Title	Two-Dimensional Transport Modeling of Tokamak Plasmas(Dissertation_全文)
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Citation	Kyoto University (京都大学)
Issue Date	2014-03-24
URL	http://dx.doi.org/10.14989/doctor.k18275
Right	許諾条件により本文は2014-12-31に公開
Type	Thesis or Dissertation
Textversion	ETD

Two-Dimensional Transport Modeling of Tokamak Plasmas

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2014

Contents

Chapter 1

Introduction

Since around the industrial revolution in the end of 18th century, the world population has been increasing explosively and reached about 7 billion in the middle of 2013 and United Nations prospects that the world population will increase up to about 10 billion in 2050 [1]. The global energy demand has been also increasing along with the world population growth and has been supplied mainly by the fossil energy and the fission energy until the present day. However it is necessary to suppress the use of the fossil energy and the fission energy as much as possible for the sustainable development since the they are unrenewable resources.

From the aspect of the environment conservation, the use of them should be also suppressed. For the fossil energy, the fuel waste have caused the critical environment issues all over the world, for example, the extreme climate and the global warming by the greenhouse effect gas (GHG), the acid rain by the SOx and the NOx, and the air pollution by the particulate matter (PM). For the fission energy, although the GHG is not directly emitted during generating electricity, the disposal method of the high-level waste (HLW) has not been established yet. Moreover, if the severe disaster occurs in the nuclear power plant like Chernobyl disaster and Fukushima Daiichi nuclear disaster, the vast area is contaminated by the high-level radioactive matter and its decontamination requires immeasurable cost and time.

Therefore the development of the alternative energy sources is one of the most important issues for the sustainable development of the human race and the fusion energy is the one of promising candidates for its preferable features;

- **Inexhaustible fuel resource**: Fuels of the nuclear fusion reactor are deuterium (D) and tritium (T). Deuterium is contained in large amounts in the water and tritium is mainly produced by the neutron activation of lithium (1.1)-(1.2) and lithium is also contained in large amounts in the ground and the sea,

  $$^{6}\mathrm{Li} + \mathrm{n} \rightarrow {}^{4}\mathrm{He} + \mathrm{T} + 4.8\mathrm{MeV}, \tag{1.1}$$

  $$^{7}\mathrm{Li} + \mathrm{n} \rightarrow {}^{4}\mathrm{He} + \mathrm{T} + \mathrm{n} - 2.5\mathrm{MeV}, \tag{1.2}$$

  where n is the neutron.

- **High-level long-lived waste free**: Although the structural material turns into the short-lived low-level waste by the neutron exposure, there is no high-level long-lived waste unlike the fission reactor, since the nuclear fusion reactor uses fusion reactions among light nuclei.

Name	Reaction	Ratio	Energy
D-D (a)	$D + D \rightarrow T(1.01MeV) + p(3.02MeV)$	50 %	4.03 MeV
D-D (b)	$D + D \rightarrow He^3(0.82MeV) + n(2.45MeV)$	50 %	3.27 MeV
D-T	$D + T \rightarrow He^4(3.5MeV) + n(14.1MeV)$	100 %	17.6 MeV
D-He^3	$D + He^3 \rightarrow He^4(3.6MeV) + p(14.7MeV)$	100 %	18.3 MeV

Table 1.1: Promising fusion reactions for fusion reactor [2], p is proton and ratios are correct for energies near the cross section peaks

- **Critical excursion free**: Although the core of the nuclear reactor reaches the very high temperature, its stored energy is very low due to its very low energy density. Moreover the fusion reaction stops when the fusion reactor is not under the controlled condition since the condition for maintaining the fusion reaction is very severe.
- **Low environment load**: The fusion energy does not emit the GHG, SOx, NOx and PM unlike the thermal energy and also does not emit the HLW unlike the fission energy.

1.1 Nuclear fusion reaction

The nuclear fusion reaction is a nuclear reaction in which two light atomic nuclei accelerated enough to overcome the coulomb barrier collide and fuse together to produce a heavy nucleus. The mass difference between two incident nuclei and fusion products Δm is converted into the kinetic energy E according to Einstein's mass-energy equivalence formula

$$E = \Delta m c^2, \tag{1.3}$$

where c is the speed of light in vacuum.

Some of promising fusion reactions for the fusion reactor and their reaction rates are listed in Table 1.1 and Figure 1.1, respectively. Since Figure 1.1 shows that the cross section of D-T reaction is much larger than those of the other fusion reactions at a relatively low temperature ($\sim$ 10 keV), D-T reaction is expected for a top candidate and will be employed in the ITER project.

1.2 Tokamak

The thermonuclear reaction requires to keep the hydrogen atoms in the high energy state (10 keV or higher) and hydrogen atoms ionize and form a hot plasma in such a high energy state. Therefore achieving the thermonuclear reactor requires a system to keep the plasma away from the first wall. There are two main types of the plasma confinement system.

The former is the inertial confinement fusion (ICF) which creates an extremely dense plasma by laser implosion. In ICF, the plasma density reaches extremely high so that the fusion reaction accomplishes instantaneously before the plasma meets the first wall. The

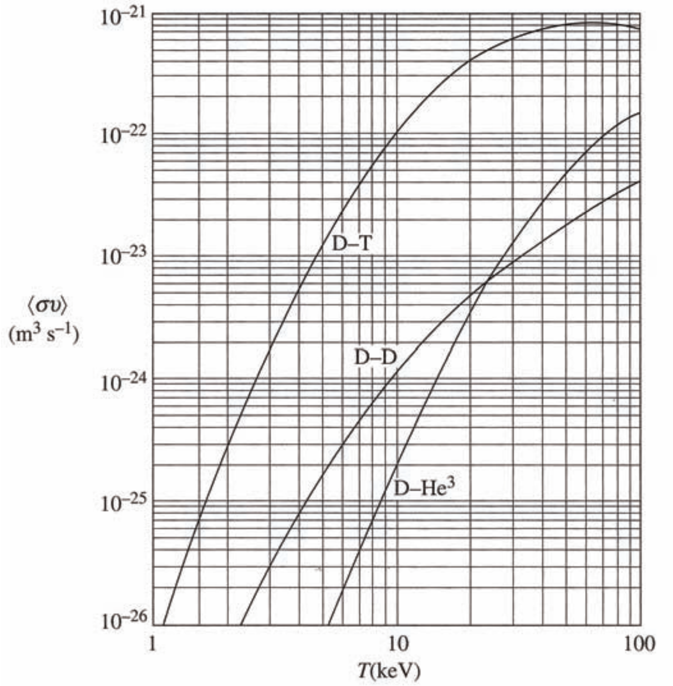

Figure 1.1: Cross sections of fusion reactions in m^3/s [3], where eV is the unit for temperature and is converted into K and J by 1eV = 11604K = 1.602×10^{-19}J.

latter is the magnetic confinement fusion (MCF) which employs the property of the charged particles moving along the magnetic field line due to the cyclotron motion. Although there are many types of magnetic confinement systems, we focus on the tokamak type magnetic confinement system which is employed in the ITER project.

Tokamak is a magnetic confinement system with axisymmetric torus geometry shown in Figure 1.2. In torus devices sustaining the plasma equilibrium requires the helical magnetic field structure. The principle magnetic field of a torus device is the toroidal magnetic field induced by external toroidal coils and the intensity of the magnetic field is inversely proportional to the distance R from the toroidal axis. This inhomogeneous magnetic field drives the drift motion of the charged particles called the ∇B drift and the curvature drift. Unfortunately the direction of these drift motions is the opposite direction for the positive and negative charged particles and then the charge separation occurs. Unless the poloidal magnetic field exist, the electric field arises in the vertical direction and this electric field drives the outward-directed drift called as $\boldsymbol{E} \times \boldsymbol{B}$ drift and the plasma equilibrium can no longer exist. If the poloidal magnetic field exist and the magnetic field has the helical structure, the charge separation can be short-circuited since the charged particle can move freely along the field line. In the tokamak device, the poloidal magnetic field is induced by the toroidal current. The helicity of magnetic field is defined by the dimensionless quantity called the safety factor defined by

$$q = \frac{\Delta\phi}{2\pi}, \tag{1.4}$$

where $\Delta\phi$ is the deviation angle of the magnetic field in the toroidal direction during the

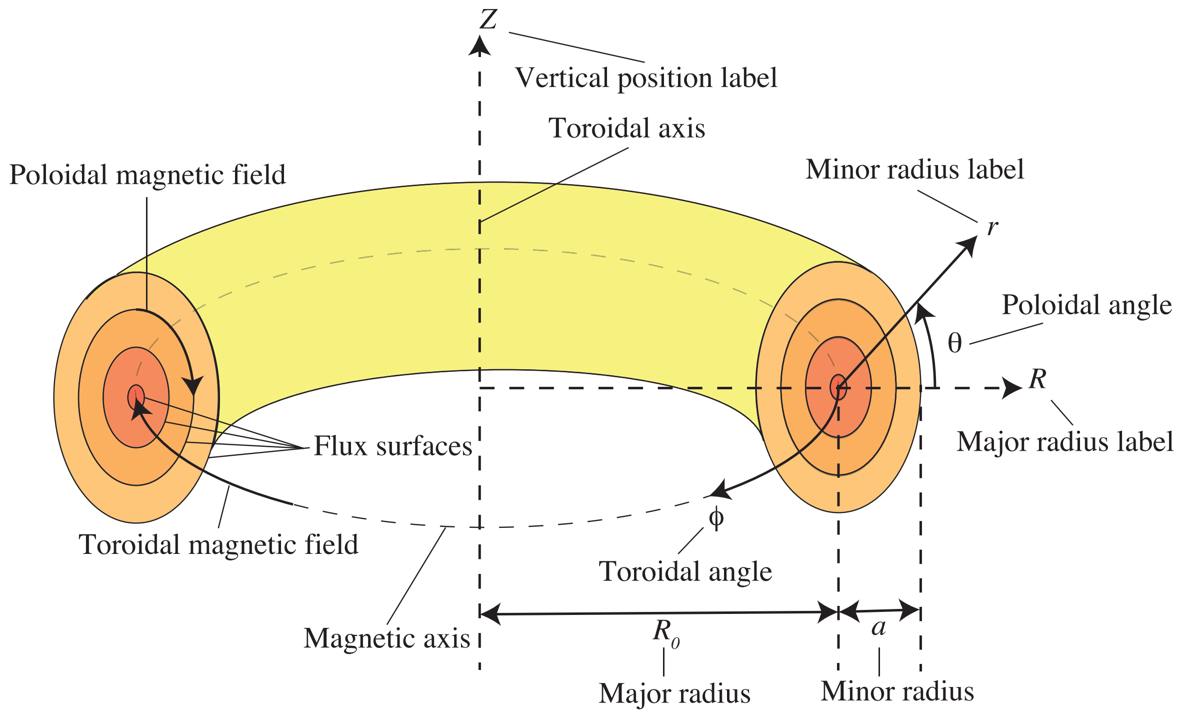

Figure 1.2: The geometry of torus fusion device, (R, Z, ϕ) and (r, θ, ϕ) are the cylindrical coordinate and the toroidal coordinate respectively

magnetic field line going around the torus in the poloidal direction. In the large aspect ratio limit $\varepsilon = r/R \to 0$, the safety factor can be approximated as

$$q = \frac{rB_\phi}{R_0 B_\theta}, \tag{1.5}$$

where $\varepsilon = r/R$ is the inverse aspect ratio.

1.3 Transport in tokamak plasma

Understanding of transport phenomena of plasmas is important to achieve a high temperature and high density enough required for the fusion reaction. There are three type of transport mechanisms in torus fusion devices; classical transport, neoclassical transport and turbulent transport. In this section we will explain them briefly.

1.3.1 Classical transport theory

The classical transport theory is based on the Coulomb collision in a homogeneous magnetic field. Since a charged particle moves along the magnetic field line due to the cyclotron motion, the transport in the radial direction is driven by the Coulomb collision with another particle. The classical transport can be estimated by the random work diffusion whose step size is the Larmor radius ϱ as

$$D_{\rm cl} = \nu \varrho^2, \tag{1.6}$$

where ν is the collision frequency.

1.3.2 Neoclassical transport theory

The neoclassical transport theory is based on the Coulomb collision in the inhomogeneous magnetic field and its property is classified into three regimes which are the Pfirsh-Schlüter regime, the banana regime and the plateau regime respectively.

The Pfirsh-Schlüter regime corresponds to the strongly collisional case described by

$$\nu \gg \frac{v_T}{qR}, \tag{1.7}$$

where v_T is the thermal velocity of the particle. Since the magnetic field line in tokamak a device has a helical structure, the drift surface is away from the magnetic surface by

$$\delta \sim \pm \varrho q. \tag{1.8}$$

If the plasma is strongly collisional, the neoclassical transport can be therefore estimated by the random work diffusion whose step size is the Larmor radius δ as

$$D_{\mathrm{P.S.}} = \nu \varrho^2 q^2, \tag{1.9}$$

which is larger than that of the classical diffusion (1.6) by the factor q^2.

The banana regime correspond to the weakly collisional case described by

$$\nu \ll \epsilon^{3/2} \frac{v_T}{qR}. \tag{1.10}$$

In the tokamak configuration, the magnetic field intensity B is inversely proportional to the major radius R. This inhomogeneous magnetic field results in the two kind of particles. One is the trapped particle which reflects at the points where its parallel velocity becomes 0 and whose guiding center describes the banana orbit due to the magnetic mirror effect. The other is the passing particle which completes its circular orbit. If the plasma is collisionless so that charged particles complete these orbits, the neoclassical transport is driven by the Coulomb collision between the trapped particle and the passing particle. Since the ratio of the trapped particle can be estimated by $\epsilon^{1/2}$, the neoclassical transport can be estimated by the random work diffusion whose step size by the banana width $\Delta_{\mathrm{b}} \sim q\varrho/\sqrt{\epsilon}$ and the effective collision frequency by $\nu_{\mathrm{eff}} \sim \nu/\epsilon$,

$$D_{\mathrm{b}} \sim \epsilon^{1/2} \Delta_{\mathrm{b}}^2 \nu_{\mathrm{eff}} \sim \epsilon^{-3/2} q^2 \varrho^2 \nu \tag{1.11}$$

Since $\epsilon \ll 1$ in the ordinary tokamak, the neoclassical transport is much larger than the classical transport in the weakly collisional regime.

The plateau regime correspond to the intermediate regime between the Pfirsh-Schlüter regime and the banana regime described by

$$\epsilon^{3/2} \frac{v_T}{qR} \ll \nu \ll \frac{v_T}{qR}, \tag{1.12}$$

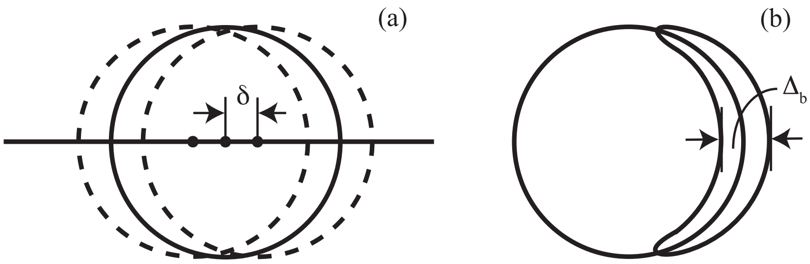

Figure 1.3: Diagram of the collisional transport in tokamak configuration. (a) is the diagram of the diffusive transport in the case of high collisionality and (b) the diagram of the diffusive transport in the case of low collisionality

and the estimation of the transport in the plateau regime is more complicated than the other regimes. From the kinetic theory however the transport in the plateau regime can be roughly estimated by

$$D_{\rm p} \sim q^2 \varrho^2 \frac{v_T}{qR} \tag{1.13}$$

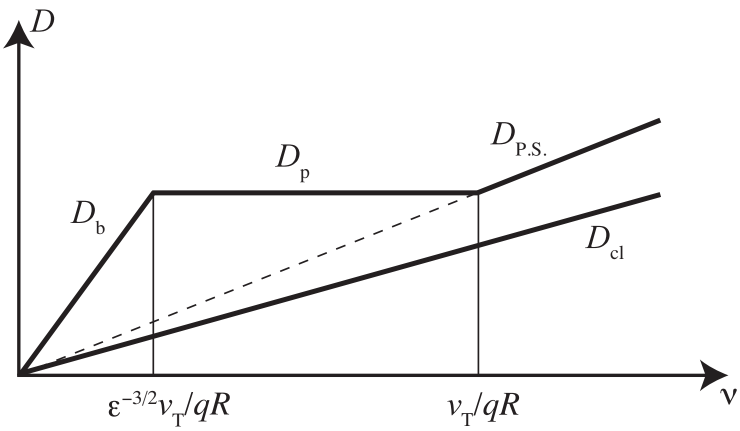

Figure 1.4: Dependent of classical and neoclassical diffusion coefficient on collisionality at large aspect ratio

1.3.3 Turbulent transport theory

The turbulent transport theory is based on the turbulence driven by micro-instabilities whose wavelength is comparable to the Larmor radii of ion or electron. Since the radial transport driven by turbulence is much larger than the other transport mechanism, the suppression of the turbulent transport is an important issue for good plasma confinement.

1.4 H-mode plasma

The performance of the plasma confinement is evaluated by the energy confinement time τ_E defined by

$$\tau_E = W/P, \tag{1.14}$$

where W is the total stored energy of the plasma and P is the total power input. The experimental energy confinement time is much shorter than that of the neoclassical prediction due to the turbulent transport and it has an undesirable dependence on the heating power. If the higher heating power is applied, the plasma pressure becomes higher and the turbulent transport is also enhanced; as a result, the confinement performance becomes worse.

It was discovered, however, that under certain operation conditions there is a discontinuous improvement in confinement when the heating power is increased. This improved confinement mode is called the H-mode, on the other hand the previous low confinement mode is called the L-mode and this discontinuous improvement in confinement is called as the L-H transition.

The H-mode was discovered in ASDEX tokamak [4] for the first time and was confirmed in various tokamak devices. It has been found that in the L-H transition, the turbulent transport is suppressed to the neoclassical level and the pedestal density and temperature profiles with steep gradient are formed in the edge region, which is called the edge transport barrier (ETB). Although the physics of ETB formation has not been well understood, some experimental studies indicated that a strongly sheared radial electric field and plasma rotation contribute to the ETB formation.

The steep gradient of density and temperature profiles in the ETB may induce plasma instabilities called the edge localized mode (ELM). The ELM triggers an emission of the energy stored in the edge region and an extraordinary heat load on the divertor plate which may cause the critical erosion of divertor plates. From the aspect of the divertor designing, a singular large ELM has to be avoided. Therefore the understanding of the H-mode physics is indispensable for not only the achievement of high confinement performance but also the divertor design.

1.5 Transport modeling in tokamak plasma

The concept of the single-null divertor tokamak configuration is shown in Figure 1.5, where "single-null" indicates that there is one point where the poloidal magnetic field vanishes in the poloidal cross section. In the single-null divertor configuration, the magnetic field structure is quite different inside and outside of the separatrix and therefore the key physics of transport is also different inside and outside of the separatrix.

The transport in the core and the peripheral regions have been therefore analyzed separately until recently owing to the difference in modeling configurations in spite of the fact that the core and peripheral plasmas are strongly coupled with each other.

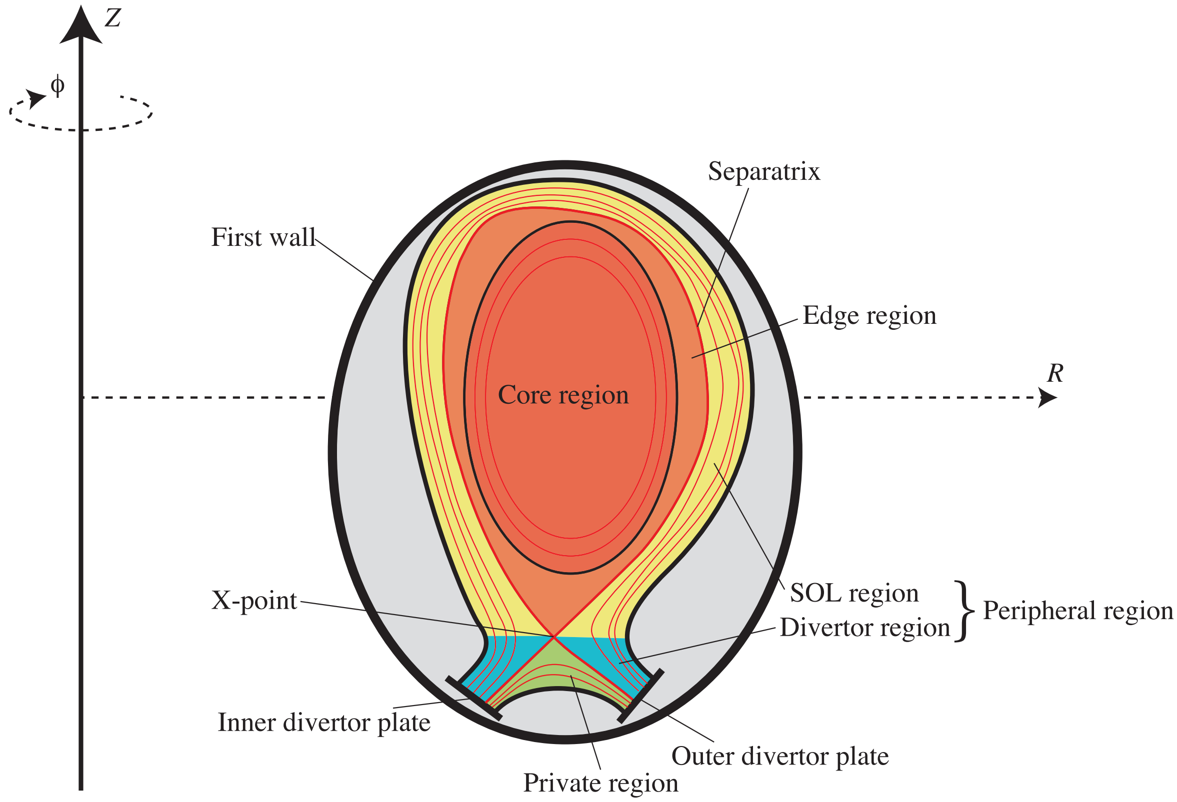

Figure 1.5: Concept of single-null divertor configuration of tokamak plasmas

1.5.1 Transport modeling in the core region

In the core region, the field lines are closed and their puncture plot on the poloidal cross section is nested surface structure called as the flux surface. Since the transport along the field lines is much faster than that in the radial direction, the poloidal and toroidal dependences of the plasma quantities such as the particle density and the temperature are negligibly small. The transport in the core region is therefore treated as one-dimensional problem in the radial direction by the use of the magnetic flux surface average for the quantities which have poloidal asymmetry. The core transport is explained by the neoclassical transport theory and turbulent transport theory.

A standard core transport modeling consists of diffusion equations for particle, toroidal momentum and energy transport as well as poloidal magnetic field [5, 6]. The force and the energy-weighted force balances in the parallel direction are employed to determine the poloidal particle and heat fluxes in the neoclassical theory [7, 8] and the charge neutrality is assumed. A new core transport modeling [9, 10] has been introduced for the analysis of plasma rotation. It includes the equation of motion and the radial electric field in addition to those mentioned above and the charge neutrality is not assumed, since the rotation and the radial electric field are strongly coupled.

One-dimensional (1D) core transport codes or 1.5D core transport codes composed of an one-dimensional core transport module and a two-dimensional MHD equilibrium module have been used for analyzing various transport issues [9–16], comparison of turbulent transport models [15], analysis of edge transport mechanism [16], analysis of plasma rotation [10] and so on.

1.5.2 Transport modeling in the peripheral region

In the peripheral region, the field line does not close and strikes at divertor plate so that the variation of quantities along field line is relatively large and important to understand the transport mechanism in the peripheral region. The peripheral transport therefore is usually described as an one dimensional problem in the parallel direction or a two-dimensional problem on the poloidal cross-section. The peripheral transport is explained by the classical transport theory and turbulent transport theory.

A standard peripheral transport modeling consists of advection-diffusion equations for particle, parallel momentum and energy transport [5]. These are based on the Braginskii's equations [17] extended to multi-species plasma [18, 19]. A new modeling [20] has been introduced for smooth extension to the weakly collision regime. It includes the contribution of the heat flux to the parallel viscosity term, since the contribution of the heat flux is comparable to that of the particle flux and important in the weakly collisional regime.

1D peripheral transport codes or 2D peripheral transport codes are used for various peripheral transport issues, impurity transport analysis [21], divertor designing [22], and so on.

1.5.3 Transport modeling in the edge region

Since the understanding of the edge transport physics is one of the critical issues as we have mentioned before, integrated core-edge-peripheral transport simulations on the whole tokamak plasma have been done by coupling a 1.5D core transport code with a 2D peripheral transport code. The simulation with TOPICS-IB [16] and SONIC [21] analyzed a L-H transition in JT-60SA [23] and that with a integrated suite JINTRAC [24] also analyzed a consecutive ELM-crash in JET [25]. There is an ambiguity, however, in the connection at a computational boundary which is an appropriately chosen flux surface inside and near the last closed flux surface (LCFS). In order to resolve this issue, the overlap computational domain in the edge region has been proposed [23]. However there are three problems in the conventional transport analysis for the edge transport.

The first issue is the applicability of the core transport modeling. Although the core transport modeling is developed on the assumption that the poloidal symmetry of some physical quantities, this assumption is violated in the edge region lying in the vicinity of the separatrix and interacting strongly with the peripheral plasma.

The second issue is the applicability of the peripheral transport modeling. In the H-mode discharge, the temperature reaches a few keV in the edge region and the edge plasma becomes weakly collisional. Since the peripheral transport modeling is developed on the assumption that the temperature is so low that the plasma is collisional.

The third issue is that the computational boundary in or near the edge region. Since simulation results may depend on the choice of the location of the boundary and the connection rule, it is desirable to remove computational boundaries near and in the edge region.

In order to resolve these three issues, it requires a two-dimensional transport modeling based on the neoclassical transport theory and the turbulent transport theory.

1.6 Contents of this thesis

The objective of this thesis is to develop a new two-dimensional transport modeling and a two-dimensional transport code applicable to the core, edge and peripheral regions of tokamaks to study the transport in the edge region and whole reactor plasmas self-consistently. The final goal of this study is to understand the ETB formation mechanism, since the understanding of the ETB formation mechanism is indispensable for improving the confinement performance and evaluating the heat load due to the ELM burst as we mentioned before.

In chapter 2, a set of equations describing the two-dimensional transport in a whole tokamak plasma is derived from the multi-fluid equations and Maxwell's equations. We show that our transport modeling is consistent with the neoclassical transport theory in the core region and with the classical transport theory in the strongly collisional limit.

Description of a new fluid-type two-dimensional transport code TASK/T2 is given in chapter 3. In TASK/T2, transport equations obtained in chapter 2 are implemented as a simultaneous advection-diffusion equations by the use of the finite element method. We discuss the numerical schemes of TASK/T2; a spatial discretization scheme, a time advancing scheme, and a computational grid, boundary conditions and initial conditions for a limiter configuration and the concept of a computational grid for a single null divertor configuration.

Finally the summary of this thesis and the future perspective are given in chapter 4.

Chapter 2

Formulation of two-dimensional multi-fluid transport

The core and peripheral plasmas are strongly coupled with each other in tokamaks. The particle and heat fluxes from the core determine the behavior of the peripheral plasma, while the peripheral plasma determines the edge density and temperature, boundary conditions of the core plasma. The transport in the core and the peripheral regions, however, have been analyzed separately until recently owing to the difference in modeling configurations.

In most conventional transport analyses in the core region, transport is usually described as one-dimensional problem in the radial direction based on the magnetic flux surface average, since the transport along the field lines is so fast that the poloidal and toroidal dependences of the plasma quantities such as the particle density and the temperature are small. One-dimensional (1D) core transport codes or 1.5D core transport codes composed of an one-dimensional core transport module and a two-dimensional MHD equilibrium module have been used for analyzing various transport issues [9–16], comparison of turbulent transport models [15], analysis of edge transport mechanism [16], analysis of plasma rotation [10] and so on.

A standard core transport modeling consists of diffusion equations for particle, toroidal momentum and energy transport as well as poloidal magnetic field [5, 6]. The force and the energy-weighted force balances in the parallel direction are employed to determine the poloidal particle and heat fluxes in the neoclassical theory [7, 8] and the charge neutrality is assumed. A new core transport modeling [9, 10] has been introduced for the analysis of plasma rotation. It includes the equation of motion and the radial electric field in addition to those mentioned above and the charge neutrality is not assumed, since the rotation and the radial electric field are strongly coupled.

On the other hand, in the peripheral region, the transport is usually described as a two-dimensional problem on the poloidal cross-section, since variation of physical quantities along a field line is relatively large and important to understand the transport mechanism in the peripheral region. Two-dimensional (2D) peripheral transport codes, for example B2 [18], B2.5 [19], EDGE2D [26], UEDGE [27] and SONIC [21, 22] have been developed and integrated with the neutral particle transport code and atomic process data. They are used for various peripheral transport issues, impurity transport analysis [21], divertor designing [22], and so on. Since these analyses are mainly based on the collisional transport model, they are not directly applicable to the weakly collisional core plasmas.

A standard peripheral transport modeling consists of advection-diffusion equations for particle, parallel momentum and energy transport [5]. These are based on the Braginskii's equations [17] extended to multi-species plasma [18]. A new modeling [20] has been introduced for smooth extension to the weakly collision regime. It includes the contribution of the heat flux to the parallel viscosity term, since the contribution of the heat flux is comparable to that of the particle flux and important in the weakly collisional regime.

Recently, integrated core-peripheral transport simulations on the whole tokamak plasma have been done by coupling a 1.5D core transport code with a 2D peripheral transport code. The simulation with TOPICS-IB [16] and SONIC [21] analyzed a L-H transition in JT-60SA [23] and that with a integrated suite JINTRAC [24] also analyzed a consecutive ELM-crash in JET [25]. There is an ambiguity, however, in the connection at the computational boundary which is an appropriately chosen flux surface inside and near the last closed flux surface (LCFS).

In order to resolve this issue, the overlap computational domain in the edge region has been proposed [23]. Since simulation results may depend on the choice of the location of the boundary and the connection rule, a transport code applicable to a whole plasma is desired for consistent transport simulation in both core and peripheral plasmas. Some efforts have been devoted to two-dimensional transport modeling, though they have not been published yet.

In this paper, we formulate two-dimensional fluid transport equations including the neoclassical transport [7] in the magnetic surface coordinate system. Our model is applicable to both core and peripheral plasmas in the axisymmetric tokamak configuration.

This paper is organized as follows. In section 2, the property and advantage of the magnetic surface coordinate system are described. The orderings used in this paper is discussed in section 3. The set of the multi-fluid equations and its closure are discussed in section 4. The set of two-dimensional transport equations is derived and it is confirmed that our two-dimensional transport model is consistent with the conventional one-dimensional neoclassical transport model in section 5. In section 6, the set of the electromagnetic equations is derived from Maxwell's equations. In section 7, the procedure for coupling a 2D transport solver with a 2D equilibrium solver is discussed. Summary and discussion are given in section 8.

2.1 Assumptions and coordinate system

In this paper, we assume axisymmetry of the system and the existence of flux surfaces with two-dimensional equilibrium magnetic field. Based on these assumptions, we employ a magnetic surface coordinate system (MSCS) (ρ, χ, ζ) in order to develop a two-dimensional transport model applicable to both the core and the peripheral regions. Here ρ is the radial coordinate label, χ is the poloidal angle, and ζ is the toroidal angle. In our MSCS, ρ is defined as the direction perpendicular to the magnetic field $\boldsymbol{B}$ and constructed by the toroidal flux function ϕ,

$$\phi \equiv \int_0^\rho d\rho' \int d\chi \sqrt{g} B^\zeta / \int d\chi, \tag{2.1}$$

χ is defined by the normalized length of the field line projected on a constant-ζ surface and ζ is defined by the geometrical toroidal angle.

Since the poloidal and toroidal angles are defined independently of the magnetic flux functions, MSCS is a kind of the non-flux coordinate system and is applicable even outside the separatrix on which the safety factor $q \equiv d\phi/d\psi$ diverges to infinity, where ψ is the poloidal flux function

$$\psi \equiv \int_0^\rho d\rho' \sqrt{g} B^\chi. \tag{2.2}$$

The axisymmetric magnetic field $\boldsymbol{B}$ can be written by the use of the two flux functions ψ and $I = B_\zeta$ [28],

$$\boldsymbol{B} = \nabla\zeta \times \nabla\psi + I\nabla\zeta. \tag{2.3}$$

The contravariant basis vectors ($\boldsymbol{e}^{\xi_i} \equiv \nabla\xi_i$) for the MSCS ($\xi_i = \rho, \chi, \zeta$) are $\boldsymbol{e}^\rho \equiv \nabla\rho$, $\boldsymbol{e}^\chi \equiv \nabla\chi$, $\boldsymbol{e}^\zeta \equiv \nabla\zeta$. The covariant basis vectors ($\boldsymbol{e}_{\xi_i} \equiv \partial\boldsymbol{x}/\partial\xi_i$) are $\boldsymbol{e}_\rho \equiv \sqrt{g}\nabla\chi \times \nabla\zeta$, $\boldsymbol{e}_\chi \equiv \sqrt{g}\nabla\zeta \times \nabla\rho$, $\boldsymbol{e}_\zeta \equiv \sqrt{g}\nabla\rho \times \nabla\chi$. The Jacobian is $\sqrt{g}^{-1} \equiv \nabla\rho \cdot \nabla\chi \times \nabla\zeta$. Since the geometrical toroidal angle is employed, the constant-ζ surface is orthogonal to both the constant-ρ and -χ surfaces so that $\boldsymbol{e}_\zeta$ and $\boldsymbol{e}^\zeta$ are parallel to one another,

$$\boldsymbol{e}_\zeta = R^2\nabla\zeta = R^2\boldsymbol{e}^\zeta, \quad B_\zeta = B^\zeta R^2, \tag{2.4}$$

where R is the major radius.

In this paper, the time evolution of the direction of the magnetic field and that of the metric tensor are neglected by assuming the slow change of magnetic flux surface. This assumption will be satisfied in most of phenomena with transport time scale, while it is not satisfied in rapid phenomena with Alfvén time scale.

