}$ corresponding to $\lambda$ regarded as a $\bar{\mathfrak{a}}$-module by the natural projection $\bar{\mathfrak{a}} \twoheadrightarrow \mathfrak{h}$. By Lemma 2.5 we have

$$W(\lambda) \cong US(\mathfrak{a}) \text{ as } \mathfrak{a}\text{-modules,} \tag{25}$$

and hence

$$H^{\frac{\infty}{2}+i}(\mathfrak{a}, W(\lambda)) \cong \begin{cases} \mathbb{C}_\lambda & \text{if } i=0, \\ 0 & \text{otherwise} \end{cases} \quad \text{as } \mathfrak{h}\text{-modules,} \tag{26}$$

$$\operatorname{ch} W(\lambda) = \operatorname{ch} M(\lambda). \tag{27}$$

In particular $W(\lambda)$ is an object of $\mathcal{O}^{\mathfrak{g}}$.

Theorem 4.7 below shows that the above definition of Wakimoto module coincides with that of Feigin and Frenkel [FF2, Fre2].

4.5. **Wakimoto modules as inductive limits of twisted Verma modules.** Let $y, w, u \in \mathcal{W}$ such that $w = yu$ and $\ell(w) = \ell(y) + \ell(u)$. Then $T_w = T_y T_u$ and $S_w \cong S_y \otimes_U \phi_y(S_u)$. Let

$$j_{w,y} : S_y \longrightarrow S_w$$

be the homomorphism of left $U$-modules which maps $s \in S_y$ to $s \otimes \mathbf{1}_u^* \in S_y \otimes_U \phi_y(S_u) = S_w$. Define $\nu_{w,y}^\lambda \in \operatorname{Hom}_{\mathfrak{g}}(M^y(\lambda), M^w(\lambda))$ by

$$\nu_{w,y}^\lambda(s \otimes v_{y^{-1}\circ\lambda}) = j_{w,y}(s) \otimes v_{w^{-1}\circ\lambda} \quad \text{for } s \in S_y,$$

where $v_\mu$ denotes the highest weight vector of $M(\mu)$ for $\mu \in \mathfrak{h}^*$. Then

$$\operatorname{Hom}_{\mathfrak{g}}(M^y(\lambda), M^w(\lambda)) = \mathbb{C}\nu_{w,y}^\lambda$$

by (23). We have

$$\nu_{w_3,w_2}^\lambda \circ \nu_{w_2,w_1}^\lambda = \nu_{w_3,w_1}^\lambda \tag{28}$$

if $w_3 = w_2 u_2$, $w_2 = w_1 u_1$ with $\ell(w_1) = \ell(w_2) + \ell(u_2)$, $\ell(w_2) = \ell(w_1) + \ell(u_1)$.

Let $\{\gamma_1, \gamma_2, \dots\}$ be a sequence in $\mathring{P}_+^\vee$ such that $\gamma_i - \gamma_{i-1} \in \mathring{P}_+^\vee$ and $\lim_{n\to\infty} \alpha(\gamma_n) = \infty$ for all $\alpha \in \mathring{\Delta}_+$. Then $t_{-\gamma_{i+1}} = t_{-\gamma_i} t_{-(\gamma_{i+1}-\gamma_i)}$ with $\ell(t_{-\gamma_{i+1}}) = \ell(t_{-\gamma_i}) + \ell(t_{-(\gamma_{i+1}-\gamma_i)})$ for all $i$. It follows that $\{M^{-\gamma_n}(\lambda) : \nu_{-\gamma_m,-\gamma_n}^\lambda\}$ forms an inductive system of $\mathfrak{g}$-modules.

**Proposition 4.3** ([Ark1, Lemma 6.1.7])**.** *There is an isomorphism of* $\mathfrak{g}$*-modules*

$$W(\lambda) \cong \varinjlim_n M^{-\gamma_n}(\lambda).$$

*Proof.* For the reader's convenience we shall give a proof of Proposition 4.3 here. Set $W(\lambda)' = \varinjlim_n M^{-\gamma_n}(\lambda)$. First note that

$$t_{-\gamma_i}(\mathfrak{n}_{-\gamma_i}) = t_{-\gamma_i}(\mathfrak{n}_-) \cap \mathfrak{n}_+ = \operatorname{span}_{\mathbb{C}}\{x_\alpha t^n; \alpha \in \Delta_+,\ 0 \le n < \alpha(\gamma_i)\},$$

$$t_{-\gamma_i}(\mathfrak{n}_-) \cap \mathfrak{n}_- = (\mathring{\mathfrak{h}} \oplus \mathring{\mathfrak{n}})[t^{-1}]t^{-1} \oplus \operatorname{span}_{\mathbb{C}}\{x_{-\alpha} t^{-n}; \alpha \in \Delta_+,\ n > \alpha(\gamma_i)\},$$

where $x_\alpha$ is a root vector of $\mathring{\mathfrak{g}}$ of root $\alpha$. Thus we have $t_{-\gamma_1}(\mathfrak{n}_{-\gamma_1}) \subset t_{-\gamma_2}(\mathfrak{n}_{-\gamma_2}) \subset \dots \subset \mathfrak{a}_+$ and $\mathfrak{a}_+ = \bigcup_{i \ge 1} t_{-\gamma_i}(\mathfrak{n}_{-\gamma_i})$. The map $j_{-\gamma_i,-\gamma_j} : S_{-\gamma_i} \to S_{-\gamma_j}$ restricts to the embedding $j_{-\gamma_i,-\gamma_j} : N_{-\gamma_i}^* \hookrightarrow N_{-\gamma_j}^*$ for $i < j$, and we have

$$U(\mathfrak{a}_+)^* \cong \varinjlim_i \phi_{-\gamma_i}(N_{-\gamma_i}^*)$$

as left $\mathfrak{a}_+$-modules. Let $j_{-\gamma_i} : \phi_{-\gamma_i}(N_{-\gamma_i}^*) \hookrightarrow U(\mathfrak{a}_+)^*$ be the embedding of left $\phi_{-\gamma_i}(N_{-\gamma_i})$-modules under the above identification.

Since $t_{-\gamma_i}(\mathfrak{n}_{-\gamma_i}) = \mathrm{span}_{\mathbb{C}}\{x_\alpha t^{-n}; \alpha \in \Delta_+,\ 0 < n \leq \alpha(\gamma_i)\} \subset \mathfrak{a}$,

$$W(\lambda) \cong T_{-\gamma_i} G_{-\gamma_i}(W(\lambda))$$

by Lemma 4.1 (ii). Hence

$$\mathrm{Hom}_{\mathfrak{g}}(M^{-\gamma_i}(\lambda), W(\lambda)) \cong \mathrm{Hom}_{\mathfrak{g}}(M(t_{\gamma_i} \circ \lambda), G_{-\gamma_i}(W(\lambda))).$$

As $\mathrm{ch}\, G_{-\gamma_i}(W(\lambda)) = \mathrm{ch}\, M(t_{\gamma_i} \circ \lambda)$, there exists a unique $\mathfrak{g}$-module homomorphism $\psi_i : M(t_{\gamma_i} \circ \lambda) \to G_{-\gamma_i}(M)$ which sends $v_{t_{\gamma_i} \circ \lambda}$ to $w_i$, a vector of $G_{-\gamma_i}(W(\lambda))$ of weight $t_{\gamma_i} \circ \lambda$. Up to a non-zero constant multiplication, $w_i$ equals to the the element of $G_{-\gamma_i}(W(\lambda)) = \mathcal{H}om_{N_{-\gamma_i}}(N^*_{-\gamma_i}, \phi^{-1}_{-\gamma_i}(W(\lambda)))$ which sends $f \in N^*_{t_{-\gamma_i}}$ to $j_{-\gamma_i}(f) \otimes 1_\lambda \in US(\mathfrak{a}) \otimes \mathbb{C}_\lambda = W(\lambda)$. The corresponding homomorphism $T_{-\gamma_i}(\psi_i) : M^{-\gamma_i}(\lambda) \to W(\lambda)$ is given by

$$T_{-\gamma_i}(\psi_i)(f \otimes v_{t_{\gamma_i} \circ \lambda}) = j_{-\gamma_i}(f) \otimes 1_\lambda \quad \text{for } f \in N^*_{-\gamma_i}\ . \tag{29}$$

It follows that $T_{-\gamma_i}(\psi_j) \circ \nu^\lambda_{\gamma_j,\gamma_i} = T_{-\gamma_i}(\psi_i)$ for $i < j$ , and the sequence $\{T_{-\gamma_i}(\psi_j)\}$ yields a $\mathfrak{g}$-module homomorphism

$$\Phi : W(\lambda)' = \varinjlim_i M^{-\gamma_i}(\lambda) \longrightarrow W(\lambda).$$

Fix $\mu \in \mathfrak{h}^*$. Since $W(\lambda) \cong US(\mathfrak{a})$ as an $\mathfrak{a}$-module, it follows from (22) that $T_{-\gamma_i}$ restricts to the isomorphism $M^{-\gamma_i}(\lambda)_\mu \xrightarrow{\sim} W(\lambda)_\mu$ for a sufficiently large $i$. This completes the proof. □

4.6. **Endmorphisms of Wakimoto modules.**

**Proposition 4.4.** *Let* $\alpha \in \overset{\circ}{P}{}^\vee_+$, $\lambda \in \mathfrak{h}^*$.

(i) $T_{-\alpha} W(\lambda) \cong W(t_{-\alpha} \circ \lambda)$.
(ii) $G_{-\alpha} W(\lambda) \cong W(t_\alpha \circ \lambda)$.

*Proof.* (i) Let $\{\gamma_1, \gamma_2, \dots\}$ be a sequence in $\overset{\circ}{P}{}^\vee_+$ such that $\gamma_i - \gamma_{i-1} \in \overset{\circ}{P}{}^\vee_+$ and $\lim_{n\to\infty} \beta(\gamma_n) = \infty$ for all $\beta \in \overset{\circ}{\Delta}_+$. Set $\gamma'_i = \gamma_i + \alpha$. Then the sequence $\{\gamma'_1, \gamma'_2, \dots\}$ satisfies the same property. Hence by Proposition 4.3 and the fact that a homology functor commutes with inductive limits we have $T_{-\alpha} W(\lambda) \cong T_{-\alpha}(\varinjlim M^{-\gamma_i}(\lambda)) = \varinjlim T_{-\alpha} M^{-\gamma_i}(\lambda) = \varinjlim T_{-\alpha} T_{-\gamma_i} M(t_{\gamma_i} \circ \lambda) = \varinjlim T_{-\gamma'_i} M(t_{\gamma_i} \circ \lambda) = \varinjlim M^{-\gamma'_i}(t_\alpha \circ \lambda) \cong W(t_\alpha \circ \lambda)$. (ii) Since $\mathfrak{n}_{t_{-\alpha}} \subset \mathfrak{a}_-$, $W(\lambda)$ is free over $\mathfrak{n}_{t_{-\alpha}}$. Hence $W(t_\alpha \circ \lambda) = G_{-\alpha} T_{-\alpha} W(t_\alpha \circ \lambda) \cong G_{-\alpha} W(\lambda)$ by Lemma 4.1 and (i). □

**Corollary 4.5.** *Let* $\alpha \in \overset{\circ}{P}{}^\vee_+$. *The functor* $G_{-\alpha}$ *gives the isomorphism*

$$\mathrm{Hom}_{\mathfrak{g}}(W(\lambda), W(\mu)) \cong \mathrm{Hom}_{\mathfrak{g}}(W(t_\alpha \circ \lambda), W(t_\alpha \circ \mu)).$$

*for* $\lambda, \mu \in \mathfrak{h}^*$.

**Proposition 4.6.** *For* $\lambda \in \mathfrak{h}^*$ *we have* $\mathrm{End}_{\mathfrak{g}}(W(\lambda)) = \mathbb{C}$.

*Proof.* Let $\{\gamma_1, \gamma_2, \dots, \}$ be in Subsection 4.5. Then

$$\begin{aligned}
\mathrm{End}_{\mathfrak{g}}(W(\lambda)) &= \mathrm{Hom}_{\mathfrak{g}}(\varinjlim_i M^{-\gamma_i}(\lambda), W(\lambda)) \quad \text{(by Proposition 4.3)} \\
&= \varprojlim_i \mathrm{Hom}_{\mathfrak{g}}(M^{-\gamma_i}(\lambda), W(\lambda)) \cong \varprojlim_i \mathrm{Hom}_{\mathfrak{g}}(M(t_{\gamma_i} \circ \lambda), G_{-\gamma_i} W(\lambda)) \\
&\cong \varprojlim_i \mathrm{Hom}_{\mathfrak{g}}(M(t_{\gamma_i} \circ \lambda), W(t_{\gamma_i} \circ \lambda)) \quad \text{(by Proposition 4.4)}.
\end{aligned}$$

As we have seen in the proof of Proposition 4.3, the space $\mathrm{Hom}_{\mathfrak{g}}(M(t_{\gamma_i}\circ\lambda), W(t_{\gamma_i}\circ\lambda))$ is one-dimensional and $\nu^{\lambda}_{-\gamma_m,\gamma_n}$ induces the isomorphism

$$\mathrm{Hom}_{\mathfrak{g}}(M^{-\gamma_m}(\lambda), W(\lambda)) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{g}}(M^{-\gamma_n}(\lambda), W(\lambda)).$$

This completes the proof. □

4.7. **Uniqueness of Wakimoto modules.** A finite filtration $0 = M_0 \subset M_1 \subset M_2 \subset M_r = M$ of a $\mathfrak{g}$-module $M$ is called a *Wakimoto flag* if each successive quotient $M_i/M_{i-1}$ is isomorphic to $W(\lambda_i)$ for some $\lambda_i$.

**Theorem 4.7.** *Suppose that $k$ is non-critical, that is, $k \neq -h^{\vee}$. For an object $M$ of $\mathcal{O}_k^{\mathfrak{g}}$ the following conditions are equivalent.*

(i) *$M$ admits a Wakimoto flag.*
(ii) *$H^{\frac{\infty}{2}+i}(\mathfrak{a}, M) = 0$ for $i \neq 0$ and $H^{\frac{\infty}{2}+0}(\mathfrak{a}, M)$ is finite-dimensional.*

*If this is the case the multiplicity $(M : W(\lambda))$ of $W(\lambda)$ in a Wakimoto flag of $M$ equals to $\dim H^{\frac{\infty}{2}+0}(\mathfrak{a}, M)_{\lambda}$. In particular if*

$$H^{\frac{\infty}{2}+i}(\mathfrak{a}, M) \cong \begin{cases} \mathbb{C}_{\lambda} & \text{if } i = 0,\\ 0 & \text{otherwise}\end{cases}$$

*as $\mathfrak{h}$-modules, $M$ is isomorphic to $W(\lambda)$.*

The proof of Theorem 4.7 will be given in Subsection 4.8.

We put on record some of consequences of Theorem 4.7:

**Proposition 4.8.** *A tilting module in $\mathcal{O}^{\mathfrak{g}}$ at a non-critical level admits a Wakimoto flag.*

*Proof.* By definition a tilting module $M$ admits both a Verma flag and a dual Verma flag. It follows that $M$ is free over $\mathfrak{n}_-$ and cofree over $\mathfrak{n}_+$. In particular $M$ is free over $\mathring{\mathfrak{n}}[t^{-1}]t^{-1}$ and cofree over $\mathring{\mathfrak{n}}[t]$. Hence by [Vor1, Theorem 2.1], we have $H^{\frac{\infty}{2}+i}(\mathfrak{a}, M) = 0$ for $i \neq 0$. The fact that $H^{\frac{\infty}{2}+0}(\mathfrak{a}, M)$ is finite-dimensional follows from the Euler-Poincaré principle. □

**Proposition 4.9.** *Suppose that $\langle \lambda+\rho, K\rangle \notin \mathbb{Q}_{\geq 0}$. Then $W(t_{\alpha}\circ\lambda) \cong M(t_{\alpha}\circ\lambda)$ for a sufficiently large $\alpha \in \mathring{P}^{\vee}_{+}$.*

*Proof.* Let $\alpha$ be sufficiently large. By the hypothesis $\langle t_{\alpha}(\lambda+\rho), \beta^{\vee}\rangle \notin \mathbb{N}$ for all $\beta \in \Delta^{re}_{+}$ such that $\bar{\beta} \in \mathring{\Delta}_{+}$. It follows from [A1, Theorem 3.1] that $M(t_{\alpha}\circ\lambda)$ is cofree over $\mathring{\mathfrak{n}}[t] = \mathfrak{a}_+$. Because $M(t_{\alpha}\circ\lambda)$ is obviously free over $\mathfrak{a}_-$ we have $H^{\frac{\infty}{2}+i}(\mathfrak{a}, M(t_{\alpha}\circ\lambda)) \cong \begin{cases} \mathbb{C}_{t_{\alpha}\circ\lambda} & \text{for } i = 0,\\ 0 & \text{otherwise.}\end{cases}$ □

The following assertion follows from Proposition 4.9 and Corollary 4.5.