The relation between the time derivatives in a fixed laboratory frame and in a moving magnetic surface frame can be expressed with a drift velocity of the flux surface $\boldsymbol{u}_g$ as

$$\left.\frac{\partial}{\partial t}\right|_{\boldsymbol{x}} = \left.\frac{\partial}{\partial t}\right|_{\rho,\chi,\zeta} - \boldsymbol{u}_g \cdot \nabla, \tag{2.5}$$

where the subscript $\boldsymbol{x}$ indicates the time derivative in the laboratory frame and the subscript ρ, χ, ζ in the magnetic surface frame. In the following discussion, the latter subscript is dropped for simplicity. The drift velocity of the magnetic surface is defined by

$$\boldsymbol{u}_g \equiv -\left.\frac{\partial\rho}{\partial t}\right|_{\boldsymbol{x}} \boldsymbol{e}_\rho - \left.\frac{\partial\chi}{\partial t}\right|_{\boldsymbol{x}} \boldsymbol{e}_\chi = u_g^\rho \boldsymbol{e}_\rho + u_g^\chi \boldsymbol{e}_\chi \tag{2.6}$$

and its actual expression depends on the definitions of ρ and χ. From the conservation of volume, Eq.(2.5) can be transformed as

$$\left.\frac{\partial f}{\partial t}\right|_{\boldsymbol{x}} = \frac{1}{\sqrt{g}}\frac{\partial}{\partial t}\left(\sqrt{g}f\right) - \nabla \cdot \left(\boldsymbol{u}_g f\right). \tag{2.7}$$

2.2 Multi-fluid equations

We consider the multi-fluid equations which describe the time evolution of macroscopic quantities, such as the particle density n_a, the momentum $m_a n_a \boldsymbol{u}_a$, the pressure p_a and the total heat flux $\boldsymbol{Q}_a$, derived from the kinetic equation for each plasma species,

$$\left.\frac{\partial f_a}{\partial t}\right|_{\boldsymbol{x}} + \boldsymbol{v}_a \cdot \nabla f_a + \frac{e_a}{m_a}\left(\boldsymbol{E} + \boldsymbol{v}_a \times \boldsymbol{B}\right) \cdot \frac{\partial f_a}{\partial \boldsymbol{v}} = C(f_a) + D_{\mathrm{QL}}(f_a) + S(f_a). \tag{2.8}$$

where f_a is the distribution function in six-dimensional phase space, $\boldsymbol{v}_a$ is the particle velocity, C is the collision operator, D_{QL} represents the quasi-linear interaction with waves, and S is the kinetic source. The multi-fluid equations are obtained by taking velocity moments $(1, m\boldsymbol{v}, mv^2/2, mv^2\boldsymbol{v}/2)$ of the kinetic equation.

Define the velocity moment of the distribution function with respect to $g_a(\boldsymbol{r}, \boldsymbol{v}_a, t)$ by

$$\langle g_a \rangle_f \equiv \frac{1}{n_a} \int g_a f_a d\boldsymbol{v}, \tag{2.9}$$

$$n_a \equiv \int f_a d\boldsymbol{v}. \tag{2.10}$$

By the use of Eq.(2.9), the velocity moment of the kinetic equation (2.8) can be written as

$$\begin{aligned} &\left.\frac{\partial}{\partial t}\left(n_a \langle g_a \rangle_f\right)\right|_{\boldsymbol{x}} + \nabla \cdot \left(n_a \langle \boldsymbol{v}_a g_a \rangle_f\right) - \frac{e_a}{m_a} n_a \left\langle (\boldsymbol{E} + \boldsymbol{v}_a \times \boldsymbol{B}) \cdot \frac{\partial g_a}{\partial \boldsymbol{v}_a} \right\rangle_f \\ &- n_a \left[\left.\left\langle \frac{\partial g_a}{\partial t} \right\rangle_f \right|_{\boldsymbol{x}} + \langle \nabla \cdot (n_a g_a) \rangle_f \right] = \int g_a C(f_a) d\boldsymbol{v} + \int g_a D_{\mathrm{QL}}(f_a) d\boldsymbol{v} + \int g_a S(f_a) d\boldsymbol{v}, \end{aligned} \tag{2.11}$$

where the following three relations has been used in the calculation of Eq.(2.11),

$$\int g_a \left.\frac{\partial f_a}{\partial t}\right|_{\boldsymbol{x}} d\boldsymbol{v} = \left.\frac{\partial}{\partial t}\left(n_a \langle g_a \rangle_f\right)\right|_{\boldsymbol{x}} - n \left.\frac{\partial}{\partial t}\left(\langle g_a \rangle_f\right)\right|_{\boldsymbol{x}}, \tag{2.12}$$

$$\int g_a \boldsymbol{v} \cdot \nabla f_a d\boldsymbol{v} = \nabla \cdot \left(n_a \langle g_a \boldsymbol{v}_a \rangle_f\right) - n_a \langle \nabla \cdot (g_a \boldsymbol{v}_a) \rangle_f, \tag{2.13}$$

$$\int g_a \frac{e_a}{m_a}\left(\boldsymbol{E} + \boldsymbol{v}_a \times \boldsymbol{B}\right) \cdot \frac{\partial f_a}{\partial \boldsymbol{v}_a} d\boldsymbol{v} = -\frac{e_a}{m_a} n_a \left\langle (\boldsymbol{E} + \boldsymbol{v}_a \times \boldsymbol{B}) \cdot \frac{\partial g_a}{\partial \boldsymbol{v}_a} \right\rangle_f. \tag{2.14}$$

The fourth term in the LHS of Eq.(2.11) vanishes if g_a is independent of the time and the position.

2.2.1 Equation of continuity

In the case of $g_a = 1$, Eq.(2.11) corresponds to the equation of continuity

$$\left.\frac{\partial n_a}{\partial t}\right|_{\boldsymbol{x}} + \nabla \cdot (n_a \boldsymbol{u}_a) = S_{na}, \tag{2.15}$$

where $\boldsymbol{u}_a \equiv \langle v_a \rangle_f$ is the flow velocity and S_{na} is the particle source

$$S_{na} = \int S(f_a) d\boldsymbol{v}. \tag{2.16}$$

Note that the collision term and the quasi-linear wave interaction term vanish in the case of $g = 1$, if the effect of the atomic process, e.g. the ionization and the recombination, are regarded as the particle source.

2.2.2 Equation of motion

In the case of $g_a = m_a \boldsymbol{v}_a$, Eq.(2.11) corresponds to the equation of motion

$$\frac{\partial}{\partial t}\left(n_a \langle m \boldsymbol{v}_a \rangle_f\right)\bigg|_{\boldsymbol{x}} + \nabla \cdot \left(n_a \langle m_a \boldsymbol{v}_a \boldsymbol{v}_a \rangle_f\right) - e_a n_a \left\langle (\boldsymbol{E} + \boldsymbol{v}_a \times \boldsymbol{B}) \cdot \overleftrightarrow{I} \right\rangle_f$$
$$= \int m_a v_a C(f_a) d\boldsymbol{v} + \int m_a v_a D^{\mathrm{QL}}(f_a) d\boldsymbol{v} + \int m_a v_a S(f_a) d\boldsymbol{v}, \tag{2.17}$$

where $n_a \langle m \boldsymbol{v}_a \rangle_f$ is the momentum $m_a n_a \boldsymbol{u}_a$ and the total stress tensor $\overleftrightarrow{P}_a$, the Lorentz force $\boldsymbol{F}_a^{\mathrm{Lor}}$, the friction force $\boldsymbol{F}_a^{\mathrm{fri}}$, the force driven by the interaction with waves $\boldsymbol{F}_a^{\mathrm{QL}}$ and the momentum source $\boldsymbol{S}_{ma}$ are introduced respectively by

$$\overleftrightarrow{P}_a \equiv n_a \langle m_a \boldsymbol{v}_a \boldsymbol{v}_a \rangle_f, \tag{2.18}$$

$$\boldsymbol{F}_a^{\mathrm{Lor}} \equiv e_a n_a \left\langle (\boldsymbol{E} + \boldsymbol{v}_a \times \boldsymbol{B}) \cdot \overleftrightarrow{I} \right\rangle_f = e_a n_a (\boldsymbol{E} + \boldsymbol{u}_a \times \boldsymbol{B}) \tag{2.19}$$

$$\boldsymbol{F}_a^{\mathrm{fri}} \equiv \int m_a v_a C(f_a) d\boldsymbol{v}, \tag{2.20}$$

$$\boldsymbol{F}_a^{\mathrm{QL}} \equiv \int m_a v_a D_{\mathrm{QL}}(f_a) d\boldsymbol{v}, \tag{2.21}$$

$$\boldsymbol{S}_{ma} \equiv \int m_a v_a S(f_a) d\boldsymbol{v}. \tag{2.22}$$

In order to evaluate the total stress tensor $\overleftrightarrow{P}_a$ with the macroscopic quantities, the particle velocity $\boldsymbol{v}_a$ is decomposed into the macroscopic velocity $\boldsymbol{u}_a$ and the random velocity $\boldsymbol{w}_a$,

$$\boldsymbol{v}_a = \boldsymbol{u}_a + \boldsymbol{w}_a, \tag{2.23}$$

where $\langle \boldsymbol{w}_a \rangle_f = \boldsymbol{0}$ from the definition of $\boldsymbol{u}_a$. Substituting Eq.(2.27) into Eq.(2.18), the total stress tensor can be decomposed into the inertial stress tensor $m_a n_a \boldsymbol{u}_a \boldsymbol{u}_a$ and the pressure tensor $m_a n_a \langle \boldsymbol{w}_a \boldsymbol{w}_a \rangle_f$,

$$\overleftrightarrow{P}_a = n_a \langle m_a \boldsymbol{v}_a \boldsymbol{v}_a \rangle_f = n_a \langle m_a (\boldsymbol{u}_a + \boldsymbol{w}_a)(\boldsymbol{u}_a + \boldsymbol{w}_a) \rangle_f$$
$$= m_a n_a \boldsymbol{u}_a \boldsymbol{u}_a + m_a n_a \langle \boldsymbol{w}_a \rangle_f \boldsymbol{u}_a + m_a n_a \boldsymbol{u}_a \langle \boldsymbol{w}_a \rangle_f + m_a n_a \langle \boldsymbol{w}_a \boldsymbol{w}_a \rangle_f$$
$$= m_a n_a \boldsymbol{u}_a \boldsymbol{u}_a + m_a n_a \langle \boldsymbol{w}_a \boldsymbol{w}_a \rangle_f. \tag{2.24}$$

Furthermore, the pressure tensor can be decomposed as

$$m_a n_a \langle \boldsymbol{w}_a \boldsymbol{w}_a \rangle_f \equiv p_a \overleftrightarrow{I} + \overleftrightarrow{\pi}_a, \tag{2.25}$$

where p_a is the isotropic pressure and $\overleftrightarrow{\pi}_a$ is the viscous stress tensor,

$$p_a \equiv \frac{1}{3} n_a m_a \left\langle w_a^2 \right\rangle_f, \tag{2.26}$$

$$\overleftrightarrow{\pi}_a \equiv n_a \left\langle m_a \left(\boldsymbol{w}_a \boldsymbol{w}_a - \frac{1}{3} w_a^2 \overleftrightarrow{I} \right) \right\rangle_f. \tag{2.27}$$

The expression of $\overleftrightarrow{\pi}_a$ in the strongly magnetized toroidal plasma is discussed as well as that of the friction force $\boldsymbol{F}_a^{\rm fri}$ in section 2.4.1. Therefore the total stress tensor can be written as

$$\overleftrightarrow{P}_a \equiv p_a \overleftrightarrow{I} + \overleftrightarrow{\pi}_a + m_a n_a \boldsymbol{u}_a \boldsymbol{u}_a. \tag{2.28}$$

The turbulent transport is induced by the interaction with low-frequency fluctuations. In the present framework, the quasi-linear term in the kinetic equation generates the force $F_a^{\rm QL}$ and this force induces particle and heat flux in the perpendicular direction. In the case of the electrostatic fluctuation, the poloidal force acting on electrons can be expresses in the toroidal coordinate (r, θ, ϕ) as [30, 31]

$$F_{e\theta}^{\rm QL} = eB_\phi n_e D_e \left[-\frac{1}{n_e} \frac{\partial n_e}{\partial r} + \frac{e}{T_e} E_r - \left\langle \frac{\omega}{m} \right\rangle_e r \frac{eB_\phi}{T_e} - \left(\frac{\mu_e}{D_e} - \frac{1}{2} \right) \frac{1}{T_e} \frac{\partial T_e}{\partial r} \right] \tag{2.29}$$

where ω and m are the mode frequency and poloidal mode number respectively, and $\langle \omega/m \rangle$ denotes the spectrum average of the phase velocity in the poloidal direction. In the above expression, we have assumed a symmetric wave spectrum with respect to $k_\parallel$ and weak velocity shear. The factor D_e is proportional to the square of the wave amplitude and corresponds to the ordinary diffusion coefficient. If the momentum is conserved between charged particles, the particle flux is intrinsically ambipolar. This particle transport model has been successfully implemented in the TASK/TX code [9]. The momentum and heat flux can be similarly implemented. The parallel component of the turbulence-induced force is neglected for simplicity, since the neoclassical term is considered to be dominant in the parallel direction.

Therefore the equation for motion becomes

$$\left. \frac{\partial}{\partial t} \left(m_a n_a \boldsymbol{u}_a \right) \right|_{\boldsymbol{x}} + \nabla \cdot \overleftrightarrow{P}_a = \boldsymbol{F}_a^{\rm Lor} + \boldsymbol{F}_a^{\rm fri} + \boldsymbol{F}_a^{\rm QL} + \boldsymbol{S}_{ma}, \tag{2.30}$$

2.2.3 Equation for energy transport

In the case of $g = m_a v_a^2/2$, Eq.(2.11) corresponds to the equation for energy transport

$$\left. \frac{\partial}{\partial t} \left(n_a \left\langle \frac{1}{2} m_a v_a^2 \right\rangle_f \right) \right|_{\boldsymbol{x}} + \nabla \cdot \left(n_a \left\langle \frac{1}{2} m_a v_a^2 \boldsymbol{v}_a \right\rangle_f \right) - e_a n_a \left\langle (\boldsymbol{E} \cdot \boldsymbol{v}_a \times \boldsymbol{B}) \cdot \boldsymbol{v}_a \right\rangle_f$$
$$= \int \frac{1}{2} m v_a^2 C(f_a) d\boldsymbol{v} + \int \frac{1}{2} m v_a^2 D_{\rm QL}(f_a) d\boldsymbol{v} + \int \frac{1}{2} m v_a^2 S(f_a) d\boldsymbol{v}, \tag{2.31}$$

where $n_a \left\langle \frac{1}{2} m_a v_a^2 \right\rangle_f$ is the total kinetic energy,

$$
\begin{aligned}
n_a \left\langle \frac{1}{2} m_a v_a^2 \right\rangle_f &= n_a \left\langle \frac{1}{2} m_a \left(\boldsymbol{u}_a + \boldsymbol{w}_a\right) \cdot \left(\boldsymbol{u}_a + \boldsymbol{w}_a\right) \right\rangle_f \\
&= \frac{1}{2} m_a n_a u_a^2 + \frac{1}{2} m_a n_a \boldsymbol{u}_a \cdot \langle \boldsymbol{w}_a \rangle_f + \frac{1}{2} m_a n_a \langle \boldsymbol{w}_a \rangle_f \cdot \boldsymbol{u}_a + \frac{1}{2} m_a n_a \left\langle w_a^2 \right\rangle_f \\
&= \frac{1}{2} m_a n_a u_a^2 + \frac{3}{2} p_a, \qquad (2.32)
\end{aligned}
$$

and the total heat flux $\boldsymbol{Q}_a$, the work by Lorentz force W_a^{Lor}, the energy exchange by the collision W_a^{col}, the work by the interaction with waves W_a^{QL}, the total energy source S_{Ea} are introduced respectively by

$$\boldsymbol{Q}_a \equiv n_a \left\langle \frac{1}{2} m_a v_a^2 \boldsymbol{v}_a \right\rangle_f, \qquad (2.33)$$

$$W_a^{\mathrm{Lor}} \equiv e_a n_a \left\langle \left(\boldsymbol{E} \cdot \boldsymbol{v}_a \times \boldsymbol{B}\right) \cdot \boldsymbol{v}_a \right\rangle_f, \qquad (2.34)$$

$$W_a^{\mathrm{col}} \equiv \int \frac{1}{2} m v_a^2 C(f_a) d\boldsymbol{v}, \qquad (2.35)$$

$$W_a^{\mathrm{QL}} \equiv \int \frac{1}{2} m v_a^2 D_{\mathrm{QL}}(f_a) d\boldsymbol{v}, \qquad (2.36)$$

$$S_{Ea} \equiv \int \frac{1}{2} m v_a^2 S(f_a) d\boldsymbol{v}. \qquad (2.37)$$

By the use of commutative law of the scalar product, the total heat flux $\boldsymbol{Q}_a$ is described as

$$
\begin{aligned}
\boldsymbol{Q}_a &= n_a \left\langle \frac{1}{2} m_a \boldsymbol{u}_a \cdot \boldsymbol{v}_a \boldsymbol{v}_a \right\rangle_f + n_a \left\langle \frac{1}{2} m_a \boldsymbol{u}_a \cdot \boldsymbol{w}_a \boldsymbol{v}_a \right\rangle_f + n_a \left\langle \frac{1}{2} m_a w_a^2 \boldsymbol{v}_a \right\rangle_f \\
&= \boldsymbol{u}_a \cdot \Big(n_a \left\langle m_a \boldsymbol{v}_a \boldsymbol{v}_a \right\rangle_f \Big) - \frac{1}{2} m_a n_a u_a^2 \boldsymbol{u}_a + n_a \left\langle \frac{1}{2} m_a w_a^2 \right\rangle_f \boldsymbol{u}_a + n_a \left\langle \frac{1}{2} m_a w_a^2 \boldsymbol{w}_a \right\rangle_f \\
&= \boldsymbol{q}_a + \frac{5}{2} p_a \boldsymbol{u}_a + \frac{1}{2} m_a n_a u_a^2 \boldsymbol{u}_a + \overleftrightarrow{\pi}_a \cdot \boldsymbol{u}_a, \qquad (2.38)
\end{aligned}
$$

where $\boldsymbol{q}_a$ is the heat flux defined by

$$\boldsymbol{q}_a \equiv \frac{1}{2} n_a m_a \left\langle w_a^2 \boldsymbol{w}_a \right\rangle_f. \qquad (2.39)$$

The work by the Lorentz force W_a^{Lor} becomes

$$W_a^{\mathrm{Lor}} = e_a n_a \left\langle \left(\boldsymbol{E} + \boldsymbol{v}_a \times \boldsymbol{B}\right) \cdot \boldsymbol{v}_a \right\rangle_f = e_a n_a \left\langle \boldsymbol{E} \cdot \boldsymbol{v}_a \right\rangle_f = e_a n_a \boldsymbol{E} \cdot \boldsymbol{u}_a. \qquad (2.40)$$

The energy exchange by the collision W_a^{col} can be expressed with the friction force $\boldsymbol{F}_a^{\mathrm{fri}}$ and the energy equipartition term $Q_{\Delta a}$

$$
\begin{aligned}
W_a^{\mathrm{col}} &= \frac{1}{2} m_a u_a^2 \int C(f_a) d\boldsymbol{v} + \boldsymbol{u}_a \cdot \int m \boldsymbol{v}_a C(f_a) d\boldsymbol{v} + \int \frac{1}{2} m_a w_a^2 C(f_a) d\boldsymbol{v} \\
&= \boldsymbol{u}_a \cdot \boldsymbol{F}_a + Q_{\Delta a}, \qquad (2.41)
\end{aligned}
$$

$$Q_{\Delta a} \equiv \sum_b \frac{3}{2} n_a \frac{T_b - T_a}{\tau_{ab}}, \qquad (2.42)$$

where $T_a \equiv p_a/n_a$ is the temperature. In Eq.(2.42), τ_{ab} is the heat exchange time defined by

$$\tau_{ab} \equiv \frac{3\sqrt{2}\pi^{3/2}\varepsilon_0^2 m_a m_b}{n_b e^4 Z_a^2 Z_b^2 \ln\Lambda_{ab}} \left(\frac{T_a}{m_a} + \frac{T_b}{m_b}\right)^{3/2}, \tag{2.43}$$

where $\ln\Lambda_{ab}$ is the Coulomb logarithm and provided by the following formulas [3]:

- electron-electron collisions

$$\ln\Lambda_{\mathrm{ee}} = 14.9 - \frac{1}{2}\ln n_{\mathrm{e}} + \ln T_{\mathrm{e}} \quad n_{\mathrm{e}} \text{ in } 10^{20}\mathrm{m}^{-3}\ T_{\mathrm{e}} \text{ in keV}, \tag{2.44}$$

- electron-ion collisions

$$\ln\Lambda_{\mathrm{ei}} = 15.2 - \frac{1}{2}\ln n_{\mathrm{e}} + \ln T_{\mathrm{e}} \quad n_{\mathrm{e}} \text{ in } 10^{20}\mathrm{m}^{-3}\ T_{\mathrm{e}} \text{ in keV}, \tag{2.45}$$

- ion-ion collision (singly charged ions, $T_{\mathrm{i}} \leq 10(m_{\mathrm{i}}/m_{\mathrm{p}})$ keV)

$$\ln\Lambda_{\mathrm{ii}} = 17.3 - \frac{1}{2}\ln n_{\mathrm{e}} + \frac{3}{2}\ln T_{\mathrm{i}} \quad n_{\mathrm{e}} \text{ in } 10^{20}\mathrm{m}^{-3}\ T_{\mathrm{i}} \text{ in keV}, \tag{2.46}$$

  where m_{p} is the proton mass.

The work by the interaction with waves W_a^{QL} can be expressed with $\boldsymbol{F}_a^{\mathrm{QL}}$ and the energy exchange by the interaction with waves $Q_{\Delta a}^{\mathrm{QL}}$,

$$\begin{aligned} W_a^{\mathrm{QL}} &= \frac{1}{2}m_a u_a^2 \int D_{\mathrm{QL}}(f_a)d\boldsymbol{v} + \boldsymbol{u}_a \cdot \int m\boldsymbol{v}_a D_{\mathrm{QL}}(f_a)d\boldsymbol{v} + \int \frac{1}{2}m_a w_a^2 D_{\mathrm{QL}}(f_a)d\boldsymbol{v} \\ &= \boldsymbol{u}_a \cdot \boldsymbol{F}_a^{\mathrm{QL}} + Q_{\Delta a}^{\mathrm{QL}}, \end{aligned} \tag{2.47}$$

where the specific expression of $Q_{\Delta a}^{\mathrm{QL}}$ is not described further in this thesis.

The equation for energy transport therefore can be written as

$$\frac{\partial}{\partial t}\left(\frac{3}{2}p_a + \frac{1}{2}m_a n_a \boldsymbol{u}_a\right)\bigg|_{\boldsymbol{x}} + \nabla \cdot \boldsymbol{Q}_a = W_a^{\mathrm{Lor}} + W_a^{\mathrm{fri}} + W_a^{\mathrm{QL}} + S_{Ea}. \tag{2.48}$$

The energy transport equation for internal energy (2.49) is employed instead of Eq.(2.48) in this thesis,

$$\frac{\partial}{\partial t}\left(\frac{3}{2}p_a\right)\bigg|_{\boldsymbol{x}} + \nabla \cdot \left(\boldsymbol{Q}_a - \frac{1}{2}m_a n_a u_a^2 \boldsymbol{u}_a\right) = \boldsymbol{u}_a \cdot \nabla p_a + \boldsymbol{u}_a \cdot \nabla \cdot \overleftrightarrow{\pi}_a + Q_{\Delta a} + Q_{\Delta a}^{\mathrm{QL}} + S_{pa}, \tag{2.49}$$

where S_{pa} is the internal energy source defined by

$$S_{pa} \equiv S_{Ea} - \boldsymbol{u}_a \cdot \boldsymbol{S}_{ma} + \frac{1}{2}m_a u_a^2 S_{na}. \tag{2.50}$$

In Eq.(2.50), the expression of internal energy source S_{pa} differs from the ordinary expression $S_{pa} \equiv S_{Ea} - \frac{1}{2}m_a u_a^2 S_{na}$, since S_{pa} in Eq.(2.50) includes contributions from not only particle source S_{na} but also momentum source $\boldsymbol{S}_{ma}$.

2.2.4 Equation for total heat flux

In the case of $g = m_a v_a^2 \boldsymbol{v}_a/2$, Eq.(2.11) corresponds to the equation for total heat flux,

$$\frac{\partial}{\partial t}\left(n_a\left\langle\frac{1}{2}m_a v_a^2\boldsymbol{v}_a\right\rangle_f\right)\Bigg|_{\boldsymbol{x}} + \nabla\cdot\left(n_a\left\langle\frac{1}{2}m_a v_a^2\boldsymbol{v}_a\boldsymbol{v}_a\right\rangle_f\right)$$
$$- e_a n_a\left\langle(\boldsymbol{E}+\boldsymbol{v}_a\times\boldsymbol{B})\cdot\left(\boldsymbol{v}_a\boldsymbol{v}_a+\frac{1}{2}v_a^2\overleftrightarrow{I}\right)\right\rangle_f$$
$$= \int\frac{1}{2}m_a v_a^2\boldsymbol{v}_a C(f_a)d\boldsymbol{v} + \int\frac{1}{2}m_a v_a^2\boldsymbol{v}_a D_{\rm QL}(f_a)d\boldsymbol{v} + \int\frac{1}{2}m_a v_a^2\boldsymbol{v}_a S(f_a)d\boldsymbol{v} \quad (2.51)$$

where the energy weighted (EW) total stress tensor $\overleftrightarrow{R}_a$, the EW Lorentz force $\boldsymbol{G}_a^{\rm Lor}$, the EW friction force $\boldsymbol{G}_a^{\rm fri}$, the total heat flux source $\boldsymbol{S}_{qa}$ are defined respectively by

$$\overleftrightarrow{R}_a \equiv n_a\left\langle\frac{1}{2}m_a v_a^2\boldsymbol{v}_a\boldsymbol{v}_a\right\rangle_f \quad (2.52)$$

$$\boldsymbol{G}_a^{\rm Lor} \equiv e_a n_a\left\langle(\boldsymbol{E}+\boldsymbol{v}_a\times\boldsymbol{B})\cdot\left(\boldsymbol{v}_a\boldsymbol{v}_a+\frac{1}{2}v_a^2\overleftrightarrow{I}\right)\right\rangle_f \quad (2.53)$$

$$\boldsymbol{G}_a^{\rm fri} \equiv \int\frac{1}{2}m_a v_a^2\boldsymbol{v}_a C(f_a)d\boldsymbol{v} \quad (2.54)$$

$$\boldsymbol{G}_a^{\rm QL} \equiv \int\frac{1}{2}m_a v_a^2\boldsymbol{v}_a D_{\rm QL}(f_a)d\boldsymbol{v} \quad (2.55)$$

$$\boldsymbol{S}_{qa} \equiv \int\frac{1}{2}m_a v_a^2\boldsymbol{v}_a S(f_a)d\boldsymbol{v}. \quad (2.56)$$

The EW total stress tensor can be decomposed into the EW inertial stress tensor part and the EW pressure tensor part,

$$\overleftrightarrow{R}_a = -\left(n_a\left\langle\frac{1}{2}m_a v_a^2\right\rangle_f\right)\boldsymbol{u}_a\boldsymbol{u}_a + \left(n_a\left\langle\frac{1}{2}m_a v_a^2\boldsymbol{v}_a\right\rangle_f\right)\boldsymbol{u}_a$$
$$+ \boldsymbol{u}_a\left(n_a\left\langle\frac{1}{2}m_a v_a^2\boldsymbol{v}_a\right\rangle_f\right) + n_a\left\langle\frac{1}{2}m_a v_a^2\boldsymbol{w}_a\boldsymbol{w}_a\right\rangle_f$$
$$= -\left(\frac{1}{2}m_a n_a u_a^2 + \frac{3}{2}p_a\right)\boldsymbol{u}_a\boldsymbol{u}_a + \boldsymbol{Q}_a\boldsymbol{u}_a + \boldsymbol{u}_a\boldsymbol{Q}_a + n_a\left\langle\frac{1}{2}m_a v_a^2\boldsymbol{w}_a\boldsymbol{w}_a\right\rangle_f. \quad (2.57)$$

For the formulation of the EW pressure tensor, we introduce the dimensionless parameter related to the particle velocity $x_a \equiv v_a/v_{Ta}$ and decompose the EW pressure tensor into the EW isotropic pressure and the EW viscous tensor similarly as the pressure tensor,

$$\begin{aligned}
n_a \left\langle \frac{1}{2} m_a v_a^2 \boldsymbol{w}_a \boldsymbol{w}_a \right\rangle_f &= \frac{T_a}{m_a} \left[\frac{5}{2} n_a \left\langle m_a \left(\boldsymbol{w}_a \boldsymbol{w}_a - \frac{w_a^2}{3} \overleftrightarrow{I} \right) \right\rangle_f \right. \\
&\quad \left. + n_a \left\langle m_a \left(\boldsymbol{w}_a \boldsymbol{w}_a - \frac{w_a^2}{3} \overleftrightarrow{I} \right) \left(x_a^2 - \frac{5}{2} \right) \right\rangle_f \right] \\
&\quad + \frac{T_a}{m_a} \left[\frac{5}{2} n_a \left\langle \frac{1}{3} m_a w_a^2 \right\rangle_f + n_a \left\langle \frac{1}{3} m_a w_a^2 \left(x_a^2 - \frac{5}{2} \right) \right\rangle_f \right] \overleftrightarrow{I} \\
&= \overleftrightarrow{r}_a + \left(\frac{5}{2} + c_a \right) \frac{T_a}{m_a} p_a \overleftrightarrow{I},
\end{aligned} \tag{2.58}$$

where $\overleftrightarrow{r}_a$ is the EW viscous tensor consisting of the viscous tensor $\overleftrightarrow{\pi}_a$ and the heat viscous tensor $\overleftrightarrow{\theta}_a$ and c_a is the dimensionless coefficient related to the higher moment of the isotropic pressure,

$$\overleftrightarrow{r}_a \equiv \frac{T_a}{m_a} \left(\frac{5}{2} \overleftrightarrow{\pi}_a + \overleftrightarrow{\theta}_a \right) \tag{2.59}$$

$$\overleftrightarrow{\theta}_a \equiv n_a \left\langle m_a \left(\boldsymbol{w}_a \boldsymbol{w}_a - \frac{1}{3} w_a^2 \overleftrightarrow{I} \right) \left(x_a^2 - \frac{5}{2} \right) \right\rangle_f \tag{2.60}$$

$$c_a \equiv \frac{1}{T_a} \left\langle \frac{1}{3} m_a w_a^2 \left(x_a^2 - \frac{5}{2} \right) \right\rangle_f . \tag{2.61}$$

Therefore the EW total stress tensor $\overleftrightarrow{R}_a$ becomes

$$\overleftrightarrow{R}_a \equiv \left(\frac{5}{2} + c_a \right) \frac{T_a}{m_a} p_a \overleftrightarrow{I} + \overleftrightarrow{r}_a + \boldsymbol{Q}_a \boldsymbol{u}_a + \boldsymbol{u}_a \boldsymbol{Q}_a - \left(\frac{3}{2} p_a + \frac{1}{2} m_a n_a u_a^2 \right) \boldsymbol{u}_a \boldsymbol{u}_a . \tag{2.62}$$

The EW Lorentz force $\boldsymbol{G}_a^{\mathrm{Lor}}$ becomes

$$\begin{aligned}
\boldsymbol{G}_a^{\mathrm{Lor}} &= e_a n_a \left\langle (\boldsymbol{E} + \boldsymbol{v}_a \times \boldsymbol{B}) \cdot \left(\boldsymbol{v}_a \boldsymbol{v}_a + \frac{1}{2} v_a^2 \overleftrightarrow{I} \right) \right\rangle_f \\
&= \frac{e_a}{m_a} \left[\boldsymbol{E} \cdot \left(n_a \langle m_a \boldsymbol{v}_a \boldsymbol{v}_a \rangle_f + n_a \left\langle \frac{1}{2} m_a v_a^2 \right\rangle_f \overleftrightarrow{I} \right) + n_a \left\langle \frac{1}{2} m_a v_a^2 \boldsymbol{v}_a \right\rangle_f \times \boldsymbol{B} \right] \\
&= \frac{e_a}{m_a} \left\{ \boldsymbol{E} \cdot \left[m_a n_a \boldsymbol{u}_a \boldsymbol{u}_a + \left(\frac{5}{2} p_a + \frac{1}{2} m_a n_a u_a^2 \right) \overleftrightarrow{I} + \overleftrightarrow{\pi}_a \right] + \boldsymbol{Q}_a \times \boldsymbol{B} \right\} .
\end{aligned} \tag{2.63}$$

The EW friction force $\boldsymbol{G}_a^{\mathrm{fri}}$ can be written by the use of the friction force $\boldsymbol{F}_a^{\mathrm{fri}}$ and the heat friction force $\boldsymbol{H}_a^{\mathrm{fri}}$,

$$\begin{aligned}
\boldsymbol{G}_a^{\mathrm{fri}} &\equiv \frac{T_a}{m_a} \left[\int m_a \boldsymbol{v}_a C(f_a) \left(x_a^2 - \frac{5}{2} \right) d\boldsymbol{v} + \frac{5}{2} \int m_a \boldsymbol{v}_a C(f_a) d\boldsymbol{v} \right] \\
&= \frac{T_a}{m_a} \left(\frac{5}{2} \boldsymbol{F}_a^{\mathrm{fri}} + \boldsymbol{H}_a^{\mathrm{fri}} \right)
\end{aligned} \tag{2.64}$$

$$\boldsymbol{H}_a^{\mathrm{fri}} \equiv \int m_a \boldsymbol{v}_a C(f_a) \left(x_a^2 - \frac{5}{2} \right) d\boldsymbol{v} \tag{2.65}$$

Therefore the equation for total heat flux becomes

$$\left.\frac{\partial \boldsymbol{Q}_a}{\partial t}\right|_{\boldsymbol{x}} + \nabla \cdot \overset{\leftrightarrow}{R}_a = \boldsymbol{G}_a^{\mathrm{Lor}} + \boldsymbol{G}_a^{\mathrm{fri}} + \boldsymbol{G}_a^{\mathrm{QL}} + \boldsymbol{S}_{qa}, \tag{2.66}$$

where the EW Lorentz force and the EW total stress tensor will be reduced to the extent that they keep consistency with the neoclassical theory in section 2.4.2.

2.3 Small gyro-radius ordering

In this paper, we employ the small gyro-radius ordering in order to formulate the two-dimensional transport model. In this ordering, a small expansion parameter δ_a for particle species a

$$\delta_a \equiv \frac{\varrho_a}{L_\perp} \ll 1 \tag{2.67}$$

is introduced, where $\varrho_a \equiv v_{Ta}/\omega_{ca}$ is the Larmor radius, $L_\perp$ is the macroscopic characteristic length in the perpendicular direction, $v_{Ta} \equiv \sqrt{2T_a/m_a}$ is the thermal velocity, $\omega_{ca} \equiv |e_a|B/m_a$ is the cyclotron frequency, e_a is the charge and m_a is the mass. Since $\delta_\mathrm{i} \sim \sqrt{m_i/m_e}\delta_\mathrm{e}$ in general, we consider $\delta \sim \delta_\mathrm{i}$ as the most severe restriction for the small gyro-radius ordering.

2.4 Modeling of the fluid closures

2.4.1 Friction forces and viscous tensors

The viscosity tensor $\overset{\leftrightarrow}{\pi}_a$, the heat viscosity tensor $\overset{\leftrightarrow}{\theta}_a$, the friction force $\boldsymbol{F}_a^{\mathrm{fri}}$, and the heat friction force $\boldsymbol{H}_a^{\mathrm{fri}}$ must be modeled in order to complete the multi-fluid equations. According to the moment approach [8], the lowest order friction force $\boldsymbol{F}_a^{\mathrm{fri}}$ and heat friction force $\boldsymbol{H}_a^{\mathrm{fri}}$ can be expressed in terms of flows

$$\boldsymbol{F}_a^{\mathrm{fri}} = \sum_b \left(l_{11}^{ab}\boldsymbol{u}_b - l_{12}^{ab}\frac{2\boldsymbol{q}_b}{5p_b} \right), \tag{2.68}$$

$$\boldsymbol{H}_a^{\mathrm{fri}} = \sum_b \left(-l_{21}^{ab}\boldsymbol{u}_b + l_{22}^{ab}\frac{2\boldsymbol{q}_b}{5p_b} \right), \tag{2.69}$$

where the coefficients l_{ij}^{ab} can be expressed in terms of the Braginskii's matrix elements of the collision operator [8]. Since the equation for total heat flux $\boldsymbol{Q}_a$ is employed instead of the equation for heat flux $\boldsymbol{q}_a$ in our formulation, the following approximation for the heat flux $\boldsymbol{q}_a$ up to $\mathcal{O}(\delta)$ have been employed for simplicity,

$$\boldsymbol{q}_a = \boldsymbol{Q}_a - \frac{5}{2}p_a\boldsymbol{u}_a + \mathcal{O}(\delta^2). \tag{2.70}$$

where Eq.(2.70) has sufficient accuracy for the modeling of the neoclassical friction force, since Eq.(2.68) and Eq.(2.69) describe the friction forces up to $\mathcal{O}(\delta)$.