**Proposition 4.10.** *Let $\lambda, \mu \in \mathfrak{h}^*$ be of level $k$, and suppose that $k + h^{\vee} \notin \mathbb{Q}_{\geq 0}$. Then*

$$\mathrm{Hom}_{\mathfrak{g}}(W(\lambda), W(\mu)) \cong \mathrm{Hom}_{\mathfrak{g}}(M(t_{\alpha}\circ\lambda), M(t_{\alpha}\circ\mu))$$

*for a sufficiently large* $\alpha \in \overset{\circ}{P}{}^{\vee}_{+}$. *In particular if* $\lambda \in \mathfrak{h}^*$ *is integral, regular anti-dominant, then*

$$\dim_{\mathbb{C}} \operatorname{Hom}_{\mathfrak{g}}(W(w \circ \lambda), W(y \circ \lambda)) = \begin{cases} 1 & \text{if } w \preceq_{\frac{\infty}{2}} y \\ 0 & \text{else} \end{cases}$$

*for* $w, y \in \mathcal{W}$.

*Conjecture* 4.11. Let $\lambda \in \mathfrak{h}^*$ be integral, regular dominant. Then

$$\dim_{\mathbb{C}} \operatorname{Hom}_{\mathfrak{g}}(W(w \circ \lambda), W(y \circ \lambda)) = \begin{cases} 1 & \text{if } w \succeq_{\frac{\infty}{2}} y \\ 0 & \text{else} \end{cases}$$

for $w, y \in \mathcal{W}$.

In Theorem 6.11 below we prove Conjecture 4.11 in the case that $w \rhd_{\frac{\infty}{2}} y$ (in a slightly more general setting).

4.8. **Proof of Theorem 4.7.** Let

$$\mathcal{H} = \overset{\circ}{\mathfrak{h}}[t, t^{-1}] \oplus \mathbb{C}K \subset \mathfrak{g},$$

the Heisenberg subalgebra. Denote by $\pi_\lambda$ the irreducible representation of $\mathcal{H}$ with highest weight $\lambda$. We have $\pi_\lambda \cong U(\overset{\circ}{\mathfrak{h}}[t^{-1}]t^{-1})$ as a module over $\overset{\circ}{\mathfrak{h}}[t^{-1}]t^{-1} \subset \mathcal{H}$ provided that $\lambda(K) \neq 0$.

For $M \in \mathcal{O}_k^{\mathfrak{g}}$ one knows that $H^{\frac{\infty}{2}+\bullet}(L\overset{\circ}{\mathfrak{n}}, M)$ is naturally an $\mathcal{H}$-module of level $k + h^\vee$ ([FF2]).

**Lemma 4.12.** *Let* $M$ *be an object of* $\mathcal{O}_k^{\mathfrak{g}}$ *with* $k \neq -h^\vee$. *Then the following conditions are equivalent:*

(i) $H^{\frac{\infty}{2}+i}(\mathfrak{a}, M) = 0$ *for* $i \neq 0$;

(ii) $H^{\frac{\infty}{2}+i}(L\overset{\circ}{\mathfrak{n}}, M) = 0$ *for* $i \neq 0$.

*Proof.* The assumption that $k \neq -h^\vee$ implies that $H^{\frac{\infty}{2}+\bullet}(L\overset{\circ}{\mathfrak{n}}, M)$ is semi-simple as an $\mathcal{H}$-module and is a direct sum of $\pi_\mu$s. Consider the Hochschild-Serre spectral sequence for the ideal $L\overset{\circ}{\mathfrak{n}} \subset \mathfrak{a}$ to compute $H^{\frac{\infty}{2}+\bullet}(\mathfrak{a}, M)$. By definition, we have

$$E_2^{p,q} = \begin{cases} H_{-p}(\overset{\circ}{\mathfrak{h}}[t^{-1}]t^{-1}, H^{\frac{\infty}{2}+q}(L\overset{\circ}{\mathfrak{n}}, M)) & \text{for } p \leq 0, \\ 0 & \text{for } p > 0. \end{cases}$$

By the above mentioned fact $H^{\frac{\infty}{2}+q}(L\overset{\circ}{\mathfrak{n}}, M)$ is free over $U(\overset{\circ}{\mathfrak{h}}[t^{-1}]t^{-1})$. Hence

$$E_2^{p,q} = \begin{cases} H^{\frac{\infty}{2}+q}(L\overset{\circ}{\mathfrak{n}}, M))/\overset{\circ}{\mathfrak{h}}[t^{-1}]t^{-1}(H^{\frac{\infty}{2}+q}(L\overset{\circ}{\mathfrak{n}}, M))) & \text{for } p = 0. \\ 0 & \text{for } p \neq 0. \end{cases}$$

Therefore the spectral sequence collapses at $E_2 = E_\infty$, and $H^{\frac{\infty}{2}+i}(\mathfrak{a}, M) = 0$ for $i \neq 0$ if and only if $H^{\frac{\infty}{2}+i}(L\overset{\circ}{\mathfrak{n}}, M) = 0$ for $i \neq 0$. This completes the proof. □

**Proposition 4.13.** *Let* $M$ *be an object of* $\mathcal{O}_k$ *at a non-critical level* $k$ *such that* $H^{\frac{\infty}{2}+i}(\mathfrak{a}, M) = 0$ *for* $i \neq 0$. *Then*

$$M \cong US(\mathfrak{a}) \otimes_{\mathbb{C}} H^{\frac{\infty}{2}+0}(\mathfrak{a}, M)$$

*as* $\mathfrak{a}$*-modules and* $\mathfrak{h}$*-modules, where* $\mathfrak{a}$ *acts only on the first factor* $US(\mathfrak{a})$ *and* $\mathfrak{h}$ *acts as* $h(s \otimes m) = \operatorname{ad}(h)(s) \otimes m + s \otimes hm$.

*Proof.* By Proposition 2.3 it suffices to show that $\text{S-ind}_{\mathfrak{a}}^{\mathfrak{a}} M \cong US(\mathfrak{a})\otimes_{\mathbb{C}} H^{\frac{\infty}{2}+0}(\mathfrak{a}, M)$. As in the proof of Lemma 4.12, we shall consider the Hochschild-Serre spectral sequence for the ideal $L\mathring{\mathfrak{n}} \subset \mathfrak{a}$ to compute $H^{\frac{\infty}{2}+\bullet}(\mathfrak{a}, US(\mathfrak{a})\otimes M)$. By definition we have

$$E_1^{\bullet,q} = H^{\frac{\infty}{2}+q}(L\mathring{\mathfrak{n}}, US(\mathfrak{a})\otimes_{\mathbb{C}} M)\otimes_{\mathbb{C}} \bigwedge\nolimits^{\bullet}(\mathring{\mathfrak{h}}[t^{-1}]t^{-1}), \tag{30}$$

$$E_2^{p,q} = H_{-p}(\mathring{\mathfrak{h}}[t^{-1}]t^{-1}, H^{\frac{\infty}{2}+q}(L\mathring{\mathfrak{n}}, US(\mathfrak{a})\otimes_{\mathbb{C}} M)). \tag{31}$$

To compute the $E_1$-term set

$$F^p US(\mathfrak{a}) = \bigoplus_{\langle \mu, \mathring{\rho}^\vee\rangle \geq p} US(\mathfrak{a})_\mu,$$

where $US(\mathfrak{a})$ is considered as an $\mathfrak{h}$-module by the adjoint action. Then

$$US(\mathfrak{a}) = F^0 US(\mathfrak{a}) \supset F^1 US(\mathfrak{a}) \supset \dots, \quad \bigcap F^p US(\mathfrak{a}) = 0,$$
$$F^p US(\mathfrak{a}) \cdot L\mathring{\mathfrak{n}} \subset F^{p+1} US(\mathfrak{a}).$$

Define the filtration $F^\bullet(US(\mathfrak{a})\otimes_{\mathbb{C}} M \otimes_{\mathbb{C}} \bigwedge^{\frac{\infty}{2}+\bullet}(L\mathring{\mathfrak{n}}))$ by setting

$$F^p(US(\mathfrak{a})\otimes_{\mathbb{C}} M\otimes_{\mathbb{C}} \bigwedge\nolimits^{\frac{\infty}{2}+\bullet}(L\mathring{\mathfrak{n}})) = F^p US(\mathfrak{a})\otimes_{\mathbb{C}} M\otimes_{\mathbb{C}} \bigwedge\nolimits^{\frac{\infty}{2}+\bullet}(L\mathring{\mathfrak{n}}).$$

This defines a decreasing, weight-wise regular filtration of the complex. Consider the associated spectral sequence $E_r' \Rightarrow H^{\frac{\infty}{2}+\bullet}(L\mathring{\mathfrak{n}}, US(\mathfrak{a})\otimes_{\mathbb{C}} M)$. Because the associated graded space $\operatorname{gr} US(\mathfrak{a})$ with respect to this filtration is a trivial $L\mathring{\mathfrak{n}}$-module the $E_1$-term of the spectral sequence $E_r'$ is isomorphic to $US(\mathfrak{a})\otimes_{\mathbb{C}} H^{\frac{\infty}{2}+\bullet}(L\mathring{\mathfrak{n}}, M)$. Hence by the hypothesis and Lemma 4.12 the spectral sequence $E_r'$ collapses at $E_1' = E_\infty'$ and we obtain the isomorphism of $\mathfrak{h}$-modules

$$H^{\frac{\infty}{2}+i}(L\mathring{\mathfrak{n}}, US(\mathfrak{a})\otimes_{\mathbb{C}} M) \cong \begin{cases} US(\mathfrak{a})\otimes_{\mathbb{C}} H^{\frac{\infty}{2}+0}(L\mathring{\mathfrak{n}}, M) & \text{for } i = 0,\\ 0 & \text{for } i \neq 0.\end{cases} \tag{32}$$

This is also an isomorphism of $\mathfrak{a}$-modules since $US(\mathfrak{a}) \cong \operatorname{gr} US(\mathfrak{a})$ as *left* $\mathfrak{a}$-modules, where $x_\alpha t^n \in \mathfrak{a}$ is considered as an operator on $\operatorname{gr} US(\mathfrak{a}) = \bigoplus_p F^p US(\mathfrak{a})/F^{p+1} US(\mathfrak{a})$ which maps $F^p US(\mathfrak{a})/F^{p+1} US(\mathfrak{a})$ to $F^{p+\alpha(\mathring{\rho}^\vee)} US(\mathfrak{a})/F^{p+\alpha(\mathring{\rho}^\vee)+1} US(\mathfrak{a})$. We have computed the $E_1$-term (30):

$$E_1^{\bullet,q} \cong \begin{cases} US(\mathfrak{a})\otimes_{\mathbb{C}} H^{\frac{\infty}{2}+0}(L\mathring{\mathfrak{n}}, M)\otimes_{\mathbb{C}} \bigwedge^{\bullet}(\mathring{\mathfrak{h}}[t^{-1}]t^{-1}) & \text{for } q = 0,\\ 0 & \text{for } q \neq 0.\end{cases}$$

It follows that

$$E_2^{p,q} \cong \begin{cases} US(\mathfrak{a})\otimes_{\mathbb{C}} H^{\frac{\infty}{2}+0}(\mathfrak{a}, M) & \text{for } p = q = 0,\\ 0 & \text{otherwise}\end{cases} \tag{33}$$

as $\mathfrak{h}$-modules and $\mathfrak{a}$-modules, see the proof of Lemma 4.12. The spectral sequence collapses at $E_2 = E_\infty$ and we obtain the required isomorphism. □

Set

$$Q_{\frac{\infty}{2},+} = \sum_{\substack{\alpha\in\Delta^{\mathrm{re}}\\ \bar{\alpha}\in\mathring{\Delta}_-}} \mathbb{Z}_{\geq 0}\alpha + \mathbb{Z}_{\geq 0}\delta \subset \mathfrak{h}^*,$$

and define the partial ordering $\leq_{\frac{\infty}{2}}$ on $\mathfrak{h}^*$ by $\mu \leq_{\frac{\infty}{2}} \lambda \iff \lambda - \mu \in Q_{\frac{\infty}{2},+}$. Note that $\mu \leq_{\frac{\infty}{2}} \lambda$ if and only if $t_\alpha \circ \mu \leq t_\alpha \circ \lambda$ for a sufficiently large $\alpha \in \mathring{Q}^\vee$.

*Theorem 4.7.* Since The direction (i) ⇒ (ii) in Theorem 4.7 is obvious by (26), we shall prove that (ii) implies (i). Let $\{\lambda_1, \dots, \lambda_r\}$ be the set of weights of $H^{\frac{\infty}{2}+0}(\mathfrak{a}, M)$ with multiplicities counted, so that

$$M \cong \bigoplus_{i=1}^{r} US(\mathfrak{a}) \otimes_{\mathbb{C}} \mathbb{C}_{\lambda_i} \tag{34}$$

as $\mathfrak{a}$-modules and $\mathfrak{h}$-modules by Proposition 4.13. We may assume that if $\lambda_i \leq_{\frac{\infty}{2}} \lambda_j$ then $j < i$.

Set $\lambda = \lambda_1$. We shall show that there is a $\mathfrak{g}$-module embedding $W(\lambda) \hookrightarrow M$. Let $\{\gamma_1, \gamma_2, \dots\}$ be a sequence in $\mathring{P}_+^\vee$ such that $\gamma_i - \gamma_{i-1} \in \mathring{P}_+^\vee$ and $\lim_{n\to\infty} \alpha(\gamma_n) = \infty$ for all $\alpha \in \mathring{\Delta}_+$, so that $W(\lambda) = \varinjlim_n M^{-\gamma_n}(\lambda)$ by Proposition 4.3. By Lemma 4.1 (ii) we have $M \cong T_{-\gamma_i} G_{-\gamma_i}(M)$, and hence,

$$\mathrm{Hom}_{\mathfrak{g}}(M^{-\gamma_i}(\lambda), M) \cong \mathrm{Hom}_{\mathfrak{g}}(M(t_{\gamma_i} \circ \lambda), G_{-\gamma_i}(M)).$$

By (34), $\mathrm{ch}\, G_{-\gamma_i}(M) = \sum_{i=1}^r \mathrm{ch}\, M(t_{\gamma_i} \circ \lambda)$. Let $i$ be sufficiently large so that $t_{\gamma_i} \circ \lambda$ is maximal in $G_{-\gamma_i}(M)$. Denote by $\Phi_i$ the $\mathfrak{g}$-module homomorphism $\psi_i : M(t_{\gamma_i} \circ \lambda) \to G_{-\gamma_i}(M)$ which sends $v_{t_{\gamma_i} \circ \lambda}$ to a vector of $G_{-\gamma_i}(M)$ of weight $t_{\gamma_i} \circ \lambda$. As in the proof of Proposition 4.3 $\{T_{-\gamma_i}(\psi_i) : M^{-\gamma_i}(\lambda) \mapsto M\}$ yield an injective $\mathfrak{g}$-module homomorphism

$$\Phi : W(\lambda) = \varinjlim_i M^{-\gamma_i}(\lambda) \hookrightarrow M.$$

The map $\Phi$ induces the homomorphism $H^{\frac{\infty}{2}+0}(\mathfrak{a}, W(\lambda)) = \mathbb{C}_\lambda \to H^{\frac{\infty}{2}+0}(\mathfrak{a}, M)$ which is certainly injective. It follows from the long exact sequence associated with the exact sequence $0 \to W(\lambda) \xrightarrow{\Phi} M \to M/W(\lambda) \to 0$ we obtain that $H^{\frac{\infty}{2}+i}(\mathfrak{a}, M/W(\lambda)) = 0$ for $i \neq 0$ and $\dim H^{\frac{\infty}{2}+0}(\mathfrak{a}, M/W(\lambda)) = \dim H^{\frac{\infty}{2}+0}(\mathfrak{a}, M) - 1$. Theorem 4.7 follows by the induction on $\dim H^{\frac{\infty}{2}+0}(\mathfrak{a}, M)$. □

4.9. **Twisted Wakimoto modules.** For $w \in \mathring{W}$ we have the decomposition $\mathfrak{g} = w(\mathfrak{a}) \oplus w(\bar{\mathfrak{a}})$, and $2\rho$ defines a semi-infinite 1-cochain of the graded subalgebra $w(\bar{\mathfrak{a}})$. Hence we can define the *twisted Wakimoto module* $W^w(\lambda)$ with highest weight $\lambda$ and twist $w \in \mathring{W}$ by

$$W^w(\lambda) = \text{S-ind}_{w(\bar{\mathfrak{a}})}^{\mathfrak{g}} \mathbb{C}_\lambda,$$

where $\mathbb{C}_\lambda$ is the one-dimensional representation of $\mathfrak{h}$ corresponding to $\lambda$ regarded as a $\bar{\mathfrak{a}}$-module by the projection $\bar{\mathfrak{a}} \to \mathfrak{h}$. We have

$$W^w(\lambda) \cong US(w(\mathfrak{a})) \text{ as } w(\mathfrak{a})\text{-modules and } \mathrm{ch}\, W^w(\lambda) = \mathrm{ch}\, M(\lambda),$$
$$H^{\frac{\infty}{2}+i}(w(\mathfrak{a}), W^w(\lambda)) \cong \begin{cases} \mathbb{C}_\lambda & \text{for } i = 0, \\ 0 & \text{otherwise,} \end{cases} \text{ as } \mathfrak{h}\text{-modules.}$$

Let $\{\gamma_1, \gamma_2, \dots\}$ be a sequence in $\mathring{P}_+^\vee$ such that $\gamma_i - \gamma_{i-1} \in \mathring{P}_+^\vee$ and $\lim_{n\to\infty} \alpha(\gamma_n) = \infty$ for all $\alpha \in \mathring{\Delta}_+$. The following assertion can be proved in the same manner as Proposition 4.3.