In the strongly magnetized toroidal plasma, the viscous stress tensor $\overleftrightarrow{\pi}_a$ is decomposed into the parallel viscous tensor $\overleftrightarrow{\pi}_{\|a}$, the gyro viscous tensor $\overleftrightarrow{\pi}_{\wedge a}$ and the perpendicular viscous tensor $\overleftrightarrow{\pi}_{\perp a}$ and these three terms are scaled in the small gyro-radius ordering as

$$\overleftrightarrow{\pi}_a \equiv \overleftrightarrow{\pi}_{\|a} + \overleftrightarrow{\pi}_{\wedge a} + \overleftrightarrow{\pi}_{\perp a} \tag{2.71}$$

$$\overleftrightarrow{\pi}_{\|a} \sim \mathcal{O}(\delta p_a), \quad \overleftrightarrow{\pi}_{\wedge a} \sim \mathcal{O}(\delta^2 p_a), \quad \overleftrightarrow{\pi}_{\perp a} \sim \mathcal{O}(\delta^3 p_a) \tag{2.72}$$

In this paper, we take only the parallel viscous tensor into account for simplicity and In the lowest order of the drift ordering $\mathcal{O}(\delta)$, the viscosity tensor $\overleftrightarrow{\pi}_a$ and the heat viscosity tensor $\overleftrightarrow{\theta}_a$ are in the CGL form as

$$\overleftrightarrow{\pi}_a = \pi_{\|a}\left(\boldsymbol{e}_\|\boldsymbol{e}_\| - \frac{1}{3}\overleftrightarrow{I}\right) + \mathcal{O}(\delta^2), \tag{2.73}$$

$$\overleftrightarrow{\theta}_a = \theta_{\|a}\left(\boldsymbol{e}_\|\boldsymbol{e}_\| - \frac{1}{3}\overleftrightarrow{I}\right) + \mathcal{O}(\delta^2), \tag{2.74}$$

where $\boldsymbol{e}_\| \equiv \boldsymbol{B}/B$ is the unit vector in the parallel direction. In this paper, we define the parallel viscosities $\pi_{\|a}$ and $\theta_{\|a}$ in terms of the neoclassical parallel viscosity coefficients μ_{ai} and the parallel-parallel components of the rate-of-strain tensors W^{ua}_{zz} and W^{qa}_{zz} as

$$\begin{bmatrix} \pi_{\|a} \\ \theta_{\|a} \end{bmatrix} = -\frac{3}{2}\begin{bmatrix} \mu_{a1} & \mu_{a2} \\ \mu_{a2} & \mu_{a3} \end{bmatrix}\begin{bmatrix} W^{ua}_{zz} \\ W^{qa}_{zz} \end{bmatrix}, \tag{2.75}$$

where

$$W^{ua}_{zz} = 2\left(\nabla_\| u_{a\|} - \boldsymbol{u}_a \cdot \boldsymbol{\kappa}\right), \tag{2.76}$$

$$W^{qa}_{zz} = 2\left[\nabla_\|\left(\frac{2q_{a\|}}{5p_a}\right) - \frac{2\boldsymbol{q}_a}{5p_a}\cdot\boldsymbol{\kappa}\right]. \tag{2.77}$$

In Eqs.(2.76) and (2.77), the incompressibility of flows,

$$\nabla \cdot \boldsymbol{u}_a = 0, \tag{2.78}$$

$$\nabla \cdot (2\boldsymbol{q}_a/5p_a) = 0, \tag{2.79}$$

have been assumed for simplicity and $\boldsymbol{\kappa} = \boldsymbol{e}_\| \cdot \boldsymbol{\nabla}\boldsymbol{e}_\|$ is the magnetic curvature. For the axisymmetric magnetic field (2.3), the curvature component of the vector $V_\kappa \equiv \boldsymbol{V}\cdot\boldsymbol{\kappa}$ becomes

$$\begin{aligned} V_\kappa = \boldsymbol{V}\cdot\boldsymbol{\kappa} &= \left[V^\rho \boldsymbol{e}_\rho + \frac{V^\chi}{B^\chi}\boldsymbol{B} + \left(V_\zeta - \frac{I}{B^\chi}V^\chi\right)\nabla\zeta\right]\cdot\boldsymbol{\kappa}, \\ &= \left[V^\rho \kappa_\rho - \left(V_\zeta - \frac{I}{B^\chi}V^\chi\right)\frac{B^\zeta}{B^2}\nabla_\| B\right], \end{aligned} \tag{2.80}$$

where we have used the following relations

$$\boldsymbol{e}_\| \cdot \boldsymbol{\kappa} = 0, \tag{2.81}$$

$$\nabla\zeta\cdot\boldsymbol{\kappa} = -\frac{B^\zeta}{B^2}\nabla_\| B. \tag{2.82}$$

Since the flows in the radial direction are much slower than those in the parallel and the toroidal direction,

$$u_a^\rho \sim q_a^\rho/p_a \sim \mathcal{O}(\delta^2) \ll u_a^\chi \sim q_a^\chi/p_a \sim u_{a\zeta} \sim q_{a\zeta}/p_a \sim \mathcal{O}(\delta), \tag{2.83}$$

Eq.(2.80) can be reduced to

$$V_\kappa = -\left(V_\zeta - \frac{I}{B^\chi}V^\chi\right)\frac{B^\zeta}{B^2}\nabla_\| B, \quad \text{for } \boldsymbol{V} = \boldsymbol{u}_a, \boldsymbol{q}_a. \tag{2.84}$$

Therefore Eq.(2.76) and Eq.(2.77) can be expressed as

$$W_{zz}^{ua} = 2\left[\nabla_\| u_{a\|} + \left(u_{a\zeta} - \frac{I}{B^\chi}u_a^\chi\right)\frac{B^\zeta}{B^2}\nabla_\| B\right], \tag{2.85}$$

$$W_{zz}^{qa} = \frac{4}{5}\left[\nabla_\|\left(\frac{q_{a\|}}{p_a}\right) + \left(\frac{q_{a\zeta}}{p_a} - \frac{I}{B^\chi}\frac{q_a^\chi}{p_a}\right)\frac{B^\zeta}{B^2}\nabla_\| B\right]. \tag{2.86}$$

It is easily shown that Eq.(2.75) is equivalent to the Hirshman-type parallel viscosities inside the LCFS in the sense of the flux averaged viscous forces $\left\langle \boldsymbol{B}\cdot\nabla\cdot\overleftrightarrow{\pi}_a\right\rangle$ and $\left\langle \boldsymbol{B}\cdot\nabla\cdot\overleftrightarrow{\theta}_a\right\rangle$ and also equivalent to the Braginskii-type parallel viscosity outside the LCFS [32].

Since the parallel flows have great influence on tokamak transport, we consider three components of vector quantities, $(\rho, \|, \zeta)$ in the radial, the parallel to the field line, and the toroidal, rather than those of MSCS (ρ, χ, ζ). The contravariant poloidal component V^χ is therefore expressed by

$$V^\chi = \frac{B^\chi}{B_p^2}\left(V_\| B - B^\zeta V_\zeta\right). \tag{2.87}$$

Unfortunately the poloidal magnetic field intensity B_p vanishes at the magnetic axis in toroidal configurations and Eq.(2.87) has singularity at the magnetic axis so that we do not employ Eq.(2.87) and leave V^χ in Eq.(2.84). A singularity free procedure for calculating V^χ from $V_\|$ and V_ζ will be described in chapter 3.

In the above discussion, the neoclassical parallel coefficients μ_{ai} obtained from the bounce-averaged drift kinetic equation are assumed, which means that μ_{ai} in Eq.(2.75) are flux functions and lose their poloidal dependence. In the core region, the equilibrium return flows are formed and transport is essentially reduced to one-dimensional problem. Therefore the poloidal dependence of μ_{ai} is assumed to be small enough to be negligible. In the edge region where the plasma is weakly collisional, μ_{ai} should have weak poloidal dependence. We assume, however, that the effect of the poloidal non-uniformity of the plasma density and temperature on the viscosity is small and use the bounce averaged μ_{ai} as an approximation. In the peripheral region where the plasma is collisional, Eq.(2.75) is reduce to the Braginskii's expression [8, 32] so that Eq.(2.75) recovers its poloidal dependence.

2.4.2 Equation for total heat flux

Completing the multi-fluid equations also requires that the EW total stress tensor $\overleftrightarrow{R}_a$ and the EW Lorentz force $\boldsymbol{G}_a^{\mathrm{Lor}}$ are simplified to the extent that they keep consistency with the neoclassical theory. As for the EW total stress tensor, the leading terms of the EW isotropic pressure, the EW viscous tensor and the EW inertial stress tensor in the small gyroradius ordering are taken into account as

$$\left(\frac{5}{2}+c_a\right)\frac{T_a}{m_a}p_a\overleftrightarrow{I}=\frac{5}{2}\frac{T_a}{m_a}p_a\overleftrightarrow{I}+\mathcal{O}(\delta), \tag{2.88}$$

$$\overleftrightarrow{r}_a=\frac{T_a}{m_a}\left(\frac{5}{2}\pi_{\|a}+\theta_{\|a}\right)\left(\boldsymbol{e}_\|\boldsymbol{e}_\|-\frac{1}{3}\overleftrightarrow{I}\right)+\mathcal{O}(\delta^2), \tag{2.89}$$

$$\boldsymbol{Q}_a\boldsymbol{u}_a+\boldsymbol{u}_a\boldsymbol{Q}_a-\left(\frac{3}{2}p_a+\frac{1}{2}m_an_au_a^2\right)\boldsymbol{u}_a\boldsymbol{u}_a=\boldsymbol{Q}_a\boldsymbol{u}_a+\boldsymbol{u}_a\boldsymbol{Q}_a-\frac{3}{2}p_a\boldsymbol{u}_a\boldsymbol{u}_a+\mathcal{O}(\delta^3). \tag{2.90}$$

Therefore the EW total stress tensor is reduced to

$$\overleftrightarrow{R}_a=\frac{5}{2}\frac{T_a}{m_a}p_a\overleftrightarrow{I}+\overleftrightarrow{r}_{\|a}+\boldsymbol{Q}_a\boldsymbol{u}_a+\boldsymbol{u}_a\boldsymbol{Q}_a-\frac{3}{2}p_a\boldsymbol{u}_a\boldsymbol{u}_a. \tag{2.91}$$

As for the EW Lorentz force, the terms can be evaluated as

$$\frac{e_a}{m_a}\boldsymbol{E}\cdot m_an_a\boldsymbol{u}_a\boldsymbol{u}_a\sim\mathcal{O}(\delta^2), \tag{2.92}$$

$$\frac{e_a}{m_a}\boldsymbol{E}\cdot\frac{5}{2}p_a\overleftrightarrow{I}\sim\mathcal{O}(\delta^0), \tag{2.93}$$

$$\frac{e_a}{m_a}\boldsymbol{E}\cdot\frac{1}{2}m_an_au_a^2\overleftrightarrow{I}\sim\mathcal{O}(\delta^2), \tag{2.94}$$

$$\frac{e_a}{m_a}\boldsymbol{E}\cdot\overleftrightarrow{\pi}_{\|a}\sim\mathcal{O}(\delta), \tag{2.95}$$

$$\frac{e_a}{m_a}\boldsymbol{Q}_a\times\boldsymbol{B}\sim\mathcal{O}(\delta). \tag{2.96}$$

Since the consistency with the neoclassical theory requires terms up to $\mathcal{O}(\delta)$, the EW Lorentz force can be therefore reduced to

$$\boldsymbol{G}_a^{\rm Lor}=\frac{e_a}{m_a}\left[\boldsymbol{E}\cdot\left(\frac{5}{2}p_a\overleftrightarrow{I}+\overleftrightarrow{\pi}_{\|a}\right)+\boldsymbol{Q}_a\times\boldsymbol{B}\right]. \tag{2.97}$$

From Eq.(2.91) and Eq.(2.97), the reduced equation for the total heat flux is obtained as

$$\left.\frac{\partial\boldsymbol{Q}_a}{\partial t}\right|_{\boldsymbol{x}}+\nabla\cdot\left[\frac{5}{2}\frac{T_a}{m_a}p_a\overleftrightarrow{I}+\overleftrightarrow{r}_{\|a}+\boldsymbol{Q}_a\boldsymbol{u}_a+\boldsymbol{u}_a\boldsymbol{Q}_a-\frac{3}{2}p_a\boldsymbol{u}_a\boldsymbol{u}_a\right]$$
$$=\frac{e_a}{m_a}\left[\boldsymbol{E}\cdot\left(\frac{5}{2}p_a\overleftrightarrow{I}+\overleftrightarrow{\pi}_{\|a}\right)+\boldsymbol{Q}_a\times\boldsymbol{B}\right]+\boldsymbol{G}_a^{\rm Lor}+\boldsymbol{S}_{qa}. \tag{2.98}$$

2.5 Derivation of two-dimensional transport equations

In this section, we derive the two-dimensional transport modeling equations composed of the equations for particle density, momentum in the three direction (radial, parallel and toroidal), internal energy, and total heat flux in the three direction (radial, parallel and toroidal) for each species, and Maxwell's equations for electromagnetic field. Since the toroidal symmetry is assumed, the spatial variation of quantities are two-dimensional, in the radial and poloidal directions. Since the parallel flows have great influence on tokamak transport, we consider three components of vector quantities, $(\rho, \|, \zeta)$ in the radial, the parallel to the field line, and the toroidal, rather than those of MSCS (ρ, χ, ζ).

2.5.1 Equation for particle density

In this paper the equation of continuity (2.15) is employed as the equation for particle density,

$$\left.\frac{\partial n_a}{\partial t}\right|_{\boldsymbol{x}} + \nabla\cdot(n_a \boldsymbol{u}_a) = S_{na}. \tag{2.99}$$

2.5.2 Equation of motion in the parallel direction

We formulate the evolution equation for the parallel momentum by taking a scalar product of the equation of motion (2.30) and $\boldsymbol{B}$:

$$\begin{aligned}\frac{\partial}{\partial t}\left(m_a n_a u_{a\|} B\right)\big|_{\boldsymbol{x}} &+ \boldsymbol{B}\cdot\nabla\cdot(m_a n_a \boldsymbol{u}_a\boldsymbol{u}_a) \\ &+ B\nabla_\| p_a + \boldsymbol{B}\cdot\nabla\cdot\overleftrightarrow{\pi}_a = e_a n_a E_\| B + F_{a\|}^{\rm fri} B + F_{a\|}^{\rm QL} B + S_{ma\|} B.\end{aligned} \tag{2.100}$$

The time derivative term in Eq.(2.100) is reduced, since the time variation of magnetic field is much slower than that of momentum, where we have evaluated $\boldsymbol{u}_a \sim \mathcal{O}(\delta)$, $\partial\boldsymbol{B}/\partial t|_{\boldsymbol{x}} \sim \mathcal{O}(\delta^2)$ and $\partial(m_a n_a u_{a\|}B)/\partial t|_{\boldsymbol{x}} \sim \mathcal{O}(\delta^2)$. Though the inertial force driven by the drift of the flux surfaces $\boldsymbol{u}_g$ included in the time derivative term in Eq.(2.100) is $\mathcal{O}(\delta^3)$, we retain it from the aspect of volume conservation.

Next, we evaluate the inertial force in the parallel direction. To obtain a simple expression, we split the flow velocity into the parallel and the perpendicular components, $\boldsymbol{u}_a = \boldsymbol{u}_{a\|} + \boldsymbol{u}_{a\perp}$. The inertial stress tensor $m_a n_a \boldsymbol{u}_a\boldsymbol{u}_a$ now is split into 4 terms and we keep terms up to $\mathcal{O}(\delta^2)$ in our transport model:

$$m_a n_a \boldsymbol{u}_a\boldsymbol{u}_a = m_a n_a \boldsymbol{u}_{a\|}\boldsymbol{u}_{a\|} + \mathcal{O}(\delta^3). \tag{2.101}$$

Therefore the inertial force in the parallel direction $F_{a\|}^{\rm ine}$ is rewritten in a simple form:

$$\begin{aligned}F_{a\|}^{\rm ine} B &= \boldsymbol{B}\cdot\nabla\cdot\left(m_a n_a \boldsymbol{u}_{a\|}\boldsymbol{u}_{a\|}\right) \\ &= B\nabla_\|\left(m_a n_a u_{a\|} u_{a\|}\right) - m_a n_a u_{a\|} u_{a\|} \nabla_\| B,\end{aligned} \tag{2.102}$$

where we have used the following relation

$$\boldsymbol{B}\cdot\nabla\cdot\left(f\boldsymbol{e}_\|\boldsymbol{e}_\|\right) = B\nabla_\| f - f\nabla_\| B. \tag{2.103}$$

The viscous force in the parallel direction $F_{a\|}^{\rm vis}$ can be written as

$$F_{a\|}^{\rm vis} B = \boldsymbol{B}\cdot\nabla\cdot\overleftrightarrow{\pi}_a = -\pi_{\|a}\nabla_\| B + \frac{2}{3} B\nabla_\|\pi_{\|a}, \tag{2.104}$$

since the parallel viscosity tensor is in the CGL form.

From Eq.(2.100), the force due to the pressure gradient in the parallel direction $F^{\nabla p}_{a\|}$, the Lorentz force in the parallel direction $F^{\rm Lor}_{a\|}$ and the friction force in the parallel direction $F^{\rm fri}_{a\|}$ can be written respectively as

$$F^{\nabla p}_{a\|}B = B\nabla_\| p_a, \tag{2.105}$$

$$F^{\rm Lor}_{a\|}B = e_a n_a E_\| B = e_a n_a \left(\frac{\psi'}{\sqrt{g}}E_\chi + \frac{I}{R^2}E_\zeta\right), \tag{2.106}$$

$$F^{\rm fri}_{a\|}B = \sum_b \left(l^{ab}_{11}u_{b\|} - l^{ab}_{12}\frac{2q_{b\|}}{5p_b}\right)B. \tag{2.107}$$

Therefore the equation for the parallel momentum is obtained as

$$\frac{\partial}{\partial t}\left(m_a n_a u_{a\|}B\right)\bigg|_{\boldsymbol{x}} + F^{\rm ine}_{a\|}B + F^{\nabla p}_{a\|}B + F^{\rm vis}_{a\|}B = F^{\rm Lor}_{a\|}B + F^{\rm fri}_{a\|}B + F^{\rm QL}_{a\|}B + S_{ma\|}B \tag{2.108}$$

2.5.3 Equation of motion in the toroidal direction

Taking the scalar product of the equation of motion (2.30) and the covariant toroidal basis $\boldsymbol{e}_\zeta$, we obtain

$$\frac{\partial}{\partial t}\left(m_a n_a u_{a\zeta}\right)\bigg|_{\boldsymbol{x}} + \nabla\cdot\left(\boldsymbol{e}_\zeta\cdot\overleftrightarrow{P}_a\right) = e_a n_a\left(E_\zeta + \psi' u^\rho_a\right) + F_{a\zeta} + F^{\rm QL}_{a\zeta} + S_{ma\zeta}. \tag{2.109}$$

where ζ is defined geometrically so that its time derivative is identically zero and ψ' indicates the derivative of ψ with respect to ρ.

Since the total stress tensor $\overleftrightarrow{P}_a$ is symmetric, the following useful identity of the second-rank symmetric tensor $\overleftrightarrow{S}$ has been used in taking the toroidal projection of total stress $\boldsymbol{e}_\zeta\cdot\nabla\cdot\overleftrightarrow{P}_a$:

$$\boldsymbol{e}_\zeta\cdot\nabla\cdot\overleftrightarrow{S} = \nabla\cdot\left(\boldsymbol{e}_\zeta\cdot\overleftrightarrow{S}\right). \tag{2.110}$$

The inertial force in the toroidal direction $F^{\rm ine}_{a\zeta}$ and the viscous force in the toroidal direction $F^{\rm vis}_{a\zeta}$ therefore can be expressed as

$$F^{\rm ine}_{a\zeta} = \nabla\cdot\left(m_a n_a u_{a\zeta}\boldsymbol{u}_a\right) \tag{2.111}$$

$$F^{\rm vis}_{a\zeta} = B\nabla_\|\left(\frac{I}{B^2}\pi_{\|a}\right). \tag{2.112}$$

Note that the parallel viscous force in the toroidal direction may not vanish in two-dimensional transport modeling in contrast to the traditional one-dimensional transport modeling. It is easily confirmed that the flux-surface-averaged value of Eq.(2.112) vanishes as $\langle\boldsymbol{B}\cdot\nabla f\rangle = 0$, which is consistent with the one-dimensional transport theory.

The Lorentz force in the toroidal direction $F_{a\zeta}^{\mathrm{Lor}}$ and the friction force in the toroidal direction $F_{a\zeta}^{\mathrm{fri}}$ can be written as

$$F_{a\zeta}^{\mathrm{Lor}} = e_a n_a E_\zeta + e_a n_a \psi' u_a^\rho, \tag{2.113}$$

$$F_{a\zeta}^{\mathrm{fri}} = \sum_b \left(l_{11}^{ab} u_{b\zeta} - l_{12}^{ab} \frac{2q_{b\zeta}}{5p_b} \right). \tag{2.114}$$

Therefore, the equation for the toroidal momentum is obtained as follows:

$$\frac{\partial}{\partial t} \left(m_a n_a u_{a\zeta} \right) \bigg|_{\boldsymbol{x}} + F_{a\zeta}^{\mathrm{ine}} + F_{a\zeta}^{\mathrm{vis}} = F_{a\zeta}^{\mathrm{Lor}} + F_{a\zeta}^{\mathrm{fri}} + F_{a\zeta}^{\mathrm{QL}} + S_{ma\zeta}. \tag{2.115}$$

2.5.4 Equation of radial force balance

Since the time derivative of the radial momentum is $\mathcal{O}(\delta^3)$ and small enough to be negligible, we assume the lowest order $\mathcal{O}(1)$ force balance in the radial direction for simplicity:

$$\nabla\rho \cdot \nabla p_a = e_a n_a E^\rho + \nabla\rho \cdot \left(e_a n_a \boldsymbol{u}_a \times \boldsymbol{B} \right). \tag{2.116}$$

From Eq.(2.116), the force due to the pressure gradient in the radial direction $F_a^{\nabla p\ \rho}$ and the Lorentz force in the radial direction $F_a^{\mathrm{Lor}\ \rho}$ can be written as

$$F_a^{\nabla p\ \rho} = g^{\rho\rho} \frac{\partial p_a}{\partial \rho} + g^{\rho\chi} \frac{\partial p_a}{\partial \chi}, \tag{2.117}$$

$$F_a^{\mathrm{Lor}\ \rho} = e_a n_a E_a^\rho + e_a \frac{IB}{\psi'} n_a u_{a\parallel} - e_a \frac{B^2}{\psi'} n_a u_{a\zeta}, \tag{2.118}$$

where the following relation have been used in Eq.(2.118):

$$\nabla\rho \cdot \left(\boldsymbol{f} \times \boldsymbol{B} \right) = \frac{IB}{\psi'} f_\parallel - \frac{B^2}{\psi'} f_\zeta. \tag{2.119}$$

Therefore, we obtain the equation of the force balance in the radial direction:

$$F_a^{\nabla p\ \rho} = F_a^{\mathrm{Lor}\ \rho} \tag{2.120}$$

2.5.5 Equation for energy transport

The energy transport equation for internal energy does not change from Eq.(2.49), since the equation for total heat flux $\boldsymbol{Q}_a$ is solved simultaneously. We substitute Eq.(2.38) into Eq.(2.49), however, in order to evaluate the terms in Eq.(2.49) in terms of δ

$$\frac{\partial}{\partial t} \left(\frac{3}{2} p_a \right) \bigg|_{\boldsymbol{x}} + \nabla \cdot \left(\boldsymbol{q}_a + \frac{5}{2} p_a \boldsymbol{u}_a + \overleftrightarrow{\pi}_a \cdot \boldsymbol{u}_a \right) = \boldsymbol{u}_a \cdot \nabla p_a + \boldsymbol{u}_a \cdot \nabla \cdot \overleftrightarrow{\pi}_a + S_{pa}. \tag{2.121}$$

Moreover, Eq.(2.121) can be transformed to the expression for the adiabatic entropy $\sqrt{g}^{5/3} p_a$. All terms in Eq.(2.121) are $\mathcal{O}(\delta^2)$ in the equilibrium state.

The viscous heating term by the parallel viscous force $Q_a^{\rm vis} \equiv \boldsymbol{u}_a \cdot \nabla \cdot \overleftrightarrow{\pi}_a$ can be written as

$$Q_a^{\rm vis} = B\nabla_\parallel \left(\frac{u_{a\parallel}\pi_{\parallel a}}{B}\right) - \pi_{\parallel a}\left(\nabla_\parallel u_{a\parallel} - \boldsymbol{u}_a \cdot \boldsymbol{\kappa}\right) - \frac{1}{3}\boldsymbol{u}_a \cdot \nabla \pi_{\parallel a}, \tag{2.122}$$

since $\overleftrightarrow{\pi}_a$ is in the CGL form. Now we will show that Eq.(2.122) is consistent with the result of one-dimensional modeling. Substituting the equilibrium return flows,

$$\bar{\boldsymbol{u}}_a \equiv \omega_{ua} R^2 \nabla\zeta + L_{ua}\boldsymbol{B}, \tag{2.123}$$

$$\bar{\boldsymbol{q}}_a \equiv \omega_{qa} R^2 \nabla\zeta + L_{qa}\boldsymbol{B}, \tag{2.124}$$

into Eq.(2.122) and averaging it over the flux surfaces, we obtain

$$\left\langle Q_a^{\rm vis}\right\rangle = L_{ua}\left\langle \boldsymbol{B}\cdot\nabla\cdot\overleftrightarrow{\pi}_a\right\rangle, \tag{2.125}$$

where ω_{ua} and ω_{qa} are the toroidal angular frequencies and L_{ua} and L_{qa} are quantities related to the equilibrium poloidal flows. We have used $\langle \boldsymbol{B}\cdot\nabla f\rangle = 0$ in the derivation of Eq.(2.125). Eq.(2.125) is consistent with the viscous heating term in the one-dimensional transport modeling [33].

Therefore, the equation for internal energy is

$$\frac{3}{2}\left.\frac{\partial p_a}{\partial t}\right|_{\boldsymbol{x}} + \nabla\cdot\left(\boldsymbol{Q}_a - \frac{1}{2}m_a n_a u_a^2 \boldsymbol{u}_a\right) = \boldsymbol{u}_a\cdot\nabla p_a + Q_a^{\rm vis} + Q_{\Delta a} + S_{pa}. \tag{2.126}$$

2.5.6 Equations for total heat flux

The equations for total heat flux can be derived by analogy with the derivation of the equation for momentum.

Taking a scalar product of the equation for total heat flux (2.66) and $\boldsymbol{B}$, we obtain the equation for total heat flux in the parallel direction

$$\left.\frac{\partial}{\partial t}\left(Q_{a\parallel}B\right)\right|_{\boldsymbol{x}} + G_{a\parallel}^{\rm ine}B + G_{a\parallel}^{\nabla p}B + G_{a\parallel}^{\rm vis}B = G_{a\parallel}^{\rm Lor}B + G_{a\parallel}^{\rm fri}B + G_{a\parallel}^{\rm QL}B + S_{qa\parallel}B, \tag{2.127}$$

where $G_{a\parallel}^{\rm ine}$, $G_{a\parallel}^{\nabla p}$, $G_{a\parallel}^{\rm vis}$, $G_{a\parallel}^{\rm Lor}$ and $G_{a\parallel}^{\rm fri}$ are the EW inertial force, the EW force due to the EW pressure gradient, the EW viscous force, the EW Lorentz force and the EW friction force in the parallel direction respectively and defined as

$$G_{a\parallel}^{\rm ine}B \equiv B\nabla_\parallel\left(Q_{a\parallel}u_{a\parallel} + u_{a\parallel}Q_{a\parallel} - \frac{3}{2}p_a u_{a\parallel}u_{a\parallel}\right) - \left(Q_{a\parallel}u_{a\parallel} + u_{a\parallel}Q_{a\parallel} - \frac{3}{2}p_a u_{a\parallel}u_{a\parallel}\right)\nabla_\parallel B, \tag{2.128}$$

$$G_{a\parallel}^{\nabla p}B \equiv B\nabla_\parallel\left(\frac{5T_a}{2m_a}p_a\right), \tag{2.129}$$

$$G_{a\parallel}^{\rm vis}B \equiv -r_{\parallel a}\nabla_\parallel B + \frac{2}{3}B\nabla_\parallel r_{\parallel a}, \tag{2.130}$$

$$G_{a\parallel}^{\rm Lor}B \equiv \frac{e_a}{m_a}\left(\frac{5}{2}p_a + \frac{2}{3}\pi_{a\parallel}\right)E_\parallel B, \tag{2.131}$$

$$G_{a\parallel}^{\rm fri}B \equiv \frac{5T_a}{2m_a}\sum_b\left(l_{11}^{ab}u_{a\parallel} - l_{12}^{ab}\frac{2q_{b\parallel}}{5p_b}\right)B + \frac{T_a}{m_a}\sum_b\left(-l_{21}^{ab}u_{a\parallel} + l_{22}^{ab}\frac{2q_{b\parallel}}{5p_b}\right)B, \tag{2.132}$$

where $r_{\parallel a}$ is the EW parallel viscosity,

$$r_{\parallel a} \equiv \frac{T_a}{m_a}\left(\frac{5}{2}\pi_{\parallel a} + \theta_{\parallel a}\right). \tag{2.133}$$

Taking a scalar product of the equation for total heat flux (2.66) and $\boldsymbol{e}_\zeta$, we obtain the equation for total heat flux in the toroidal direction,

$$\left.\frac{\partial Q_{a\zeta}}{\partial t}\right|_{\boldsymbol{x}} + G_{a\zeta}^{\text{ine}} + G_{a\zeta}^{\text{vis}} = G_{a\zeta}^{\text{Lor}} + G_{a\zeta}^{\text{fri}} + G_{a\zeta}^{\text{QL}} + S_{qa\zeta}, \tag{2.134}$$

where $G_{a\zeta}^{\text{ine}}$, $G_{a\zeta}^{\text{vis}}$, $G_{a\zeta}^{\text{Lor}}$ and $G_{a\zeta}^{\text{fri}}$ are the EW inertial force, the EW viscous force, the EW Lorentz force and the EW friction force in the toroidal direction respectively and defined as

$$G_{a\zeta}^{\text{ine}} \equiv \nabla\cdot\left(Q_{a\zeta}\boldsymbol{u}_a + u_{a\zeta}\boldsymbol{Q}_a - \frac{3}{2}p_a u_{a\zeta}\boldsymbol{u}_a\right), \tag{2.135}$$

$$G_{a\zeta}^{\text{vis}} \equiv B\nabla_\parallel\left(\frac{I}{B^2}r_{a\parallel}\right), \tag{2.136}$$

$$G_{a\zeta}^{\text{Lor}} \equiv \frac{e_a}{m_a}\left[\left(\frac{5}{2}p_a - \frac{1}{3}\pi_{a\parallel}\right)E_\zeta + \frac{I}{B}\pi_{a\parallel}E_\parallel + \psi' Q_a^\rho\right], \tag{2.137}$$

$$G_{a\zeta}^{\text{fri}} \equiv \frac{5T_a}{2m_a}\sum_b\left(l_{11}^{ab}u_{a\zeta} - l_{12}^{ab}\frac{2q_{b\zeta}}{5p_b}\right) + \frac{T_a}{m_a}\sum_b\left(-l_{21}^{ab}u_{a\zeta} + l_{22}^{ab}\frac{2q_{b\zeta}}{5p_b}\right). \tag{2.138}$$

The equation for total heat flux in the radial direction in the lowest order is given by

$$G_a^{\nabla p\ \rho} = G_a^{\text{Lor}\ \rho}, \tag{2.139}$$

where $F_{qa}^{\nabla p\ \rho}$ and $F_{qa}^{\text{Lor}\ \rho}$ are the force due to the EW pressure gradient and the EW Lorentz force in the radial direction respectively and defined as

$$G_a^{\nabla p\ \rho} \equiv g^{\rho\rho}\frac{\partial}{\partial\rho}\left(\frac{5T_a}{2m_a}p_a\right) + g^{\rho\chi}\frac{\partial}{\partial\chi}\left(\frac{5T_a}{2m_a}p_a\right) \tag{2.140}$$

$$G_a^{\text{Lor}\ \rho} \equiv \frac{5}{2}\frac{T_a}{m_a}e_a n_a E^\rho + \frac{e_a}{m_a}\frac{IB}{\psi'}Q_{a\parallel} - \frac{e_a}{m_a}\frac{B^2}{\psi'}Q_{a\zeta} \tag{2.141}$$

2.5.7 Consistency with the conventional neoclassical transport theory

We will show that our two-dimensional transport model is consistent with the ordinary flux-surface-averaged neoclassical transport theory [7, 8]. Assuming the equilibrium state inside of the LCFS and the force balance up to $\mathcal{O}(\delta)$ in Eq.(2.108) and averaging on the flux surfaces, we obtain

$$\left\langle F_{a\parallel}^{\nabla p}B\right\rangle + \left\langle F_{a\parallel}^{\text{vis}}B\right\rangle = \left\langle F_{a\parallel}^{\text{Lor}}B\right\rangle + \left\langle F_{a\parallel}^{\text{fri}}B\right\rangle. \tag{2.142}$$

Substituting Eqs.(2.123) and (2.124) into Eq.(2.142), we obtain

$$\left\langle 3(\nabla_{\|}B)^2 \right\rangle \left(\mu_{a1} L_{ua} + \mu_{a2} \frac{2L_{qa}}{5p_a} \right) = \sum_b \left(l_{11}^{ab} \left\langle u_{b\|} B \right\rangle - l_{12}^{ab} \left\langle q_{b\|} B \right\rangle \right) + e_a n_a \left\langle E_{\|} B \right\rangle , \quad (2.143)$$

where we have used $\langle \boldsymbol{B} \cdot \nabla f \rangle = 0$. The flux-surface-averaged parallel force balance up to $\mathcal{O}(\delta)$ in Eq.(2.127) also becomes

$$\left\langle G_{a\|}^{\nabla p} B \right\rangle + \left\langle G_{a\|}^{\mathrm{vis}} B \right\rangle = \left\langle G_{a\|}^{\mathrm{Lor}} B \right\rangle + \left\langle G_{a\|}^{\mathrm{fri}} B \right\rangle \quad (2.144)$$

and we obtain

$$\left\langle 3(\nabla_{\|}B)^2 \right\rangle \left(\mu_{a2} L_{ua} + \mu_{a3} \frac{2L_{qa}}{5p_a} \right) = \sum_b \left(-l_{21}^{ab} \left\langle u_{b\|} B \right\rangle + l_{22}^{ab} \left\langle q_{b\|} B \right\rangle \right) , \quad (2.145)$$

where we have used $\langle \boldsymbol{B} \cdot \nabla f \rangle = 0$ and Eq.(2.143). The flux-surface-averaged parallel flows $\left\langle u_{a\|} B \right\rangle$ and $\left\langle q_{a\|} B \right\rangle$ are decomposed by the use of Eqs.(2.123) and (2.124)

$$\left\langle u_{a\|} B \right\rangle = V_{1a} B + L_{ua} \left\langle B^2 \right\rangle , \quad V_{1a} \equiv \frac{I}{B} \omega_{ua} \quad (2.146)$$

$$\left\langle q_{a\|} B \right\rangle = V_{2a} B + \frac{2L_{qa}}{5p_a} \left\langle B^2 \right\rangle , \quad V_{2a} \equiv \frac{I}{B} \omega_{qa} \quad (2.147)$$

Substituting Eqs.(2.146) and (2.147) into Eqs.(2.143) and (2.145), the equations for poloidal rotations in the conventional neoclassical theory is obtained as

$$\begin{aligned} &\left\langle 3 \left(\nabla_{\|} B \right)^2 \right\rangle \begin{pmatrix} \mu_{1a} & \mu_{2a} \\ \mu_{2a} & \mu_{3a} \end{pmatrix} \begin{pmatrix} L_{ua} \\ \dfrac{2L_{qa}}{5p_a} \end{pmatrix} \\ &= \sum_b \begin{pmatrix} l_{11}^{ab} & -l_{12}^{ab} \\ -l_{21}^{ab} & l_{22}^{ab} \end{pmatrix} \begin{pmatrix} V_{1b} B + L_{ub} \left\langle B^2 \right\rangle \\ V_{2b} B + \dfrac{2L_{qb}}{5p_b} \left\langle B^2 \right\rangle \end{pmatrix} + \begin{pmatrix} e_a n_a \left\langle E_{\|} B \right\rangle \\ 0 \end{pmatrix} . \end{aligned} \quad (2.148)$$

2.6 Derivation of electromagnetic equations

In this section, the electromagnetic equations are derived from Maxwell's equations:

$$\left. \frac{\partial \boldsymbol{B}}{\partial t} \right|_{\boldsymbol{x}} + \nabla \times \boldsymbol{E} = \boldsymbol{0} \quad (2.149)$$

$$\frac{1}{c^2} \left. \frac{\partial \boldsymbol{E}}{\partial t} \right|_{\boldsymbol{x}} - \nabla \times \boldsymbol{B} + \mu_0 \boldsymbol{j} = \boldsymbol{0} \quad (2.150)$$

$$\nabla \cdot \boldsymbol{B} = 0 \quad (2.151)$$

$$\nabla \cdot \boldsymbol{E} = \frac{\rho_c}{\varepsilon_0}, \quad (2.152)$$

where ρ_c is the electric charge density. Gauss's law for magnetism (2.151) has already been taken into account through the expression of the equilibrium magnetic field.