**Proposition 4.14.** *Let $\lambda \in \mathfrak{h}^*$, $w \in \mathring{\mathcal{W}}$. There is an isomorphism of $\mathfrak{g}$-modules*

$$W^w(\lambda) \cong \varinjlim_n M^{-w(\gamma_n)}(\lambda).$$

The following assertion can be proved in the same manner as Theorem 4.7.

**Theorem 4.15.** *Let $\lambda \in \mathfrak{h}^*$ be non-critical, $w \in \mathring{\mathcal{W}}$. Let $M$ be an object of $\mathcal{O}^{\mathfrak{g}}$ such that*

$$H^{\frac{\infty}{2}+i}(w(\mathfrak{a}), M) \cong \begin{cases} \mathbb{C}_\lambda & \text{if } i = 0, \\ 0 & \text{otherwise,} \end{cases}$$

*as $\mathfrak{h}$-modules. Then $M$ is isomorphic to $W^w(\lambda)$.*

## 5. Borel-Weil-Bott vanishing property of Twisting functors

5.1. **Left derived functors of twisting functors.** The functor $T_w$, $w \in \mathcal{W}^e$, admits the left derived functor $\mathcal{L}_\bullet T_w$ in the category $\mathcal{O}^{\mathfrak{g}}$ since it is a Lie algebra homology functor:

$$\mathcal{L}_i T_w(M) = \phi_w(H_i(\mathfrak{g}, S_w \otimes_{\mathbb{C}} M)),$$

where $\mathfrak{g}$ acts on $N_w^* \otimes_{\mathbb{C}} M$ by $X(f \otimes m) = -fX \otimes m + f \otimes Xm$. Because

$$\mathcal{L}_i T_w(M) \cong \phi_w(H_i(\mathfrak{n}_w, N_w^* \otimes_{\mathbb{C}} M)) \tag{35}$$

as $w(\mathfrak{n}_w)$-modules, we have the following assertion.

**Lemma 5.1.** *Suppose $M \in \mathcal{O}^{\mathfrak{g}}$ is free over $\mathfrak{n}_w$. Then $\mathcal{L}_i T_w(M) = 0$ for $i \geq 1$.*

Let $\{e_i, h_i, f_i; i \in I\}$, $e_i \in \mathfrak{g}_{\alpha_i}$, $f_i \in \mathfrak{g}_{-\alpha_i}$, be the Chevalley generators of $\mathfrak{g}$. For $i \in I$, let $\mathfrak{sl}_2^{(i)}$ denote the copy of $\mathfrak{sl}_2$ in $\mathfrak{g}$ spanned by $\{e_i, h_i, f_i\}$

**Proposition 5.2.** *Let $M \in \mathcal{O}^{\mathfrak{g}}$, $i \in I$. Denote by $N$ the largest $\mathfrak{sl}_2^{(i)}$-integrable submodule of $M$. Then $T_i(M) \cong T_i(M/N)$, $\operatorname{ch} \mathcal{L}_1 T_i(M) \cong \operatorname{ch} N$ and $\mathcal{L}_p T_i(M) = 0$ for $p \geq 2$.*

*Proof.* Let $T_i^{(i)}$ denote the twisting functor for $\mathfrak{sl}_2^{(i)}$ corresponding to the reflection $s_{\alpha_i}$. Because $T_i(M) \cong T_i^{(i)}(M)$ as $\mathfrak{sl}_2^{(i)}$-modules and $\mathfrak{h}$-modules, we have

$$\mathcal{L}_p T_i(M) \cong \mathcal{L}_p T_i^{(i)}(M) \quad \text{as } \mathfrak{sl}_2^{(i)}\text{-modules and } \mathfrak{h}\text{-modules.} \tag{36}$$

In particular $\mathcal{L}_p T_i(M) = 0$ for $p \geq 2$. It follows that the exact sequence

$$0 \to N \to M \to M/N \to 0$$

yields the long exact sequence

$$\begin{aligned} 0 \to \mathcal{L}_1 T_i(N) \to \mathcal{L}_1 T_i(M) \to \mathcal{L}_1 T_i(M/N) \\ \to T_i(N) \to T_i(M) \to T_i(M/N) \to 0. \end{aligned}$$

Since $M/N$ is free as $\mathbb{C}[f_i]$-module $\mathcal{L}_1 T_i(M/N) = 0$ by Lemma 5.1. Also, $T_i(N) = 0$ and $\mathcal{L}_1 T_i(N) \cong N$ as $\mathfrak{h}$-modules by [AS, Theorem 6.1] and (36). This completes the proof. ☐

Let $L(\lambda) \in \mathcal{O}^{\mathfrak{g}}$ be the irreducible highest weight representation of $\mathfrak{g}$ with highest weight $\lambda \in \mathfrak{h}^*$.

**Theorem 5.3** ([AS, Theorem 6.1])**.** *Let $\lambda \in \mathfrak{h}^*$ and suppose that $\langle \lambda, \alpha_i^\vee \rangle \in \mathbb{Z}_{\geq 0}$ with $i \in I$. Then*

$$\mathcal{L}_p T_i(L(\lambda)) \cong \begin{cases} L(\lambda) & \text{if } p = 1, \\ 0 & \text{if } p \neq 1. \end{cases}$$

*Proof.* The hypothesis implies that $L(\lambda)$ is $\mathfrak{sl}_2^{(i)}$-integrable. Therefore $\mathcal{L}_p T_i(L(\lambda)) = 0$ for $p \neq 1$ and $\operatorname{ch} \mathcal{L}_1 T_i(L(\lambda)) = \operatorname{ch} L(\lambda)$ by Proposition 5.2. □

5.2. **Twisting functors associated with integral Weyl group.**

**Lemma 5.4.** *Let $\lambda \in \mathfrak{h}^*$, $\alpha \in \Pi(\lambda)$. There exists $x \in \mathcal{W}$ and $\alpha_i \in \Pi$ such that $s_\alpha = x s_i x^{-1}$, $\ell(s_\alpha) = 2\ell(x) + 1$ and $\Delta_+^{re} \cap x(\Delta_-^{re}) \cap \Delta(\lambda) = \emptyset$.*

*Proof.* Let $s_\alpha = s_{j_l} s_{j_{l-1}} \dots s_{j_1}$ be a reduced expression of $s_\alpha$ in $\mathcal{W}$. Then

$$\Delta_+^{re} \cap s_\alpha(\Delta_-^{re}) = \{\alpha_1, s_{j_1}(\alpha_{j_2}), \dots, s_{j_1} \dots s_{j_{l-1}}(\alpha_{j_l})\}$$

Since $\ell_\lambda(\alpha) = 1$, $\Delta_+^{re} \cap s_\alpha(\Delta_-^{re}) \cap \Delta(\lambda) = \{\alpha\}$. Thus there exists $r$ such that $\alpha = s_{j_1} \dots s_{j_{r-1}}(\alpha_{j_r})$. Set $x = s_{j_1} \dots s_{j_{r-1}}$, $i = j_r$. Then $s_\alpha = s_{x(\alpha_i)} = x s_i x^{-1}$. It follows that $s_{j_l} \dots s_{j_{r+1}} = x$ and $\ell(s_\alpha) = 2\ell(x)+1$. Also $\Delta_+^{re} \cap s_\alpha(\Delta_-^{re}) \cap \Delta(\lambda) = \{\alpha\}$ implies that $\Delta_+^{re} \cap x(\Delta_-^{re}) \cap \Delta(\lambda) = \emptyset$. □

Note that if $\lambda$, $\alpha$, $\alpha_i$, $x$ are as in Lemma 5.4 then

$$T_\alpha = T_x \circ T_i \circ T_{x^{-1}}.$$

Let $\mathcal{O}^{\mathfrak{g}}_{[\lambda]}$ be the block of $\mathcal{O}^{\mathfrak{g}}$ corresponding to $\lambda$, that is, the full subcategory of $\mathcal{O}^{\mathfrak{g}}$ consisting of objects $M$ such that $[M : L(\mu)] \neq 0 \Rightarrow \mu \in \mathcal{W}(\lambda) \circ \mu$, where $[M : L(\mu)]$ is the multiplicity of $L(\mu)$ in the local composition factor of $M$.

**Lemma 5.5.** *Let $\lambda \in \mathfrak{h}^*$, $y \in \mathcal{W}$, and suppose that $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}$ for all $\alpha \in \Delta_+^{re} \cap y^{-1}(\Delta_-^{re})$. Then $T_y M(w \circ \lambda) \cong M(yw \circ \lambda)$, $T_y L(w \circ \lambda) \cong L(yw \circ \lambda)$ for $w \in \mathcal{W}(\lambda)$. Moreover $T_w$ gives an equivalence of categories $\mathcal{O}^{\mathfrak{g}}_{[\lambda]} \xrightarrow{\sim} \mathcal{O}^{\mathfrak{g}}_{[w \circ \lambda]}$. The same is true for $G_w$.*

*Proof.* First note that the assumption implies that $\mathcal{W}(y \circ \lambda) = y \mathcal{W}(\lambda) y^{-1}$.

We prove by induction on $\ell(y)$. Let $\ell(y) = 1$, so that $y = s_i$ for $i \in I$. Then the fact that $T_i M(w\lambda) \cong M(s_i w \circ \lambda)$ with $w \in \mathcal{W}(\lambda)$ follow from (24). By [A1, Theorems 3.1, 3.2] any object of $\mathcal{O}^{\mathfrak{g}}_{[\lambda]}$ and $\mathcal{O}^{\mathfrak{g}}_{[s_i \circ \lambda]}$ is free over $\mathbb{C}[f_i]$ and cofree over $\mathbb{C}[e_i]$. Hence by Lemma 4.1 $T_i$ gives an equivalence of categories $\mathcal{O}^{\mathfrak{g}}_{[\lambda]} \xrightarrow{\sim} \mathcal{O}^{\mathfrak{g}}_{[s_i \circ \lambda]}$ with a quasi-inverse $G_i$. It follows that $T_i L(\lambda)$ is a simple $\mathfrak{g}$-module which is a quotient of $T_i M(\lambda) = M(s_i \circ \lambda)$, and hence is isomorphic to $L(s_i \circ \lambda)$. Next let $y = s_i z$ with $z \in \mathcal{W}$, $\ell(y) = \ell(z) + 1$. Then $\Delta_+^{re} \cap y^{-1}(\Delta_-^{re}) = \{z^{-1}(\alpha_i)\} \sqcup (\Delta_+^{re} \cap z^{-1} \Delta_-^{re})$. The assertion follows from the induction hypothesis. □

**Corollary 5.6.** *Let $\lambda$, $\alpha$, $\alpha_i$, $x$ be as in Lemma 5.4. Then $T_x$ give an equivalence of categories $\mathcal{O}^{\mathfrak{g}}_{[x^{-1} \circ \lambda]} \xrightarrow{\sim} \mathcal{O}^{\mathfrak{g}}_{[\lambda]}$ such that $T_x M(\mu) \cong M(x \circ \mu)$, $T_x L(\mu) \cong M(x \circ \mu)$ for $\mu \in \mathcal{W}(x^{-1} \circ \lambda) \circ x^{-1} \lambda = x^{-1} \mathcal{W}(\lambda) \circ \lambda$.*

**Lemma 5.7.** *Let $\lambda \in \mathfrak{h}^*$, $\alpha_i \in \Pi$ such that $\langle \lambda + \rho, \alpha_i^\vee \rangle \notin \mathbb{Z}$. Then $T_i M^w(\lambda) \cong M^{s_i w s_i}(s_i \circ \lambda)$ for $w \in \mathcal{W}(\lambda)$.*

*Proof.* By Lemma 5.5, $T_i M^w(\lambda) \cong T_i T_w M(w^{-1}\circ\lambda) \cong T_i T_w T_i M(s_i w^{-1}\circ\lambda) \cong T^{s_i w s_i} M(s_i w^{-1} s_i s_i \circ \lambda)$ . □

**Lemma 5.8.** *Let $\lambda \in \mathfrak{h}^*$, $\alpha_i \in \Pi$ such that $\langle \lambda+\rho, \alpha_i^\vee\rangle \notin \mathbb{Z}$. Then $T_i^2 : \mathcal{O}^{\mathfrak{g}}_{[\lambda]} \to \mathcal{O}^{\mathfrak{g}}_{[\lambda]}$ is isomorphic to the identity functor, and so is $G_i^2 : \mathcal{O}^{\mathfrak{g}}_{[\lambda]} \to \mathcal{O}^{\mathfrak{g}}_{[\lambda]}$.*

*Proof.* By Lemma 5.5 $T_i^2$ induces an auto-equivalence of the category $\mathcal{O}^{\mathfrak{g}}_{[\lambda]}$ such that $T_i^2 M(w\circ\lambda) \cong M(w\circ\lambda)$ and $T_i^2(L(w\circ\lambda)) \cong L(w\circ\lambda)$ for all $w \in \mathcal{W}(\lambda)$. The standard argument shows that such a functor must be isomorphic to the identify functor. □

**Corollary 5.9.** *Let $\lambda \in \mathfrak{h}^*$, $w = s_\alpha y \in \mathcal{W}(\lambda)$, $\alpha \in \Pi(\lambda)$, $y \in \mathcal{W}(\lambda)$, $\ell_\lambda(w) = \ell_\lambda(y)+1$. Then $T_w : \mathcal{O}^{\mathfrak{g}}_{[\lambda]} \to \mathcal{O}^{\mathfrak{g}}_{[w\circ\lambda]}$ is isomorphic to the functor $T_{s_\alpha}\circ T_y : \mathcal{O}^{\mathfrak{g}}_{[\lambda]} \to \mathcal{O}^{\mathfrak{g}}_{[w\circ\lambda]}$.*

**Proposition 5.10.** *Let $\lambda \in \mathfrak{h}^*$, $w \in \mathcal{W}(\lambda)$, $\alpha \in \Pi(\lambda)$ and suppose that $\langle w(\lambda+\rho), \alpha^\vee\rangle \notin \mathbb{N}$. Then the following sequence is exact:*

$$0 \to M(s_\alpha w\circ\lambda) \xrightarrow{\varphi_1} M(w\circ\lambda) \xrightarrow{\varphi_2} M^{s_\alpha}(w\circ\lambda) \xrightarrow{\varphi_3} M^{s_\alpha}(s_\alpha w\circ\lambda) \to 0,$$

*where $\varphi_1,\varphi_2,\varphi_3$ are any non-trivial $\mathfrak{g}$-homomorphisms.*

*Proof.* First observe that $\mathrm{Hom}_{\mathfrak{g}}(M(s_\alpha w\circ\lambda), M(w\circ\lambda))$, $\mathrm{Hom}_{\mathfrak{g}}(M(w\circ\lambda), M^{s_\alpha}(w\circ\lambda))$ and $\mathrm{Hom}_{\mathfrak{g}}(M^{s_\alpha}(w\circ\lambda), M^{s_\alpha}(s_\alpha w\circ\lambda))$ are all one-dimensional. (The first and the third are one-dimensional by Theorem 4.2.) By Lemma 5.4 there exists $x \in \mathcal{W}$ and $\alpha_i \in \Pi$ such that $s_\alpha = x s_i x^{-1}$, $\ell(s_\alpha) = 2\ell(x)+1$, and $\Delta_+^{re} \cap x(\Delta_-^{re}) \cap \Delta(\lambda) = \emptyset$. We have

$$M(y\circ\lambda) \cong T_x M(x^{-1}y\circ\lambda),$$
$$M^{s_\alpha}(y\circ\lambda) = T_x T_i T_{x^{-1}} M(x s_i x^{-1} y\circ\lambda) \cong T_x T_i M(s_i x^{-1}y\circ\lambda) \cong T_x M^{s_i}(x^{-1}y\circ\lambda)$$

for $y \in \mathcal{W}(\lambda)$ by Lemma 5.5. Since $\langle x^{-1}w(\lambda+\rho), \alpha_i^\vee\rangle = \langle w(\lambda+\rho), \alpha^\vee\rangle \in \mathbb{N}$ there is an exact sequence

$$0 \to M(s_i x^{-1} w\circ\lambda) \to M(x^{-1}w\circ\lambda) \to M^{s_i}(x^{-1}w\circ\lambda) \to M^{s_i}(s_i x^{-1}w\circ\lambda) \to 0$$

by [AL, Propostion 6.2]. The required exact sequence is obtained by applying the exact functor $T_x : \mathcal{O}^{\mathfrak{g}}_{[x^{-1}\circ\lambda]} \to \mathcal{O}^{\mathfrak{g}}_{[\lambda]}$ to the above. □

**Proposition 5.11.** *Let $\lambda \in \mathfrak{h}^*$, $\alpha \in \Pi(\lambda)$, $M \in \mathcal{O}^{\mathfrak{g}}_{[\lambda]}$. Take $\alpha_i \in \Pi$, $x \in \mathcal{W}$ such that $\alpha = x(\alpha_i)$ and $x^{-1}\Delta(\lambda)_+ \subset \Delta_+^{re}$ as in Lemma 5.4. Let $N'$ be the largest $\mathfrak{sl}_2^{(i)}$-integrable submodule of $T_{x^{-1}}(M)$ and set $N = T_x(N') \subset M$. Then $T_\alpha(M) \cong T_{s_\alpha}(M/N)$, $\mathrm{ch}\,\mathcal{L}_1 T_{s_\alpha}(M) = \mathrm{ch}\,N$ and $\mathcal{L}_p T_{s_\alpha}(M) = 0$ for $p \geq 2$.*

*Proof.* We have $T_\alpha = T_x T_i T_{x^{-1}}$ and $T_{x^{-1}} : \mathcal{O}^{\mathfrak{g}}_{[\lambda]} \to \mathcal{O}^{\mathfrak{g}}_{[x^{-1}\circ\lambda]}$, $T_x : \mathcal{O}^{\mathfrak{g}}_{x^{-1}\circ\lambda} \to \mathcal{O}^{\mathfrak{g}}_{[\lambda]}$ are exact functors by Corollary 5.6. Therefore

$$\mathcal{L}_p T_{s_\alpha}(M) = T_x(\mathcal{L}_p T_i(T_{x^{-1}} M)). \tag{37}$$

Hence Proposition 5.2 gives that

$$T_{s_\alpha}(M) = T_x T_i T_{x^{-1}}(M) \cong T_x T_i(T_{x^{-1}}(M)/N') \cong T_x T_i T_{x^{-1}}(M/N) = T_{s_\alpha}(M/N),$$
$$\mathrm{ch}\,\mathcal{L}_1 T_{s_\alpha}(M) = \mathrm{ch}\,T_x T_{x^{-1}}(N) = \mathrm{ch}\,N,$$
$$\mathcal{L}_p T_{s_\alpha}(M) = 0 \quad \text{for } p \geq 0.$$

This completes the proof. □

**Theorem 5.12.** *Let $\lambda \in \mathfrak{h}^*$ be regular dominant weight, $w \in \mathcal{W}(\lambda)$. Then*

$$\mathcal{L}_p T_w(L(\lambda)) \cong \begin{cases} L(\lambda) & \text{if } p = \ell(w), \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* Let $\alpha \in \Pi(\lambda)$. Since $T_{x^{-1}} L(\lambda) = L(x^{-1} \circ \lambda)$ and $\langle x^{-1} \circ \lambda + \rho, \alpha_i^\vee \rangle = \langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{N}$, $T_{x^{-1}} L(\lambda)$ is $\mathfrak{sl}_2^{(i)}$-integrable. Thus,

$$\mathcal{L}_p T_i T_{x^{-1}} L(\lambda) \cong \begin{cases} T_{x^{-1}} L(\lambda) & \text{if } p = 1, \\ 0 & \text{if } p \neq 0 \end{cases}$$

by Theorem 5.3. It follows from (37) that

$$\mathcal{L}_p T_{s_\alpha}(L(\lambda)) \cong \begin{cases} L(\lambda) & \text{if } p = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{38}$$

Finally the assertion follows in the same manner as in [AS, Corollary 6.2] by Corollary 5.9. □

## 6. Two-sided BGG resolutions of admissible representations

6.1. **Admissible representations.** A weight $\lambda \in \mathfrak{h}^*$ is called *admissible* if it is regular dominant and

$$\mathbb{Q}\Delta(\lambda) = \mathbb{Q}\Delta^{re}.$$

The irreducible representation $L(\lambda)$ is called admissible if $\lambda$ is admissible. A complex number $k$ is called an *admissible number* for $\mathfrak{g}$ if the weight $k\Lambda_0$ is admissible.

Let $r^\vee$ be the lacing number of $\mathring{\mathfrak{g}}$, that is, the maximal number of the edges of the Dynkin digram of $\mathring{\mathfrak{g}}$. Also, let $h$ be the Coxeter number of $\mathring{\mathfrak{g}}$.

**Proposition 6.1** ([KW2, KW3])**.** *A complex number $k$ is admissible if and only if*

$$k + h^\vee = \frac{p}{q} \quad \text{with } p, q \in \mathbb{N},\ (p,q) = 1,\ p \geq \begin{cases} h^\vee & \text{if } (r^\vee, q) = 1, \\ h & \text{if } (r^\vee, q) = r^\vee. \end{cases} \tag{39}$$

A complex number $k$ of the form (39) is called an *admissible number with denominator $q$*. For an an admissible number $k$ with denominator $q$, we have

$\Delta(k\Lambda_0) = \{\alpha + nq\delta; \alpha \in \Delta,\ n \in \mathbb{Z}\} \cong \Delta^{re}$ and $\mathcal{W}(k\Lambda_0) \cong \mathcal{W}$ if $(r^\vee, q) = 1$,
$\Delta(k\Lambda_0)^\vee = \{\alpha^\vee + nq\delta; \alpha \in \Delta,\ n \in \mathbb{Z}\} \cong {}^L\Delta^{re}$ and $\mathcal{W}(k\Lambda_0) \cong {}^L\mathcal{W}$ if $(r^\vee, q) = r^\vee$,

where $\Delta(\lambda)^\vee = \{\alpha^\vee; \alpha \in \Delta(\lambda)\}$ and ${}^L\Delta^{re}$ and ${}^L\mathcal{W}$ are the real root system and the Weyl group of the non-twisted affine Kac-Moody algebra ${}^L\mathfrak{g}$ associated with the Langlands dual ${}^L\mathring{\mathfrak{g}}$ of $\mathring{\mathfrak{g}}$, respectively. Set

$$\dot{\alpha}_0 = \begin{cases} -\theta + q\delta & \text{if } (r^\vee, q) = 1, \\ -\theta_s + \frac{q}{r^\vee}\delta & \text{if } (r^\vee, q) = r^\vee. \end{cases}$$

Then $\Pi(k\Lambda_0) = \{\alpha_1, \dots, \alpha_\ell, \dot{\alpha}_0\}$. Put $\dot{s}_0 = s_{\dot{\alpha}_0} \in \mathcal{W}(k\Lambda_0)$, so that $\mathcal{W}(k\Lambda_0) = \langle s_1, \dots, s_\ell, \dot{s}_0 \rangle$.

For an admissible number $k$ let $Pr_k^+$ be the set of admissible weights $\lambda$ of level $k$ such that $\lambda(\alpha^\vee) \in \mathbb{Z}_{\geq 0}$ for all $\alpha \in \mathring{\Delta}_+$. Then $\{L(\lambda); \lambda \in Pr_k^+\}$ is the set of irreducible admissible representations of level $k$ which are integrable over $\mathring{\mathfrak{g}} \subset \mathfrak{g}$. We have $\Delta(\lambda) = \Delta(k\Lambda_0)$ for $\lambda \in Pr_k^+$.

For an admissible number $k$ denote by $Pr_k$ the set of admissible weights $\lambda$ of level $k$ such that $\Delta(\lambda) \cong \Delta(k\Lambda_0)$ as root systems. Then [KW2]

$$Pr_k = \bigcup_{\substack{y \in \mathcal{W}^e \\ y(\Delta(k\Lambda_0) \subset \Delta_+^{re}}} Pr_{k,y}, \quad Pr_{k,y} = y \circ Pr_k^+. \tag{40}$$

Note that

$$\mathcal{W}(\lambda) = y\mathcal{W}(k\Lambda_0)y^{-1} \quad \text{for } \lambda \in Pr_{k,y}. \tag{41}$$

For $\lambda \in Pr_k$, let $\ell_\lambda^{\frac{\infty}{2}}(?)$ be the semi-infinite length function of the affine Weyl group $\mathcal{W}(\lambda)$. The semi-infinite Bruhat ordering $\preceq_{\lambda,\frac{\infty}{2}}$ are also defined for $\mathcal{W}(\lambda)$. We will use the symbol $w \rhd_{\lambda,\frac{\infty}{2}} w'$ to denote a covering in the twisted Bruhat order $\succeq_{\lambda,\frac{\infty}{2}}$.

*Remark* 6.2. The admissible weight $\lambda \in Pr_k$ is called the *principal admissible weight* [KW2] if $\Delta(\lambda) \cong \Delta^{re}$, that is, if the denominator $q$ of $k$ is prime to $r^\vee$.

6.2. **Fiebig's equivalence and BGG resolution of admissible representations.** The following theorem is the special case of a result of Fiebig [Fie, Theorem 11].

**Theorem 6.3** ([Fie])**.** *Let $\lambda$ be regular dominant. Suppose that there exists a symmetrizable Kac-Moody algebra $\mathfrak{g}'$ whose Weyl group $\mathcal{W}'$ is isomorphic to $\mathcal{W}(\lambda)$. Let $\lambda'$ be an integral dominant weight of $\mathfrak{g}'$, $\mathcal{O}^{\mathfrak{g}'}_{[\lambda']}$ the block of $\mathcal{O}^{\mathfrak{g}'}$ containing the irreducible highest weight representation $L^{\mathfrak{g}'}(\lambda')$ of $\mathfrak{g}$ with highest weight $\lambda'$. Then there is an equivalence of categories*

$$\mathcal{O}^{\mathfrak{g}}_{[\lambda]} \cong \mathcal{O}^{\mathfrak{g}'}_{[\lambda']}$$

*which maps $M(w\circ\lambda)$ and $L(w\circ\lambda)$, $w \in \mathcal{W}(\lambda)$, to $M^{\mathfrak{g}'}(\phi(w)\circ\lambda')$ and $L^{\mathfrak{g}}(\phi(w)\circ\lambda')$, respectively. Here $M^{\mathfrak{g}'}(\lambda')$ is the Verma module of $\mathfrak{g}'$ with highest weight $\lambda'$ and $\phi : \mathcal{W}(\lambda) \xrightarrow{\sim} \mathcal{W}'$ is the isomorphism.*

Let $k$ be an admissible number with denominator $q$, $\lambda \in Pr_k$. By Theorem 6.3 the block $\mathcal{O}^{\mathfrak{g}}_{[\lambda]}$ is equivalent to a block of the category $\mathcal{O}$ of $\mathfrak{g}$ or ${}^L\mathfrak{g}$ containing an integrable representation. In particular the existence of a BGG resolution of an integrable representation of an affine Kac-Moody algebra [GL, RCW] implies the existence of a BGG resolution for $L(\lambda)$:

**Theorem 6.4.** *Let $k$ be an admissible number, $\lambda \in Pr_k$. Then there exists a complex*

$$\mathcal{B}_\bullet(\lambda) : \cdots \xrightarrow{d_3} \mathcal{B}_2(\lambda) \xrightarrow{d_2} \mathcal{B}_1(\lambda) \xrightarrow{d_1} \mathcal{B}_0(\lambda) \xrightarrow{d_0} 0$$

*of the form $\mathcal{B}_i(\lambda) = \bigoplus_{\substack{w \in \mathcal{W}(\lambda) \\ \ell_\lambda(w)=i}} M(w\circ\lambda)$, $d_i = \sum_{\substack{w,w' \in \mathcal{W}(\lambda) \\ \ell_\lambda(w)=i,\ w \rhd_\lambda w'}} d_{w',w}$, $d_{w',w} \in \mathrm{Hom}_{\mathfrak{g}}(M(w\circ\lambda), M(w'\circ\lambda))$, such that*

$$H_i(\mathcal{B}_\bullet(\lambda)) \cong \begin{cases} L(\lambda) & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

The resolution of $L(\lambda)$ in Theorem 6.4 can be combinatorially constructed as follows [BGG]: Fix a $\mathfrak{g}$-homomorphisms

$$i^{\lambda}_{w',w} : M(w \circ \lambda) \to M(w' \circ \lambda)$$

for $w, w' \in \mathcal{W}(\lambda)$ with $w \succeq_\lambda w'$ in such a way that $i^{\lambda}_{w'',w'} \circ i^{\lambda}_{w',w} = i^{\lambda}_{w'',w}$ if $w \succeq_\lambda w' \succeq_\lambda w$.

A quadruple $(w_1, w_2, w_3, w_4)$ in $\mathcal{W}(\lambda)$ is called a *square* if $w_1 \rhd_\lambda w_2 \rhd_\lambda w_4$, $w_1 \rhd_\lambda w_3 \rhd_\lambda w_4$ and $w_2 \neq w_3$.

**Theorem 6.5.** *Let $k$ be an admissible number, $\lambda \in Pr_k$. Assign $\epsilon_{w_2,w_1} \in \mathbb{C}^*$ for every pair $(w_1, w_2)$ in $\mathcal{W}(\lambda)$ with $w_1 \rhd_\lambda w_2$ in such a way that $\epsilon_{w_4,w_2}\epsilon_{w_2,w_1} + \epsilon_{w_4,w_3}\epsilon_{w_3,w_1} = 0$ for every square $(w_1, w_2, w_3, w_4)$ of $\mathcal{W}(\lambda)$ (such an assignment is possible by [BGG]). Set $d_{w',w} = \epsilon_{w',w} i^{\lambda}_{w',w}$, $d_i = \sum_{\substack{w,w' \in \mathcal{W}(\lambda) \\ \ell_\lambda(w)=i,\ w \rhd_\lambda w'}} d_{w',w}$. Then*

$$\mathcal{B}_\bullet(\lambda) : \cdots \xrightarrow{d_3} \mathcal{B}_2(\lambda) \xrightarrow{d_2} \mathcal{B}_1(\lambda) \xrightarrow{d_1} \mathcal{B}_0(\lambda) \xrightarrow{d_0} 0,$$

*where $\mathcal{B}_i(\lambda) = \bigoplus_{\substack{w \in \mathcal{W}(\lambda) \\ \ell_\lambda(w)=i}} M(w \circ \lambda)$, is a resolution of $L(\lambda)$.*

6.3. **Twisted BGG resolution.** For $w_1, w_2, y \in \mathcal{W}(\lambda)$ with $w_1 \succeq_y w_2$, set

$$\varphi^{\lambda,y}_{w_2,w_1} = T_y(i^{\lambda}_{y^{-1}w_2, y^{-1}w_1}) : M^y(w_1 \circ \lambda) \to M^y(w_2 \circ \lambda).$$

A quadruple $(w_1, w_2, w_3, w_4)$ in $\mathcal{W}(\lambda)$ is called a *$y$-twisted square* if $w_1 \rhd_y w_2 \rhd_y w_4$, $w_1 \rhd_y w_3 \rhd_y w_4$ and $w_2 \neq w_3$.