Variables used to describe the electromagnetic field are chosen as follows. For the magnetic field $\boldsymbol{B}$, the contravariant poloidal component $B^\chi(=\psi'\sqrt{g}^{-1})$ and the covariant toroidal component $B_\zeta(=I)$ are suitable for describing the magnetic field $\boldsymbol{B}$ in Eq.(2.3). For the electric field $\boldsymbol{E}$, the covariant components are suitable for taking a scalar product of $\boldsymbol{E}$ and $\boldsymbol{B}$. We should note that the distinction between the covariant and the contravariant components is not essential in the toroidal direction in MSCS owing to its orthogonality in the toroidal direction. Therefore, the following five variables are employed to describe the evolution of the electromagnetic field, ψ', I, E_ρ, E_χ and E_ζ.

Since the existence of magnetic surfaces is assumed, ψ' and I are the flux functions. From Faraday's law, E_ζ is also the flux function as is shown later. Taking account of the consistency with these properties, we introduce flux-surface-average for some of electromagnetic field equations. This approximation is necessary for the compatibility of the two-dimensional transport analysis with the existence of magnetic surfaces. The validity of this approximation has to be examined a posteriori.

For Faraday's law (2.149), the contravariant poloidal direction $\nabla\chi$ and the toroidal direction $\nabla\zeta$ are chosen for the direction of projection, since there is no contravariant radial component of the magnetic field in MSCS. For Ampère's law (2.150), the projection in the parallel direction $\boldsymbol{B}$ and the toroidal direction $\nabla\zeta$ are used owing to the compatibility with the direction of the current density $\boldsymbol{j}$ derived from the equation of motion. Instead of the contravariant radial component of Ampère's law, we solve Gauss's law (2.152) which is the time integral of the divergence of Ampère's law.

2.6.1 Equations for magnetic field

In this section, we will derive the equations for ψ' and I from Faraday's law (2.149). Substituting Eq.(2.3) into Faraday's law (2.149), we obtain

$$\left.\frac{\partial\psi'}{\partial t}\right|_{\boldsymbol{x}}\nabla\zeta\times\nabla\rho+\left.\frac{\partial I}{\partial t}\right|_{\boldsymbol{x}}\boldsymbol{e}^\zeta+\nabla\times\boldsymbol{E}=\boldsymbol{0}. \tag{2.153}$$

Taking a scalar product of (2.153) and $\nabla\chi$, we obtain the equation for ψ'

$$\left.\frac{\partial\psi'}{\partial t}\right|_{\boldsymbol{x}}-\frac{\partial E_\zeta}{\partial\rho}=0. \tag{2.154}$$

Since ψ' is the flux function, E_ζ is also the flux function.

Since the covariant toroidal magnetic field $B_\zeta(=I)$ is the flux function, we take a scalar product of (2.153) and $\boldsymbol{e}_\zeta$ and the $\boldsymbol{\nabla}\times\boldsymbol{E}$ term is averaged over the flux surfaces to obtain

$$\left.\frac{\partial I}{\partial t}\right|_{\boldsymbol{x}}+\left\langle\frac{R^2}{\sqrt{g}}\left(\frac{\partial E_\chi}{\partial\rho}-\frac{\partial E_\rho}{\partial\chi}\right)\right\rangle=0. \tag{2.155}$$

2.6.2 Equations for electric field

In this section the equations for the covariant toroidal electric field E_ζ and the covariant poloidal electric field E_χ are derived from Ampère's law and the equation for the covariant radial electric field E_ρ from Gauss's law. The rotation of the magnetic field can be expressed as

$$\begin{aligned}\nabla \times \boldsymbol{B} &= \nabla \times (\nabla\zeta \times \nabla\psi + I\nabla\zeta) \\ &= \nabla \cdot (\nabla\psi\nabla\zeta - \nabla\zeta\nabla\psi) + \nabla I \times \nabla\zeta \\ &= \nabla \cdot \left(\frac{1}{R^2}\nabla\psi\right) R^2\nabla\zeta + \nabla I \times \nabla\zeta, \end{aligned} \tag{2.156}$$

where $\nabla\psi\nabla\zeta - \nabla\zeta\nabla\psi$ is a 2nd-rank antisymmetric tensor and the following tensor identities for any vectors $\boldsymbol{f}$ and $\boldsymbol{g}$ and any second-rank antisymmetric tensor $\overleftrightarrow{A}$ have been employed:

$$\nabla \times (\boldsymbol{f} \times \boldsymbol{g}) = \nabla \cdot (\boldsymbol{g}\boldsymbol{f} - \boldsymbol{f}\boldsymbol{g}) \tag{2.157}$$

$$\nabla \cdot \overleftrightarrow{A} = \sum_{\xi_i=\rho,\chi,\zeta} \nabla \cdot \left(\overleftrightarrow{A} \cdot \boldsymbol{e}^{\xi_i}\right) \boldsymbol{e}_{\xi_i}, \tag{2.158}$$

Substituting Eq.(2.156) into Eq.(2.150), we obtain the equation for the electric field in the axisymmetric system,

$$\frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}\bigg|_{\boldsymbol{x}} - \nabla \cdot \left(\frac{1}{R^2}\nabla\psi\right) R^2\nabla\zeta - \nabla I \times \nabla\zeta + \mu_0 \boldsymbol{j} = \boldsymbol{0}. \tag{2.159}$$

Taking a scalar product of Eq.(2.159) and $\boldsymbol{e}_\zeta$, we obtain the equation for the covariant toroidal electric field E_ζ,

$$\frac{1}{c^2}\frac{\partial E_\zeta}{\partial t}\bigg|_{\boldsymbol{x}} - R^2\nabla \cdot \left(\frac{1}{R^2}\nabla\psi\right) + \mu_0 j_\zeta = 0. \tag{2.160}$$

This equation reduces to the Grad-Shafranov equation in a stationary state. Since E_ζ is the flux function, we employ the flux-surface-average of the second and the third terms to obtain,

$$\frac{1}{c^2}\frac{\partial E_\zeta}{\partial t}\bigg|_{\boldsymbol{x}} - \left\langle R^2\nabla \cdot \left(\frac{\psi'}{R^2}\nabla\rho\right)\right\rangle + \mu_0 \langle j_\zeta \rangle = 0. \tag{2.161}$$

This equation corresponds to the flux-surface-averaged Grad-Shafranov equation employed in the Flux Conserving Tokamak (FCT) scheme [34–36].

Taking a scalar product of Eq.(2.159) and $\boldsymbol{B}$, we obtain

$$\frac{1}{c^2}\left(\frac{\psi'}{\sqrt{g}}\frac{\partial E_\chi}{\partial t}\bigg|_{\boldsymbol{x}} + I\,\frac{\partial E^\zeta}{\partial t}\bigg|_{\boldsymbol{x}}\right) - \nabla \cdot \left(\frac{1}{R^2}\nabla\psi\right) I + \frac{g^{\rho\rho}}{R^2}\psi'\frac{dI}{d\rho} + \mu_0 j_\parallel B = 0. \tag{2.162}$$

Substituting Eq.(2.160) into Eq.(2.162), we obtain the equation for the covariant poloidal electric field E_χ,

$$\frac{1}{c^2}\frac{\partial E_\chi}{\partial t}\bigg|_{\boldsymbol{x}} + \frac{g_{\chi\chi}}{\sqrt{g}}\frac{dI}{d\rho} + \mu_0\frac{\sqrt{g}\left(j_\parallel B - j^\zeta I\right)}{\psi'} = 0. \tag{2.163}$$

Finally the covariant radial electric field E_ρ is obtained by solving Gauss's law,

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial\rho}\left[\sqrt{g}\left(g^{\rho\rho}E_\rho + g^{\rho\chi}E_\chi\right)\right] + \frac{1}{\sqrt{g}}\frac{\partial}{\partial\chi}\left[\sqrt{g}\left(g^{\chi\rho}E_\rho + g^{\chi\chi}E_\chi\right)\right] = \frac{\rho_c}{\varepsilon_0}. \tag{2.164}$$

2.7 Connection between transport and equilibrium solver

In this section we briefly describe the procedure for coupling the transport solver with an equilibrium solver. At the beginning, MSCS is calculated by solving the Grad-Shafranov equation for initial profiles.

At the first step, in the transport solver, the set of transport equations, Eqs.(2.99), (2.108), (2.115), (2.120), (2.126), (2.127), (2.134) and (2.139), and the set of electromagnetic equations, Eqs.(2.154) , (2.155), (2.161), (2.163) and (2.164) are solved simultaneously in MSCS in an implicit way. Since the transport coefficients and the source terms depend on the plasma quantities, densities, temperatures, and flows, this procedure has to be repeated until the solutions are converged.

At the second step, the two-dimensional toroidal current density profile $j_\zeta(\rho,\chi)$ and the toroidal component of the displacement current density profile $j_\zeta^{dc}(\rho) = 1/(\mu_0 c^2)\partial E_\zeta/\partial t|_{\boldsymbol{x}}$ calculated by the transport solver in MSCS, are converted to the two-dimensional profiles $j_\zeta(R,Z)$ and $j_\zeta^{dc}(R,Z)$ in the cylindrical coordinate system (R,φ,Z) and sent to the free-boundary equilibrium solver.

At the third step, in the free-boundary equilibrium solver, Eq.(2.160) is solved with given $j_\zeta(R,Z)$ and $j_\zeta^{dc}(R,Z)$ to calculate $\psi(R,Z)$,

$$\frac{1}{R}\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2\psi}{\partial Z^2} = \frac{\mu_0}{R^2}\left(j_\zeta + j_\zeta^{dc}\right). \tag{2.165}$$

In this recalculation of the equilibrium magnetic field, we employ the FCT scheme [34–36] in which the toroidal and poloidal fluxes are conserved; therefore $q(\rho)$ is unchanged. The particle density, the momentum, the pressure and the heat flux are also changed adiabatically according to the change of volume. In order to obtain the equilibrium satisfying these constrains, Eqs.(2.161) and (2.165) are solved iteratively. In the 1.5D transport modeling, Eq.(2.161) without the displacement current is solved for fixed $q(\rho)$ and $p(\rho)$ with adiabatic constraint to obtain $I(\rho)$, which is related to the plasma volume as well as the toroidal magnetic field. In the present 2D transport modeling, the safety factor $q(\rho)$ or $\psi'(\rho)$ and the toroidal current density $j_\zeta(\rho,\chi)$ are fixed in solving Eq.(2.161) to calculate the derivative of the volume $dV/d\rho$. This quantity is used to calculate $j_\zeta(R,Z)$ from $j_\zeta(\rho,\chi)$ before solving Eq.(2.165)

2.8 Summary and discussion

The set of equations describing the two-dimensional transport in a whole tokamak plasma has been derived in MSCS from the multi-fluid equations and Maxwell's equations, where the flux-surface-average has been applied on Eq.(2.155) and Eq.(2.161) in order to meet the constraint for the existence of the magnetic surface. The set of the fluid equations consists of the equation for the particle density n_a (2.99), the parallel momentum $m_a n_a u_{a\|}$ (2.108), the toroidal momentum $m_a n_a u_{a\zeta}$ (2.115), the radial momentum $m_a n_a u_a^\rho$ (2.120), the pressure p_a (2.126), the parallel total heat flux $Q_{a\|}$ (2.127), the toroidal total heat flux $Q_{a\zeta}$ (2.134) and the radial total heat flux Q_a^ρ (2.139) for each particle species. The set of equations for electromagnetic field includes the poloidal magnetic field ψ' (2.154), the toroidal magnetic field I (2.155), the toroidal electric field E_ζ (2.161), the poloidal electric field E_χ (2.163) and the radial electric field E_ρ (2.164).

The neoclassical parallel viscosity and heat viscosity have been rewritten in order to be applicable in the open field region outside the LCFS. We have shown that our parallel viscosity is consistent with the Hirshman-type parallel viscosity inside the LCFS and the Braginskii-type one outside the LCFS.

We have shown that our fluid equations are consistent with the neoclassical transport theory by yielding the neoclassical force balance equations from the equations for the parallel momentum and the parallel total heat flux. These equations are expected to provide a better description of the time evolution of the tokamak plasma, especially that of the poloidal and toroidal rotation.

We have emphasized the extension of the neoclassical transport in this article. The turbulent transport induced by the interaction with wave fluctuations will be included similarly as discussed in [9].

Chapter 3

Numerical scheme of TASK/T2

In this chapter we will describe the methodology for implementing the two-dimensional transport equations derived in chapter 2 into TASK/T2 component. In the TASK/T2 the set of two-dimensional transport equations is solved in both the core and the peripheral region as the advection-diffusion equations by finite element method (FEM).

FEM has some advantages for two-dimensional transport analysis in the tokamak plasma including both the core and the peripheral region such as, 1) the high flexibility for the structure of the numerical grid, 2) the easy implementation of stabilization scheme for advection driven numerical instability for example SUPG [37], BTD [38], GLS [39] and so on, and 3) the easy implementation of boundary conditions. Especially the first advantage is very important for two-dimensional transport in the both core and peripheral plasmas, since the characteristic length of the transport in the radial direction is quite different in the core region and the peripheral region and the topological structure of the magnetic field is different in the core region and the peripheral region. In addition, the transport parallel and perpendicular to the magnetic field line are different in several orders of magnitude and this anisotropy may cause the numerical instability and degrade the computational accuracy. In order to resolve these issues, we employ a hierarchical rectangular grid in MSCS which highly separates the parallel and perpendicular transport in order to suppress the numerical instability by the strong anisotropy and keeps the spatial resolution in the poloidal direction in the outer region.

This chapter is organized as follows. A coordinate system and dependent variables in TASK/T2 are discussed in section 1. In section 2, a numerical formulation of the transport equations as advection diffusion form is described. In section 3, properties and advantages of FEM, a formulation of finite element equations of advection diffusion equation and a concept of hierarchical rectangular grid are described. A flux surface averaging scheme in TASK/T2 are shown in section 4. In section 5, A time discretization scheme is described. In section 6, a concept of a computational grid, boundary conditions and initial conditions for limiter configuration are discussed. A concept of a computational grid for single-null divertor configuration is described in section 7. Summary and discussion are given in section 8.

3.1 Coordinates and dependent variables in TASK/T2

In this section we will discuss a methodology for implementing transport equations in a torus coordinate system (TCS) (σ, χ, ζ) in TASK/T2 as a preliminary step for the MSCS. Although the TCS does not take the self-consistent magnetic equilibrium configuration, it is useful to discuss the singularity of the equation system at the magnetic axis.

Since both the derivative of poloidal flux function with respect to the radial label ψ' and the contravariant poloidal magnetic field B^χ have to be definable, a TCS in TASK/T2 has to have a non-zero Jacobian at the magnetic axis. We therefore employ the torus coordinate defined by,

$$R = R_0 + a\sqrt{\sigma}\cos\chi \tag{3.1}$$
$$\phi = \zeta \tag{3.2}$$
$$Z = -a\sqrt{\sigma}\sin\chi \tag{3.3}$$

where σ is defined by the area and has the relation, $\sigma = r^2$, with the minor radius r. This torus coordinate system is orthogonal and has metric coefficients defined by

$$g_{\sigma\sigma} = \frac{a^2}{4\sigma} \tag{3.4}$$
$$g_{\chi\chi} = a^2\sigma \tag{3.5}$$
$$g_{\zeta\zeta} = R^2 \tag{3.6}$$
$$g^{\sigma\sigma} = \frac{4\sigma}{a^2} \tag{3.7}$$
$$g^{\chi\chi} = \frac{1}{a^2\sigma} \tag{3.8}$$
$$g^{\zeta\zeta} = \frac{1}{R^2} \tag{3.9}$$
$$\sqrt{g} = a^2 R. \tag{3.10}$$

Its covariant radial-radial geometrical coefficient $g_{\sigma\sigma}$ and contravariant poloidal-poloidal geometrical coefficient $g^{\chi\chi}$ have $1/\sigma$-singularity while its Jacobian is non-zero at the magnetic axis. In order to eliminate these singularity at the magnetic axis, we employ $n_a\bar{u}_a^\chi$, $\bar{Q}_a^\sigma$ and $\bar{E}_\chi$ as dependent variables instead of $n_a u_a^\chi$, Q_a^σ and E_χ, which are defined by

$$n_a\bar{u}_a^\sigma \equiv \sigma^{-1} n_a u_a^\sigma \tag{3.11}$$
$$\bar{Q_a}^\sigma \equiv \sigma^{-1} Q^\sigma \tag{3.12}$$
$$\bar{E}_\chi \equiv \sigma^{-1} E_\chi. \tag{3.13}$$

Note that the definitions (3.11)-(3.13) imply that $n_a u_a^\chi$, Q_a^σ and E_χ always vanish at the magnetic axis.

There is also a singularity at the magnetic axis due to the poloidal magnetic field. In the toroidal configuration, the poloidal magnetic field B_p essentially vanishes at the magnetic axis, since there is no loop current inside the flux surface corresponding to the magnetic axis. Therefore the transformation relation from the parallel flow $V_\parallel$ and the covariant

Maxwell's equations (1D)	ψ', I, E_ζ
Maxwell's equations (2D)	$\bar{E}_\chi$, E_σ
Transport equations (2D)	n_a, $n_a\bar{u}_a^\sigma$, $n_a u_{a\parallel}$, $n_a u_{a\zeta}$, $n_a u_a^\chi$, p_a, $\bar{Q}_a^\sigma$, $Q_{a\parallel}$, $Q_{a\zeta}$, Q_a^χ

Table 3.1: The dependent variables in TASK/T2

toroidal flow V_ζ to contravariant poloidal flow V^χ obtained by the parallel projection of the flow vector,

$$V^\chi = \frac{B^\chi}{B_p^2}\left(V_\parallel B - V_\zeta B^\zeta\right), \quad \boldsymbol{V} = \boldsymbol{u}_a, \boldsymbol{Q}_a \tag{3.14}$$

is no longer available at the magnetic axis. In the derivation of Eq.(3.14), the ordering,

$$V^\sigma \sim \mathcal{O}(\delta^2) \ll V_\parallel \sim V_\zeta \sim \mathcal{O}(\delta), \tag{3.15}$$

has been employed for simplicity. In order to avoid this singularity, we employ the contravariant poloidal flows as additional dependent variables and solve the equations expressed by

$$n_a u_a^\chi g_{\chi\chi} B^\chi = n_a u_{a\parallel} B - n_a u_{a\zeta} B^\zeta \tag{3.16}$$

$$Q_a^\chi g_{\chi\chi} B^\chi = Q_{a\parallel} B - Q_{a\zeta} B^\zeta. \tag{3.17}$$

The resulting set of the dependent variables in TASK/T2 is therefore summarized in Table 3.1.

3.2 Numerical formulation of transport equations in TASK/T2

Although we derived the set of equations for two-dimensional transport in tokamak plasmas in chapter 2, some of them are required to be deformed into advection-diffusion form. In TASK/T2, the two-dimensional transport equations are solved as the simultaneous advection-diffusion equations,

$$\sum_b \left[\frac{1}{\sqrt{g}} \frac{\partial}{\partial t} \left(\sqrt{g} M_{ab} f_b \right) + \nabla \cdot \left(\boldsymbol{V}_{ab} f_b \right) - \nabla \cdot \left(\overleftrightarrow{D}_{ab} \cdot \nabla f_b \right) + \boldsymbol{A}_{ab} \cdot \nabla f_b + C_{ab} f_b \right] = S_a, \tag{3.18}$$

and some of them are flux surface averaged as

$$\sum_b \left[\frac{1}{V'} \frac{\partial}{\partial t} \left(V' \langle M_{ab} \rangle f_b \right) + \langle \nabla \cdot \left(\boldsymbol{V}_{ab} f_b \right) \rangle - \left\langle \nabla \cdot \left(\overleftrightarrow{D}_{ab} \cdot \nabla f_b \right) \right\rangle + \langle \boldsymbol{A}_{ab} \cdot \nabla f_b \rangle + \langle C_{ab} f_b \rangle \right]$$
$$= \langle S_a \rangle, \tag{3.19}$$

where a, b is the variable index, f_b is the unknown variables, M_{ab} is the mass scalar coefficient, $\boldsymbol{V}_{ab}$ is the advection vector coefficient, $\overleftrightarrow{D}_{ab}$ is the diffusion tensor coefficient, $\boldsymbol{A}_{ab}$ is the gradient vector coefficient, C_{ab} is the excitation scalar coefficient and S_a is the source term. Since some of the coefficients of the two-dimensional transport equations consist of the spatial derivatives, $\boldsymbol{V}_{ab}$, $\boldsymbol{A}_{ab}$ and C_{ab} in Eq.(3.19) and Eq.(3.18) are required to be decomposed as

$$\boldsymbol{V}_{ab} = \boldsymbol{V}^1_{ab} - \sum_x \overleftrightarrow{V}^2_{abx} \cdot \nabla g_x, \tag{3.20}$$

$$\boldsymbol{A}_{ab} = \boldsymbol{A}^1_{ab} + \sum_x \nabla g_x \cdot \overleftrightarrow{A}^2_{abx}, \tag{3.21}$$

$$C_{ab} = C^1_{ab} + \sum_x \nabla g_x \cdot \boldsymbol{C}^2_{abx} + \sum_{x,y} \nabla g_x \cdot \overleftrightarrow{C}^3_{abxy} \cdot \nabla g_y, \tag{3.22}$$

in order to ensure the C^0 continuity of numerical coefficients at the interface of the element, where g_x and g_y are the known integrands. Therefore the governing equations of the TASK/T2 code can be expressed as

$$\sum_b \frac{1}{\sqrt{g}} \frac{\partial}{\partial t} \left(\sqrt{g} M_{ab} f_b \right) + \sum_b \nabla \cdot \left[\left(\boldsymbol{V}^1_{ab} - \sum_x \overleftrightarrow{V}^2_{abx} \cdot \nabla g_x \right) f_b \right]$$
$$- \sum_b \nabla \cdot \left(\overleftrightarrow{D}_{ab} \cdot \nabla f_b \right) + \sum_b \left(\boldsymbol{A}^1_{ab} + \sum_x \nabla g_x \cdot \overleftrightarrow{A}^2_{abx} \right) \cdot \nabla f_b$$
$$+ \sum_b \left(C^1_{ab} + \sum_x \nabla g_x \cdot \boldsymbol{C}^2_{abx} + \sum_{x,y} \nabla g_x \cdot \overleftrightarrow{C}^3_{abxy} \cdot \nabla g_y \right) f_b = S_a. \tag{3.23}$$

After the lengthy and tedious manipulation, the advection-diffusion forms of the two-dimensional transport equations are obtained respectively, where the derivation process and the specific expressions of the coefficients are summarized in Appendix A.

- Equation for poloidal magnetic flux function

$$\frac{1}{V'}\frac{\partial}{\partial t}\left(V'\left\langle M_{01.01}\right\rangle\psi'\right)+\left\langle\nabla\cdot\left(\boldsymbol{V}^{1}_{01.01}\psi'\right)\right\rangle+\left\langle\boldsymbol{A}^{1}_{01.03}\cdot\nabla E_{\zeta}\right\rangle=0, \tag{3.24}$$

- Equation for poloidal current function

$$\begin{aligned}\frac{1}{V'}\frac{\partial}{\partial t}\left(V'\left\langle M_{02.02}\right\rangle I\right)+\left\langle\nabla\cdot\left(\boldsymbol{V}^{1}_{02.01}I\right)\right\rangle+\left\langle\boldsymbol{A}^{1}_{02.04}\cdot\nabla\bar{E}_{\chi}\right\rangle+\left\langle\boldsymbol{A}^{1}_{02.05}\cdot\nabla E_{\sigma}\right\rangle\\+\left\langle C^{1}_{02.04}\bar{E}_{\chi}\right\rangle=0\end{aligned} \tag{3.25}$$

- Equation for covariant toroidal electric field

$$\begin{aligned}\frac{1}{V'}\frac{\partial}{\partial t}\left(V'\left\langle M_{03.03}\right\rangle E_{\zeta}\right)+\left\langle\nabla\cdot\left(\boldsymbol{V}^{1}_{03.01}\psi'\right)\right\rangle+\left\langle\nabla\cdot\left(\boldsymbol{V}^{1}_{03.03}E_{\zeta}\right)\right\rangle\\+\sum_{a}\left\langle C^{1}_{03.09a}n_{a}u_{a\zeta}\right\rangle+\left\langle\left(\nabla R\cdot\boldsymbol{C}^{2}_{03.01.02}\right)\psi'\right\rangle=0\end{aligned} \tag{3.26}$$

- Equation for covariant poloidal electric field

$$\begin{aligned}\frac{1}{\sqrt{g}}\frac{\partial}{\partial t}\left(\sqrt{g}M_{04.04}\bar{E}_{\chi}\right)+\nabla\cdot\left(\boldsymbol{V}^{1}_{04.04}E_{\chi}\right)+\boldsymbol{A}^{1}_{04.02}\nabla I\\+\sum_{a}C^{1}_{04.07a}n_{a}\bar{u}_{a\parallel}+\sum_{a}C^{1}_{04.10a}n_{a}u_{a\zeta}=0\end{aligned} \tag{3.27}$$

- Equation for covariant radial electric field

$$\nabla\cdot\left(\boldsymbol{V}^{1}_{05.04}\bar{E}_{\chi}\right)+\nabla\cdot\left(\boldsymbol{V}^{1}_{05.05}E_{\sigma}\right)+\sum_{a}C^{1}_{05.06a}n_{a}=0 \tag{3.28}$$

- Equation for particle density

$$\frac{1}{\sqrt{g}}\frac{\partial}{\partial t}\left(\sqrt{g}M_{06a.06a}n_{a}\right)+\nabla\cdot\left(\boldsymbol{V}^{1}_{06a.06a}n_{a}\right)=S_{06a.06a} \tag{3.29}$$

- Equation for contravariant radial particle flux

$$\boldsymbol{A}^{1}_{07a.11a}\cdot\nabla p_{a}+C^{1}_{07a.04}\bar{E}_{\chi}+C^{1}_{07a.05}E_{\sigma}+C^{1}_{07a.08a}n_{a}u_{a\parallel}+C^{1}_{07a.09a}n_{a}u_{a\zeta}=0 \tag{3.30}$$

- Equation for parallel particle flux

$$
\begin{aligned}
&\frac{1}{\sqrt{g}}\frac{\partial}{\partial t}\left(\sqrt{g}M_{08a.08a}n_a u_{a\|}\right)+\nabla\cdot\left(\boldsymbol{V}^1_{08a.08a}n_a u_{a\|}\right)+\nabla\cdot\left(\boldsymbol{V}^1_{08a.11a}p_a\right)\\
&\quad-\nabla\cdot\left[\left(\overleftrightarrow{V}^1_{08a.09a.01}\cdot\nabla B\right)n_a u_{a\zeta}\right]-\nabla\cdot\left[\left(\overleftrightarrow{V}^1_{08a.10a.01}\cdot\nabla B\right)n_a u_a^{\chi}\right]\\
&\qquad-\nabla\cdot\left[\left(\overleftrightarrow{V}^1_{08a.14a.01}\cdot\nabla B\right)Q_{a\zeta}\right]-\nabla\cdot\left[\left(\overleftrightarrow{V}^1_{08a.15a.01}\cdot\nabla B\right)Q_a^{\chi}\right]\\
&\quad-\nabla\cdot\left(\overleftrightarrow{D}_{08a.06a}\cdot\nabla n_a\right)-\nabla\cdot\left[\overleftrightarrow{D}_{08a.08a}\cdot\nabla\left(n_a u_{a\|}\right)\right]\\
&\qquad-\nabla\cdot\left(\overleftrightarrow{D}_{08a.11a}\cdot\nabla p_a\right)-\nabla\cdot\left(\overleftrightarrow{D}_{08a.13a}\cdot\nabla Q_{a\|}\right)\\
&\quad+\left(\nabla B\cdot\overleftrightarrow{A}^1_{08a.06a.01}\right)\cdot\nabla n_a+\left(\nabla B\cdot\overleftrightarrow{A}^1_{08a.08a.01}\right)\cdot\nabla\left(n_a u_{a\|}\right)\\
&\qquad+\left(\nabla B\cdot\overleftrightarrow{A}^1_{08a.11a.01}\right)\cdot\nabla p_a+\left(\nabla B\cdot\overleftrightarrow{A}^1_{08a.13a.01}\right)\cdot\nabla Q_{a\|}\\
&\quad+C^1_{08a.03}E_\zeta+C^1_{08a.04}\bar{E}_\chi+\sum_b C^1_{08a.08b}n_a u_{b\|}+\sum_b C^1_{08a.12b}Q_{b\|}\\
&\quad+\left(\boldsymbol{C}^2_{08a.08a.01}\cdot\nabla B\right)n_a u_{a\|}\\
&\quad+\left(\nabla B\cdot\overleftrightarrow{C}^3_{08a.09a.01.01}\cdot\nabla B\right)n_a u_{a\zeta}+\left(\nabla B\cdot\overleftrightarrow{C}^3_{08a.10a.01.01}\cdot\nabla B\right)n_a u_a^{\chi}\\
&\qquad+\left(\nabla B\cdot\overleftrightarrow{C}^3_{08a.14a.01.01}\cdot\nabla B\right)Q_{a\zeta}+\left(\nabla B\cdot\overleftrightarrow{C}^3_{08a.15a.01.01}\cdot\nabla B\right)Q_a^{\chi}\\
&=S_{08a.08a}
\end{aligned}
\tag{3.31}
$$

- Equation for covariant toroidal particle flux

$$
\begin{aligned}
&\frac{1}{\sqrt{g}}\frac{\partial}{\partial t}\left(\sqrt{g}M_{09a.09a}n_a u_{a\zeta}\right)+\nabla\cdot\left(\boldsymbol{V}^1_{09a.09a}n_a u_{a\zeta}\right)\\
&\quad-\nabla\cdot\left[\left(\overleftrightarrow{V}^2_{09a.09a.01}\cdot\nabla B\right)n_a u_{a\zeta}\right]-\nabla\cdot\left[\left(\overleftrightarrow{V}^2_{09a.10a.01}\cdot\nabla B\right)n_a u_a^{\chi}\right]\\
&\qquad-\nabla\cdot\left[\left(\overleftrightarrow{V}^2_{09a.14a.01}\cdot\nabla B\right)Q_{a\zeta}\right]-\nabla\cdot\left[\left(\overleftrightarrow{V}^2_{09a.15a.01}\cdot\nabla B\right)Q_a^{\chi}\right]\\
&\quad-\nabla\cdot\left(\overleftrightarrow{D}_{09a.06a}\cdot\nabla n_a\right)-\nabla\cdot\left[\overleftrightarrow{D}_{09a.08a}\cdot\nabla\left(n_a u_{a\|}\right)\right]\\
&\qquad-\nabla\cdot\left(\overleftrightarrow{D}_{09a.11a}\cdot\nabla p_a\right)-\nabla\cdot\left(\overleftrightarrow{D}_{09a.13a}\cdot\nabla Q_{a\|}\right)\\
&\quad+C^1_{09a.03}E_\zeta+C^1_{09a.07a}n_a\bar{u}_a^{\sigma}+\sum_b C^1_{09a.09b}n_a u_{b\zeta}+\sum_b C^1_{09a.14b}Q_{b\zeta}\\
&=S_{09a.09a}.
\end{aligned}
\tag{3.32}
$$

- Expression of contravariant poloidal particle flux

$$
C^1_{10a.08a}n_a u_{a\|}+C^1_{10a.09a}n_a u_{a\zeta}+C^1_{10a.010a}n_a u_a^{\chi}=0 \tag{3.33}
$$