**Theorem 6.6.** *Let $k$ be an admissible number, $\lambda \in Pr_k$, $y \in \mathcal{W}(\lambda)$. Assign $\epsilon^y_{w_2,w_1} \in \mathbb{C}^*$ for every pair $(w_1, w_2)$ with $w_1 \rhd_{\lambda,y} w_2$ in $\mathcal{W}(\lambda)$ in such a way that $\epsilon^y_{w_4,w_2}\epsilon^y_{w_2,w_1} + \epsilon^y_{w_4,w_3}\epsilon^y_{w_3,w_1} = 0$ for every $y$-twisted square $(w_1, w_2, w_3, w_4)$ of $\mathcal{W}(\lambda)$. Set $\mathcal{B}^y_i(\lambda) = \bigoplus_{\substack{w \in \mathcal{W}(\lambda) \\ \ell^y_\lambda(w)=i}} M^y(w \circ \lambda)$, $d^y_{w',w} = \epsilon^y_{w',w}\varphi^{\lambda,y}_{w',w}$, $d_i = \sum_{\substack{w,w' \in \mathcal{W}(\lambda) \\ \ell^y_\lambda(w)=i,\ w \rhd_{\lambda,y} w'}} d_{w',w} :$ $\mathcal{B}^y_i(\lambda) \to \mathcal{B}^y_{i-1}(\lambda)$. Then*

$$\mathcal{B}^y_\bullet(\lambda) : \cdots \xrightarrow{d_3} \mathcal{B}^y_2(\lambda) \xrightarrow{d_2} \mathcal{B}^y_1(\lambda) \xrightarrow{d_1} \mathcal{B}^y_0(\lambda) \xrightarrow{d_0} \mathcal{B}^y_{-1}(\lambda) \to \dots \to \mathcal{B}^y_{-\ell(y)}(\lambda) \to 0$$

*is a complex of $\mathfrak{g}$-modules such that*

$$H_i(B^y_\bullet(\lambda)) \cong \begin{cases} L(\lambda) & \text{for } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* Set $\epsilon_{y^{-1}w_1, y^{-1}w_2} = \epsilon^y_{w_1,w_2}$. Then $\{\epsilon^y_{w_1,w_2}\}$ satisfies the condition in Theorem 6.6 if and only if $\{\epsilon_{y^{-1}w_1,y^{-1}w_2}\}$ satisfies the condition in Theorem 6.4. In particular such an assignment is possible. Consider the BGG resolution $\mathcal{B}_\bullet(\lambda)$ of $L(\lambda)$ in Theorem 6.5 associated with this assignment. We have $\mathcal{B}^y_\bullet(\lambda) = T_y(B_\bullet(\lambda))[-\ell(y)]$, where $[-\ell(y)]$ denotes the shift of the degree. Therefore the assertion follows from Theorem 5.12. □

6.4. **System of twisted BGG resolutions.**

**Proposition 6.7.** *Let $\lambda \in \mathfrak{h}^*$ be regular dominant, $y = s_{\beta_1}s_{\beta_2}\dots s_{\beta_l}$ a reduced expression of $y \in \mathcal{W}(\lambda)$ with $\beta_i \in \Pi(\lambda)$. Set $y_i = s_{\beta_1}s_{\beta_2}\dots s_{\beta_i}$ for $i = 0, 1, \dots, l$ and fix a non-zero $\mathfrak{g}$-homomorphism $\phi^{y_i}_w : M^{y_i}(w \circ \lambda) \to M^{y_{i+1}}(w \circ \lambda)$ for $w \in \mathcal{W}(\lambda)$,*

$i = 1, \dots, l$. *One can assign* $\epsilon^i_{w_2,w_1} \in \mathbb{C}^*$ *for each pair* $(w_1, w_2)$ *with* $w_1 \rhd_{\lambda,y_i} w_2$ *for all* $i = 1, \dots, l$ *in such a way that the following hold:*

(i) $\epsilon^i_{w_4,w_2}\epsilon^i_{w_2,w_1} + \epsilon^i_{w_4,w_3}\epsilon^i_{w_3,w_1} = 0$ *for every* $y_i$*-twisted square* $(w_1, w_2, w_3, w_4)$ *of* $\mathcal{W}(\lambda)$,

(ii) *If* $w_1 \rhd_{\lambda,y_i} w_2$, $w_1 \rhd_{\lambda,y_{i-1}} w_2$, $\ell^{y_i}_\lambda(w_1) = \ell^{y_{i-1}}_\lambda(w_1)$ *and* $\ell^{y_i}_\lambda(w_2) = \ell^{y_{i-1}}_\lambda(w_2)$, *then the the following diagram commutes.*

$$\begin{array}{ccc} M^{y_{i-1}}(w_1 \circ \lambda) & \xrightarrow{\epsilon^{i-1}_{w_2,w_1}\varphi^{\lambda,y_{i-1}}_{w_2,w_1}} & M^{y_{i-1}}(w_2 \circ \lambda) \\ \phi^{y_{i-1}}_{w_1}\Big\downarrow & & \Big\downarrow\phi^{y_{i-1}}_{w_2} \\ M^y(w_1 \circ \lambda) & \xrightarrow{\epsilon^{i}_{w_2,w_1}\varphi^{\lambda,y_i}_{w_2,w_1}} & M^y(w_2 \circ \lambda). \end{array} \tag{42}$$

**Proposition 6.8.** *Let* $\lambda \in \mathfrak{h}^*$ *be regular dominant,* $y \in \mathcal{W}(\lambda)$, $\alpha \in \Pi(\lambda)$ *such that* $\ell_\lambda(ys_\alpha) = \ell_\lambda(y) + 1$. *Set* $\beta = y(\alpha)$

(i) *Let* $w_1, w_2 \in \mathcal{W}(\lambda)$. *Suppose that* $w_1 \rhd_y w_2$, $w_1 \rhd_{ys_\alpha} w_2$ *and* $\ell^y_\lambda(w_1) = \ell^{ys_\alpha}_\lambda(w_1)$. *Then*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathfrak{g}}(M^y(w_1 \circ \lambda), M^{ys_\alpha}(w_2 \circ \lambda)) = 1.$$

*Moreover, either of the followings span the one-dimensional vector space* $\mathrm{Hom}_{\mathfrak{g}}(M^y(w_1 \circ \lambda), M^{ys_\alpha}(w_2 \circ \lambda))$*:*

(a) *the composition* $M^y(w_1 \circ \lambda) \to M^y(w_2 \circ \lambda) \to M^{ys_\alpha}(w_2 \circ \lambda)$ *of any non-trivial* $\mathfrak{g}$*-homomorphisms;*

(b) *the composition* $M^y(w_1 \circ \lambda) \to M^{ys_\alpha}(w_1 \circ \lambda) \to M^{ys_\alpha}(w_2 \circ \lambda)$ *of any non-trivial* $\mathfrak{g}$*-homomorphisms.*

(ii) *Let* $w_1, w_2 \in \mathcal{W}(\lambda)$. *Suppose that* $\ell^y_\lambda(w_1) = \ell^y_\lambda(w_2) + 2$ *and* $w_i^{-1}(\beta) \in \Delta^{re}_+$ *for* $i = 1, 2$. *Then the composition* $M^y(w_1 \circ \lambda) \to M^y(w_2 \circ \lambda) \to M^{ys_\alpha}(w_2 \circ \lambda)$ *of any non-trivial homomorphisms is non-zero.*

(iii) *Let* $w \in \mathcal{W}(\lambda)$ *and suppose that* $s_\alpha w \rhd_{\lambda,y} w$. *Then the composition* $M^y(s_\alpha w \circ \lambda) \to M^y(w \circ \lambda) \to M^{ys_\alpha}(w \circ \lambda)$ *of any* $\mathfrak{g}$*-homomorphisms is zero.*

*Proof.* (i) Since $y^{-1}w_1 \rhd y^{-1}w_2$, the Jantzen sum formula implies that

$$[M(y^{-1}w_2 \circ \lambda) : L(y^{-1}w_1 \circ \lambda)] = 1.$$

Hence $[M^{s_\alpha}(y^{-1}w_2 \circ \lambda) : L(y^{-1}w_1 \circ \lambda)] = 1$. As

$$\mathrm{Hom}(M^y(w_1 \circ \lambda), M^{ys_\alpha}(w_2 \circ \lambda)) \cong \mathrm{Hom}(M(y^{-1}w_1 \circ \lambda), M^{s_\alpha}(y^{-1}w_2 \circ \lambda)),$$

it follows that

$$\dim_{\mathbb{C}} \mathrm{Hom}(M^y(w_1 \circ \lambda), M^{ys_\alpha}(w_2 \circ \lambda)) \leq 1$$

Now we have

$$\mathrm{Hom}_{\mathfrak{g}}(M^y(w_1 \circ \lambda), M^y(w_2 \circ \lambda)) \cong \mathrm{Hom}_{\mathfrak{g}}(M(y^{-1}w_1 \circ \lambda), M(y^{-1}w_2 \circ \lambda)),$$
$$\mathrm{Hom}_{\mathfrak{g}}(M^y(w_1 \circ \lambda), M^{ys_\alpha}(w_1 \circ \lambda)) \cong \mathrm{Hom}_{\mathfrak{g}}(M(y^{-1}w_1 \circ \lambda), M^{s_\alpha}(y^{-1}w_1 \circ \lambda)),$$
$$\mathrm{Hom}_{\mathfrak{g}}(M^y(w_2 \circ \lambda), M^{ys_\alpha}(w_2 \circ \lambda)) \cong \mathrm{Hom}_{\mathfrak{g}}(M(y^{-1}w_2 \circ \lambda), M^{s_\alpha}(y^{-1}w_2 \circ \lambda)),$$
$$\mathrm{Hom}_{\mathfrak{g}}(M^{ys_\alpha}(w_1 \circ \lambda), M^{ys_\alpha}(w_2 \circ \lambda)) \cong \mathrm{Hom}_{\mathfrak{g}}(M(s_\alpha y^{-1}w_1 \circ \lambda), M(s_\alpha y^{-1}w_2 \circ \lambda)).$$

In particular they are all one-dimensional. Hence it remains to show that the compositions in (a) and (b) are non-trivial. This is equivalent to the non-triviality of the compositions

$$M(y^{-1}w_1\circ\lambda)\to M(y^{-1}w_2\circ\lambda)\to M^{s_\alpha}(y^{-1}w_2\circ\lambda)$$
$$\text{and } M(y^{-1}w_1\circ\lambda)\to M^{s_\alpha}(y^{-1}w_1\circ\lambda)\to M^{s_\alpha}(y^{-1}w_2\circ\lambda),$$

respectively. Therefore we may assume that $y=1$.

Since $\langle w_2(\lambda+\rho),\alpha^\vee\rangle\in\mathbb{N}$, we have the exact sequence

$$0\to M(s_\alpha w_2\circ\lambda)\to M(w_2\circ\lambda)\to M^{s_\alpha}(w_2\circ\lambda)\to M^{s_\alpha}(s_\alpha w_2\circ\lambda)\to 0 \tag{43}$$

by Proposition 5.10. On the other hand

$$w_1\circ\lambda\not\leq_\lambda s_\alpha w_2\circ\lambda \tag{44}$$

as we have the square $(s_\alpha w_1, w_1, s_\alpha w_2, w_2)$ by the assumption and (15). Hence (43) implies that the image of the highest weight vector of $M(w_1\circ\lambda)$ in $M(w_2\circ\lambda)$ does not lie in the kernel of the map $M(w_2\circ\lambda)\to M^{s_\alpha}(w_2\circ\lambda)$. This proves the non-triviality of the composition map in (a) for $y=1$, and thus, for all $y$. Next we show the non-triviality of the composition in (b). Consider the exact sequence

$$0\to M(s_\alpha w_1\circ\lambda)\to M(s_\alpha w_2\circ\lambda)\to N\to 0$$

in the category $\mathcal{O}^{\mathfrak{g}}_{[\lambda]}$, where $N=M(s_\alpha w_2\circ\lambda)/M(s_\alpha w_1\circ\lambda)$. Applying the functor $T_{s_\alpha}$ we obtain the exact sequence

$$0\to\mathcal{L}_1T_{s_\alpha}N\to M^{s_\alpha}(w_1\circ\lambda)\to M^{s_\alpha}(w_2\circ\lambda)\to T_iN\to 0. \tag{45}$$

By Proposition 5.11, the weights of $\mathcal{L}_1T_{s_\alpha}N$ are contained in the set of weights of $N$, and hence of $M(s_\alpha w_2\circ\lambda)$. Therefore (44) and (45) imply that the image of the highest weight vector of $M(w_1\circ\lambda)$ in $M^{s_\alpha}(w_1\circ\lambda)$ does not belong to the kernel of the map $M^{s_\alpha}(w_1\circ\lambda)\to M^{s_\alpha}(w_2\circ\lambda)$. This competes the proof of (i). (ii) Similarly as above, the problem reduces to the case $y=1$. By the assumption we have $s_\beta w_1\rhd_\lambda w_1$, $s_\beta w_2\rhd_\lambda w_2$. Thus $w_1\not\leq_\lambda s_\beta w_2$ because otherwise $(w_1, s_\beta w_1, s_\beta w_1, w_2)$ is a square. Hence (43) proves the assertion by the same argument as above. (iii) Again we may assume that $y=1$ and the assertion follows from (43). □

*Proof of Proposition 6.7.* We prove by induction on $i$ that such an assignment is possible.

As we already remarked the case $i=0$ is the well-known result of [BGG]. So let $i>0$. Suppose that $w_1\rhd_{\lambda,y_i}w_2$. Set $\beta=y_{i-1}(\alpha_i)\in\Delta_+^{re}$. The following four cases are possible. (The case $w_1^{-1}(\beta)\in\Delta_+^{re}$, $w_2^{-1}(\beta)\in\Delta_-^{re}$ does not happen by [BGG, Lemma 11.3].)

I) $w_1^{-1}(\beta), w_2^{-1}(\beta)\in\Delta_+^{re}$. In this case $w_1\rhd_{\lambda,y_{i-1}}w_2$, $\ell_\lambda^{y_i}(w_1)=\ell_\lambda^{y_{i-1}}(w_1)$ and $\ell_\lambda^{y_i}(w_2)=\ell_\lambda^{y_{i-1}}(w_2)$. By Proposition 6.8 there exists a unique $\epsilon^i_{w_2,w_1}$ which makes the diagram (42) commutes.

II) $w_1=s_\beta w_2$. In this case $w_2\rhd_{\lambda,y_{i-1}}w_1$, $\ell_\lambda^{y_i}(w_1)=\ell_\lambda^{y_{i-1}}(w_1)-2$ and $\ell_\lambda^{y_i}(w_2)=\ell_\lambda^{y_{i-1}}(w_2)$. We set $\epsilon^i_{w_2,w_1}=\epsilon^{i-1}_{w_1,w_2}$.

III) $w_1^{-1}(\beta), w_2^{-1}(\beta)\in\Delta_-^{re}$. In this case $w_1\rhd_{\lambda,y_{i-1}}w_2$, $\ell_\lambda^{y_i}(w_1)=\ell_\lambda^{y_{i-1}}(w_1)-2$ $\ell_\lambda^{y_i}(w_2)=\ell_\lambda^{y_{i-1}}(w_2)-2$, and we have the $y_i$-twisted square $(w_1, s_\beta w_1, w_2, s_\beta w_2)$. Note that $\epsilon^i_{s_\beta w_2, s_\beta w_1}$ is defined in I), and $\epsilon^i_{s_\beta w_1,w_1}$, $\epsilon^i_{s_\beta w_2,w_2}$ are defined in II). We set

$$\epsilon^i_{w_2,w_1}=-\frac{\epsilon^i_{s_\beta w_1,w_1}\epsilon^i_{s_\beta w_2,s_\beta w_1}}{\epsilon^i_{s_\beta w_2,w_2}}. \tag{46}$$

IV) $w_1^{-1}(\beta) \in \Delta_-^{re}$, $w_2^{-1}(\beta) \in \Delta_+^{re}$, $w_2 \neq s_\beta w_1$. In this case there exists a unique $w_3 \in \mathcal{W}$ such that $(s_\beta w_1, w_1, w_3, w_2)$ is a $y_i$-twisted square. Note that $w_3^{-1}(\beta) \in \Delta_+^{re}$ because $(w_3, w_2, s_\beta w_3, s_\beta w_2)$ is a $y_i$-twisted square by (15). Since $\epsilon^i_{w_3, s_\beta w_1}$, $\epsilon^i_{w_2,w_3}$ are defined in I) and $\epsilon^i_{w_1, s_\beta w_1}$ is defined in II), we can set

$$\epsilon^i_{w_1,w_1} = -\frac{\epsilon^i_{w_3,s_\beta w_1}\epsilon^i_{w_2,w_3}}{\epsilon^i_{w_1,s_\beta w_1}}. \tag{47}$$

Now let $(w_1, w_2, w_3, w_4)$ be a $y_i$-twisted square. Set

$$A_i(w_1,w_2,w_3,w_4) = \frac{\epsilon^i_{w_4,w_2}\epsilon^i_{w_2,w_1}}{\epsilon^i_{w_4,w_3}\epsilon^i_{w_3,w_1}}.$$

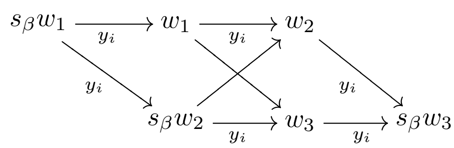
We need to show that $A_i(w_1, w_2, w_3, w_4) = -1$.

The following four cases are possible.

1) $w_2 = s_\beta w_1$, $w_4 = s_\beta w_3$. In this case the assertion follows from the definition (46).

2) $w_2 = s_\beta w_1$, $w_4 \neq s_\beta w_3$. In this case $(s_\beta w)^{-1}(\beta) \in \Delta_-^{re}$, and $w_4^{-1}(\beta) \in \Delta_+^{re}$ because otherwise $w_3 = s_\beta w_4$. Hence the assertion follows from the definition (47).

3) $w_2 \neq s_\beta w_1$, $w_4 = s_\beta w_3$. In this case $(s_\beta w_1, w_1, s_\beta w_2.w_2)$, $(s_\beta w_1, w_1, s_\beta w_2, w_3)$, $(s_\beta w_2, w_2, s_3, w_4)$ are $y_i$-twisted squares:




We have by 1)

$$A_i(s_\beta w_1, w_1, s_\beta w_2, w_2) = A_i(s_\beta w_2, w_2, w_3, s_\beta w_3) = -1$$

and by 2)

$$A_i(s_\beta w_1, w_1, s_\beta w_2, w_3) = -1.$$

But

$$\begin{aligned} &A_i(w_1, w_2, w_3, s_\beta w_3) \\ &= A_i(s_\beta w_1, w_1, s_\beta w_2, w_2) A_i(s_\beta w_2, w_2, w_3, s_\beta w_3) A_i(s_\beta w_1, s_\beta w_2, w_1, w_3). \end{aligned}$$

Hence the assertion follows.

4) $w_2 \neq s_\beta w_1$, $w_4 \neq s_\beta w_2$. we see as in [BGG, p.57, c)] that $w_4 \neq s_\beta w_2, s_\beta w_3$, and hence as in [BGG, p.56, 1)] we find that $(s_\beta w_1, s_\beta w_2, s_\beta w_3, s_\beta w_4)$ is also a $y_i$-twisted square. Hence a) $w_i^{-1}(\beta) \in \Delta_+^{re}$ for all $i$ or b) $w_i^{-1}(\beta) \in \Delta_-^{re}$ for all $i$.

a) The case $w_i^{-1}(\beta) \in \Delta_+^{re}$ for all $i$: By the definition I) we have the commutative diagram

$$\begin{array}{ccc} M^{y_{i-1}}(w_1 \circ \lambda) & \xrightarrow{\epsilon^{i-1}_{w_4,w_a}\epsilon^{i-1}_{w_a,w_1}\varphi^{\lambda,y_{i-1}}_{w_4,w_1}} & M^{y_{i-1}}(w_4 \circ \lambda) \\ \phi^{y_{i-1}}_{w_1}\downarrow & & \downarrow\phi^{y_{i-1}}_{w_4} \\ M^{y}(w_1 \circ \lambda) & \xrightarrow{\epsilon^{i}_{w_4,w_a}\epsilon^{i}_{w_a,w_1}\varphi^{\lambda,y_i}_{w_4,w_1}} & M^{y}(w_4 \circ \lambda) \end{array} \tag{48}$$

for $a=2,3$. Since $\epsilon^{i-1}_{w_4,w_2}\epsilon^{i-1}_{w_2,w_1}=-\epsilon^{i-1}_{w_4,w_3}\epsilon^{i-1}_{w_3,w_1}$ by the induction hypothesis the commutativity of the above diagram implies that $\epsilon^{i}_{w_4,w_2}\epsilon^{i}_{w_2,w_1}=-\epsilon^{i}_{w_4,w_3}\epsilon^{i}_{w_3,w_1}$ by Proposition 6.8 (ii).

b) The case that $w_i^{-1}(\beta)\in\Delta^{re}_-$ for all $i$: We have that $(s_\beta w_1,w_1,s_\beta w_2,w_2)$, $(s_\beta w_1,w_1,s_\beta w_3,w_3)$, $(s_\beta w_1,s_\beta w_2,s_\beta w_3,s_\beta w_4)$, $(s_\beta w_2,w_2,s_\beta w_4,w_4)$ and $(s_\beta w_3,w_3,s_\beta w_4,w_4)$ are all $y_i$-twisted squares. Hence the assertion follows from the equality

$$\begin{aligned}&A_i(w_1,w_2,w_3,w_4)A_i(s_\beta w_1,s_\beta w_2,w_1,w_2)A_i(s_\beta w_1,w_1,s_\beta w_3,w_3)\\ =\ &A_i(s_\beta w_1,s_\beta w_2,s_\beta w_3,s_\beta w_4)A_i(s_\beta w_2,w_2,s_\beta w_4,w_4)A_i(s_\beta w_3,s_\beta w_4,w_3,w_4).\end{aligned}$$

□

Let $k$ be an admissible number, $\lambda\in Pr_k$. Let $y\in\mathcal{W}(\lambda)$, $\{y_i\}$, $\{\phi^{y_i}_w\}$, $\{\epsilon^i_{w_2,w_1}\}$ be as in Proposition 6.7. Because $\{\epsilon^i_{w_2,w_1}\}$ satisfies the condition in Theorem 6.6 there is a corresponding twisted BGG resolution $\mathcal{B}^{y_i}_\bullet(\lambda)$ of $L(\lambda)$ for $i=0,1,\dots,l=\ell(y)$. Define

$$\Phi^{y_{i+1},y_i}_p=\bigoplus_{\substack{w\in\mathcal{W}(\lambda)\\ \ell^{y_i}_\lambda(w)=\ell^{y_{i+1}}_\lambda(w)=p}}\phi^{y_{i+1},y_i}_w:\mathcal{B}^{y_i}_p(w\circ\lambda)\to\mathcal{B}^{y_{i+1}}_p(w\circ\lambda).$$

**Proposition 6.9.** *In the above setting $\Phi^{y_{i+1},y_i}_\bullet$ gives a quasi-isomorphism $\mathcal{B}^{y_i}_\bullet(\lambda)\sim\mathcal{B}^{y_{i+1}}_\bullet(\lambda)$ of complexes for each $i=0,1,\dots,l-1$.*

**Lemma 6.10.** *Let $\lambda\in\mathfrak{h}^*$, $y$, $y_i$ be as in Proposition 6.7, $w_1,w_2\in\mathcal{W}(\lambda)$.*

(i) *Suppose that $w_1\rhd_{\lambda,y_i}w_2$, $\ell^{y_i}(w_1)=\ell^{y_{i+1}}(w_1)$. Then $w_1\rhd_{\lambda,y_{i+1}}w_2$.*
(ii) *Suppose that $w_1\rhd_{\lambda,y_i}w_2$, $\ell^{y_i}(w_2)=\ell^{y_{i+1}}(w_2)$. Then either of the following two holds.*
  (a) *$w_2=s_\beta w_1$ and $w_2\rhd_{\lambda,y_{i+1}}w_1$.*
  (b) *$w_1\rhd_{\lambda,y_{i+1}}w_2$.*

*Proof.* (1) By assumption $s_\beta w_1\rhd_{\lambda,y_i}w_2$. Therefore $(s_\beta w_1,w_1,s_\beta w_2,w_2)$ is a $y_i$-twisted square. (2) Similarly, if $w_2\neq s_\beta w_1$ then $(s_\beta w_1,w_1,s_\beta w_2,w_2)$ $y_i$-twisted square. The $w_2\neq s_\beta w_1$ case is obvious. □

*Proof of Proposition 6.9.* The fact that $\Phi^{y_i}_\bullet$ defines a homomorphism of complexes follows from the commutativity of (42), Proposition 6.8 (iii), and Lemma 6.10. Since both complexes are quasi-isomorphic to $L(\lambda)$, to show that it defines a quasi-isomorphism it suffices to check that it defines a non-trivial homomorphism between the corresponding homology spaces. This follows from the fact that $\phi^{y_i}_1:M^{y_i}(\lambda)\to M^{y_{i+1}}(\lambda)$ sends the highest weight vector of $M^{y_i}(\lambda)$ to the highest weight vector of $M^{y_{i+1}}(\lambda)$. □

6.5. **Two-sided BGG resolutions of $G$-integrable admissible representations.** For $\lambda\in Pr_k$ and $i\in\mathbb{Z}$ set

$$\mathcal{W}^i(\lambda)=\{w\in\mathcal{W}(\lambda);\ell^{\frac{\infty}{2}}_\lambda(w)=i\}.$$

We note that

$$\sharp\mathcal{W}^i(\lambda)=\begin{cases}1 & \text{if } \mathring{\mathfrak{g}}=\mathfrak{sl}_2,\\ \infty & \text{else.}\end{cases}$$

**Theorem 6.11.** *Let $k$ be an admissible number, $\lambda\in Pr^+_k$*

(i) *The space* $\mathrm{Hom}_{\mathfrak{g}}(W(w\circ\lambda), W(w'\circ\lambda))$ *is one-dimensional for* $w, w' \in \mathcal{W}(\lambda)$ *such that* $w \rhd_{\lambda,\frac{\infty}{2}} w'$.

(ii) *There exists a complex*

$$C^{\bullet}(\lambda) : \cdots \to C^{-2}(\lambda) \xrightarrow{d_{-2}} C^{-1}(\lambda) \xrightarrow{d_{-1}} C^{0}(\lambda) \xrightarrow{d_0} C^{1}(\lambda) \xrightarrow{d_1} C^{2}(\lambda) \xrightarrow{d_2} \cdots$$

*in the category* $\mathcal{O}$ *of the form*

$$C^{i}(\lambda) = \bigoplus_{w\in\mathcal{W}^{i}(\lambda)} W(w\circ\lambda), \quad d_i = \sum_{\substack{w\in\mathcal{W}^{i}(\lambda),\ w'\in\mathcal{W}^{i+1}(\lambda) \\ w\rhd_{\lambda,\frac{\infty}{2}} w'}} d_{w',w},$$

*where* $d_{w',w}$ *is a non-trivial* $\mathfrak{g}$*-homomorphism* $W(w\circ\lambda) \to W(w'\circ\lambda)$*, such that*

$$H^{i}(C^{\bullet}(\lambda)) \cong \begin{cases} L(\lambda) & \text{for } i = 0, \\ 0 & \text{for } i \neq 0. \end{cases}$$

*Proof.* (ii) Let $q$ be the denominator of $k$ and set $M = q\overset{\circ}{Q}{}^{\vee}$ if $(r^{\vee}, q) = 1$ and $M = q\overset{\circ}{Q}$ if $(r^{\vee}, q) = r^{\vee}$, so that $\mathcal{W}(\lambda) = \overset{\circ}{\mathcal{W}} \ltimes t_M$. Let $\gamma_1, \gamma_2, \ldots,$ be a sequence in $\overset{\circ}{P}{}^{\vee}_{+} \cap M$ such that $\gamma_i - \gamma \in \overset{\circ}{P}{}^{\vee}_{+} \cap M$, $\lim_{i\to\infty} \alpha(\gamma_i) = \infty$ for all $\alpha \in \overset{\circ}{\Delta}_{+}$.

By Proposition 6.9 there is an inductive system $\{\mathcal{B}^{-\gamma_i}_{\bullet}(\lambda)\}$ of twisted BGG resolutions. Let $\mathcal{B}^{\bullet}_{-\gamma_i}(\lambda)$ be the complex $\mathcal{B}^{-\gamma_i}_{\bullet}(\lambda)$ with the opposite homological grading. Thus it is a complex

$$B^{\bullet}_{-\gamma_i}(\lambda) : \cdots \xrightarrow{d_{-2}} \mathcal{B}^{-1}_{\gamma_i}(\lambda) \xrightarrow{d_{-1}} \mathcal{B}^{0}_{-\gamma_i}(\lambda) \xrightarrow{d_0} B^{1}_{\gamma_i}(\lambda) \xrightarrow{d_1} \cdots$$

of the form $\mathcal{B}^{p}_{\gamma_i}(\lambda) = \bigoplus_{\substack{w\in\mathcal{W}(\lambda) \\ \ell^{-\gamma_i}_{\lambda}(w)=-p}} M^{-\gamma_i}(w\circ\lambda)$, $d_p = \sum_{\substack{w,w'\in\mathcal{W}(\lambda) \\ \ell^{-\gamma_i}_{\lambda}(w)=-p,\, w\rhd_{\lambda,t_{-\gamma_i}} w'}} d^{\gamma_i}_{w',w}$, $d^{\gamma_i}_{w',w}$ : $M^{-\gamma_i}(w\circ\lambda) \to M^{-\gamma_i}(w'\circ\lambda)$ such that $H^{p}(B^{\bullet}_{-\gamma_i}(\lambda)) = \begin{cases} L(\lambda) & \text{if } p = 0, \\ 0 & \text{otherwise.} \end{cases}$

Let $(C^{\bullet}(\lambda), d_{\bullet})$ be the complex obtained as the inductive limit of complex $\mathcal{B}^{\bullet}_{-\gamma_i}(\lambda)$. By Lemma 3.2, Proposition 4.3 and Proposition 6.9 we have

$$C^{p}(\lambda) = \bigoplus_{w\in\mathcal{W}^{p}(\lambda)} \varinjlim_{i} M^{-\gamma_i}(w\circ\lambda) = \bigoplus_{w\in\mathcal{W}^{p}(\lambda)} W(w\circ\lambda) \quad \text{for } p \in \mathbb{Z},$$

$$H^{p}(C^{\bullet}(\lambda)) = \varinjlim_{i} H^{p}(B^{\bullet}_{-\gamma_i}(\lambda)) = \begin{cases} L(\lambda) & \text{if } p = 0, \\ 0 & \text{otherwise,} \end{cases}$$

and the differential $d_p : C^{p}(\lambda) \to C^{p+1}(\lambda)$ has the form

$$d_p = \sum_{\substack{w\in\mathcal{W}^{p}(\lambda),\ w'\in\mathcal{W}^{p+1}(\lambda) \\ w\rhd_{\lambda,\frac{\infty}{2}} w'}} d_{w',w},$$

where $d_{w',w} : W(w\circ\lambda) \to W(w'\circ\lambda)$ is induced by the homomorphisms $d^{-\gamma_i}_{w',w}$ : $M^{-\gamma_i}(w\circ\lambda) \to M^{-\gamma_i}(w'\circ\lambda)$ with $i = 1, 2, \ldots,$. To complete the proof of (ii) it remains to show that the map $d_{w',w}$ is nonzero for $w \rhd_{\lambda,\frac{\infty}{2}} w'$.

Let $w', w \in \mathcal{W}(\lambda)$ such that $w \rhd_{\lambda, \frac{\infty}{2}} w'$. We have the commutative diagram

$$\begin{array}{ccc} M^{-\gamma_i}(w' \circ \lambda) & \xrightarrow{d^{-\gamma_i}_{w,w'}} & M^{-\gamma_i}(w \circ \lambda) \\ \downarrow{\scriptstyle \phi^{w'\circ\lambda}_{-\gamma_i}} & & \downarrow{\scriptstyle \phi^{w\circ\lambda}_{-\gamma_i}} \\ W(w' \circ \lambda) & \xrightarrow{d_{w,w'}} & W(w \circ \lambda) \end{array}$$

for all $i$. By applying the functor $G_{-\gamma_i}$ we obtain the commutative diagram

$$\begin{array}{ccc} M(t_{\gamma_i} w' \circ \lambda) & \xrightarrow{G_{-\gamma_i}(d^{-\gamma_i}_{w,w'})} & M(t_{\gamma_i} w \circ \lambda) \\ \downarrow{\scriptstyle G_{-\gamma_i}(\phi^{w'\circ\lambda}_{-\gamma_i})} & & \downarrow{\scriptstyle G_{-\gamma_i}(\phi^{w'\circ\lambda}_{-\gamma_i})} \\ W(t_{\gamma_i} w' \circ \lambda) & \xrightarrow{G_{-\gamma_i}(d_{w,w'})} & W(t_{\gamma_i} w \circ \lambda). \end{array}$$

By Corollary 4.5 $d_{w,w'} \neq 0$ if and only if $G_{-\gamma_i}(d_{w,w'}) \neq 0$. Therefore it is sufficient to show that $G_{-\gamma_i}(\phi^{w'\circ\lambda}_{-\gamma_i}) \circ G_{-\gamma_i}(d^{-\gamma_i}_{w,w'}) : M(t_{\gamma_i} w' \circ \lambda) \to W(t_{\gamma_i} w \circ \lambda)$ is non-zero for a sufficiently large $i$.

Write $w' = s_\alpha w$ with $\alpha \in \Delta^{re}$, $\bar{\alpha} \in \mathring{\Delta}_-$. (This is possible because $s_\alpha = s_{-\alpha}$.) Then, for a sufficiently large $i$, $\beta := t_{\gamma_i}(\alpha) \in \Delta^{re}_+$ and $t_{\gamma_i} s_\alpha w = s_\beta t_{\gamma_i} w \to t_{\gamma_i} w$. The determinant formula [Fre1, Proposition 2 (2)] shows that the image of the highest weight vector of $M(t_{\gamma_i} w' \circ \lambda) = M(s_\beta t_{\gamma_i} w \circ \lambda)$ in $M(t_{\gamma_i} w \circ \lambda)$ is not in the kernel of the map $G_{\gamma_i}(\phi^{w',\lambda}_{\gamma_i}); M(t_{\gamma_i} w \circ \lambda) \to W(t_{\gamma_i} w \circ \lambda)$. Therefore $G_{\gamma_i}(\phi^{w',\lambda}_{\gamma_i}) \circ G_{\gamma_i}(d^{\gamma_i}_{w,w'})$ is non-zero, and hence so is $d_{w,w'}$.