- Equation for pressure

$$
\begin{aligned}
&\frac{1}{\sqrt{g}}\frac{\partial}{\partial t}\left(\sqrt{g}M_{11a.11a}p_a\right)+\nabla\cdot\left(\boldsymbol{V}^1_{11a.11a}p_a\right)\\
&\quad-\nabla\cdot\left[\left(\overleftrightarrow{V}^2_{11a.09a.01}\cdot\nabla B\right)n_a u_{a\zeta}\right]-\nabla\cdot\left[\left(\overleftrightarrow{V}^2_{11a.10a.01}\cdot\nabla B\right)n_a u^\chi_a\right]\\
&\qquad-\nabla\cdot\left[\left(\overleftrightarrow{V}^2_{11a.14a.01}\cdot\nabla B\right)Q_{a\zeta}\right]-\nabla\cdot\left[\left(\overleftrightarrow{V}^2_{11a.15a.01}\cdot\nabla B\right)Q^\chi_a\right]\\
&\quad-\nabla\cdot\left(\overleftrightarrow{D}_{11a.06a}\cdot\nabla n_a\right)-\nabla\cdot\left[\overleftrightarrow{D}_{11a.08a}\cdot\nabla\left(n_a u_{a\|}\right)\right]\\
&\qquad-\nabla\cdot\left(\overleftrightarrow{D}_{11a.11a}\cdot\nabla p_a\right)-\nabla\cdot\left(\overleftrightarrow{D}_{11a.13a}\cdot\nabla Q_{a\|}\right)\\
&\quad+\boldsymbol{A}^1_{11a.11a}\cdot\nabla p_a\\
&\quad+\left(\nabla B\cdot\overleftrightarrow{A}^2_{11a.06a.01}\right)\cdot\nabla n_a+\left(\nabla u_{a\|}\cdot\overleftrightarrow{A}^2_{11a.06a.03a}\right)\cdot\nabla n_a\\
&\qquad+\left(\nabla B\cdot\overleftrightarrow{A}^2_{11a.08a.01}\right)\cdot\nabla\left(n_a u_{a\|}\right)+\left(\nabla u_{a\|}\cdot\overleftrightarrow{A}^2_{11a.08a.03a}\right)\cdot\nabla\left(n_a u_{a\|}\right)\\
&\qquad+\left(\nabla B\cdot\overleftrightarrow{A}^2_{11a.11a.01}\right)\cdot\nabla p_a+\left(\nabla u_{a\|}\cdot\overleftrightarrow{A}^2_{11a.11a.03a}\right)\cdot\nabla p_a\\
&\qquad+\left(\nabla B\cdot\overleftrightarrow{A}^2_{11a.13a.01}\right)\cdot\nabla Q_{a\|}+\left(\nabla u_{a\|}\cdot\overleftrightarrow{A}^2_{11a.13a.03a}\right)\cdot\nabla Q_{a\|}\\
&\quad+C^1_{11a.11a}p_a\\
&\quad+\left(\nabla B\cdot\overleftrightarrow{C}^3_{11a.09a.01.01}\cdot\nabla B\right)n_a u_{a\zeta}+\left(\nabla u_{a\|}\cdot\overleftrightarrow{C}^3_{11a.09a.03a.01}\cdot\nabla B\right)n_a u_{a\zeta}\\
&\qquad+\left(\nabla B\cdot\overleftrightarrow{C}^3_{11a.10a.01.01}\cdot\nabla B\right)n_a u^\chi_a+\left(\nabla u_{a\|}\cdot\overleftrightarrow{C}^3_{11a.10a.03a.01}\cdot\nabla B\right)n_a u^\chi_a\\
&\qquad+\left(\nabla B\cdot\overleftrightarrow{C}^3_{11a.14a.01.01}\cdot\nabla B\right)Q_{a\zeta}+\left(\nabla u_{a\|}\cdot\overleftrightarrow{C}^3_{11a.14a.03a.01}\cdot\nabla B\right)Q_{a\zeta}\\
&\qquad+\left(\nabla B\cdot\overleftrightarrow{C}^3_{11a.15a.01.01}\cdot\nabla B\right)Q^\chi_a+\left(\nabla u_{a\|}\cdot\overleftrightarrow{C}^3_{11a.15a.03a.01}\cdot\nabla B\right)Q^\chi_a\\
&=S_{11a.11a}
\end{aligned}
\tag{3.34}
$$

- Equation for contravariant radial total heat flux

$$
\begin{aligned}
&\boldsymbol{A}^1_{12a.06a}\cdot\nabla n_a+\boldsymbol{A}^1_{12a.11a}\cdot\nabla p_a\\
&\quad+C^1_{12a.04}\bar{E}_\chi+C^1_{12a.05}E_\sigma+C^1_{12a.13a}Q_{a\|}+C^1_{12a.14a}Q_{a\zeta}=0
\end{aligned}
\tag{3.35}
$$

- Equation for parallel total heat flux

$$
\begin{aligned}
&\frac{1}{\sqrt{g}}\frac{\partial}{\partial t}\left(\sqrt{g}M_{13a.13a}Q_{a\|}\right)+\nabla\cdot\left(\boldsymbol{V}^1_{13a.08a}n_a u_{a\|}\right)+\nabla\cdot\left(\boldsymbol{V}^1_{13a.11a}p_a\right)+\nabla\cdot\left(\boldsymbol{V}^1_{13a.13a}Q_{a\|}\right)\\
&\quad-\nabla\cdot\left[\left(\nabla B\cdot\overleftrightarrow{V}^2_{13a.06a.01}\right)n_a u_{a\zeta}\right]-\nabla\cdot\left[\left(\nabla B\cdot\overleftrightarrow{V}^2_{13a.10a.01}\right)n_a u^\chi_a\right]\\
&\qquad-\nabla\cdot\left[\left(\nabla B\cdot\overleftrightarrow{V}^2_{13a.14a.01}\right)Q_{a\zeta}\right]-\nabla\cdot\left[\left(\nabla B\cdot\overleftrightarrow{V}^2_{13a.15a.01}\right)Q^\chi_a\right]\\
&\quad-\nabla\cdot\left(\overleftrightarrow{D}_{13a.06a}\cdot\nabla n_a\right)-\nabla\cdot\left[\overleftrightarrow{D}_{13a.08a}\cdot\nabla\left(n_a u_{a\|}\right)\right]\\
&\qquad-\nabla\cdot\left(\overleftrightarrow{D}_{13a.11a}\cdot\nabla p_a\right)-\nabla\cdot\left(\overleftrightarrow{D}_{13a.13a}\cdot\nabla Q_{a\|}\right)\\
&\quad+\left(\nabla B\cdot\overleftrightarrow{A}^2_{13a.06a.01}\right)\cdot\nabla n_a+\left(\nabla B\cdot\overleftrightarrow{A}^2_{13a.08a.01}\right)\cdot\nabla\left(n_a u_{a\|}\right)\\
&\qquad+\left(\nabla B\cdot\overleftrightarrow{A}^2_{13a.11a.01}\right)\cdot\nabla p_a+\left(\nabla B\cdot\overleftrightarrow{A}^2_{13a.13a.01}\right)\cdot\nabla Q_{a\|}\\
&\quad+C^1_{13a.03}E_\zeta+C^1_{13a.04}\bar{E}_\chi+\sum_b C^1_{13a.08b}n_b u_{b\|}+\sum_b C^1_{13a.13b}Q_{b\|}\\
&\quad+\left(\nabla B\cdot\boldsymbol{C}^2_{13a.03.01}\right)E_\zeta+\left(\nabla u_{a\|}\cdot\boldsymbol{C}^2_{13a.03.03a}\right)E_\zeta+\left(\nabla w_{a\|}\cdot\boldsymbol{C}^2_{13a.03.04a}\right)E_\zeta\\
&\qquad+\left(\nabla B\cdot\boldsymbol{C}^2_{13a.04.01}\right)\bar{E}_\chi+\left(\nabla u_{a\|}\cdot\boldsymbol{C}^2_{13a.04.03a}\right)\bar{E}_\chi+\left(\nabla w_{a\|}\cdot\boldsymbol{C}^2_{13a.04.04a}\right)\bar{E}_\chi\\
&\qquad+\left(\nabla B\cdot\boldsymbol{C}^2_{13a.08a.01}\right)n_a u_{a\|}+\left(\nabla u_{a\|}\cdot\boldsymbol{C}^2_{13a.13a.01}\right)Q_{a\|}\\
&\quad+\left(\nabla B\cdot\overleftrightarrow{C}^3_{13a.09a.01.01}\cdot\nabla B\right)n_a u_{a\zeta}+\left(\nabla B\cdot\overleftrightarrow{C}^3_{13a.10a.01.01}\cdot\nabla B\right)n_a u^\chi_a\\
&\qquad+\left(\nabla B\cdot\overleftrightarrow{C}^3_{13a.14a.01.01}\cdot\nabla B\right)Q_{a\zeta}+\left(\nabla B\cdot\overleftrightarrow{C}^3_{13a.15a.01.01}\cdot\nabla B\right)Q^\chi_a\\
&=S_{13a.13a}
\end{aligned}
\tag{3.36}
$$

- Equation for covariant toroidal total heat flux

$$
\begin{aligned}
&\frac{1}{\sqrt{g}}\frac{\partial}{\partial t}\left(\sqrt{g}M_{14a.14a}n_a u_{a\zeta}\right)+\nabla\cdot\left(\boldsymbol{V}^1_{14a.09a}n_a u_{a\zeta}\right)+\nabla\cdot\left(\boldsymbol{V}^1_{14a.14a}Q_{a\zeta}\right)\\
&\quad-\nabla\cdot\left[\left(\overleftrightarrow{V}^2_{14a.09a.01}\cdot\nabla B\right)n_a u_{a\zeta}\right]-\nabla\cdot\left[\left(\overleftrightarrow{V}^2_{14a.10a.01}\cdot\nabla B\right)n_a u^\chi_a\right]\\
&\qquad-\nabla\cdot\left[\left(\overleftrightarrow{V}^2_{14a.14a.01}\cdot\nabla B\right)Q_{a\zeta}\right]-\nabla\cdot\left[\left(\overleftrightarrow{V}^2_{14a.15a.01}\cdot\nabla B\right)Q^\chi_a\right]\\
&\quad-\nabla\cdot\left(\overleftrightarrow{D}_{14a.06a}\cdot\nabla n_a\right)-\nabla\cdot\left[\overleftrightarrow{D}_{14a.08a}\cdot\nabla\left(n_a u_{a\|}\right)\right]\\
&\qquad-\nabla\cdot\left(\overleftrightarrow{D}_{14a.11a}\cdot\nabla p_a\right)-\nabla\cdot\left(\overleftrightarrow{D}_{14a.13a}\cdot\nabla Q_{a\|}\right)\\
&\quad+C^1_{14a.03}E_\zeta+C^1_{14a.12a}\bar{Q}^\sigma_a+\sum_b C^1_{14a.09b}n_b u_{b\zeta}+\sum_b C^1_{14a.14b}Q_{b\zeta}\\
&\quad+\left(\nabla B\cdot\boldsymbol{C}^2_{14a.03.01}\right)E_\zeta+\left(\nabla u_{a\|}\cdot\boldsymbol{C}^2_{14a.03.03a}\right)E_\zeta+\left(\nabla w_{a\|}\cdot\boldsymbol{C}^2_{14a.03.04a}\right)E_\zeta\\
&\qquad+\left(\nabla B\cdot\boldsymbol{C}^2_{14a.04.01}\right)\bar{E}_\chi+\left(\nabla u_{a\|}\cdot\boldsymbol{C}^2_{14a.04.03a}\right)\bar{E}_\chi+\left(\nabla w_{a\|}\cdot\boldsymbol{C}^2_{14a.04.04a}\right)\bar{E}_\chi\\
&=S_{14a.14a}
\end{aligned}
\tag{3.37}
$$

- Expression of contravariant poloidal particle flux

$$
C^1_{15a.13a}Q_{a\|}+C^1_{15a.14a}Q_{a\zeta}+C^1_{15a.15a}Q^\chi_a=0 \tag{3.38}
$$

3.3 Finite element method

3.3.1 Weighted residual method and weak formulation

Finite element method (FEM) is a kind of numerical techniques for finding an approximate solution for a partial differential equation and has some advantages such as the high flexibility for the numerical domain configuration and the direct treatment of the boundary conditions. In this section We will briefly explain the principle of the weighted residual method (WRM) which is the basis of FEM by taking Poisson equation for instance,

$$\nabla^2 f - S = 0. \tag{3.39}$$

We introduce spatial coordinate (ξ_1, ξ_2, ξ_3) and bounded domain Ω whose boundary $\partial\Omega$ consists of the fixed boundary Γ_1 and free boundary Γ_2.

$$\nabla^2 f - S = 0 \qquad \text{on } \Omega \ , \tag{3.40}$$

$$f = f_b \qquad \text{on } \Gamma_1, \tag{3.41}$$

$$(\nabla f)_n = q \qquad \text{on } \Gamma_2, \tag{3.42}$$

where S is the known source term on Ω, f_b is the known value of f on Γ_1, q is the the known value of the normal gradient of f on Γ_2 and the subscript n is the projection in the normal direction of the boundary Γ_2.

In WRM, a residual $r(f)$ is defined in terms of f satisfying Eqs.(3.40)-(3.42),

$$r(\boldsymbol{\xi}) = \nabla^2 f(\boldsymbol{\xi}) - S(\boldsymbol{\xi}), \tag{3.43}$$

where $r(\boldsymbol{\xi})$ vanishes at any point of Ω if f satisfies Eq.(3.40) rigorously. In the weighted residual method, the functional I is introduced,

$$I = \int w(\boldsymbol{\xi}) r(\boldsymbol{\xi}) d\Omega, \tag{3.44}$$

where w is the arbitrary weighting function and I corresponds to the weighted average of the residual r. Since the functional I vanishes for the arbitrary weighting function w in the case where $r(\boldsymbol{\xi}) = 0$ for $\boldsymbol{\xi} \in \Omega$, the following weighted residual equation corresponding to Eq.(3.40) is obtained by substituting Eq.(3.43) to Eq.(3.44),

$$I = \int w(\boldsymbol{\xi}) \left[\nabla^2 f - S(\boldsymbol{\xi})\right] d\Omega = 0. \tag{3.45}$$

In the WRM, f satisfying Eq.(3.45) is calculated instead of f satisfying Eq.(3.40).

In the WRM, the boundary conditions of the weighting function w and the unknown function f is applied on the fixed boundary Γ_1 and the free boundary Γ_2 respectively,

$$f = f_b, \quad w = 0 \qquad \text{on } \Gamma_1, \tag{3.46}$$

$$\int_{\Gamma_2} \left[(\nabla f)_n - q\right] d\Gamma = 0 \qquad \text{on } \Gamma_2. \tag{3.47}$$

The problem finding f satisfying Eqs.(3.40)-(3.41) is therefore reduced to the problem finding f satisfying Eq.(3.48),

$$\int_\Omega w\left(\nabla^2 f - S\right) d\Omega + \int_{\Gamma_2} w\left[(\nabla f)_n - q\right] d\Gamma = 0. \tag{3.48}$$

Since there is a second order differential of f in Eq.(3.48), the solution f is required to be of C^2. By the use of the partial integral in Eq.(3.48), we can eliminate the second order differential of f and then the restriction for the continuity of the solution f can be reduced to C^1,

$$\int_\Omega \nabla w \cdot \nabla f d\Omega - \int_\Omega w S d\Omega - \int_{\Gamma_2} w q d\Gamma = 0, \tag{3.49}$$

where Eq.(3.49) is known as the weak form of the advection diffusion equation and Eq.(3.49) also requires that w is of C^1.

3.3.2 Finite element equation

In FEM, the solution of Eq.(3.49) is limited in the function space V^h spanned by N known functions ϕ_i which vanish on the boundary Γ_1,

$$V^h(f_b) = \left\{ v^h(\boldsymbol{\xi}); \quad v^h = f_b(\boldsymbol{\xi}) + \sum_{i=1}^{N} f_i \phi_i(\boldsymbol{\xi}) \right\}, \tag{3.50}$$

where $f_b(\boldsymbol{\xi})$ is a function satisfying the fixed boundary condition (3.46) and f_i is the set of the coefficients. Therefore the problem is reduced to find the best approximate function $f^h(\boldsymbol{\xi})$

$$f^h(\boldsymbol{\xi}) = f_b(\boldsymbol{\xi}) + \sum_{i=1}^{N} f_i \phi_i(\boldsymbol{\xi}). \tag{3.51}$$

The weighting function w is limited to N known functions φ_i which also vanish on the boundary Γ_1, which means not that Eq.(3.47) is solved in order to make the residual vanish at any point of Γ_2 and Ω but that the solution of Eq.(3.47) is solved in order to make the N-points weighted average of the residual vanish. If Eq.(3.49) is valid for $w = \varphi_1, \cdots, w = \varphi_N$, Eq.(3.49) is also valid for any element of the function space W^h,

$$W^h = \left\{ w(\boldsymbol{\xi}); \quad w(\boldsymbol{\xi}) = \sum_{i=1}^{N} w_i \varphi_i(\boldsymbol{\xi}) \right\}. \tag{3.52}$$

Substituting Eq.(3.52) into Eq.(3.49), the problem to solve Eq.(3.49) is reduce to the problem to find the best approximation function f^h for $i = 1, \ldots, N$,

$$\int_\Omega \nabla \varphi_i \cdot \nabla f^h d\Omega - \int_\Omega \varphi_i S d\Omega - \int_{\Gamma_2} \varphi_i q d\Gamma = 0, \tag{3.53}$$

where Eq.(3.53) is called as finite element equation. Substituting Eq.(3.51) into Eq.(3.53), the following simultaneous equations are obtained as

$$\sum_{j=1}^{N} f_j \int_{\Omega} \nabla\varphi_i \cdot \nabla\phi_j d\Omega = \int_{\Omega} \varphi_i S d\Omega + \int_{\Gamma_2} \varphi_i q d\Gamma - \int_{\Omega} \nabla\varphi_i \cdot \nabla f_g d\Omega. \tag{3.54}$$

The following symbols are introduced in order to express the integral quantities in Eq.(3.54)

$$\mathsf{A}_{ij} = \int_{\Omega} \nabla\varphi_i \cdot \nabla\phi_j d\Omega \tag{3.55}$$

$$\mathsf{b}_i = \int_{\Omega} \varphi_i S d\Omega + \int_{\Gamma_2} \varphi_i q d\Gamma - \int_{\Omega} \nabla\varphi_i \cdot \nabla f_g d\Omega. \tag{3.56}$$

By the use of Eq.(3.55) and Eq.(3.56), Eq.(3.54) can be written in matrix equation form as

$$\mathsf{Af} = \mathsf{b}, \quad \mathsf{f}^{\mathsf{T}} = \begin{bmatrix} f_1 & \cdots & f_N \end{bmatrix}. \tag{3.57}$$

Therefore the problem solving Eq.(3.39) is reduced to the problem solving the matrix equation (3.57) in FEM.

The two sets of known functions ϕ_i and φ have been introduced and the choice of these sets has some freedom. For the symmetrical system with respect to f and w such as Poisson equation and diffusion equation, the same set of known functions should be applied to f and w in order to keep the symmetry of the system, which is called as Galerkin FEM,

$$\varphi_i = \phi_i. \tag{3.58}$$

On the other hand, in the asymmetrical system with respect to f and w such as advection equation and advection-diffusion equation, the use of the same set of known functions for f and w may cause the numerical instability. In order to suppress this numerical instability, the different set of function are used for f and w, which is called as Petrov-Galerkin (PG) FEM,

$$\varphi_i \neq \phi_i. \tag{3.59}$$

In this thesis we employ Stream Upwind/Petrov Galerkin (SUPG) method which is a kind of PG-FEM for the advection diffusion equation when the set of our transport equations are formulated into the finite element equations,

$$\varphi_i = \phi_i + \tau \boldsymbol{u} \cdot \nabla\phi_i \quad \text{(SUPG)}, \tag{3.60}$$

where τ and $\boldsymbol{u}$ is the stabilization parameters in SUPG-FEM and there are some models determining them.

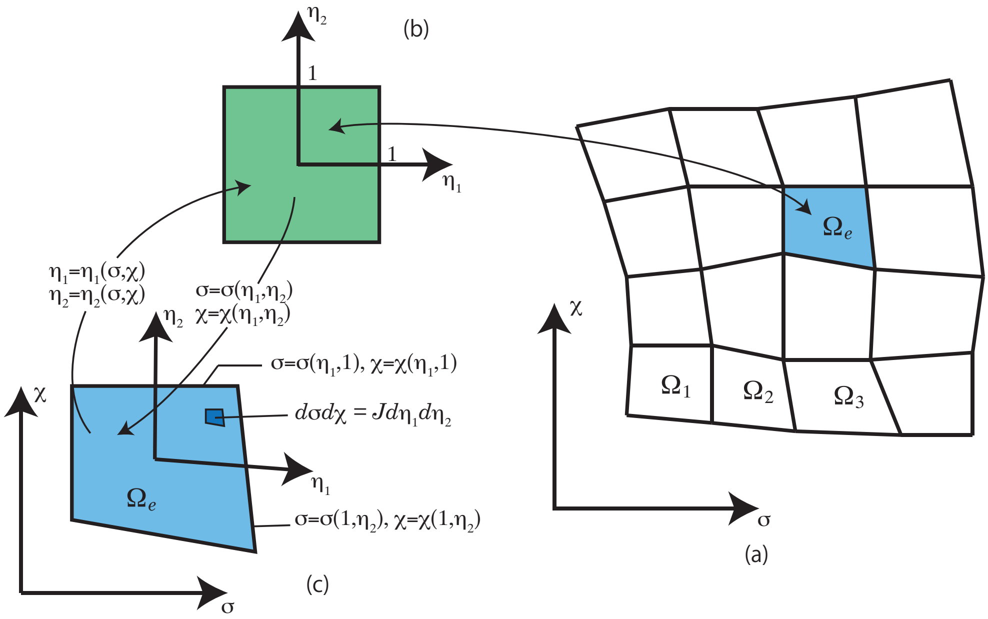

Figure 3.1: Concept of master rectangular element

3.3.3 Domain decomposition and basis functions

In FEM the computational domain Ω is decomposed into N_{elm} small sub-domains called as the finite element $\Omega_1, \Omega_2, \ldots, \Omega_{N_{\text{elm}}}$ and N_{node} nodal points P_i as Figure 3.1, where there must not be any overlapped domain and uncovered domain. Fig 1-(a) is the concept of the domain decomposition by the quadrangle element with arbitrary shape. Since the basis functions called the known function are more easily derivable and it is easier to evaluate the area integral over the square domain than that over the arbitrary quadrangle domain, the finite element Ω_e in MSCS is transformed into the normalized square element in the local coordinate space (LCS) (η_1, η_2) in Figure 3.1-(b). The coordinate transformation between MSCS and LCS is summarized in Figure 3.1-(c).

Domain decomposition

We introduce the basis function ϕ_i satisfying

$$\phi_i(P_j) = \delta_{ij} \quad (j = 1, \ldots, N_{\text{node}}) \tag{3.61}$$

as the known function in Eq.(3.49). By the use of Eq.(3.61), The approximate function f is expressed as

$$f^h(\boldsymbol{r}) = f_g(\boldsymbol{r}) + \sum_{P_i \in \Omega \cup \Gamma_2} f_i \phi_i(\boldsymbol{r}), \tag{3.62}$$

where f_i is a value of the function f^h at a nodal point P_i. The function $f_b(\boldsymbol{r})$ is also expressed as

$$f_b(\boldsymbol{r}) = \sum_{P_i \in \Gamma_1} f_b(P_i) \phi_i(\boldsymbol{r}). \tag{3.63}$$

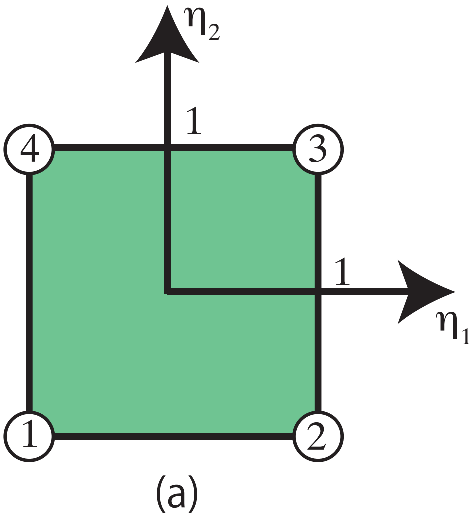

Figure 3.2: The master rectangular elements; (a) linear Lagrange element and (b) quadratic Lagrange element element

In the case of Galarkin-FEM, the matrix coefficient A_{ij} (3.55) and the vector coefficient b_j therefore become respectively

$$\mathsf{A}_{ij} = \int_{\Omega} \nabla\phi_i(\boldsymbol{r}) \cdot \nabla\phi_j(\boldsymbol{r}) d\Omega = \sum_{\text{elm}} \int_{\Omega_{\text{elm}}} \nabla\phi_i(\boldsymbol{r}) \cdot \nabla\phi_j(\boldsymbol{r}) d\Omega \tag{3.64}$$

$$\begin{aligned} \mathsf{b}_i &= \int_{\Omega} \phi_i(\boldsymbol{r}) S(\boldsymbol{r}) d\Omega + \int_{\Gamma_2} \phi_i(\boldsymbol{r}) q(\boldsymbol{r}) d\Gamma - \int_{\Omega} \nabla\phi_i(\boldsymbol{r}) \cdot \nabla f_b(\boldsymbol{r}) d\Omega \\ &= \sum_{\text{elm}} \int_{\Omega_{\text{elm}}} \phi_i(\boldsymbol{r}) S(\boldsymbol{r}) d\Omega + \sum_{\text{elm}} \int_{\Gamma_{2\text{elm}}} \phi_i(\boldsymbol{r}) q(\boldsymbol{r}) d\Gamma - \sum_{\text{elm}} \int_{\Omega_{\text{elm}}} \nabla\phi_i(\boldsymbol{r}) \cdot \nabla f_b(\boldsymbol{r}) d\Omega, \end{aligned} \tag{3.65}$$

where i, j are indices satisfying $P_i, P_j \in \Omega \cup \Gamma_2$.

Basis functions and Coordinate transformation

As we have mentioned in the previous section, the master element and their master basis functions are preferable to evaluate the area integral and the line integral in Eq.(3.64) and Eq.(3.65). The typical master rectangular elements are shown in Figure 3.2 and their basis functions are in Table3.2. In this section we will discuss how the derivatives in MSCS are expressed in LCS. By the use of the interpolation functions in Table 3.2 instead of the basis functions, Eq.(3.64) becomes

$$\mathsf{A}_{IJ}^{\text{elm}} = \sum_{i=1}^{2} \sum_{j=1}^{2} \int_{\Omega_e} g^{\xi_i \xi_j}(\sigma, \chi) \frac{\partial \phi_I^{\text{elm}}}{\partial \xi_i} \frac{\partial \phi_J^{\text{elm}}}{\partial \xi_j} \sqrt{g}(\sigma, \chi) d\sigma d\chi, \tag{3.66}$$

where n_{node} is the number of nodal point in the element and Eq.(3.66) requires the coordinate transformation from MSCS to LCS. At first, the map between Ω_{elm} and Ω_M is written

Element type	Interpolation functions	Remarks
Linear	$\phi_i^{\text{elm}} = \frac{1}{4}\left(1+\eta_1^\dagger\right)\left(1+\eta_2^\dagger\right)$	Node $i = 1, \ldots, 4$
Quadratic	$\phi_i^{\text{elm}} = \frac{1}{4}\eta_1^\dagger\left(1+\eta_1^\dagger\right)\eta_2^\dagger\left(1+\eta_2^\dagger\right)$	Corner node i
	$\phi_i^{\text{elm}} = \frac{1}{2}\left(1-\eta_1^2\right)\eta_2^\dagger\left(1+\eta_2^\dagger\right)$	Side node i, $\eta_{1,i} = 0$
	$\phi_i^{\text{elm}} = \frac{1}{2}\eta_1^\dagger\left(1+\eta_1^\dagger\right)\left(1-\eta_2^2\right)$	Side node i, $\eta_{2,i} = 0$
	$\phi_i^{\text{elm}} = \left(1-\eta_1^2\right)\left(1-\eta_2^2\right)$	Interior node i
		where $\eta_1^\dagger \equiv \eta_{1,i}\eta_1$, $\eta_2^\dagger \equiv \eta_{2,i}\eta_2$

Table 3.2: Interpolation functions for rectangular elements [40]

by a coordinate transformation expressed by

$$\sigma = \sum_{i=1}^{n_{\text{node}}} \sigma_i^{\text{elm}} \phi_i^{\text{elm}}(\eta_1, \eta_2), \quad \chi = \sum_{i=1}^{n_{\text{node}}} \chi_i^{\text{elm}} \phi_i^{\text{elm}}(\eta_1, \eta_2), \quad \text{in } \Omega^{\text{elm}} \tag{3.67}$$

where $(\sigma_i^{\text{elm}}, \chi_i^{\text{elm}})$ is the position at the nodal point P_i in the subdomain Ω^{elm}. By the use of chain rule, the derivative of the interpolation function ϕ_i^{elm} with respect to the LCS can be expressed as

$$\frac{\partial \phi_i^{\text{elm}}}{\partial \eta_1} = \frac{\partial \phi_i^{\text{elm}}}{\partial \sigma}\frac{\partial \sigma}{\partial \eta_1} + \frac{\partial \phi_i^{\text{elm}}}{\partial \chi}\frac{\partial \chi}{\partial \eta_1} \tag{3.68}$$

$$\frac{\partial \phi_i^{\text{elm}}}{\partial \eta_2} = \frac{\partial \phi_i^{\text{elm}}}{\partial \sigma}\frac{\partial \sigma}{\partial \eta_2} + \frac{\partial \phi_i^{\text{elm}}}{\partial \chi}\frac{\partial \chi}{\partial \eta_2}. \tag{3.69}$$

Eq.(3.68) and Eq.(3.70) can be also expressed in the matrix form as

$$\begin{bmatrix} \dfrac{\partial \phi_i}{\partial \eta_1} \\ \dfrac{\partial \phi_i}{\partial \eta_2} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial \sigma}{\partial \eta_1} & \dfrac{\partial \chi}{\partial \eta_1} \\ \dfrac{\partial \sigma}{\partial \eta_2} & \dfrac{\partial \chi}{\partial \eta_2} \end{bmatrix} \begin{bmatrix} \dfrac{\partial \sigma}{\partial \eta_1} \\ \dfrac{\partial \chi}{\partial \eta_2} \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \begin{bmatrix} \dfrac{\partial \phi_i}{\partial \sigma} \\ \dfrac{\partial \phi_i}{\partial \chi} \end{bmatrix}, \tag{3.70}$$

where J_{ij} is the Jacobi matrix from MSCS to LCS defined by

$$\begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \begin{bmatrix} \displaystyle\sum_{i=1}^{n_{\text{node}}} \sigma_i \frac{\partial \phi_i}{\partial \eta_1} & \displaystyle\sum_{i=1}^{n_{\text{node}}} \chi_i \frac{\partial \phi_i}{\partial \eta_1} \\ \displaystyle\sum_{i=1}^{n_{\text{node}}} \sigma_i \frac{\partial \phi_i}{\partial \eta_2} & \displaystyle\sum_{i=1}^{n_{\text{node}}} \chi_i \frac{\partial \phi_i}{\partial \eta_2} \end{bmatrix}. \tag{3.71}$$

The Jacobian J is defined as

$$J = J_{11}J_{22} - J_{12}J_{21} > 0. \tag{3.72}$$

Therefore the volume element is expressed as

$$dV = \sqrt{g} d\sigma d\chi = \sqrt{g} J d\eta_1 d\eta_2, \tag{3.73}$$

and the derivative of the interpolation function ϕ_i^{elm} with respect to the MSCS can be expressed as

$$\begin{bmatrix} \dfrac{\partial \phi_i}{\partial \sigma} \\ \dfrac{\partial \phi_i}{\partial \chi} \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}^{-1} \begin{bmatrix} \dfrac{\partial \phi_i}{\partial \eta_1} \\ \dfrac{\partial \phi_i}{\partial \eta_2} \end{bmatrix} = \begin{bmatrix} J_{11}^* & J_{12}^* \\ J_{21}^* & J_{22}^* \end{bmatrix} \begin{bmatrix} \dfrac{\partial \phi_i}{\partial \eta_1} \\ \dfrac{\partial \phi_i}{\partial \eta_2} \end{bmatrix}, \tag{3.74}$$

where J_{ij}^* is the inverse matrix of J_{ij}. By the use of Eq.(3.73) and Eq.(3.74), Eq.(3.66) can be reduced to the area integral over the square element in LCS as

$$\mathsf{A}_{IJ}^{\mathrm{elm}} = \sum_{k=1}^{2}\sum_{l=1}^{2}\sum_{m=1}^{2}\sum_{n=1}^{2} \int_{\Omega_M} J_{ik}^* \frac{\partial \phi_I^{\mathrm{elm}}}{\partial \eta_k} J_{jl}^* \frac{\partial \phi_J^{\mathrm{elm}}}{\partial \eta_l} \phi_K^{\mathrm{elm}} J d\eta_1 d\eta_2 \left\{ \sqrt{g} g^{\xi_i \xi_j} \right\}_K^{\mathrm{elm}} \tag{3.75}$$

$$= \int_{-1}^{1}\int_{-1}^{1} F(\eta_1, \eta_2) d\eta_1 d\eta_2 \tag{3.76}$$

The area integral in Eq.(3.76) is evaluated by the Gaussian quadrature as

$$\mathsf{A}_{ij}^{\mathrm{elm}} = \sum_{i=1}^{N_{\mathrm{abs}}}\sum_{j=1}^{N_{\mathrm{abs}}} F(\eta_{1,i}, \eta_{2,j}) W_i W_j, \tag{3.77}$$

where $\eta_{1,i}$ and $\eta_{2,j}$ are abscissas and W_i and W_j are corresponding weighting factors respectively.

3.3.4 Finite element equation of advection-diffusion equation

The finite element equation of Eq.(3.23)

$$
\begin{aligned}
&\sum_b \iint \tilde{w}_a \frac{1}{\sqrt{g}} \frac{\partial}{\partial t} \left(\sqrt{g} M_{ab} f_b\right) \sqrt{g} d\sigma d\chi \\
&\quad + \sum_b \iint \tilde{w}_a \nabla \cdot \left(\boldsymbol{V}^1_{ab} f_b\right) \sqrt{g} d\sigma d\chi - \sum_b \sum_x \iint \tilde{w}_a \nabla \cdot \left[\left(\overleftrightarrow{V}^2_{abx} \cdot \nabla g_x\right) f_b\right] \sqrt{g} d\sigma d\chi \\
&\quad - \sum_b \iint \tilde{w}_a \nabla \cdot \left(\overleftrightarrow{D}_{ab} \cdot \nabla f_b\right) \sqrt{g} d\sigma d\chi \\
&\quad + \sum_b \iint \tilde{w}_a \boldsymbol{A}^1_{ab} \cdot \nabla f_b \sqrt{g} d\sigma d\chi + \sum_b \sum_x \iint \tilde{w}_a \left(\nabla g_x \cdot \overleftrightarrow{A}^2_{abx}\right) \cdot \nabla f_b \sqrt{g} d\sigma d\chi \\
&\quad + \sum_b \iint \tilde{w}_a C^1_{ab} f_b \sqrt{g} d\sigma d\chi + \sum_b \sum_x \iint \tilde{w}_a \left(\nabla g_x \cdot \boldsymbol{C}^2_{abx}\right) f_b \sqrt{g} d\sigma d\chi \\
&\quad + \sum_b \sum_{x,y} \iint \tilde{w}_a \left(\nabla g_x \cdot \overleftrightarrow{C}^3_{abxy} \cdot \nabla g_y\right) f_b \sqrt{g} d\sigma d\chi = \iint \tilde{w}_a S_a \sqrt{g} d\sigma d\chi, \qquad (3.78)
\end{aligned}
$$

can be therefore written in the matrix form as

$$
\begin{aligned}
&\sum_b \frac{\partial}{\partial t} \left[\left(\mathsf{M}_{ab} + \mathsf{M}^s_{ab}\right) \mathsf{f}_b\right] + \sum_b \left(\mathsf{V}^1_{ab} + \mathsf{V}^{1s}_{ab} + \mathsf{V}^2_{ab} - \mathsf{V}^{2s}_{ab}\right) \mathsf{f}_b + \sum_b \left(\mathsf{D}_{ab} - \mathsf{D}^s_{ab}\right) \mathsf{f}_b \\
&\quad + \sum_b \left(\mathsf{A}^1_{ab} + \mathsf{A}^{1s}_{ab} + \mathsf{A}^2_{ab} + \mathsf{A}^{2s}_{ab}\right) \mathsf{f}_b + \sum_b \left(\mathsf{C}^1_{ab} + \mathsf{C}^{1s}_{ab} + \mathsf{C}^2_{ab} + \mathsf{C}^{2s}_{ab} + \mathsf{C}^3_{ab} + \mathsf{C}^{3s}_{ab}\right) \mathsf{f}_b \\
&= \mathsf{S}_a + \mathsf{S}^s_a + \sum_b \mathsf{F}_{ab}. \qquad (3.79)
\end{aligned}
$$

where

$$\iint \tilde{w}_a \nabla \cdot \left(\boldsymbol{V}^1_{ab} f_b\right) \sqrt{g} d\sigma d\chi = \mathsf{w}^{\mathsf{T}}_a \left(\mathsf{V}^1_{ab} + \mathsf{V}^{1s}_{ab}\right) \mathsf{f}_b, \qquad (3.80)$$

$$\sum_x \iint \tilde{w}_a \nabla \cdot \left[\left(\overleftrightarrow{V}^2_{abx} \cdot \nabla g_x\right) f_b\right] \sqrt{g} d\sigma d\chi = \mathsf{w}^{\mathsf{T}}_a \left(-\mathsf{V}^2_{ab} + \mathsf{V}^{2s}_{ab}\right) \mathsf{f}_b, \qquad (3.81)$$

$$\iint \tilde{w}_a \nabla \cdot \left(\overleftrightarrow{D}_{ab} \cdot \nabla f_b\right) \sqrt{g} d\sigma d\chi = \mathsf{w}^{\mathsf{T}}_a \left(-\mathsf{D}_{ab} + \mathsf{D}^s_{ab}\right) \mathsf{f}_b + \mathsf{w}^{\mathsf{T}}_a \mathsf{F}_{ab}, \qquad (3.82)$$

$$\iint \tilde{w}_a \boldsymbol{A}^1_{ab} \cdot \nabla f_b \sqrt{g} d\sigma d\chi = \mathsf{w}^{\mathsf{T}}_a \left(\mathsf{A}^1_{ab} + \mathsf{A}^{1s}_{ab}\right) \mathsf{f}_b, \qquad (3.83)$$

$$\sum_x \iint \tilde{w}_a \left(\nabla g_x \cdot \overleftrightarrow{A}^2_{abx}\right) \cdot \nabla f_b \sqrt{g} d\sigma d\chi = \mathsf{w}^{\mathsf{T}}_a \left(\mathsf{A}^2_{ab} + \mathsf{A}^{2s}_{ab}\right) \mathsf{f}_b, \qquad (3.84)$$

$$\iint \tilde{w}_a C^1_{ab} f_b \sqrt{g} d\sigma d\chi = \mathsf{w}^{\mathsf{T}}_a \left(\mathsf{C}^1_{ab} + \mathsf{C}^{1s}_{ab}\right) \mathsf{f}_b, \qquad (3.85)$$

$$\sum_x \iint \tilde{w}_a \left(\nabla g_x \cdot \boldsymbol{C}^2_{abx}\right) f_b \sqrt{g} d\sigma d\chi = \mathsf{w}^{\mathsf{T}}_a \left(\mathsf{C}^2_{ab} + \mathsf{C}^{2s}_{ab}\right) \mathsf{f}_b, \qquad (3.86)$$

$$\sum_{x,y} \iint \tilde{w}_a \left(\nabla g_x \cdot \overleftrightarrow{C}^3_{abxy} \cdot \nabla g_y \right) f_b \sqrt{g} d\sigma d\chi = \mathsf{w}_a^\mathsf{T} \left(\mathsf{C}^3_{ab} + \mathsf{C}^{3s}_{ab} \right) \mathsf{f}_b, \tag{3.87}$$

$$\iint \tilde{w}_a S_a V' d\sigma = \mathsf{w}_a^\mathsf{T} \left(\mathsf{S}_a + \mathsf{S}^s_a \right), \tag{3.88}$$

where w_a is a vector of nodal values of the weighting function w_a. The derivation of integral matrices are summarized in Appendix B.

3.3.5 FEM with hierarchical computational grid

The rectangular grid in MSCS is employed in order to decompose the fluxes parallel and perpendicular to the field line for the computational stability and accuracy in TASK/T2. If the computational grid whose rectangular elements are placed at regular interval in the poloidal direction is employed, it causes not only the over constraint of the behavior of the solution at the vicinity of the magnetic axis but also the lack of the spatial resolution in the poloidal direction in the outer region. Therefore the hierarchical rectangular grid shown in Figure 3.3 is employed in TASK/T2.

FEM requires C^0-level continuity on the interface between different roughness domains corresponding to the different colored domains in Figure 3.4 the hierarchical rectangular grid and then we employ the constrained node method to keep C^0-level continuity on the interface between different roughness. In the constrained node method the same master element is employ in both the rough and the fine domain and the value of the dependent variable at the constrained node lying on a edge of the rough element $f_{\mathrm{constrained}}$ is constrained by the interpolation in the more rough element as

$$f_{\mathrm{constrained}} - \sum_{i}^{n_{\mathrm{node}}} f_i^{\mathrm{elm}} \phi_i^{\mathrm{elm}} (\eta_{1,\mathrm{constrained}}, \eta_{2,\mathrm{constrained}}) = 0, \tag{3.89}$$

where $(\eta_{1,\mathrm{constrained}}, \eta_{2,\mathrm{constrained}})$ is the position of the constrained node in the LCS of the rough element.

The direct elimination method is employed in order to implement the constraint (3.89) into the FEM analysis in TASK/T2. In the following discussion we use the fact that the minimization of the functional $\mathcal{J}$ is equivalent to the FEM analysis in the matrix form as

$$\mathrm{Min} \mathcal{J}(\mathsf{f}) = \frac{1}{2} \mathsf{f}^\mathsf{T} \mathsf{A} \mathsf{f} - \mathsf{f}^\mathsf{T} \mathsf{b} \Leftrightarrow \mathsf{A} \mathsf{f} - \mathsf{b} \tag{3.90}$$

Superposing all constraint (3.89), we obtain the constraint of the system in the matrix form as

$$\mathsf{C} \mathsf{f} = 0, \tag{3.91}$$

where we introduce the number of constrained nodes M and the number of nodes including free nodes and constrained nodes N and then C is a $M \times N$ matrix and f is a $N \times 1$ matrix. Decomposing f into f_1 consisting M constrained value and f_2 consisting $N - M$ free value, we can transform Eq.(3.91) as

$$\mathsf{C} \mathsf{f} = \left[\begin{array}{cc} \mathsf{C}_1 & \mathsf{C}_2 \end{array} \right] \left[\begin{array}{c} \mathsf{f}_1 \\ \mathsf{f}_2 \end{array} \right] = \mathbf{0} \Leftrightarrow \mathsf{C}_1 \mathsf{f}_1 + \mathsf{C}_2 \mathsf{f}_2 = \mathbf{0} \Leftrightarrow \mathsf{f}_1 = -\mathsf{C}_1^{-1} \mathsf{C}_2 \mathsf{f}_2, \tag{3.92}$$

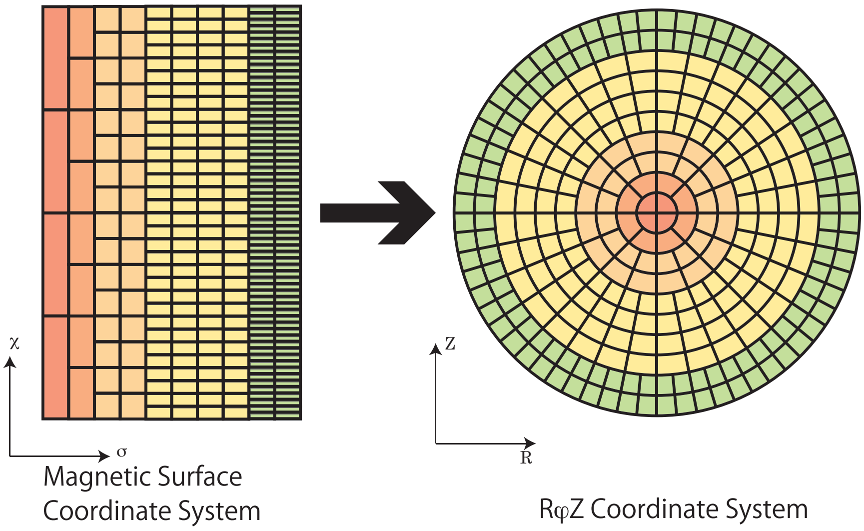

Figure 3.3: Concept of the hierarchical rectangular grid in TASK/T2

where C_1 and C_2 are the submatrices of C. By the use of Eq.(3.92), the dependent variable matrix f can be described only by the submatrix f_2 as

$$\mathsf{f} = \begin{bmatrix} \mathsf{f}_1 \\ \mathsf{f}_2 \end{bmatrix} = \begin{bmatrix} -\mathsf{C}_1^{-1}\mathsf{C}_2 \\ \mathsf{I} \end{bmatrix} \mathsf{f}_2 = \mathsf{B}\mathsf{f}_2, \tag{3.93}$$

where B is called the condensation matrix. Substituting Eq.(3.93) into Eq.(3.90), the functional with respect to f can be converted to that with respect to f_2 as

$$\begin{aligned} \mathrm{Min}\mathcal{J}(\mathsf{f}) &= \frac{1}{2}\mathsf{f}_2^\mathsf{T}\mathsf{B}^\mathsf{T}\mathsf{A}\mathsf{B}\mathsf{f}_2 - \mathsf{f}_2^\mathsf{T}\mathsf{B}^\mathsf{T}\mathsf{b} \\ &= \frac{1}{2}\mathsf{f}_2^\mathsf{T}\hat{\mathsf{A}}\mathsf{f}_2 - \mathsf{f}_2^\mathsf{T}\hat{\mathsf{b}}, \end{aligned} \tag{3.94}$$

where the $(N-M)\times(N-M)$ matrix $\hat{\mathsf{A}}$ and $(N-M)\times 1$ matrix $\hat{\mathsf{b}}$ are defined respectively as

$$\hat{\mathsf{A}} = \mathsf{B}^\mathsf{T}\mathsf{A}\mathsf{B} \tag{3.95}$$

$$\hat{\mathsf{b}} = \mathsf{B}^\mathsf{T}\mathsf{b}. \tag{3.96}$$

Therefore the matrix equation corresponding to Eq.(3.96) is finally obtained as

$$\hat{\mathsf{A}}\mathsf{f}_2 = \hat{\mathsf{b}}, \tag{3.97}$$

where the solution of Eq.(3.97) f_2 satisfies the constraint (3.91) rigorously. In TASK/T2 the direct elimination method is implemented as the element-by-element process.

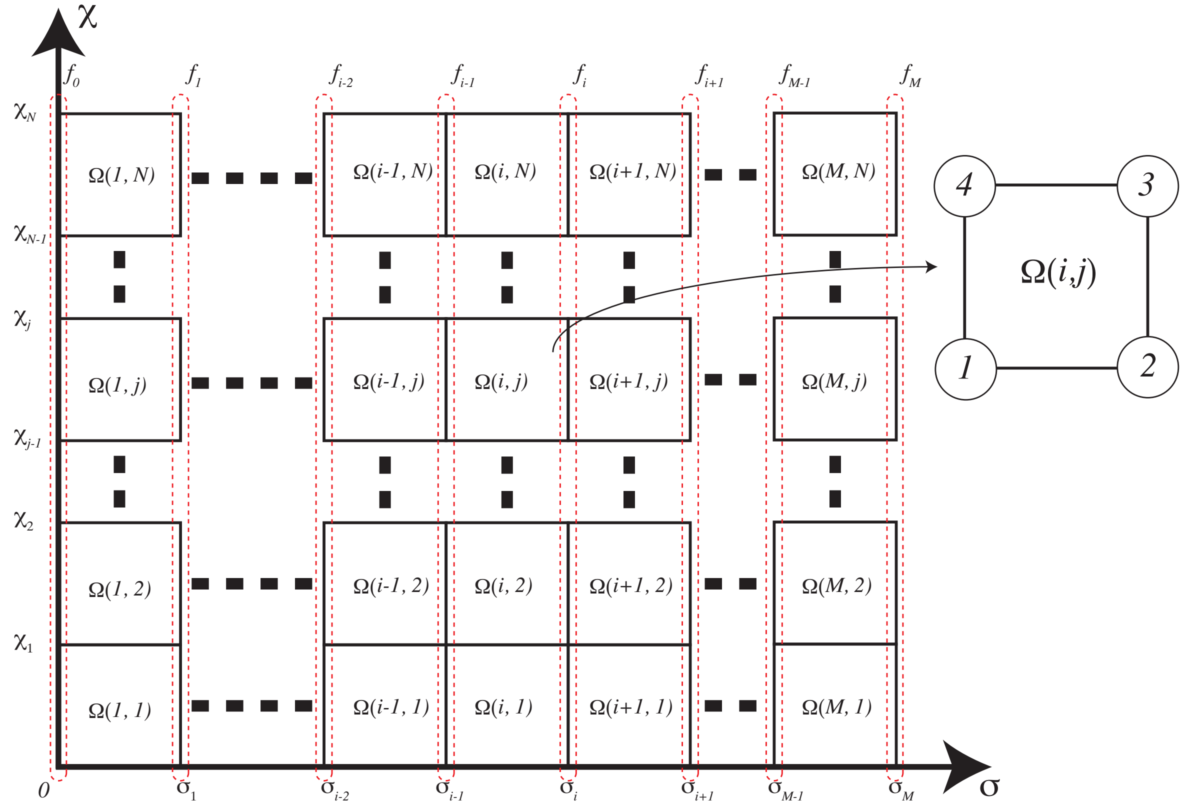

Figure 3.4: Concept of FSA in TASK/T2, red dashed line at σ_i shows the integral route of FSA at σ_i respectively.

3.4 Flux surface averaging

Three of dependent variables in TASK/T2 are constant on the flux surfaces and their time evolutions are described approximately by the flux surface averaged equations. In this chapter we will briefly introduce the flux surface averaging scheme in TASK/T2 by taking the case of the equation for covariant toroidal electric field with 4-points rectangular elements shown in Figure 3.4 for instance.

Multiplying Eq.(3.24) by a weighting function w_{03} and integrating it over the radial direction, we obtain

$$\int w_{03}\frac{\partial}{\partial t}\left(\left[\oint M_{03.03}\sqrt{g}d\chi\right]E_\zeta\right)d\sigma + \int w_{03}\left[\oint \nabla\cdot\left(\boldsymbol{V}^1_{03.01}\psi'\right)\sqrt{g}d\chi\right]d\sigma$$
$$+\int w_{03}\left[\oint \nabla\cdot\left(\boldsymbol{V}^1_{03.03}E_\zeta\right)\sqrt{g}d\chi\right]d\sigma + \sum_a\int w_{03}\left[\oint C^1_{03.09a}n_a u_{a\zeta}\sqrt{g}d\chi\right]d\sigma$$
$$+\int w_{03}\left[\oint\left(\nabla R\cdot\boldsymbol{C}^2_{03.01.02}\psi'\right)d\chi\right]d\sigma = 0. \tag{3.98}$$

Noting that w_{03} is independent of the poloidal angle χ, since it is the weighting function of E_ζ, we obtain

$$\iint w_{03}\frac{\partial}{\partial t}\left(M_{03.03}E_\zeta\sqrt{g}\right)d\sigma d\chi + \iint w_{03}\nabla\cdot\left(\boldsymbol{V}^1_{03.01}\psi'\right)\sqrt{g}d\sigma d\chi$$
$$+\iint w_{03}\nabla\cdot\left(\boldsymbol{V}^1_{03.03}E_\zeta\right)\sqrt{g}d\sigma d\chi + \sum_a \iint w_{03}C^1_{03.09a}n_a u_{a\zeta}\sqrt{g}d\sigma d\chi$$
$$+\iint w_{03}\left(\nabla R\cdot\boldsymbol{C}^2_{03.01.02}\psi'\right)d\sigma d\chi = 0. \tag{3.99}$$

If the interpolation functions of FEM are employ for the integration in the poloidal direction in Eq.(3.99), the finite element equation corresponding to Eq.(3.99) in the line segment between σ_i and σ_{i+1} becomes

$$\sum_{j=1}^{N}\int_{\Omega_{(i,j)}} w_{03}\frac{\partial}{\partial t}\left(M_{03.03}E_\zeta\sqrt{g}\right)d\Omega + \sum_{j=1}^{N}\int_{\Omega_{(i,j)}} w_{03}\nabla\cdot\left(\boldsymbol{V}^1_{03.01}\psi'\right)\sqrt{g}d\Omega$$
$$+\sum_{j=1}^{N}\int_{\Omega_{(i,j)}} w_{03}\nabla\cdot\left(\boldsymbol{V}^1_{03.03}E_\zeta\right)\sqrt{g}d\Omega + \sum_a\sum_{j=1}^{N}\int_{\Omega_{(i,j)}} w_{03}C^1_{03.09a}n_a u_{a\zeta}\sqrt{g}d\Omega$$
$$+\sum_{j=1}^{N}\int_{\Omega_{(i,j)}} w_{03}\left(\nabla R\cdot\boldsymbol{C}^2_{03.01.02}\psi'\right)d\Omega = 0$$
$$\sum_{j=1}^{N}\mathsf{w}^{\Omega(i,j)}_{03}\frac{\partial}{\partial t}\left(\mathsf{M}^{\Omega(i,j)}_{03.03}\mathsf{f}^{\Omega(i,j)}_{03}\right) + \sum_{j=1}^{N}\mathsf{w}^{\Omega(i,j)}_{03}\mathsf{V}^{1,\Omega(i,j)}_{03.01}\mathsf{f}^{\Omega(i,j)}_{01} + \sum_{j=1}^{N}\mathsf{w}^{\Omega(i,j)}_{03}\mathsf{V}^{1,\Omega(i,j)}_{03.03}\mathsf{f}^{\Omega(i,j)}_{03}$$
$$+\sum_a\sum_{j=1}^{N}\mathsf{w}^{\Omega(i,j)}_{03}\mathsf{C}^{1,\Omega(i,j)}_{03.09a}\mathsf{f}^{\Omega(i,j)}_{09a} + \sum_a\sum_{j=1}^{N}\mathsf{w}^{\Omega(i,j)}_{03}\mathsf{C}^{2,\Omega(i,j)}_{03.01.02}\mathsf{f}^{\Omega(i,j)}_{01} = 0. \tag{3.100}$$

Since node 1 and node 4 lie on the σ-constant line ($\sigma = \sigma_i$) and node 2 and node 3 lie on the σ-constant line ($\sigma = \sigma_{i+1}$) in each element, information of E_ζ, w_{03} and ψ' are condensed and then submatrices in Eq.(3.100) are also condensed respectively. 4×4-matrix $\mathsf{M}^{\Omega(i,j)}_{03.03}$

$$\mathsf{M}^{\Omega(i,j)}_{03.03} = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{bmatrix}^{\Omega(i,j)}_{03.03} \tag{3.101}$$

is condensed to 2×2-matrix $\bar{\mathsf{M}}^{\Omega(i,j)}_{03.03}$,

$$\bar{\mathsf{M}}^{\Omega(i,j)}_{03.03} = \begin{bmatrix} \bar{M}_{11} & \bar{M}_{12} \\ \bar{M}_{21} & \bar{M}_{22} \end{bmatrix}^{\Omega(i,j)}_{03.03}$$
$$= \begin{bmatrix} M_{11}+M_{14}+M_{41}+M_{44} & M_{12}+M_{13}+M_{42}+M_{43} \\ M_{21}+M_{24}+M_{31}+M_{34} & M_{22}+M_{23}+M_{32}+M_{33} \end{bmatrix}^{\Omega(i,j)}_{03.03}, \tag{3.102}$$

4×4-matrix $\mathsf{V}_{03.01}^{1,\Omega(i,j)}$,

$$\mathsf{V}_{03.01}^{1,\Omega(i,j)} = \begin{bmatrix} V_{11} & V_{12} & V_{13} & V_{14} \\ V_{21} & V_{22} & V_{23} & V_{24} \\ V_{31} & V_{32} & V_{33} & V_{34} \\ V_{41} & V_{42} & V_{43} & V_{44} \end{bmatrix}_{03.01}^{1,\Omega(i,j)} \tag{3.103}$$

is condensed to 2×2-matrix $\bar{\mathsf{V}}_{03.01}^{1,\Omega(i,j)}$,

$$\begin{aligned} \bar{\mathsf{V}}_{03.01}^{1,\Omega(i,j)} &= \begin{bmatrix} \bar{V}_{11} & \bar{V}_{12} \\ \bar{V}_{21} & \bar{V}_{22} \end{bmatrix}_{03.01}^{1,\Omega(i,j)} \\ &= \begin{bmatrix} V_{11}+V_{14}+V_{41}+V_{44} & V_{12}+V_{13}+V_{42}+V_{43} \\ V_{21}+V_{24}+V_{31}+V_{34} & V_{22}+V_{23}+V_{32}+V_{33} \end{bmatrix}_{03.01}^{1,\Omega(i,j)}, \end{aligned} \tag{3.104}$$

4×4-matrix $\mathsf{V}_{03.03}^{1,\Omega(i,j)}$

$$\mathsf{V}_{03.03}^{1,\Omega(i,j)} = \begin{bmatrix} V_{11} & V_{12} & V_{13} & V_{14} \\ V_{21} & V_{22} & V_{23} & V_{24} \\ V_{31} & V_{32} & V_{33} & V_{34} \\ V_{41} & V_{42} & V_{43} & V_{44} \end{bmatrix}_{03.03}^{1,\Omega(i,j)} \tag{3.105}$$

is condensed to 2×2-matrix $\bar{\mathsf{V}}_{03.03}^{\Omega(i,j)}$,

$$\begin{aligned} \bar{\mathsf{V}}_{03.03}^{1,\Omega(i,j)} &= \begin{bmatrix} \bar{V}_{11} & \bar{V}_{12} \\ \bar{V}_{21} & \bar{V}_{22} \end{bmatrix}_{03.03}^{1,\Omega(i,j)} \\ &= \begin{bmatrix} V_{11}+V_{14}+V_{41}+V_{44} & V_{12}+V_{13}+V_{42}+V_{43} \\ V_{21}+V_{24}+V_{31}+V_{34} & V_{22}+V_{23}+V_{32}+V_{33} \end{bmatrix}_{03.03}^{1,\Omega(i,j)} \end{aligned} \tag{3.106}$$

4×4-matrix $\mathsf{C}_{03.09a}^{1,\Omega(i,j)}$

$$\mathsf{C}_{03.09a}^{1,\Omega(i,j)} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & C_{34} \\ C_{41} & C_{42} & C_{43} & C_{44} \end{bmatrix}_{03.09a}^{1,\Omega(i,j)} \tag{3.107}$$

is condensed to 2×4-matrix $\bar{\mathsf{C}}_{03.09a}^{1,\Omega(i,j)}$,

$$\begin{aligned} \bar{\mathsf{C}}_{03.09a}^{1,\Omega(i,j)} &= \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} & \bar{C}_{13} & \bar{C}_{14} \\ \bar{C}_{21} & \bar{C}_{22} & \bar{C}_{23} & \bar{C}_{24} \end{bmatrix}_{03.09a}^{1,\Omega(i,j)} \\ &= \begin{bmatrix} C_{11}+C_{41} & C_{12}+C_{42} & C_{13}+C_{43} & C_{14}+C_{44} \\ C_{21}+C_{31} & C_{22}+C_{32} & C_{23}+C_{33} & C_{24}+C_{44} \end{bmatrix}_{03.09a}^{1,\Omega(i,j)}, \end{aligned} \tag{3.108}$$

and 4×4-matrix $\mathsf{C}^{2,\Omega(i,j)}_{03.01}$

$$\mathsf{C}^{2,\Omega(i,j)}_{03.01.02} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} \\ C_{21} & C_{22} & C_{23} & C_{24} \\ C_{31} & C_{32} & C_{33} & C_{34} \\ C_{41} & C_{42} & C_{43} & C_{44} \end{bmatrix}^{2,\Omega(i,j)}_{03.01.02} \tag{3.109}$$

is condensed 2×2-matrix $\bar{\mathsf{C}}^{2,\Omega(i,j)}_{03.01.02}$,

$$\begin{aligned} \bar{\mathsf{C}}^{2,\Omega(i,j)}_{03.01.02} &= \begin{bmatrix} \bar{C}_{11} & \bar{C}_{12} \\ \bar{C}_{21} & \bar{C}_{22} \end{bmatrix}^{2,\Omega(i,j)}_{03.01} \\ &= \begin{bmatrix} C_{11}+C_{14}+C_{41}+C_{44} & C_{12}+C_{13}+C_{42}+C_{43} \\ C_{21}+C_{24}+C_{31}+C_{34} & C_{22}+C_{23}+C_{32}+C_{33} \end{bmatrix}^{2,\Omega(i,j)}_{03.01.02}. \end{aligned} \tag{3.110}$$

The finite element equation (3.100) is therefore reduced to

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\hat{\mathsf{M}}^{i}_{03.03}\begin{bmatrix} \{E_\zeta\}_{\sigma_i} \\ \{E_\zeta\}_{\sigma_{i+1}} \end{bmatrix}\right) + \left(\hat{\mathsf{V}}^{1,i}_{03.01} + \hat{\mathsf{C}}^{2,i}_{03.01.02}\right)\begin{bmatrix} \{\psi'\}_{\sigma_i} \\ \{\psi'\}_{\sigma_{i+1}} \end{bmatrix} \\ &\quad + \hat{\mathsf{V}}^{1,i}_{03.03}\begin{bmatrix} \{E_\zeta\}_{\sigma_i} \\ \{E_\zeta\}_{\sigma_{i+1}} \end{bmatrix} + \sum_a \sum_{j=1}^{N} \bar{\mathsf{C}}^{1,\Omega(i,j)}_{03.09a}\begin{bmatrix} \{n_a u_{a\zeta}\}_1 \\ \{n_a u_{a\zeta}\}_2 \\ \{n_a u_{a\zeta}\}_3 \\ \{n_a u_{a\zeta}\}_4 \end{bmatrix}^{\Omega(i,j)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \end{aligned} \tag{3.111}$$

where $\hat{\mathsf{M}}^{i}_{03.03}$, $\hat{\mathsf{V}}^{1,i}_{03.01}$, $\hat{\mathsf{V}}^{1,i}_{03.03}$ and $\hat{\mathsf{C}}^{2,i}_{03.01.02}$ are defined by

$$\hat{\mathsf{M}}^{i}_{03.03} = \sum_{j=1}^{N} \bar{\mathsf{M}}^{\Omega(i,j)}_{03.03} \tag{3.112}$$

$$\hat{\mathsf{V}}^{1,i}_{03.01} = \sum_{j=1}^{N} \bar{\mathsf{V}}^{1,\Omega(i,j)}_{03.01} \tag{3.113}$$

$$\hat{\mathsf{V}}^{1,i}_{03.03} = \sum_{j=1}^{N} \bar{\mathsf{V}}^{1,\Omega(i,j)}_{03.03} \tag{3.114}$$

$$\hat{\mathsf{C}}^{2,i}_{03.01.02} = \sum_{j=1}^{N} \bar{\mathsf{C}}^{2,\Omega(i,j)}_{03.01.02}. \tag{3.115}$$

Since the fourth term in the left hand side of Eq.(3.111) is connected to the all local values of $n_a u_{a\zeta}$ at the nodal points lying on both $\sigma = \sigma_i$ and $\sigma = \sigma_{i+1}$, we introduce the intermediate variables $X^{i,j}_{03}$ defined by

$$\begin{bmatrix} X^{i,j}_{03,1} \\ X^{i,j}_{03,2} \end{bmatrix} = \begin{bmatrix} X^{i,j-1}_{03,1} \\ X^{i,j-1}_{03,2} \end{bmatrix} + \sum_a \bar{\mathsf{C}}^{1,\Omega(i,j)}_{03.09a}\begin{bmatrix} \{n_a u_{a\zeta}\}_1 \\ \{n_a u_{a\zeta}\}_2 \\ \{n_a u_{a\zeta}\}_3 \\ \{n_a u_{a\zeta}\}_4 \end{bmatrix}^{\Omega(i,j)}_{03.09a} \quad (j = 1,\cdots,N-1) \tag{3.116}$$

$$\begin{bmatrix} X^{i,0}_{03,1} \\ X^{i,0}_{03,2} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{3.117}$$

in order to improve the locality of the matrix equation. By the use of $X_{03}^{i,j}$, Eq.(3.111) is reduced to

$$\frac{\partial}{\partial t}\left(\hat{\mathsf{M}}_{03.03}^{i}\begin{bmatrix}\{E_\zeta\}_{\sigma_i}\\ \{E_\zeta\}_{\sigma_{i+1}}\end{bmatrix}\right)+\left(\hat{\mathsf{V}}_{03.01}^{1,i}+\hat{\mathsf{C}}_{03.01.02}^{2,i}\right)\begin{bmatrix}\{\psi'\}_{\sigma_i}\\ \{\psi'\}_{\sigma_{i+1}}\end{bmatrix}+\hat{\mathsf{V}}_{03.03}^{1,i}\begin{bmatrix}\{E_\zeta\}_{\sigma_i}\\ \{E_\zeta\}_{\sigma_{i+1}}\end{bmatrix}$$

$$+\begin{bmatrix}X_{03,1}^{i,N-1}\\ X_{03,2}^{i,N-1}\end{bmatrix}+\sum_a\begin{bmatrix}\bar{C}_{11} & \bar{C}_{12} & \bar{C}_{13} & \bar{C}_{14}\\ \bar{C}_{21} & \bar{C}_{22} & \bar{C}_{23} & \bar{C}_{24}\end{bmatrix}_{03.09a}^{2,\Omega(i,N)}\begin{bmatrix}\{n_a u_{a\zeta}\}_1\\ \{n_a u_{a\zeta}\}_2\\ \{n_a u_{a\zeta}\}_3\\ \{n_a u_{a\zeta}\}_4\end{bmatrix}_{03.09a}^{\Omega(i,N)}=0. \quad (3.118)$$

Introducing the intermediate variable $X_{03}^{i,j}$ defined by

$$X_{03}^{i,j}\equiv X_{03,1}^{i,j}+X_{03,2}^{i-1,j}\quad(2\le i\le M),\quad X_{03}^{1,j}\equiv X_{03,1}^{1,j},\quad X_{03}^{M+1,j}\equiv X_{03,2}^{M,j}, \quad (3.119)$$

we finally obtain N simultaneous equations including $M+1$ equations as

$$\frac{\partial}{\partial t}\left(\tilde{\mathsf{M}}_{03.03}\tilde{\mathsf{f}}_{03}\right)+\left(\tilde{V}_{03.01}^{1}+\tilde{C}_{03.01.02}^{2}\right)\tilde{\mathsf{f}}_{01}+\tilde{V}_{03.03}^{1}\tilde{\mathsf{f}}_{03}+\tilde{\mathsf{I}}\tilde{\mathsf{X}}_{03}^{N-1}+\sum_a\tilde{\mathsf{C}}_{03.09a}^{N}\tilde{\mathsf{f}}_{09a}^{N}=0 \quad (3.120)$$

$$-\tilde{\mathsf{I}}\tilde{\mathsf{X}}_{03}^{j}+\tilde{\mathsf{I}}\tilde{\mathsf{X}}_{03}^{j-1}+\sum_a\tilde{\mathsf{C}}_{03.09a}^{1,j}\tilde{\mathsf{f}}_{09a}^{j}=0\quad(1\le j\le N-1), \quad (3.121)$$

where

$$\tilde{\mathsf{f}}_{01}=\begin{bmatrix}\{\psi\}_{\sigma_1}\\ \vdots\\ \{\psi\}_{\sigma_{M+1}}\end{bmatrix},\quad \tilde{\mathsf{f}}_{03}=\begin{bmatrix}\{E_\zeta\}_{\sigma_1}\\ \vdots\\ \{E_\zeta\}_{\sigma_{M+1}}\end{bmatrix},\quad \tilde{\mathsf{f}}_{09a}^{j}=\begin{bmatrix}\{n_a u_{a\zeta}\}_{\sigma_1,\chi_j}\\ \vdots\\ \{n_a u_{a\zeta}\}_{\sigma_{M+1},\chi_j}\end{bmatrix}$$

$$\tilde{\mathsf{X}}_{03}^{j}=\begin{bmatrix}X_{03}^{1,j}\\ \vdots\\ X_{03}^{M+1,j}\end{bmatrix}$$

$$\tilde{\mathsf{M}}_{03.03}=\sum_{i=1}^{M}\hat{M}_{03.03}^{i},\quad \tilde{\mathsf{V}}_{03.01}^{1}=\sum_{i=1}^{M}\hat{V}_{03.01}^{1,i},\quad \tilde{\mathsf{V}}_{03.03}^{1}=\sum_{i=1}^{M}\hat{V}_{03.03}^{1,i},$$

$$\tilde{\mathsf{C}}_{03.01.02}^{2}=\sum_{i=1}^{M}\hat{C}_{03.01.02}^{i},\quad \tilde{\mathsf{C}}_{03.09a}^{1}=\sum_{i=1}^{M}\hat{C}_{03.09a}^{i}.$$

The other flux surface averaged equations can be discretized by the same procedure.

3.5 Time discretization

Although we have described the derivation of the finite element equations of (FSA) advection-diffusion equations by SUPG-FEM in the previous section, we have to also discretize these equations in the time direction. Since the interpolation function which we employ is independent of time and the time dependence of the dependent values are expressed as the time evolution of their nodal values, we employ the finite difference method as the discretization method in the time direction.

We introduce the following time-dependent nonlinear equation

$$\frac{\partial}{\partial t}\left[\mathsf{M}(\mathsf{f})\mathsf{f}\right] + \mathsf{X}(\mathsf{f})\mathsf{f} = \mathsf{S}(\mathsf{f}). \tag{3.122}$$

Eq.(3.122) is a simplified form of Eq.(3.79), where f is the dependent variable vector depending only on time, M and X are coefficient matrices and S is a source vector. Applying the finite difference approximation in the time direction on Eq.(3.122), we obtain

$$\frac{\mathsf{M}(\mathsf{f}^{n+1})\mathsf{f}^{n+1} - \mathsf{M}(\mathsf{f}^{n})\mathsf{f}^{n}}{\Delta t} + (1-\alpha)\,\mathsf{X}(\mathsf{f}^{n})\mathsf{f}^{n} + \alpha\mathsf{X}(\mathsf{f}^{n+1})\mathsf{f}^{n+1} = (1-\alpha)\,\mathsf{S}(\mathsf{f}^{n}) + \alpha\mathsf{S}(\mathsf{f}^{n+1}), \tag{3.123}$$

where the subscript n indicates the n-th time step, Δt is the times step width between n-th and $(n+1)$-th time step and α is an arbitrary parameter taking $0 \leq \alpha \leq 1$, where $\alpha = 0$ is the full explicit method, $\alpha = 1/2$ is the Crank-Nicolson method and $\alpha = 1$ is the full implicit method.