Finally we shall prove (i). Note that

$$\mathrm{Hom}_{\mathfrak{g}}(W(w' \circ \lambda), W(w \circ \lambda)) = \varprojlim_i \mathrm{Hom}_{\mathfrak{g}}(M^{-\gamma_i}(w' \circ \lambda), W(w \circ \lambda))$$

and that $\mathrm{Hom}_{\mathfrak{g}}(M^{-\gamma_i}(w' \circ \lambda), W(w \circ \lambda))$ is at most one-dimensional by the Jantzen sum formula since $w' \rhd_\lambda w$. It follows from (the proof of) (ii) that $\mathrm{Hom}_{\mathfrak{g}}(W(w' \circ \lambda), W(w \circ \lambda))$ is spanned by $d_{w,w'}$. This completes the proof. □

*Remark* 6.12. By Theorem 6.11 (i) the resolution in Theorem 6.11 (ii) may be described in terms of screening operators as in [BF] provided that the existence of corresponding cycles is established, see e.g. [TK].

The following assertion is an immediate consequence of Theorem 6.11 which generalizes [FF2, Theorem 4.1].

**Theorem 6.13.** *Let $k$ be an admissible number, $\lambda \in Pr^+_k$, $p \in \mathbb{Z}$. We have*

$$H^{\frac{\infty}{2}+p}(\mathfrak{a}, L(\lambda)) = \bigoplus_{w \in \mathcal{W}^p(\lambda)} \mathbb{C}_{w\circ\lambda} \quad \text{as } \mathfrak{h}\text{-modules},$$

$$H^{\frac{\infty}{2}+p}(L\mathring{\mathfrak{n}}, L(\lambda)) = \bigoplus_{w \in \mathcal{W}^p(\lambda)} \pi_{w\circ\lambda + h^\vee \Lambda_0} \quad \text{as } \mathcal{H}\text{-modules}.$$

6.6. **A description of vacuum admissible representation.** Let $V^k(\mathring{\mathfrak{g}})$ be the universal affine vertex algebra associated with $\mathring{\mathfrak{g}}$ at level $k$:

$$V^k(\mathring{\mathfrak{g}}) = U(\mathfrak{g}) \otimes_{U(\mathring{\mathfrak{g}}[t] \oplus \mathbb{C}K)} \mathbb{C}_k,$$

where $\mathbb{C}_k$ is the one-dimensional representations of $\mathring{\mathfrak{g}}[t]\oplus\mathbb{C}K$ on which $\mathring{\mathfrak{g}}[t]$ acts trivially and $K$ acts as the multiplication by $k$. By [Fre2, Proposition 5.2] we have an injective homomorphism of vertex algebras

$$V^k(\mathring{\mathfrak{g}})\hookrightarrow W(k\Lambda_0)$$

for all $k\in\mathbb{C}$. Hence $V^k(\mathring{\mathfrak{g}})$ may be regarded as a vertex subalgebra of $W(k\Lambda_0)$.

Note that $L(k\Lambda_0)$ is the unique simple quotient of $V^k(\mathring{\mathfrak{g}})$.

**Proposition 6.14.** *Let $k$ be an admissible number, $\Psi: W(\dot{s}_0\circ k\Lambda_0)\to W(k\Lambda_0)$ a non-zero $\mathfrak{g}$-homomorphism, which exists uniquely up to a nonzero constant multiplication by Theorem 6.11 (i). Then the image of the highest weight vector of $W(\dot{s}_0\circ k\Lambda_0)$ generates the maximal submodule of $V^k(\mathring{\mathfrak{g}})\subset W(k\Lambda_0)$.*

*Proof.* By [KW1] the maximal submodule of $V^k(\mathring{\mathfrak{g}})$ is generated by a singular vector $v$ of weight $\dot{s}_0\circ k\Lambda_0$. Consider the two-sided resolution $C^\bullet(k\Lambda_0)$ of $L(k\Lambda_0)$ in Theorem 6.11 (ii). Because it is a resolution of $L(k\Lambda_0)$ and $V^k(\mathring{\mathfrak{g}})\subset W(k\lambda_0)$, the vector $v$ must be in the image of $d_{1,w}: W(w\circ k\Lambda_0)\to W(k\lambda_0)$ for some $w\in\mathcal{W}^{-1}(k\Lambda_0)$. Since the weight $w\circ k\Lambda_0$ is strictly smaller than $\dot{s}_0\circ k\Lambda_0$ for $w\in\mathcal{W}^{-1}(k\Lambda_0)\backslash\{\dot{s}_0\}$, the only possibility is that $v$ is the image of the highest weight vector of $W(\dot{s}_0\circ k\Lambda_0)$. □

6.7. **Two-sided BGG resolutions of more general admissible representations.** Let $\lambda\in Pr_{k,y}$ with $y=\bar{y}t_\eta$, $\bar{y}\in\mathring{\mathcal{W}}$, $\eta\in\mathring{Q}^\vee$. Then there exists $\lambda_1\in Pr_k^+$ such that $\lambda=y\circ\lambda_1$. Since $y(\Delta(\lambda_1)_+)\subset\Delta_+^{re}$, $T_y:\mathcal{O}^{\mathfrak{g}}_{[\lambda_1]}\to\mathcal{O}^{\mathfrak{g}}_{[\lambda]}$ is exact,

$$\begin{aligned}&T_yL(\lambda_1)\cong L(\lambda),\\ &T_yW(w\circ\lambda_1)\cong T_y\varinjlim_i M^{-\gamma_i}(w\circ\lambda_1)\cong\varinjlim_i T_yM^{-\gamma_i}(w\circ\lambda_1)\\ &\cong\varinjlim_i M^{-y(\gamma_i)}(ywy^{-1}\circ\lambda)\cong W^{\bar{y}}(ywy^{-1}\circ\lambda)\end{aligned}$$

for $w\in\mathcal{W}(\lambda_1)=y^{-1}\mathcal{W}(\lambda)y$ by Proposition 4.14, Lemmas 5.5 and 5.7, where $(\gamma_1,\gamma_2,\dots,)$ is a sequence as in proof of Theorem 6.11. Therefore the following assertion follows immediately from Theorem 6.6.

**Theorem 6.15.** . *Let $k$ be an admissible number, $\lambda\in Pr_{k,y}$ with $y=\bar{y}t_\eta$, $\bar{y}\in\mathring{\mathcal{W}}$, $\eta\in\mathring{P}^\vee$. Then there exists a complex*

$$C^\bullet(\lambda):\cdots\xrightarrow{d_{-3}}C^{-2}(\lambda)\xrightarrow{d_{-2}}C^{-1}(\lambda)\xrightarrow{d_{-1}}C^0(\lambda)\xrightarrow{d_0}C^1(\lambda)\xrightarrow{d_1}C^2(\lambda)\xrightarrow{d_2}\cdots$$

*in the category $\mathcal{O}$ of the form $C^i=\bigoplus_{w\in\mathcal{W}^i(\lambda)}W^{\bar{y}}(w\circ\lambda)$, $d_i=\sum_{\substack{w\in\mathcal{W}^i(\lambda),\ w'\in\mathcal{W}^{i+1}(\lambda)\\ w\rhd_{\lambda,\frac{\infty}{2}}w'}}d_{w',w}$.*

*such that*

$$H^i(C^\bullet(\lambda))\cong\begin{cases}L(\lambda) & \text{for } i=0,\\ 0 & \text{for } i\neq 0.\end{cases}$$

*Remark* 6.16. If $\lambda\in Pr_{k,y}$ and $\bar{y}=1$ (that is, $y\in\mathring{P}^\vee$), then $W^{\bar{y}}(w\circ\lambda)=W(w\circ\lambda)$. Hence the above is the resolution of $L(\lambda)$ in terms of (non-twisted) Wakimoto modules as conjectured in [FKW].

## 7. Semi-infinite restriction and induction

7.1. **Feigin-Frenkel parabolic induction.** Let $\mathring{\mathfrak{p}}$ be a parabolic subalgebra of $\mathring{\mathfrak{g}}$ containing $\mathring{\mathfrak{b}}_-$, and let $\mathring{\mathfrak{p}} = \mathring{\mathfrak{l}} \oplus \mathring{\mathfrak{m}}_-$ be the direct sum decomposition of $\mathring{\mathfrak{p}}$ with the Levi subalgebra $\mathring{\mathfrak{l}}$ containing $\mathring{\mathfrak{h}}$ and the nilpotent radical $\mathring{\mathfrak{m}}_-$. Denote by $\mathring{\mathfrak{m}} \subset \mathring{\mathfrak{n}}$ the opposite algebra of $\mathring{\mathfrak{m}}_-$, so that $\mathring{\mathfrak{g}} = \mathring{\mathfrak{p}} \oplus \mathring{\mathfrak{m}}$. Let

$$\mathring{\mathfrak{l}} = \mathring{\mathfrak{l}}_0 \oplus \bigoplus_{i=1}^{s} \mathring{\mathfrak{l}}_i$$

be the decomposition of $\mathring{\mathfrak{l}}$ into direct sum of simple Lie subalgebras $\mathring{\mathfrak{l}}_i$, $i = 1, \dots, s$, and its center $\mathring{\mathfrak{l}}_0$ of $\mathring{\mathfrak{l}}$. Let $\mathring{\mathfrak{h}}_i = \mathring{\mathfrak{l}} \cap \mathring{\mathfrak{h}}$, the Cartan subalgebra of $\mathring{\mathfrak{l}}_i$, and denote by $\mathring{\Delta}_i \subset \mathring{\Delta}$ the subroot system of $\mathring{\mathfrak{g}}$ corresponding to $\mathring{\mathfrak{l}}_i$, $\mathring{\Pi}_i = \mathring{\Pi} \cap \mathring{\Delta}_i$. Let $h_i^\vee$ be the dual Coxeter number of $\mathring{\mathfrak{l}}_i$ (with a convention $h_0^\vee = 0$), $\theta_i$ the highest root of $\mathring{\Delta}_i$, $\theta_{i,s}$ the highest short roof of $\mathring{\Delta}_i$.

Let $\mathfrak{l}_i = \mathring{\mathfrak{l}}_i[t, t^{-1}] \oplus \mathbb{C}K \subset \mathfrak{g}$ for $i = 0, 1, \dots, s$. Set

$$K_i = \frac{2}{(\theta_i|\theta_i)} K,$$

and we consider $K_i$ as an element of $\mathfrak{l}_i$. Thus,

$$\mathfrak{l}_i = \mathring{\mathfrak{l}}_i[t, t^{-1}] \oplus \mathbb{C}K_i,$$

and $\mathfrak{h}_i := \mathring{\mathfrak{h}}_i \oplus \mathbb{C}K_i$ is a Cartan subalgebra of $\mathring{\mathfrak{l}}_i$.

Define

$$\mathfrak{l} = \bigoplus_{i=0}^{s} \mathfrak{l}_i. \quad \mathfrak{t} = \bigoplus_{i=0}^{s} \mathfrak{h}_i.$$

The grading of $\mathfrak{l}_i$ induces the grading of $\mathfrak{l}$.

For $k \in \mathbb{C}$ define $k_0, \dots, k_s \in \mathbb{C}$ by

$$k_0 = k + h^\vee, \quad k_i + h_i^\vee = \frac{2}{(\theta_i|\theta_i)}(k + h^\vee) \quad \text{for } i = 1, \dots, s. \tag{49}$$

**Lemma 7.1.** *Let $k$ be an admissible number for $\mathfrak{g}$. Then $k_i$, $i = 1, \dots, s$, is an admissible number for the Kac-Moody algebra $\mathfrak{l}_i$.*

Let $\mathcal{O}^{\mathfrak{l}}_{(k_0,\dots,k_s)}$ be the full subcategory of $\mathcal{O}^{\mathfrak{l}}$ consisting of objects on which $K_i$ acts as the multiplication by $k_i$, $i = 0, 1, \dots, s$. Feigin and Frenkel [FF2, 5.2], [Fre2, §6] constructed a functor

$$\text{F-ind}^{\mathfrak{g}}_{\mathfrak{l}} : \mathcal{O}^{\mathfrak{l}}_{(k_0,k_1,\dots,k_s)} \to \mathcal{O}^{\mathfrak{g}}_k, \quad M \to \text{F-ind}^{\mathfrak{g}}_{\mathfrak{l}}(M),$$

which enjoys the property

$$\text{F-ind}^{\mathfrak{g}}_{\mathfrak{l}}(M) \cong US(L\mathring{\mathfrak{m}}) \otimes_{\mathbb{C}} M \tag{50}$$

as modules over

$$L\mathring{\mathfrak{m}} = \mathring{\mathfrak{m}}[t, t^{-1}] \subset \mathfrak{g},$$

where $L\mathring{\mathfrak{m}}$ only on the first factor $US(L\mathring{\mathfrak{m}})$. In particular F-ind$^{\mathfrak{g}}_{\mathfrak{l}}$ is an exact functor.

Denote by $W_{\mathfrak{l}_i}(\lambda^{(i)})$ the Wakimoto module of the affine Kac-Moody algebra $\mathfrak{l}_i$ with highest weight $\lambda^{(i)} \in {\mathfrak{h}_i}^*$ and by $L_{\mathfrak{l}}(\lambda^{(i)})$ the irreducible highest weight representation of $\mathfrak{l}_i$ with highest weight $\lambda^{(i)}$ (with a convention that $W_{\mathfrak{l}_0}(\lambda^{(0)})$ is the irreducible representation of the Heisenberg algebra $\mathfrak{l}_0$ with highest weight $\lambda^{(0)}$). For $\lambda \in \mathfrak{t}^*$ let $W_{\mathfrak{l}}(\lambda)$ and $L_{\mathfrak{l}}(\lambda)$ be the Wakimoto module and the irreducible highest weight representation of $\mathfrak{l}$ with highest weight $\lambda$:

$$W_{\mathfrak{l}}(\lambda) = \bigotimes_{i=0}^{s} W_{\mathfrak{l}_i}(\lambda|_{\mathfrak{h}_i}), \quad L_{\mathfrak{l}}(\lambda) = \bigotimes_{i=0}^{s} L_{\mathfrak{l}_i}(\lambda|_{\mathfrak{h}_i}).$$

For $\lambda \in \mathfrak{h}^*$, define $\lambda_{\mathfrak{l}} \in \mathfrak{t}^*$ by

$$\lambda_{\mathfrak{l}}|_{\overset{\circ}{\mathfrak{h}}_i} = \lambda|_{\overset{\circ}{\mathfrak{h}}_i} \text{ and } (\lambda_{\mathfrak{l}} + \rho_i)(K_i) = \frac{2}{(\theta_i|\theta_i)}(\lambda + \rho)(K)$$

for $i = 0, 1, \ldots, s$.

**Proposition 7.2** ([FF2])**.** *For $\lambda \in \mathfrak{h}^*$ we have* $\text{F-ind}_{\mathfrak{p}}^{\mathfrak{g}} W_{\mathfrak{l}}(\lambda_{\mathfrak{l}}) \cong W(\lambda)$*.*

*Proof.* By using the Hochschild-Serre spectral sequence for $L\overset{\circ}{\mathfrak{m}} \subset \mathfrak{a}$ we see from (50) that

$$H^{\frac{\infty}{2}+i}(\mathfrak{a}, \text{F-ind}_{\mathfrak{l}}^{\mathfrak{g}} W_{\mathfrak{l}}(\lambda_{\mathfrak{l}})) \cong \begin{cases} \mathbb{C}_\lambda & \text{for } i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Hence the assertion follows from Theorem 4.7. □

7.2. **Semi-infinite restriction functors.** Let $M \in \mathcal{O}_k^{\mathfrak{g}}$. Then $H^{\frac{\infty}{2}+p}(L\overset{\circ}{\mathfrak{m}}, M)$, $p \in \mathbb{Z}$, is naturally an $\mathfrak{l}$-module on which $K_i$ acts as the multiplication by $k_i$, see e.g. [HT, Proposition 2.3]. Hence

$$\text{S-res}_{\mathfrak{l}}^{\mathfrak{g}} := H^{\frac{\infty}{2}+0}(L\overset{\circ}{\mathfrak{m}}, ?)$$

defines a functor $\mathcal{O}_k^{\mathfrak{g}} \to \mathcal{O}_{(k_0,k_1,\ldots,k_s)}^{\mathfrak{l}}$. We refer to $\text{S-res}_{\mathfrak{l}}^{\mathfrak{g}}$ as the *semi-infinite restriction functor*.

The following assertion follows from Proposition 7.2.

**Proposition 7.3.** *For $\lambda \in \mathfrak{h}^*$ we have $H^{\frac{\infty}{2}+i}(L\overset{\circ}{\mathfrak{m}}, W(\lambda)) = 0$ for $i \neq 0$ and*

$$\text{S-res}_{\mathfrak{l}}^{\mathfrak{g}} W(\lambda) \cong W_{\mathfrak{l}}(\lambda_{\mathfrak{l}}).$$

7.3. **Decomposition of integral Weyl groups.** Let $k$ be an admissible number with denominator $q$, $\lambda \in Pr_k^+$. Let $\overset{\circ}{\mathcal{W}}_{S_i}$ be the parabolic subgroup of $\overset{\circ}{\mathcal{W}}$ corresponding to $\overset{\circ}{\mathfrak{l}}_i$, $\overset{\circ}{\mathcal{W}}_S = \overset{\circ}{\mathcal{W}}_{S_1} \times \overset{\circ}{\mathcal{W}}_{S_2} \times \cdots \times \overset{\circ}{\mathcal{W}}_{S_s}$. Define $\dot{\alpha}_0^{(i)} \in \Delta(\lambda)$, $i = 1, \ldots, s$, by

$$\dot{\alpha}_0^{(i)} = -\theta_i + q\delta \quad \text{if } (r^\vee, q) = 1,$$
$$\text{and } (\dot{\alpha}_0^{(i)})^\vee = -\theta_{i,s}^\vee + q\delta \quad \text{if } (r^\vee, q) = r^\vee.$$

Set $\dot{s}_0^{(i)} = s_{\dot{\alpha}_0^{(i)}}$.

Let $\mathcal{W}(\lambda)_{S_i}$ be the subgroup of $\mathcal{W}(\lambda)$ generated by $\overset{\circ}{\mathcal{W}}_{S_i}$ and $\dot{s}_0^{(i)}$. Then

$$\mathcal{W}(\lambda)_S = \mathcal{W}(\lambda)_{S_1} \times \mathcal{W}(\lambda)_{S_2} \times \cdots \times \mathcal{W}(\lambda)_{S_s}$$

is the subgroup corresponding to $\mathring{\mathcal{W}}_S$ described in §3.4. Let $\mathcal{W}(\lambda)^S \subset \mathcal{W}(\lambda)$ be as in Theorem 3.3 so that

(51)
$$\mathcal{W}(\lambda) = \mathcal{W}(\lambda)_S \times \mathcal{W}(\lambda)^S, \quad \ell_\lambda^{\frac{\infty}{2}}(uv) = \ell_\lambda^{\frac{\infty}{2}}(u) + \ell_\lambda^{\frac{\infty}{2}}(v) \text{ for } u \in \mathcal{W}(\lambda)_S,\ v \in \mathcal{W}(\lambda)^S.$$

Let $w, w' \in \mathcal{W}(\lambda)_{S_i} \subset \mathcal{W}(\lambda)$ such that $w \rhd_{\lambda,\frac{\infty}{2}} w'$. Then $w \circ_{\mathfrak{l}_i} \lambda_{\mathfrak{l}}^{(i)} = (w \circ \lambda)_{\mathfrak{l}}^{(i)}$, where $\circ_{\mathfrak{l}_i}$ is the dot action of $\mathcal{W}(\lambda)_{S_i}$ on $\mathfrak{h}_i^*$ and $\lambda_{\mathfrak{l}_i}^{(i)} = \lambda_{\mathfrak{l}}|_{\mathfrak{h}_i}$.

**Proposition 7.4.** *Let $\lambda \in Pr_k^+$, $w, w' \in \mathcal{W}(\lambda)_{S_i}$ with $i \in \{1, 2, \dots, s\}$ such that $w \rhd_{\lambda,\frac{\infty}{2}} w'$. Then the correspondence $\Phi \mapsto$ F-ind$_{\mathfrak{l}}^{\mathfrak{g}}(\Phi)$ defines a linear isomorphism*

$$\mathrm{Hom}_{\mathfrak{l}}(W_{\mathfrak{l}}((w \circ \lambda)_{\mathfrak{l}}), W_{\mathfrak{l}}((w' \circ \lambda)_{\mathfrak{l}})) \xrightarrow{\sim} \mathrm{Hom}_{\mathfrak{g}}(W(w \circ \lambda), W(w' \circ \lambda)).$$

*The inverse map is given by $\Psi \to$ S-res$_{\mathfrak{l}}^{\mathfrak{g}}(\Psi)$.*

*Proof.* By Proposition 4.6 and Theorem 6.11 (i) both $\mathrm{Hom}_{\mathfrak{l}}(W_{\mathfrak{l}}((w \circ \lambda)_{\mathfrak{l}}), W_{\mathfrak{l}}((w' \circ \lambda)_{\mathfrak{l}}))$ and $\mathrm{Hom}_{\mathfrak{g}}(W(w \circ \lambda), W(w' \circ \lambda))$ are one-dimensional. The assertion follows since the correspondence $\Phi \mapsto$ F-ind$_{\mathfrak{l}}^{\mathfrak{g}}(\tilde{\Phi})$ is clearly injective and S-res$_{\mathfrak{l}}^{\mathfrak{g}}$(F-ind$_{\mathfrak{l}}^{\mathfrak{g}}(\Phi)) = \Phi$. □

7.4. **Semi-infinite restriction of admissible affine vertex algebras.** Since it is defined by the semi-infinite cohomology the space S-res$_{\mathfrak{l}}^{\mathfrak{g}}(V^k(\mathring{\mathfrak{g}}))$ inherits a vertex algebra structure from $V^k(\mathring{\mathfrak{g}})$, and we have a natural vertex algebra homomorphism

$$\bigotimes_{i=0}^{s} V^{k_i}(\mathring{\mathfrak{l}}_i) \to \text{S-res}_{\mathfrak{l}}^{\mathfrak{g}}(V^k(\mathring{\mathfrak{g}})),$$

where $V^{k_i}(\mathring{\mathfrak{l}}_i)$ denote the universal affine vertex algebra associated with $\mathring{\mathfrak{l}}_i$ at level $k_i$. By composing with the map S-res$_{\mathfrak{l}}^{\mathfrak{g}}(V^k(\mathring{\mathfrak{g}})) \to$ S-res$_{\mathfrak{l}}^{\mathfrak{g}}(L(k\Lambda_0))$ induced by the surjection $V^k(\mathring{\mathfrak{g}}) \twoheadrightarrow L(k\Lambda_0)$ this gives rise to a vertex algebra homomorphism

(52)
$$\bigotimes_{i=0}^{s} V^{k_i}(\mathfrak{l}_i) \to \text{S-res}_{\mathfrak{l}}^{\mathfrak{g}}(L(k\Lambda_0)).$$

On the other hand there is a natural surjective homomorphism

$$\bigotimes_{i=0}^{s} V^{k_i}(\mathfrak{l}_i) \twoheadrightarrow \bigotimes_{i=0}^{s} L_{\mathfrak{l}_i}(k_i\Lambda_0)$$

of vertex algebras, where $L_{\mathfrak{l}_i}(k_i\Lambda_0)$ is the unique simple quotient of $V^{k_i}(\mathfrak{l}_i)$.

**Theorem 7.5.** *Let $k$ be an admissible number. The vertex algebra homomorphism (52) factors through the vertex algebra homomorphism*

$$\bigotimes_{i=0}^{s} L_{\mathfrak{l}_i}(k_i\Lambda_0) \hookrightarrow \text{S-res}_{\mathfrak{l}}^{\mathfrak{g}}(L(k\Lambda_0)).$$

*Proof.* Put $\lambda = k\Lambda_0$ and let $C^\bullet(\lambda)$ be the two-sided BGG resolution of $L(k\Lambda_0)$ in Theorem 6.11. By the vanishing assertion of Proposition 7.3 the semi-infinite cohomology $H^{\frac{\infty}{2}+\bullet}(L\mathring{\mathfrak{m}}, L(\lambda))$ is isomorphic to the cohomology of the complex S-res$_{\mathfrak{l}}^{\mathfrak{g}}(C^\bullet(\lambda))$ obtained from $C^\bullet(\lambda)$ applying the functor S-res$_{\mathfrak{l}}^{\mathfrak{g}}$. Thus S-res$_{\mathfrak{l}}^{\mathfrak{g}}(L(k\Lambda_0))$ is isomorphic to the zero-th cohomology of the complex S-res$_{\mathfrak{l}}^{\mathfrak{g}}(C^\bullet(\lambda))$.

Consider the map $C^{-1}(\lambda) \supset W(\dot{s}_0^{(i)} \circ \lambda) \overset{d_{1,\dot{s}_0^{(i)}}}{\to} W(\lambda) \subset C^0(\lambda)$ for $i = 1, \dots, s$. By applying the functor S-$\mathrm{res}^{\mathfrak{g}}_{\mathfrak{l}}$ this induces a non-zero homomorphism

$$W_{\mathfrak{l}}(\dot{s}_0^{(i)} \circ_{\mathfrak{l}_i} \lambda_{\mathfrak{l}}) \to W_{\mathfrak{l}}(\lambda_{\mathfrak{l}})$$

by Proposition 7.4, and the image of the highest weight vector of $W_{\mathfrak{l}}(\dot{s}_0^{(i)} \circ_{\mathfrak{l}_i} \lambda_{\mathfrak{l}})$ generates the maximal $\mathfrak{l}_i$-submodule of $V^{k_i}(\mathfrak{l}_i) \subset W_{\mathfrak{l}}(\lambda_{\mathfrak{l}})$ by Proposition 6.14. It follows that the maximal $\mathfrak{l}$-submodule of $\bigotimes_{i=0}^{s} V^{k_i}(\mathfrak{l}_i) \subset W_{\mathfrak{l}}(\lambda)$ is in the image of S-$\mathrm{res}^{\mathfrak{g}}_{\mathfrak{l}}(d_{-1})$ : S-$\mathrm{res}^{\mathfrak{g}}_{\mathfrak{l}}(C^{-1}(\lambda)) \to$ S-$\mathrm{res}^{\mathfrak{g}}_{\mathfrak{l}}(C^0(\lambda))$. This completes the proof. □

7.5. **The case of minimal parabolic subalgebras.** Consider the case that $\mathring{\mathfrak{p}}$ is generated by $\mathring{\mathfrak{b}}_-$ and $e_i$ with $i \in \mathring{I}$. Then $\mathring{\mathfrak{l}} = \mathring{\mathfrak{l}}_0 \oplus \mathring{\mathfrak{l}}_1$, $\mathring{\mathfrak{l}}_1 = \mathfrak{sl}_2^{(i)}$ and $\mathfrak{l}_1 = \widehat{\mathfrak{sl}}_2^{(i)}$.

**Theorem 7.6** ($\mathring{\mathfrak{p}}$ minimal)**.** *Let $k$ be an admissible number and let $M$ be a module over the vertex algebra $L(k\Lambda_0)$. Then, for each $p \in \mathbb{Z}$, $H^{\frac{\infty}{2}+p}(L\mathring{\mathfrak{m}}, M)$ is a direct sum of admissible representations of level $k_1$ (see (49)) as $\widehat{\mathfrak{sl}}_2^{(i)}$-modules.*

*Proof.* By Theorem 7.5, $L_{\mathfrak{l}_1}(k_1\Lambda_0)$ is a vertex subalgebra of S-$\mathrm{res}^{\mathfrak{g}}_{\mathfrak{l}}(L(k\Lambda_0)) = H^{\frac{\infty}{2}+0}(L\mathring{\mathfrak{m}}, L(k\Lambda_0))$. If $M$ is a module over $L(k\Lambda_0)$ then $H^{\frac{\infty}{2}+p}(L\mathring{\mathfrak{m}}, M)$ is naturally a module over S-$\mathrm{res}^{\mathfrak{g}}_{\mathfrak{l}}(L(k\Lambda_0))$, and therefore, it is a module over $L_{\mathfrak{l}_1}(k_1\Lambda_0)$. The assertion follows since it is known by [AM] that any module over $L_{\mathfrak{l}_1}(k_1\Lambda_0)$ in the category $\mathcal{O}^{\mathfrak{l}_1}$ must be a direct sum of admissible representations of $\mathfrak{l}_1 \cong \widehat{\mathfrak{sl}}_2$. □

The following assertion generalizes [HT, Theorem 3.8] in the case that $\mathring{\mathfrak{p}}$ is minimal.

**Theorem 7.7** ($\mathring{\mathfrak{p}}$ minimal)**.** *Let $k$ be an admissible number, $\lambda \in Pr_k^+$. Then*

$$H^{\frac{\infty}{2}+p}(L\mathring{\mathfrak{m}}, L(\lambda)) \cong \bigoplus_{\substack{w \in \mathcal{W}(\lambda)^S \\ \ell^{\frac{\infty}{2}}(w)=p}} L_{\mathfrak{l}}((w \circ \lambda)_{\mathfrak{l}})$$

*as $\mathfrak{l}$-modules.*

*Proof.* It is known by [MF] (see also [FM]) that $L(\lambda)$ with $\lambda \in Pr_k^+$ is a module over $L(k\Lambda_0)$. Therefore $H^{\frac{\infty}{2}+\bullet}(L\mathring{\mathfrak{m}}, L(\lambda))$ is a direct sum of irreducible admissible representations as $\widehat{\mathfrak{sl}}_2^{(i)}$-modules by Theorem 7.6. Hence it is sufficient to determine the subspace $H^{\frac{\infty}{2}+\bullet}(L\mathring{\mathfrak{m}}, L(\lambda))^{\mathfrak{l}_+}$ of the singular vectors of $H^{\frac{\infty}{2}+\bullet}(L\mathring{\mathfrak{m}}, L(\lambda))$. Clearly, any weight of $H^{\frac{\infty}{2}+\bullet}(L\mathring{\mathfrak{m}}, L(\lambda))^{\mathfrak{l}_+}$ must be admissible for $\mathfrak{l}_1 = \widehat{\mathfrak{sl}}_2^{(i)}$.

As is remarked in the proof of Proposition 7.5, $H^{\frac{\infty}{2}+\bullet}(L\mathring{\mathfrak{m}}, L(\lambda))$ is the cohomology of the complex S-$\mathrm{res}^{\mathfrak{g}}_{\mathfrak{l}}(C^\bullet(\lambda))$ and we have S-$\mathrm{res}^{\mathfrak{g}}_{\mathfrak{l}}(C^p(\lambda)) = \bigoplus_{w \in \mathcal{W}^p(\lambda)} W_{\mathfrak{l}}((w \circ \lambda)_{\mathfrak{l}})$ by Proposition 7.3. Now Theorem 3.3 and Lemma 7.1 imply that

$$\begin{aligned}
&\{(w \circ \lambda)_{\mathfrak{l}}; w \in \mathcal{W}(\lambda),\ (w \circ \lambda)_{\mathfrak{l}} \text{ is an admissible weight for } \widehat{\mathfrak{sl}}_2^{(i)}\} \\
=&\{(w \circ \lambda)_{\mathfrak{l}}; w \in \mathcal{W}(\lambda),\ (w \circ \lambda)_{\mathfrak{l}} \text{ is a dominant weight for } \widehat{\mathfrak{sl}}_2^{(i)}\} \\
=&\{(w \circ \lambda)_{\mathfrak{l}}; w \in \mathcal{W}(\lambda)^S\}.
\end{aligned}$$

It follows that if a weight $\mu$ of $W_{\mathfrak{l}}((w \circ \lambda)_{\mathfrak{l}})$ is admissible for $\widehat{\mathfrak{sl}}_2^{(i)}$ then $w \in \mathcal{W}(\lambda)^S$ and $\mu = (w \circ \lambda)_{\mathfrak{l}}$. Therefore the image $[|(w \circ \lambda)_{\mathfrak{l}}\rangle]$ of the highest weight vector $|(w \circ \lambda)_{\mathfrak{l}}\rangle$

of $W_{\mathfrak{l}}((w\circ\lambda)_{\mathfrak{l}})$ is nonzero in $H^{\frac{\infty}{2}+\bullet}(L\mathring{\mathfrak{m}}, L(\lambda))$ and $\{[|(w\circ\lambda)_{\mathfrak{l}}\rangle]; w\in\mathcal{W}(\lambda)^S\}$ forms a basis of $H^{\frac{\infty}{2}+\bullet}(L\mathring{\mathfrak{m}}, L(\lambda))^{\mathfrak{l}_+}$. By Theorem 3.3, this completes the proof. □

*Remark* 7.8. In the subsequent paper [A6] we prove that for an admissible number $k$ any $L(k\Lambda_0)$-module in the category $\mathcal{O}^{\mathfrak{g}}$ must be a direct sum of admissible representations. Hence it follows from the proof that the assertion of Theorem 7.7 is valid for any parabolic subalgebra of $\mathring{\mathfrak{g}}$.

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