If the parameter α increases, the robustness of the discretized equation increases but its computational cost also increases. Since solving the transport in fusion plasma requires a robust algorithm due to its strong anisotropy and non-linearity, we employ the full implicit method. After a short calculation, Eq.(3.123) can be reduced to

$$\mathsf{A}(\mathsf{f}^{n+1})\mathsf{f}^{n+1} = \mathsf{b}(\mathsf{f}^{n+1}, \mathsf{f}^{n}), \tag{3.124}$$

where

$$\mathsf{A}(\mathsf{f}^{n+1}) = \mathsf{M}(\mathsf{f}^{n+1}) + \Delta t\mathsf{X}(\mathsf{f}^{n+1}) \tag{3.125}$$

$$\mathsf{b}(\mathsf{f}^{n+1}, \mathsf{f}^{n}) = \Delta t\mathsf{S}(\mathsf{f}^{n+1}) + \mathsf{M}(\mathsf{f}^{n})\mathsf{f}^{n}. \tag{3.126}$$

Solving Eq.(3.124) requires a successive approximation scheme, since Eq.(3.124) is a nonlinear equation. The Picard iterative method is therefore employed in TASK/T2. In the Picard iterative method, successive approximations are obtained by solving

$$\mathsf{f}^{n+1}_{l+1} = \mathsf{A}(\mathsf{f}^{n+1}_{l})^{-1}\mathsf{b}(\mathsf{f}^{n+1}_{l}, \mathsf{f}^{n}), \tag{3.127}$$

where the subscript l indicates l-th iteration and $\mathsf{f}^{n+1}_{0} = \mathsf{f}^{n}$. Eq.(3.127) is solved iteratively till the following convergence criterion is satisfied

$$\frac{\left|\mathsf{f}^{n+1}_{l+1} - \mathsf{f}^{n+1}_{l}\right|}{\left|\mathsf{f}^{n+1}_{l+1}\right|} < \varepsilon_{\mathrm{tolerance}}, \tag{3.128}$$

where $\varepsilon_{\mathrm{tolerance}}$ is the error threshold.

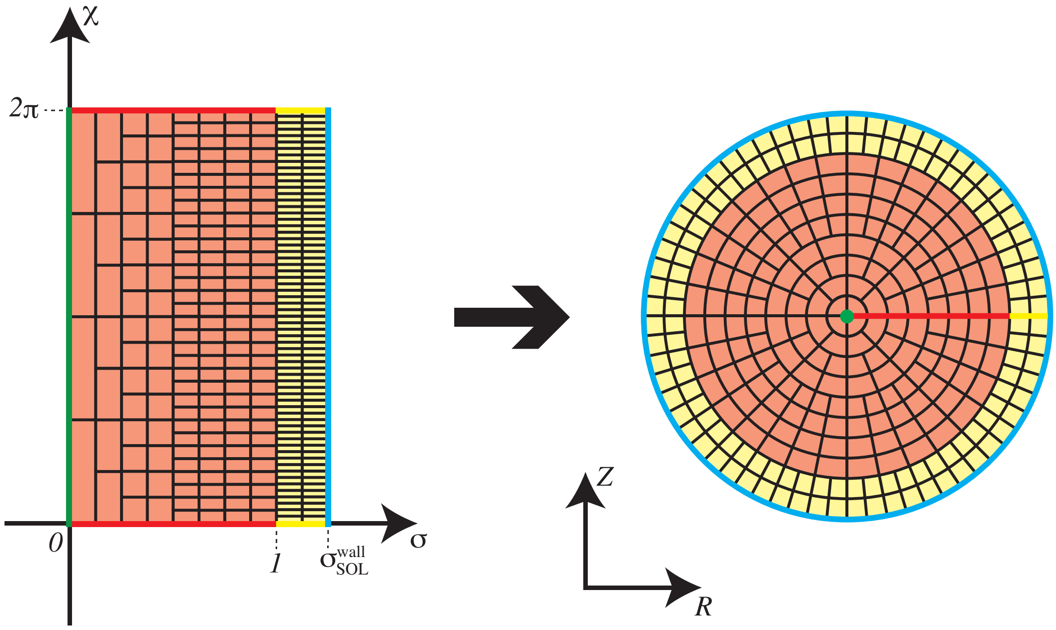

Figure 3.5: Concept of the computational grid for limiter configuration; the red colored area is core region and the yellow colored area is the peripheral region respectively. Computational boundaries are expressed by the colored heavy lines.

3.6 Computational grid, boundary conditions and initial conditions for limiter configuration

The concept of the computational grid for limiter configuration is shown in Figure 3.5. In Figure 3.5, $\sigma_{\text{wall}}^{\text{SOL}}$ is the position of the first wall normalized by the position of the LCFS. The boundary conditions for the limiter configuration are summarized as follows.

- Green heavy line ($\sigma = 0$, $0 \leq \chi \leq 2\pi$): the green heavy line is projected to the green point in the cylindrical coordinate so that the dependent variable have a same value

$$f(0, \chi) = f_0, \quad (0 \leq \chi \leq 2\pi) \tag{3.129}$$

- Red heavy lines ($0 \leq \sigma \leq 1$, $\chi = 0, 2\pi$): the red heavy lines are projected to the red line in the cylindrical coordinate so that the dependent variable have the periodic property as

$$f(\sigma, 0) = f(\sigma, 2\pi), \quad (0 \leq \sigma \leq 2\pi). \tag{3.130}$$

- Yellow heavy lines ($1 \leq \sigma \leq \sigma_{\text{wall}}^{\text{SOL}}$, $\chi = 0, 2\pi$): the yellow heavy lines correspond to limiter surfaces. As a preliminary step for the limiter tokamak analysis with boundary conditions on limiter plate, we assume a virtual limiter system where a large particle sink is assumed on the limiter plate in order to produce the sonic flow in the front of the limiter plates. In the virtual limiter tokamak, periodic boundary conditions are imposed as

$$f(\sigma, 0) = f(\sigma, 2\pi), \quad (0 \leq \sigma \leq 2\pi). \tag{3.131}$$

R_0	:	major radius
a	:	minor radius
b	:	radial position of first wall
B_c	:	magnetic field intensity at the magnetic axis
q_c	:	safety factor at the magnetic axis
n_{ac}	:	particle density at the magnetic axis
T_{ac}	:	temperature at the magnetic axis
q_s	:	safety factor at LCFS
n_{as}	:	particle density at LCFS
T_{as}	:	temperature at LCFS
n_{aw}	:	particle density at first wall
T_{aw}	:	temperature at first wall
n_n	:	parameter of shape of density profile
m_n	:	parameter of shape of density profile
n_T	:	parameter of shape of temperature profile
m_T	:	parameter of shape of temperature profile

Table 3.3: list of the given parameters for limiter configuration in TASK/T2

- Light blue heavy lines ($\sigma = \sigma_{\text{wall}}^{\text{SOL}}$, $0 \leq \chi \leq 2\pi$): the light blue heavy line corresponds to the surface of the first wall. Since the set of equations consists of $5+6N$ first-order differential equations with respect to radial direction, we need $5+6N$ boundary conditions to solve them, where N is the number of particle species. They are determined as

$$\psi = \psi^{\text{initial}}, \quad I = I^{\text{initial}}, \quad E_\sigma = 0, \quad \bar{E}_\chi = 0, \quad E_\zeta = 0,$$
$$n_a \bar{u}_a^\rho = 0, \quad n_a u_{a\parallel} = 0, \quad n_a u_{a\zeta} = 0, \quad \bar{Q}_a^\rho = 0, \quad Q_{a\parallel} = 0, \quad Q_{a\zeta} = 0$$

Since initial conditions may affect the result of nonlinear calculation, the number of given parameter should be reduced as few as possible and the other conditions should be constructed consistency from the given conditions. The given parameters in TASK/T2 are summarized in Table 3.3.

At first the initial profiles of density and pressure are given as

$$n_a = \begin{cases} (n_{ac} - n_{as})\left(1 - \rho^{n_n}\right)^{m_n} + n_{as} & (0 \leq \rho \leq 1) \\ n_{aw} + \sum_{l=1}^{4} a_{nl}\left(\rho - \rho_w\right)^l & (1 \leq \rho \leq \rho_w) \end{cases} \tag{3.132}$$

$$T_a = \begin{cases} (T_{ac} - T_{as})\left(1 - \rho^{n_T}\right)^{m_T} + T_{as} & (0 \leq \rho \leq 1) \\ T_{aw} + \sum_{l=1}^{4} a_{Tl}\left(\rho - \rho_w\right)^l & (1 \leq \rho \leq \rho_w) \end{cases} \tag{3.133}$$

$$p_a = n_a T_a, \tag{3.134}$$

where $\rho \equiv r/a$, $\rho_w \equiv b/a$ and $\sigma \equiv \rho^2$. The coefficients a_{nl} and a_{Tl} are automatically determined by following conditions,

$$
\begin{aligned}
n_a(1+0) = n_a(1-0), \quad & \frac{\partial n_a}{\partial \rho}(1+0) = \frac{\partial n_a}{\partial \rho}(1-0), \\
\frac{\partial^2 n_a}{\partial \rho^2}(1+0) = \frac{\partial^2 n_a}{\partial \rho^2}(1-0), \quad & \frac{\partial n_a}{\partial \rho}(\rho_{\mathrm{w}}) = 0 \\
T_a(1+0) = T_a(1-0), \quad & \frac{\partial T_a}{\partial \rho}(1+0) = \frac{\partial T_a}{\partial \rho}(1-0), \\
\frac{\partial^2 T_a}{\partial \rho^2}(1+0) = \frac{\partial^2 T_a}{\partial \rho^2}(1-0), \quad & \frac{\partial T_a}{\partial \rho}(\rho_{\mathrm{w}}) = 0
\end{aligned}
$$

At second, the initial profiles of ψ' and I are determined by the profile of the safety factor

$$
q = \begin{cases} (q_c - q_s)\left(1-\rho^2\right) + q_s & (0 \le \rho \le 1) \\ (q_s - q_c)\rho^2 + q_c & (1 \le \rho \le \rho_w). \end{cases} \tag{3.135}
$$

$$\tag{3.136}$$

According to the definition of the safety factor, its profile can be expressed with I and ψ' as

$$
\begin{aligned}
q \equiv \frac{d\phi}{d\psi} = \frac{\langle B^\zeta \rangle}{\langle B^\chi \rangle} = \frac{\langle R^{-2} \rangle I}{\left\langle \sqrt{g}^{-1} \right\rangle \psi'} &= \frac{I}{\psi'} \frac{1}{2\pi} \oint \frac{\sqrt{g}}{R^2} d\chi \\
&= \frac{I}{\psi'} \frac{a^2}{2} \frac{1}{2\pi R_0} \oint \frac{1}{(1+\epsilon_0 \rho \cos\chi)} d\chi \\
&= \frac{a^2 I}{2\psi' R_0} \frac{1}{\sqrt{1-\epsilon_0^2 \sigma}},
\end{aligned} \tag{3.137}
$$

where $\epsilon_0 = a/R_0$ is the inverse aspect ratio of toroidal device. In the integration in Eq.(3.137), the following formula has been employed

$$
\int_0^{2\pi} \frac{dx}{1 + a\cos x} = \frac{2\pi}{\sqrt{1-a^2}} \quad \text{for} \quad (|a| < 1). \tag{3.138}
$$

Since I in the limit of large aspect ratio is expressed as

$$
I = B_0 R_0, \tag{3.139}
$$

the initial profile of ψ' is therefore obtained as

$$
\psi' = \frac{d\psi}{d\sigma} = \frac{a^2 B_0}{2q\sqrt{1-\epsilon_0^2 \sigma}}. \tag{3.140}
$$

At third, the initial profile of current density is discussed which determines those of the electric field and the fluxes. In the case of axisymmetric configurations, the equilibrium current density profile is obtained from Ampère law as

$$
\mu_0 \boldsymbol{j} = \nabla \times \boldsymbol{B} = \nabla I \times \nabla\zeta + \nabla \cdot \left(\frac{1}{R^2} \nabla\psi \right) R^2 \nabla\zeta. \tag{3.141}
$$

From Eq.(3.139) and Eq.(3.141), there is only the toroidal current density in axisymmetric configuration,

$$j^\sigma = 0 \tag{3.142}$$

$$j_\chi = 0. \tag{3.143}$$

The covariant toroidal current density is obtained from total force balance in the radial direction as

$$\begin{aligned}
\nabla p \cdot \nabla \rho &= \left[I \left(\nabla I \times \nabla \zeta \right) \times \nabla \zeta - \left(\boldsymbol{j} \cdot \nabla \zeta \right) \nabla \psi \right] \cdot \nabla \sigma \\
\frac{dp}{d\sigma} &= \left[-\frac{1}{\mu_0 R^2} I \frac{\partial I}{\partial \sigma} - \frac{1}{R^2} j_\zeta \frac{d\psi}{d\sigma} \right] \\
j_\zeta &= -\left(R^2 \frac{dp}{d\psi} + \frac{1}{\mu_0} I \frac{dI}{d\psi} \right) = -R^2 \frac{dp}{d\psi} = -R^2 \frac{1}{\psi'} \frac{dp}{d\sigma},
\end{aligned} \tag{3.144}$$

where $p = \sum p_a$ is the total pressure. Eq.(3.144) is employed to determine the initial profiles of covariant toroidal fluxes, but Eq.(3.144) cannot be used directly to determine that of the covariant toroidal electric field profile by Ohm's law since E_ζ has to be constant on the flux surface. We then take the surface average of Eq.(3.144) and obtain

$$\begin{aligned}
\langle j \rangle_\zeta = \left\langle -R^2 \frac{1}{\psi'} \frac{dp}{d\rho} \right\rangle &= -\frac{\displaystyle\oint R^2 \sqrt{g} d\chi}{\displaystyle\oint \sqrt{g} d\chi} \frac{1}{\psi'} \frac{dp}{d\sigma} \\
&= -\left(1 + \frac{3}{2} \epsilon_0^2 \sigma \right) \frac{R_0^2}{\psi'} \frac{dp}{d\sigma},
\end{aligned} \tag{3.145}$$

where the following integration formulae have been used,

$$\int_0^{2\pi} \left(1 + a \cos x \right) dx = 2\pi \tag{3.146}$$

$$\int_0^{2\pi} \left(1 + a \cos x \right)^3 dx = 2\pi \left(1 + \frac{3}{2} a^2 \right). \tag{3.147}$$

By the use of Eq.(3.143) and Eq.(3.144), the initial profile of the parallel current density profile is obtained as

$$j_\parallel = j^\chi \frac{B^\chi}{B} + \frac{B^\zeta}{B} j_\zeta = -\frac{I}{B} \frac{1}{\psi'} \frac{dp}{d\sigma} \tag{3.148}$$

At fourth, the initial profiles of E_ζ, $\hat{E}_\chi$ and E_ρ are determined by the current density profiles and the radial force balance. The initial profiles of the covariant toroidal electric field E_ζ and the covariant poloidal electric field $\hat{E}_\chi$ are determined by Ohm's law respectively as

$$E_\zeta = \eta \left\langle j_\zeta \right\rangle = -\eta \left(1 + \frac{3}{2} \epsilon_0^2 \sigma \right) \frac{R_0^2}{\psi'} \frac{dp}{d\sigma}, \tag{3.149}$$

$$E_\chi = \sigma \hat{E}_\chi = \eta j_\chi = 0 \tag{3.150}$$

where η is the neoclassical resistivity defined as

$$\eta = 1.65 \times 10^{-9} \ln \Lambda / T_{\rm e}^{3/2} \, (1 - \sqrt{\epsilon_0 \rho})^{-2} \quad T_{\rm e} \text{ in keV.} \tag{3.151}$$

On the other hand, the initial profile of the covariant radial electric field is determined by the radial force balance for electron as

$$\begin{aligned} g^{\sigma\sigma} \frac{dp_{\rm e}}{d\sigma} &= -g^{\sigma\sigma} e n_{\rm e} E_\sigma + \frac{IB}{\psi'} j_\parallel - \frac{B^2}{\psi'} j_\zeta \\ g^{\sigma\sigma} \frac{dp_{\rm e}}{d\sigma} &= -g^{\sigma\sigma} e n_{\rm e} E_\sigma + g^{\sigma\sigma} \frac{dp}{d\sigma} \\ E_\sigma &= \frac{1}{e n_{\rm e}} \sum_{a \neq {\rm e}} \frac{dp_a}{d\sigma}, \end{aligned} \tag{3.152}$$

where we have assumed that ions are immobile and the current is driven only by the electron flow.

Finally, the initial profiles of fluxes are determined by the current profiles. Since we have assumed that ions are immobile and the current is driven only by the electron flow as previously mentioned, the initial profiles of the particle fluxes are obtained as

$$\rho n_{\rm e} \bar{u}_{\rm e}^\rho = 0 \tag{3.153}$$

$$n_{\rm e} u_{{\rm e}\parallel} = -\frac{j_\parallel}{e} \tag{3.154}$$

$$n_{\rm e} u_{{\rm e}\zeta} = -\frac{\langle j_\zeta \rangle}{e} \tag{3.155}$$

$$\rho n_a \bar{u}_a^\rho = 0 \quad \text{for} \quad a \neq {\rm e} \tag{3.156}$$

$$n_a u_{a\parallel} = 0 \quad \text{for} \quad a \neq {\rm e} \tag{3.157}$$

$$n_a u_{a\zeta} = 0 \quad \text{for} \quad a \neq {\rm e}. \tag{3.158}$$

As for the total heat fluxes, we have only taken the convective part of them into account so that the initial profiles of the total heat fluxes becomes

$$\bar{Q}_a^\rho = \frac{5}{2} p_a \bar{u}_a^\rho \tag{3.159}$$

$$Q_{a\parallel} = \frac{5}{2} p_a u_{a\parallel} \tag{3.160}$$

$$Q_{a\zeta} = \frac{5}{2} p_a u_{a\zeta} \tag{3.161}$$

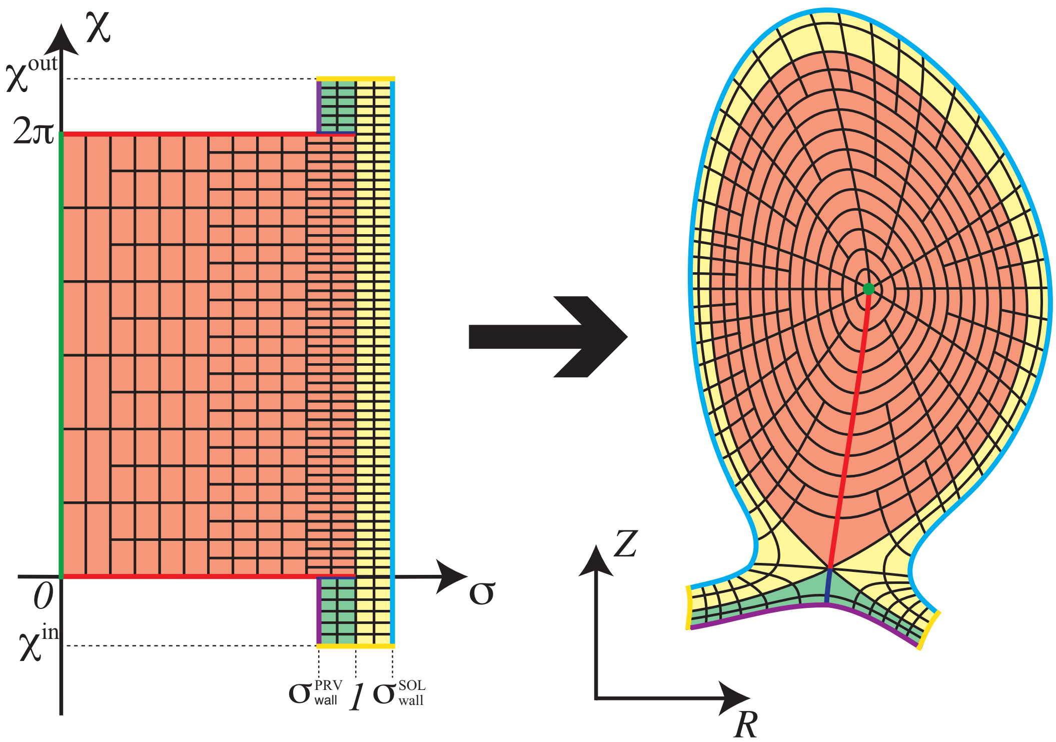

Figure 3.6: Concept of the computational grid for single-null divertor configuration; the red colored area is core region, the yellow colored area is the peripheral region and the green colored area is the private region respectively. Computational boundaries are expressed by the colored heavy lines.

3.7 Computational grid for single null divertor configuration

The concept of the computational grid for single null divertor configuration is shown in Figure 3.6. In Figure 3.6, $\sigma_{\text{wall}}^{\text{SOL}}$ and $\sigma_{\text{wall}}^{\text{PRV}}$ are the position of the first wall normalized by the position of the LCFS and χ^{in} and χ^{out} are the extended poloidal angle defined by the magnetic field length. The boundary conditions and initial conditions for the single-null divertor configuration are future works, but some findings on the boundary conditions are summarized as follows.

- Green heavy line ($\sigma = 0$, $0 \le \chi \le 2\pi$): the green heavy line is projected to the green point in the cylindrical coordinate so that the dependent variable have a same value

$$f(0, \chi) = f_0, \quad (0 \le \chi \le 2\pi) \tag{3.162}$$

- Red heavy lines ($0 \le \sigma \le 1$, $\chi = 0 + 0, 2\pi - 0$): the red heavy lines in MSCS are projected to the red line in the cylindrical coordinate and correspond to the periodic boundaries in the core region so that the dependent variable have the periodic property

$$f(\sigma, 0+0) = f(\sigma, 2\pi - 0), \quad (0 \le \sigma \le 1). \tag{3.163}$$

- Dark blue heavy lines ($\sigma_{\rm wall}^{\rm PRV} \le \sigma \le 1$, $\chi = 0, 2\pi$): the dark blue heavy lines in MSCS are projected to the dark blue line in the cylindrical coordinate correspond to the periodic boundaries in the private region so that the dependent variable have the periodic property

$$f(\sigma, 0-0) = f(\sigma, 2\pi + 0), \quad (\sigma_{wall}^{\rm PRV} \le \sigma \le 1). \tag{3.164}$$

- Yellow heavy lines ($\sigma_{\rm wall}^{\rm PRV} \le \sigma \le \sigma_{\rm wall}^{\rm SOL}$, $\chi = \chi^{\rm in}, \chi^{\rm out}$): the yellow heavy lines correspond to divertor plates and appropriate boundary conditions are future works.
- Light blue heavy lines ($\sigma = \sigma_{\rm wall}^{\rm SOL}$, $\chi^{\rm in} \le \chi \le \chi^{\rm out}$): the light blue heavy line corresponds to the surface of the first wall and appropriate boundary conditions are future works.
- Purple heavy lines ($\sigma = \sigma_{\rm wall}^{\rm PRV}$, $\chi^{\rm in} \le \chi \le 0, 2\pi \le \chi \le \chi^{\rm out}$): the purple heavy line also corresponds to the surface of the first wall and appropriate boundary conditions are future works.

3.8 Summary and discussion

we have discussed numerical schemes and developed a transport code for the two-dimensional transport modeling. We have reduced our two-dimensional transport model to advection-diffusion equations and derived their coefficients. The finite element method has been employed as a discretization scheme in space, the full implicit method has been employed as a discretization scheme in time, and the Picard iteration has been employed as a non-linear iteration method. The hierarchical rectangular grid in MSCS has been employed in order to separate the fluxes parallel and perpendicular to the field line for numerical stability and accuracy, to keep the poloidal resolution of the calculation in the outer region, and to ensure the flexibility of the grid width in the radial direction. We have discussed boundary conditions and initial conditions for limiter tokamak plasmas

Chapter 4

Conclusions and future perspectives

We have engaged in a fundamental study of two-dimensional transport modeling describing the time evolution of the core and the peripheral plasmas for the purpose of analyzing the edge transport barrier which enhances the confinement performance and the peripheral transport which determines the heat load on the divertor plate.

Firstly, we have derived a set of two-dimensional transport equations which is a starting point for two-dimensional transport modeling in tokamak plasmas for the first time. This set of equations has the following features;

- The magnetic surface coordinate system is employed to evaluate transport with strong anisotropy driven by very fast transport in the parallel direction accurately and the poloidal coordinate based on the field line length is introduced to describe the peripheral region outside of the separatrix.
- In order to describe self-consistent time evolution of the plasma rotations and the radial electric field, the two-dimensional equations have been derived from the multi-fluid equation describing the electron and ion transport separately without the assumption of charge quasi-neutrality.
- We have developed a unified method describing both the neoclassical transport in the weakly collisional core plasma and the classical transport in the strongly collisional peripheral plasma.
- In order to evaluate accurately the neoclassical heat transport playing an important role in the transport barrier, the equation for heat flux has been also employed. It has been confirmed that our transport equations reproduce the conventional diffusive-type neoclassical transport model in a stationary state.

Secondly, we have derived a set of electromagnetic equations consistent with the two-dimensional transport equation from Maxwell's equations for the first time. This set of equations has the following features;

- Since the existence of the flux surfaces is assumed, the time evolution of the electromagnetic field is described by the five components except for the contravariant radial magnetic field.

- The existence of axisymmetric flux surfaces requires that the poloidal flux, the poloidal current and the covariant toroidal electric field are flux functions.

- The equations describing the time evolution of the flux functions have to be flux-surface averaged and the equation of the covariant electric field corresponds to the flux-surface averaged Grad-Shafranov equation employed in the conventional 1.5D transport modeling that couples the two-dimensional equilibrium and the one-dimensional transport modeling.

- There is an algorithm which couples our two-dimensional transport equations and electromagnetic equations to describe the time evolution of tokamak plasmas.

In addition, we have discussed numerical schemes and developed a transport code for the two-dimensional transport modeling. We have reduced our two-dimensional transport model to advection-diffusion equations and derived their coefficients. The finite element method has been employed as a discretization scheme in space, the full implicit method has been employed as a discretization scheme in time, and the Picard iteration has been employed as a nonlinear iteration method. The hierarchical rectangular grid in MSCS has been employed in order to separate the fluxes parallel and perpendicular to the field line for numerical stability and accuracy, to keep the poloidal resolution of the calculation in the outer region, and to ensure the flexibility of the grid width in the radial direction. We have discussed boundary conditions and initial conditions for tokamak plasmas with limiter, launched the development of transport code and done preliminary calculations.

The future work extending the present study will include, the accomplishment of code development, the introduction of neutral particle transport model, the introduction of turbulent transport model and the application to divertor configurations.

Acknowledgement

The author would like to express his sincere appreciation to Professor Atsushi Fukuyama for his valuable and considerable advices, discussion and encouragement at Kyoto University. He would like to express his sincere appreciation to Associate Professor Sadayoshi Murakami for his constant encouragement. He also thanks to all members of his laboratory, especially Drs. Arimitsu Wakasa, Daniel L. Raburn, Hideo Nuga, Yoshitada Masaoka and Mr. Hiroyuki Yamaguchi for their support.

The author would like to acknowledge insightful comments from and fruitful discussion with Dr. Tomonori Takizuka of Osaka University, Drs. Katsuhiro Shimizu, Mitsuru Honda of Japan Atomic Energy Agency, Dr. Gakushi Kawamura of Nationa Institute for Fusion Science, Drs. D. Coster, V.A. Chankin, B. Scott and O. Kardaun of Max Planck Institute for Plasma Physics and Dr. D. Tskhakaya of University of Vienna.

This work was supported in part by the Grant-in-Aid for Scientific Research (S) (20226017) form Japan Society for the Promotion of Science (JSPS) and Global-COE program (J-051) from Ministry of Education, Culture, Sports, Science and Technology (MEXT).

Finally, the author would like to thank Takenaka Scholarship Society for its 8 years grant and his family for their continuing support.

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Appendix A

Two-dimensional transport equations in advection-diffusion form

A.1 Preliminaries

A.1.1 Parallel viscous coefficients

The parallel viscous coefficient $\pi_{\parallel a}$ can be written with the dependent variables in TASK/T2 as

$$\begin{aligned}
\pi_{\parallel a} &= -3\mu_{a1}\left(\nabla_\parallel u_{a\parallel} - u_{a\kappa}\right) - 3\mu_{a2}\left[\nabla_\parallel\left(\frac{2q_{a\parallel}}{5p_a}\right) - \frac{2q_{a\kappa}}{5p_a}\right] \\
&= -3\frac{\bar{\mu}_{01a}}{n_a}\left[\frac{B^\chi}{B}\frac{\partial}{\partial\chi}\left(n_a u_{a\parallel}\right) - u_{a\parallel}\frac{B^\chi}{B}\frac{\partial n_a}{\partial\chi} + \frac{B_t^2}{B^3}\frac{B^\chi}{I}\frac{\partial B}{\partial\chi}n_a u_{a\zeta} - \frac{B_t^2}{B^3}\frac{\partial B}{\partial\chi}n_a u_a^\chi\right] \\
&\quad - 3\frac{\bar{\mu}_{02a}}{p_a}\left[\frac{B^\chi}{B}\frac{\partial Q_{a\parallel}}{\partial\chi} - w_{a\parallel}\frac{B^\chi}{B}\frac{\partial p_a}{\partial\chi} + \frac{B_t^2}{B^3}\frac{B^\chi}{I}\frac{\partial B}{\partial\chi}Q_{a\zeta} - \frac{B_t^2}{B^3}\frac{\partial B}{\partial\chi}Q_a^\chi\right]
\end{aligned} \tag{A.1}$$

where

$$\bar{\mu}_{01a} = \mu_{a1} - \mu_{a2}, \tag{A.2}$$

$$\bar{\mu}_{02a} = \frac{2}{5}\mu_{a2}, \tag{A.3}$$

$$w_{a\parallel} = \frac{Q_{a\parallel}}{p_a}. \tag{A.4}$$

The EW parallel viscous coefficient $r_{\parallel a}$ can be also written as

$$\begin{aligned}
r_{\parallel a} &= -3\frac{\bar{\mu}_{03a}}{n_a}\frac{T_a}{m_a}\left[\frac{B^\chi}{B}\frac{\partial}{\partial\chi}\left(n_a u_{a\parallel}\right) - u_{a\parallel}\frac{B^\chi}{B}\frac{\partial n_a}{\partial\chi} + \frac{B_t^2}{B^3}\frac{B^\chi}{I}\frac{\partial B}{\partial\chi}n_a u_{a\zeta} - \frac{B_t^2}{B^3}\frac{\partial B}{\partial\chi}n_a u_a^\chi\right] \\
&\quad - 3\frac{\bar{\mu}_{04a}}{p_a}\frac{T_a}{m_a}\left[\frac{B^\chi}{B}\frac{\partial Q_{a\parallel}}{\partial\chi} - w_{a\parallel}\frac{B^\chi}{B}\frac{\partial p_a}{\partial\chi} + \frac{B_t^2}{B^3}\frac{B^\chi}{I}\frac{\partial B}{\partial\chi}Q_{a\zeta} - \frac{B_t^2}{B^3}\frac{\partial B}{\partial\chi}Q_a^\chi\right]
\end{aligned} \tag{A.5}$$

where

$$\bar{\mu}_{03a} = \frac{5}{2}\left(\mu_{a1} - \mu_{a2}\right) + \left(\mu_{a2} - \mu_{a3}\right) \tag{A.6}$$

$$\bar{\mu}_{04a}^\chi = \mu_{a2} + \frac{2}{5}\mu_{a3} \tag{A.7}$$

A.1.2 Friction forces

The friction force $\boldsymbol{F}_a^{\text{fri}}$ can be written with the dependent variables in TASK/T2 as

$$\begin{aligned}\boldsymbol{F}_a^{\text{fri}} &= \sum_b \left(l_{11}^{ab}\boldsymbol{u}_b - l_{12}^{ab}\frac{2\boldsymbol{q}_b}{5p_b} \right) \\ &= \sum_b \left[l_{11}^{ab}\frac{n_a\boldsymbol{u}_b}{n_b} - l_{12}^{ab}\frac{2}{5p_b}\left(\boldsymbol{Q}_b - \frac{5}{2}p_b\frac{n_b\boldsymbol{u}_b}{n_b} \right) \right] \\ &= \sum_b \left(\frac{\bar{l}_{01ab}}{n_b}n_b\boldsymbol{u}_b + \frac{\bar{l}_{02ab}}{p_b}\boldsymbol{Q}_b \right) \qquad \text{(A.8)}\end{aligned}$$

where

$$\bar{l}_{01ab} = l_{11}^{ab} + l_{12}^{ab} \qquad \text{(A.9)}$$

$$\bar{l}_{02ab} = -\frac{2}{5}l_{12}^{ab} \qquad \text{(A.10)}$$

The EW friction force $\boldsymbol{G}_a^{\text{fri}}$ can be also written as

$$\boldsymbol{G}_a^{\text{fri}} = \frac{T_a}{m_a}\left[\frac{5}{2}\sum_b \left(l_{11}^{ab}\boldsymbol{u}_b - l_{12}^{ab}\frac{2\boldsymbol{q}_b}{5p_b} \right) + \sum_b \left(-l_{21}^{ab}\boldsymbol{u}_b + l_{22}^{ab}\frac{2\boldsymbol{q}_b}{5p_b} \right) \right] \qquad \text{(A.11)}$$

$$\begin{aligned}&= \frac{T_a}{m_a}\left[\sum_b \left(\frac{5}{2}\frac{l_{11}^{ab}+l_{12}^{ab}}{n_b} - \frac{l_{21}^{ab}+l_{22}^{ab}}{n_b} \right) n_b\boldsymbol{u}_b + \sum_b \left(-\frac{5}{2}\frac{2l_{21}^{ab}}{5p_b} + \frac{2l_{22}^{ab}}{5p_b} \right)\boldsymbol{Q}_b \right] \\ &= \sum_b \frac{T_a}{m_a}\frac{\bar{l}_{03ab}}{n_a}n_a\boldsymbol{u}_b + \sum_b \frac{T_a}{m_a}\frac{\bar{l}_{04ab}}{p_a}\boldsymbol{Q}_b, \qquad \text{(A.12)}\end{aligned}$$

where

$$\bar{l}_{03ab} = \frac{5}{2}\left(l_{11}^{ab} + l_{12}^{ab} \right) - \left(l_{21}^{ab} + l_{22}^{ab} \right) \qquad \text{(A.13)}$$

$$\bar{l}_{04ab} = -l_{21}^{ab} + \frac{2}{5}l_{22}^{ab} \qquad \text{(A.14)}$$

A.2 Equation for poloidal magnetic flux function

$$\begin{aligned}&\frac{1}{V'}\frac{\partial}{\partial t}\left(V'\psi' \right) - \left\langle \nabla\cdot\left(\boldsymbol{u}_g\psi' \right) \right\rangle - \frac{\partial E_\zeta}{\partial\sigma} = 0 \\ &\frac{1}{V'}\frac{\partial}{\partial t}\left(V'\left\langle M_{01.01}\right\rangle\psi' \right) + \left\langle \nabla\cdot\left(\boldsymbol{V}_{01.01}^1\psi' \right) \right\rangle + \left\langle \boldsymbol{A}_{01.03}^1\cdot\nabla E_\zeta \right\rangle = 0, \qquad \text{(A.15)}\end{aligned}$$

where

$$\sqrt{g}M_{01.01} = \sqrt{g} \qquad \text{(A.16)}$$

$$\sqrt{g}\boldsymbol{V}_{01.01}^1 = \begin{pmatrix} -\sqrt{g}u_g^\sigma & -\sqrt{g}u_g^\chi \end{pmatrix} \qquad \text{(A.17)}$$

$$\sqrt{g}\boldsymbol{A}_{01.03}^1 = \begin{pmatrix} -\sqrt{g} & 0 \end{pmatrix} \qquad \text{(A.18)}$$

A.3 Equation for poloidal current function

$$
\begin{aligned}
&\frac{1}{V'}\frac{\partial}{\partial t}\left(V'I\right)-\left\langle\nabla\cdot\left(\boldsymbol{u}_g I\right)\right\rangle+\left\langle\frac{R^2}{\sqrt{g}}\left(\frac{\partial E_\chi}{\partial\sigma}-\frac{\partial E_\sigma}{\partial\chi}\right)\right\rangle=0\\
&\frac{1}{V'}\frac{\partial}{\partial t}\left(V'I\right)-\left\langle\nabla\cdot\left(\boldsymbol{u}_g I\right)\right\rangle+\left\langle\frac{R^2\sigma}{\sqrt{g}}\frac{\partial\bar{E}_\chi}{\partial\sigma}\right\rangle+\left\langle\frac{R^2}{\sqrt{g}}\bar{E}_\chi\right\rangle-\left\langle\frac{R^2}{\sqrt{g}}\frac{\partial E_\sigma}{\partial\chi}\right\rangle=0\\
&\frac{1}{V'}\frac{\partial}{\partial t}\left(V'\left\langle M_{02.02}\right\rangle I\right)+\left\langle\nabla\cdot\left(\boldsymbol{V}^1_{02.02}I\right)\right\rangle+\left\langle\boldsymbol{A}^1_{02.04}\cdot\nabla\bar{E}_\chi\right\rangle+\left\langle\boldsymbol{A}^1_{02.05}\cdot\nabla E_\sigma\right\rangle+\left\langle C^1_{02.04}\bar{E}_\chi\right\rangle=0
\end{aligned}
\tag{A.19}
$$

where

$$\sqrt{g}M_{02.02}=\sqrt{g} \tag{A.20}$$

$$\sqrt{g}\boldsymbol{V}^1_{02.02}=\begin{pmatrix}-\sqrt{g}u_g^\sigma & -\sqrt{g}u_g^\chi\end{pmatrix} \tag{A.21}$$

$$\sqrt{g}\boldsymbol{A}^1_{02.04}=\begin{pmatrix}R^2\sigma & 0\end{pmatrix} \tag{A.22}$$

$$\sqrt{g}\boldsymbol{A}^1_{02.05}=\begin{pmatrix}0 & -R^2\end{pmatrix} \tag{A.23}$$

$$\sqrt{g}C^1_{02.04}=R^2 \tag{A.24}$$

A.4 Equation for covariant toroidal electric field

$$
\begin{aligned}
&\frac{1}{c^2}\left.\frac{\partial E_\zeta}{\partial t}\right|_{\boldsymbol{x}}-\left\langle R^2\nabla\cdot\left(\frac{1}{R^2}\nabla\psi\right)\right\rangle+\left\langle\mu_0 j_\zeta\right\rangle=0\\
&\frac{1}{V'}\frac{\partial}{\partial t}\left(V'\frac{1}{c^2}E_\zeta\right)-\left\langle\nabla\cdot\left(\frac{\boldsymbol{u}_g}{c^2}E_\zeta\right)\right\rangle-\left\langle\nabla\cdot\left(\psi'\nabla\sigma\right)\right\rangle\\
&\qquad+\left\langle\left(\nabla R\cdot\frac{2}{R}\nabla\sigma\right)\psi'\right\rangle+\mu_0\sum_a\left\langle e_a n_a u_{a\zeta}\right\rangle=0\\
&\frac{1}{V'}\frac{\partial}{\partial t}\left(V'\frac{1}{c^2}E_\zeta\right)-\left\langle\nabla\cdot\left(\psi'\nabla\sigma\right)\right\rangle-\left\langle\nabla\cdot\left(\frac{\boldsymbol{u}_g}{c^2}E_\zeta\right)\right\rangle\\
&\qquad+\sum_a\left\langle\mu_0 e_a n_a u_{a\zeta}\right\rangle+\left\langle\left(\nabla R\cdot\frac{2}{R}\nabla\sigma\right)\psi'\right\rangle=0\\
&\frac{1}{V'}\frac{\partial}{\partial t}\left(V'\left\langle M_{03.03}\right\rangle E_\zeta\right)+\left\langle\nabla\cdot\left(\boldsymbol{V}^1_{03.01}\psi'\right)\right\rangle+\left\langle\nabla\cdot\left(\boldsymbol{V}^1_{03.03}E_\zeta\right)\right\rangle+\sum_a\left\langle C^1_{03.09a}n_a u_{a\zeta}\right\rangle\\
&\qquad+\left\langle\left(\nabla R\cdot\boldsymbol{C}^2_{03.01.02}\right)\right\rangle\psi'=0,
\end{aligned}
\tag{A.25}
$$

where

$$\sqrt{g}M_{03.03} = \frac{\sqrt{g}}{c^2} \tag{A.26}$$

$$\sqrt{g}\boldsymbol{V}^1_{03.01} = \begin{pmatrix} -\sqrt{g}g^{\sigma\sigma} & -\sqrt{g}g^{\sigma\chi} \end{pmatrix} \tag{A.27}$$

$$\sqrt{g}\boldsymbol{V}^1_{03.03} = \begin{pmatrix} -\sqrt{g}\frac{u_g^\sigma}{c^2} & -\sqrt{g}\frac{u_g^\chi}{c^2} \end{pmatrix} \tag{A.28}$$

$$\sqrt{g}C^1_{03.09a} = \sqrt{g}\mu_0 e_a \tag{A.29}$$

$$\sqrt{g}\boldsymbol{C}^2_{03.01.02} = \begin{pmatrix} \sqrt{g}\frac{2g^{\sigma\sigma}}{R} & \sqrt{g}\frac{2g^{\sigma\chi}}{R} \end{pmatrix} \tag{A.30}$$

A.5 Equation for covariant poloidal electric field

$$\begin{aligned}
&\frac{1}{c^2}\frac{\partial E_\chi}{\partial t}\bigg|_{\boldsymbol{x}} + \frac{g_{\chi\chi}}{\sqrt{g}}\frac{\partial I}{\partial \sigma} + \frac{\mu_0}{B^\chi}\left(j_\parallel B - j^\zeta I\right) = 0 \\
&\frac{1}{c^2}\frac{\partial E_\chi}{\partial t}\bigg|_{\boldsymbol{x}} + \frac{g_{\chi\chi}}{\sqrt{g}}\frac{\partial I}{\partial \sigma} + \mu_0 j_\chi = 0 \\
&\frac{1}{\sqrt{g}}\frac{\partial}{\partial t}\left(\sqrt{g}\frac{1}{c^2}\sigma\bar{E}_\chi\right) - \nabla\cdot\left(\frac{\boldsymbol{u}_g}{c^2}\sigma\bar{E}_\chi\right) + \frac{g_{\chi\chi}}{\sqrt{g}}\frac{\partial I}{\partial \sigma} + \sum_a \mu_0 e_a g_{\chi\sigma}\sigma n_a \bar{u}_a^\sigma + \sum_a \mu_0 e_a g_{\chi\chi} n_a u_a^\chi = 0 \\
&\frac{1}{\sqrt{g}}\frac{\partial}{\partial t}\left(\sqrt{g}M_{04.04}\bar{E}_\chi\right) + \nabla\cdot\left(\boldsymbol{V}^1_{04.04}\bar{E}_\chi\right) + \boldsymbol{A}^1_{04.02}\nabla I + \sum_a C^1_{04.07a} n_a \bar{u}_a^\sigma + \sum_a C^1_{04.10a} n_a u_a^\chi = 0,
\end{aligned} \tag{A.31}$$

where

$$\sqrt{g}M_{04.04} = \sqrt{g}\frac{\sigma}{c^2} \tag{A.32}$$

$$\sqrt{g}\boldsymbol{V}^1_{04.04} = \begin{pmatrix} -\sqrt{g}\frac{u_g^\sigma \sigma}{c^2} & -\sqrt{g}\frac{u_g^\chi \sigma}{c^2} \end{pmatrix} \tag{A.33}$$

$$\sqrt{g}\boldsymbol{A}^1_{04.02} = \begin{pmatrix} g_{\chi\chi} & 0 \end{pmatrix} \tag{A.34}$$

$$\sqrt{g}C^1_{04.07a} = \sqrt{g}\mu_0 e_a g_{\chi\sigma}\sigma \tag{A.35}$$

$$\sqrt{g}C^1_{04.10a} = \sqrt{g}\mu_0 e_a g_{\chi\chi} \tag{A.36}$$

A.6 Equation for covariant radial electric field

$$\begin{aligned}
&\nabla\cdot\boldsymbol{E} = \frac{\rho_c}{\varepsilon_0} \\
&\nabla\cdot\left(\sigma\bar{E}_\chi\nabla\chi\right) + \nabla\cdot\left(E_\sigma\nabla\sigma\right) - \sum_a \frac{e_a}{\varepsilon_0} n_a = 0 \\
&\nabla\cdot\left(\boldsymbol{V}^1_{05.04}E_\chi\right) + \nabla\cdot\left(\boldsymbol{V}^1_{05.05}E_\sigma\right) + \sum_a C^1_{05.06a} n_a = 0,
\end{aligned} \tag{A.37}$$

where

$$\sqrt{g}\boldsymbol{V}^1_{05.04} = \begin{pmatrix} \sqrt{g}g^{\sigma\chi}\sigma & \sqrt{g}g^{\chi\chi}\sigma \end{pmatrix} \tag{A.38}$$
$$\sqrt{g}\boldsymbol{V}^1_{05.05} = \begin{pmatrix} \sqrt{g}g^{\sigma\sigma} & \sqrt{g}g^{\chi\sigma} \end{pmatrix} \tag{A.39}$$
$$\sqrt{g}C^1_{05.06a} = -\frac{e_a\sqrt{g}}{\varepsilon_0} \tag{A.40}$$

A.7 Equation for particle density

$$\begin{aligned}
&\frac{1}{\sqrt{g}}\frac{\partial}{\partial t}\left(\sqrt{g}n_a\right) - \nabla\cdot\left(\boldsymbol{u}_g n_a\right) + \nabla\cdot\left(n_a\boldsymbol{u}_a\right) = S_{na} \\
&\frac{1}{\sqrt{g}}\frac{\partial}{\partial t}\left(\sqrt{g}n_a\right) + \nabla\cdot\left[\left(\boldsymbol{u}_a - \boldsymbol{u}_g\right)n_a\right] = S_{na} \\
&\frac{1}{\sqrt{g}}\frac{\partial}{\partial t}\left(\sqrt{g}M_{06a.06a}n_a\right) + \nabla\cdot\left(\boldsymbol{V}^1_{06a.06a}n_a\right) = S_{06a.06a},
\end{aligned} \tag{A.41}$$

where

$$\sqrt{g}M_{06a.06a} = \sqrt{g} \tag{A.42}$$
$$\sqrt{g}\boldsymbol{V}^1_{06a.06a} = \begin{pmatrix} \sqrt{g}\left(u_a^\sigma - u_g^\sigma\right) & \sqrt{g}\left(u_a^\chi - u_g^\chi\right) \end{pmatrix} \tag{A.43}$$
$$\sqrt{g}S_{06a.06a} = \sqrt{g}S_{na} \tag{A.44}$$

A.8 Equation for contravariant radial particle flux

$$\begin{aligned}
&\frac{1}{m_a}\nabla\sigma\cdot\nabla p_a = \frac{e_a}{m_a}n_aE^\sigma + \frac{e_a}{m_a}\frac{IB}{\psi'}n_au_{a\|} - \frac{e_a}{m_a}\frac{B^2}{\psi'}n_au_{a\zeta} \\
&\frac{B^\chi}{m_a}\nabla\sigma\cdot\nabla p_a - \frac{e_an_a}{m_a}g^{\sigma\chi}\sigma B^\chi\bar{E}_\chi - \frac{e_an_a}{m_a}g^{\sigma\sigma}B^\chi E_\sigma - \frac{e_a}{m_a}\frac{IB}{\psi'}B^\chi n_au_{a\|} + \frac{e_a}{m_a}\frac{B^2}{\psi'}B^\chi n_au_{a\zeta} = 0 \\
&\boldsymbol{A}^1_{07a.11a}\cdot\nabla p_a + C^1_{07a.04}\bar{E}_\chi + C^1_{07a.05}E_\sigma + C^1_{07a.08a}n_au_{a\|} + C^1_{07a.09a}n_au_{a\zeta} = 0
\end{aligned} \tag{A.45}$$

where

$$\sqrt{g}\boldsymbol{A}^1_{07a.11a} = \begin{pmatrix} \frac{\sqrt{g}g^{\sigma\sigma}}{m_a}B^\chi & \frac{\sqrt{g}g^{\sigma\chi}}{m_a}B^\chi \end{pmatrix} \tag{A.46}$$
$$\sqrt{g}C^1_{07a.04} = -\frac{\sqrt{g}e_an_a\sigma}{m_a}g^{\sigma\chi}B^\chi \tag{A.47}$$
$$\sqrt{g}C^1_{07a.05} = -\frac{\sqrt{g}e_an_a}{m_a}g^{\sigma\sigma}B^\chi \tag{A.48}$$
$$\sqrt{g}C^1_{07a.08a} = -\frac{e_a}{m_a}IB \tag{A.49}$$
$$\sqrt{g}C^1_{07a.09a} = \frac{e_a}{m_a}B^2 \tag{A.50}$$

A.9 Equation of parallel particle flux

$$\frac{\partial}{\partial t}\left(n_a u_{a\|} B\right)\Big|_{\boldsymbol{x}} + F_{a\|}^{\text{ine}}\frac{B}{m_a} + F_{a\|}^{\nabla p}\frac{B}{m_a} + F_{a\|}^{\text{vis}}\frac{B}{m_a} = F_{a\|}^{\text{Lor}}\frac{B}{m_a} + F_{a\|}^{\text{fri}}\frac{B}{m_a} + S_{ma\|}\frac{B}{m_a} \tag{A.51}$$

Inertial term

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(n_a u_{a\|} B\right)\Big|_{\boldsymbol{x}} + F_{a\|}^{\text{ine}}\frac{B}{m_a} \\
&\quad = \frac{1}{\sqrt{g}}\frac{\partial}{\partial t}\left(\sqrt{g} B n_a u_{a\|}\right) - \nabla\cdot\left(\boldsymbol{u}_g B n_a u_{a\|}\right) + B\nabla_\|\left(n_a u_{a\|} u_{a\|}\right) - \left(n_a u_{a\|} u_{a\|}\right)\nabla_\| B \\
&= \frac{1}{\sqrt{g}}\frac{\partial}{\partial t}\left(\sqrt{g} M_{08a.08a} n_a u_{a\|}\right) + \nabla\cdot\left(\boldsymbol{V}^1_{08a.08a} n_a u_{a\|}\right) + \left(\nabla B\cdot \boldsymbol{C}^2_{08a.08a.01}\right) n_a u_{a\|}
\end{aligned} \tag{A.52}$$

where

$$\sqrt{g} M_{08a.08a} = \sqrt{g} B \tag{A.53}$$

$$\sqrt{g}\boldsymbol{V}^1_{08a.08a} = \begin{pmatrix} -\sqrt{g} u_g^\sigma B & \sqrt{g}\left(u_{a\|} B^\chi - u_g^\chi B\right) \end{pmatrix} \tag{A.54}$$

$$\sqrt{g}\boldsymbol{C}^2_{08a.08a.01} = \begin{pmatrix} 0 & -\sqrt{g}\frac{B^\chi}{B} u_{a\|} \end{pmatrix} \tag{A.55}$$

Force by pressure gradient

$$F_{a\|}^{\nabla p}\frac{B}{m_a} = \frac{B}{m_a}\nabla_\| p_a = \frac{1}{\sqrt{g}}\frac{\partial}{\partial \chi}\left(\sqrt{g}\frac{B^\chi}{m_a} p_a\right) = \nabla\cdot\left(\boldsymbol{V}^1_{08a.11a} p_a\right), \tag{A.56}$$

where

$$\sqrt{g}\boldsymbol{V}^1_{08a.11a} = \begin{pmatrix} 0 & \sqrt{g}\frac{B^\chi}{m_a} \end{pmatrix} \tag{A.57}$$

Viscous force

$$
\begin{aligned}
F_{a\parallel}^{\mathrm{vis}}\frac{B}{m_a} &= \left(-\pi_{\parallel a}\nabla_\parallel B + \frac{2}{3}B\nabla_\parallel \pi_{\parallel a}\right)\frac{1}{m_a} \\
&= \frac{3\bar{\mu}_{01a}}{m_a n_a}\left[\frac{B^\chi}{B}\frac{\partial}{\partial\chi}\left(n_a u_{a\parallel}\right) - u_{a\parallel}\frac{B^\chi}{B}\frac{\partial n_a}{\partial\chi} + \frac{B_t^2}{B^3}\frac{B^\chi}{I}\frac{\partial B}{\partial\chi}n_a u_{a\zeta} - \frac{B_t^2}{B^3}\frac{\partial B}{\partial\chi}n_a u_a^\chi\right]\frac{B^\chi}{B}\frac{\partial B}{\partial\chi} \\
&\quad + \frac{3\bar{\mu}_{02a}}{m_a p_a}\left[\frac{B^\chi}{B}\frac{\partial Q_{a\parallel}}{\partial\chi} - w_{a\parallel}\frac{B^\chi}{B}\frac{\partial p_a}{\partial\chi} + \frac{B_t^2}{B^3}\frac{B^\chi}{I}\frac{\partial B}{\partial\chi}Q_{a\zeta} - \frac{B_t^2}{B^3}\frac{\partial B}{\partial\chi}Q_a^\chi\right]\frac{B^\chi}{B}\frac{\partial B}{\partial\chi} \\
&\quad - \frac{1}{\sqrt{g}}\frac{\partial}{\partial\chi}\left[\sqrt{g}\frac{2\bar{\mu}_{01a}B^\chi}{m_a n_a}\left[\frac{B^\chi}{B}\frac{\partial}{\partial\chi}\left(n_a u_{a\parallel}\right) - u_{a\parallel}\frac{B^\chi}{B}\frac{\partial n_a}{\partial\chi} + \frac{B_t^2}{B^3}\frac{B^\chi}{I}\frac{\partial B}{\partial\chi}n_a u_{a\zeta} - \frac{B_t^2}{B^3}\frac{\partial B}{\partial\chi}n_a u_a^\chi\right]\right] \\
&\quad - \frac{1}{\sqrt{g}}\frac{\partial}{\partial\chi}\left[\sqrt{g}\frac{2\bar{\mu}_{02a}B^\chi}{m_a p_a}\left[\frac{B^\chi}{B}\frac{\partial Q_{a\parallel}}{\partial\chi} - w_{a\parallel}\frac{B^\chi}{B}\frac{\partial p_a}{\partial\chi} + \frac{B_t^2}{B^3}\frac{B^\chi}{I}\frac{\partial B}{\partial\chi}Q_{a\zeta} - \frac{B_t^2}{B^3}\frac{\partial B}{\partial\chi}Q_a^\chi\right]\right] \\
&= \frac{\partial B}{\partial\chi}\left[-\frac{3\bar{\mu}_{01a}u_{a\parallel}}{m_a n_a}\left(\frac{B^\chi}{B}\right)^2\right]\frac{\partial n_a}{\partial\chi} + \frac{\partial B}{\partial\chi}\left[\frac{3\bar{\mu}_{01a}}{m_a n_a}\left(\frac{B^\chi}{B}\right)^2\right]\frac{\partial}{\partial\chi}\left(n_a u_{a\parallel}\right) \\
&\quad + \frac{\partial B}{\partial\chi}\left[\frac{3\bar{\mu}_{01a}}{m_a n_a}\frac{B_t^2}{B^3}\frac{B^\chi}{I}\frac{B^\chi}{B}\right]\frac{\partial B}{\partial\chi}n_a u_{a\zeta} + \frac{\partial B}{\partial\chi}\left[-\frac{3\bar{\mu}_{01a}}{m_a n_a}\frac{B_t^2}{B^3}\frac{B^\chi}{B}\right]\frac{\partial B}{\partial\chi}n_a u_a^\chi \\
&\quad + \frac{\partial B}{\partial\chi}\left[-\frac{3\bar{\mu}_{02a}w_{a\parallel}}{m_a p_a}\left(\frac{B^\chi}{B}\right)^2\right]\frac{\partial p_a}{\partial\chi} + \frac{\partial B}{\partial\chi}\left[\frac{3\bar{\mu}_{02a}}{m_a p_a}\left(\frac{B^\chi}{B}\right)^2\right]\frac{\partial Q_{a\parallel}}{\partial\chi} \\
&\quad + \frac{\partial B}{\partial\chi}\left[\frac{3\bar{\mu}_{02a}}{m_a p_a}\frac{B_t^2}{B^3}\frac{B^\chi}{I}\frac{B^\chi}{B}\right]\frac{\partial B}{\partial\chi}Q_{a\zeta} + \frac{\partial B}{\partial\chi}\left[-\frac{3\bar{\mu}_{02a}}{m_a p_a}\frac{B_t^2}{B^3}\frac{B^\chi}{B}\right]\frac{\partial B}{\partial\chi}Q_a^\chi \\
&\quad - \frac{1}{\sqrt{g}}\frac{\partial}{\partial\chi}\left[\sqrt{g}\left\{-\frac{2\bar{\mu}_{01a}u_{a\parallel}B}{m_a n_a}\left(\frac{B^\chi}{B}\right)^2\right\}\frac{\partial n_a}{\partial\chi}\right] \\
&\quad - \frac{1}{\sqrt{g}}\frac{\partial}{\partial\chi}\left[\sqrt{g}\left\{\frac{2\bar{\mu}_{01a}B}{m_a n_a}\left(\frac{B^\chi}{B}\right)^2\right\}\frac{\partial}{\partial\chi}\left(n_a u_{a\parallel}\right)\right] \\
&\quad - \frac{1}{\sqrt{g}}\frac{\partial}{\partial\chi}\left[\sqrt{g}\left(\frac{2\bar{\mu}_{01a}}{m_a n_a}\frac{B_t^2}{B^2}\frac{B^\chi}{B}\frac{B^\chi}{I}\right)\frac{\partial B}{\partial\chi}n_a u_{a\zeta}\right] \\
&\quad - \frac{1}{\sqrt{g}}\frac{\partial}{\partial\chi}\left[\sqrt{g}\left(-\frac{2\bar{\mu}_{01a}}{m_a n_a}\frac{B_t^2}{B^2}\frac{B^\chi}{B}\right)\frac{\partial B}{\partial\chi}n_a u_a^\chi\right] \\
&\quad - \frac{1}{\sqrt{g}}\frac{\partial}{\partial\chi}\left[\sqrt{g}\left\{-\frac{2\bar{\mu}_{02a}w_{a\parallel}B}{m_a p_a}\left(\frac{B^\chi}{B}\right)^2\right\}\frac{\partial p_a}{\partial\chi}\right] \\
&\quad - \frac{1}{\sqrt{g}}\frac{\partial}{\partial\chi}\left[\sqrt{g}\left\{\frac{2\bar{\mu}_{02a}B}{m_a p_a}\left(\frac{B^\chi}{B}\right)^2\right\}\frac{\partial Q_{a\parallel}}{\partial\chi}\right] \\
&\quad - \frac{1}{\sqrt{g}}\frac{\partial}{\partial\chi}\left[\sqrt{g}\left(\frac{2\bar{\mu}_{02a}}{m_a p_a}\frac{B_t^2}{B^2}\frac{B^\chi}{B}\frac{B^\chi}{I}\right)\frac{\partial B}{\partial\chi}Q_{a\zeta}\right] \\
&\quad - \frac{1}{\sqrt{g}}\frac{\partial}{\partial\chi}\left[\sqrt{g}\left(-\frac{2\bar{\mu}_{02a}}{m_a p_a}\frac{B_t^2}{B^2}\frac{B^\chi}{B}\right)\frac{\partial B}{\partial\chi}Q_a^\chi\right] \qquad \text{(A.58)}
\end{aligned}
$$

$$
\begin{aligned}
F_{a\|}^{\text{vis}}\frac{B}{m_a} = &-\nabla\cdot\left[\left(\nabla B\cdot\overleftrightarrow{V}^2_{08a.09a.01}\right)n_a u_{a\zeta}\right]-\nabla\cdot\left[\left(\nabla B\cdot\overleftrightarrow{V}^2_{08a.10a.01}\right)n_a u_a^\chi\right]\\
&-\nabla\cdot\left[\left(\nabla B\cdot\overleftrightarrow{V}^2_{08a.14a.01}\right)Q_{a\zeta}\right]-\nabla\cdot\left[\left(\nabla B\cdot\overleftrightarrow{V}^2_{08a.15a.01}\right)Q_a^\chi\right]\\
&-\nabla\cdot\left(\overleftrightarrow{D}_{08a.06a}\cdot\nabla n_a\right)-\nabla\cdot\left[\overleftrightarrow{D}_{08a.08a}\cdot\nabla\left(n_a u_{a\|}\right)\right]\\
&-\nabla\cdot\left(\overleftrightarrow{D}_{08a.11a}\cdot\nabla p_a\right)-\nabla\cdot\left(\overleftrightarrow{D}_{08a.13a}\cdot\nabla Q_{a\|}\right)\\
&+\left(\nabla B\cdot\overleftrightarrow{A}^2_{08a.06a.01}\right)\cdot\nabla n_a+\left(\nabla B\cdot\overleftrightarrow{A}^2_{08a.08a.01}\right)\cdot\nabla\left(n_a u_{a\|}\right)\\
&+\left(\nabla B\cdot\overleftrightarrow{A}^2_{08a.11a.01}\right)\cdot\nabla p_a+\left(\nabla B\cdot\overleftrightarrow{A}^2_{08a.13a.01}\right)\cdot\nabla Q_{a\|}\\
&+\left(\nabla B\cdot\overleftrightarrow{C}^3_{08a.09a.01.01}\cdot\nabla B\right)n_a u_{a\zeta}+\left(\nabla B\cdot\overleftrightarrow{C}^3_{08a.10a.01.01}\cdot\nabla B\right)n_a u_a^\chi\\
&+\left(\nabla B\cdot\overleftrightarrow{C}^3_{08a.14a.01.01}\cdot\nabla B\right)Q_{a\zeta}+\left(\nabla B\cdot\overleftrightarrow{C}^3_{08a.15a.01.01}\cdot\nabla B\right)Q_a^\chi
\end{aligned}
\tag{A.59}
$$

where

$$
\sqrt{g}\overleftrightarrow{V}^2_{08a.09a.01} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{g}\frac{2\bar{\mu}_{01a}}{m_a n_a}\frac{B_t^2}{B^2}\frac{B^\chi}{B}\frac{B^\chi}{I} \end{pmatrix} \tag{A.60}
$$

$$
\sqrt{g}\overleftrightarrow{V}^2_{08a.10a.01} = \begin{pmatrix} 0 & 0 \\ 0 & -\sqrt{g}\frac{2\bar{\mu}_{01a}}{m_a n_a}\frac{B_t^2}{B^2}\frac{B^\chi}{B} \end{pmatrix} \tag{A.61}
$$

$$
\sqrt{g}\overleftrightarrow{V}^2_{08a.14a.01} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{g}\frac{2\bar{\mu}_{02a}}{m_a p_a}\frac{B_t^2}{B^2}\frac{B^\chi}{B}\frac{B^\chi}{I} \end{pmatrix} \tag{A.62}
$$

$$
\sqrt{g}\overleftrightarrow{V}^2_{08a.15a.01} = \begin{pmatrix} 0 & 0 \\ 0 & -\sqrt{g}\frac{2\bar{\mu}_{02a}}{m_a p_a}\frac{B_t^2}{B^2}\frac{B^\chi}{B} \end{pmatrix} \tag{A.63}
$$

$$
\sqrt{g}\overleftrightarrow{D}_{08a.06a} = \begin{pmatrix} 0 & 0 \\ 0 & -\sqrt{g}\frac{2\bar{\mu}_{01a}u_{a\|}B}{m_a n_a}\left(\frac{B^\chi}{B}\right)^2 \end{pmatrix} \tag{A.64}
$$

$$
\sqrt{g}\overleftrightarrow{D}_{08a.08a} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{g}\frac{2\bar{\mu}_{01a}B}{m_a n_a}\left(\frac{B^\chi}{B}\right)^2 \end{pmatrix} \tag{A.65}
$$

$$
\sqrt{g}\overleftrightarrow{D}_{08a.11a} = \begin{pmatrix} 0 & 0 \\ 0 & -\sqrt{g}\frac{2\bar{\mu}_{02a}w_{a\|}B}{m_a p_a}\left(\frac{B^\chi}{B}\right)^2 \end{pmatrix} \tag{A.66}
$$

$$
\sqrt{g}\overleftrightarrow{D}_{08a.13a} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{g}\frac{2\bar{\mu}_{02a}B}{m_a p_a}\left(\frac{B^\chi}{B}\right)^2 \end{pmatrix} \tag{A.67}
$$

$$\sqrt{g}\overleftrightarrow{A}^2_{08a.06a.01} = \begin{pmatrix} 0 & 0 \\ 0 & -\sqrt{g}\frac{3\bar{\mu}_{01a}u_{a\|}}{m_a n_a}\left(\frac{B^\chi}{B}\right)^2 \end{pmatrix} \tag{A.68}$$

$$\sqrt{g}\overleftrightarrow{A}^2_{08a.08a.01} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{g}\frac{3\bar{\mu}_{01a}}{m_a n_a}\left(\frac{B^\chi}{B}\right)^2 \end{pmatrix} \tag{A.69}$$

$$\sqrt{g}\overleftrightarrow{A}^2_{08a.11a.01} = \begin{pmatrix} 0 & 0 \\ 0 & -\sqrt{g}\frac{3\bar{\mu}_{02a}w_{a\|}}{m_a p_a}\left(\frac{B^\chi}{B}\right)^2 \end{pmatrix} \tag{A.70}$$

$$\sqrt{g}\overleftrightarrow{A}^2_{08a.13a.01} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{g}\frac{3\bar{\mu}_{02a}}{m_a p_a}\left(\frac{B^\chi}{B}\right)^2 \end{pmatrix} \tag{A.71}$$

$$\sqrt{g}\overleftrightarrow{C}^3_{08a.09a.01.01} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{g}\frac{3\bar{\mu}_{01a}}{m_a n_a}\frac{B_t^2}{B^3}\frac{B^\chi}{I}\frac{B^\chi}{B} \end{pmatrix} \tag{A.72}$$

$$\sqrt{g}\overleftrightarrow{C}^3_{08a.10a.01.01} = \begin{pmatrix} 0 & 0 \\ 0 & -\sqrt{g}\frac{3\bar{\mu}_{01a}}{m_a n_a}\frac{B_t^2}{B^3}\frac{B^\chi}{B} \end{pmatrix} \tag{A.73}$$

$$\sqrt{g}\overleftrightarrow{C}^3_{08a.14a.01.01} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{g}\frac{3\bar{\mu}_{02a}}{m_a p_a}\frac{B_t^2}{B^3}\frac{B^\chi}{I}\frac{B^\chi}{B} \end{pmatrix} \tag{A.74}$$

$$\sqrt{g}\overleftrightarrow{C}^3_{08a.15a.01.01} = \begin{pmatrix} 0 & 0 \\ 0 & -\sqrt{g}\frac{3\bar{\mu}_{02a}}{m_a p_a}\frac{B_t^2}{B^3}\frac{B^\chi}{B} \end{pmatrix} \tag{A.75}$$

Lorentz force

$$\begin{aligned} F^{\mathrm{Lor}}_{a\|}\frac{B}{m_a} &= \frac{e_a}{m_a}n_a E_\| B \\ &= \frac{e_a}{m_a}n_a B^\zeta E_\zeta + \frac{e_a}{m_a}n_a B^\chi E_\chi \\ &= \frac{e_a}{m_a}n_a B^\zeta E_\zeta + \frac{e_a}{m_a}n_a B^\chi \sigma \bar{E}_\chi \\ &= -C^1_{08a.03}E_\zeta - C^1_{08a.04}\bar{E}_\chi, \end{aligned} \tag{A.76}$$

where

$$\sqrt{g}C^1_{08a.03} = -\sqrt{g}\frac{e_a n_a B^\zeta}{m_a} \tag{A.77}$$

$$\sqrt{g}C^1_{08a.04} = -\sqrt{g}\frac{e_a n_a B^\chi \sigma}{m_a} \tag{A.78}$$

Friction force

$$F_{a\|}^{\text{fri}}\frac{B}{m_a}=\sum_b\left(\frac{\bar{l}_{01ab}}{m_an_b}n_bu_{b\|}B+\frac{\bar{l}_{02ab}}{m_ap_b}Q_{b\|}B\right)$$
$$=-\sum_b\left(C^1_{08a.08b}n_bu_{b\|}+C^1_{08a.13b}Q_{b\|}\right) \tag{A.79}$$

where,

$$\sqrt{g}C^1_{08a.08b}=-\sqrt{g}\frac{\bar{l}_{01ab}B}{m_an_b} \tag{A.80}$$
$$\sqrt{g}C^1_{08a.13b}=-\sqrt{g}\frac{\bar{l}_{02ab}B}{m_ap_b} \tag{A.81}$$

Source term

$$\sqrt{g}S_{08a.08a}=\sqrt{g}\frac{S_{ma\|}B}{m_a} \tag{A.82}$$

Equation of parallel particle flux in advection-diffusion form

$$\begin{aligned}
&\frac{1}{\sqrt{g}}\frac{\partial}{\partial t}\left(\sqrt{g}M_{08a.08a}n_au_{a\|}\right)+\nabla\cdot\left(\boldsymbol{V}^1_{08a.08a}n_au_{a\|}\right)+\nabla\cdot\left(\boldsymbol{V}^1_{08a.11a}p_a\right)\\
&\quad-\nabla\cdot\left[\left(\overleftrightarrow{V}^1_{08a.09a.01}\cdot\nabla B\right)n_au_{a\zeta}\right]-\nabla\cdot\left[\left(\overleftrightarrow{V}^1_{08a.10a.01}\cdot\nabla B\right)n_au_a^{\chi}\right]\\
&\qquad-\nabla\cdot\left[\left(\overleftrightarrow{V}^1_{08a.14a.01}\cdot\nabla B\right)Q_{a\zeta}\right]-\nabla\cdot\left[\left(\overleftrightarrow{V}^1_{08a.15a.01}\cdot\nabla B\right)Q_a^{\chi}\right]\\
&\quad-\nabla\cdot\left(\overleftrightarrow{D}_{08a.06a}\cdot\nabla n_a\right)-\nabla\cdot\left[\overleftrightarrow{D}_{08a.08a}\cdot\nabla\left(n_au_{a\|}\right)\right]\\
&\qquad-\nabla\cdot\left(\overleftrightarrow{D}_{08.11a}\cdot\nabla p_a\right)-\nabla\cdot\left(\overleftrightarrow{D}_{08.13a}\cdot\nabla Q_{a\|}\right)\\
&\quad+\left(\nabla B\cdot\overleftrightarrow{A}^1_{08a.06a.01}\right)\cdot\nabla n_a+\left(\nabla B\cdot\overleftrightarrow{A}^1_{08a.08a.01}\right)\cdot\nabla\left(n_au_{a\|}\right)\\
&\qquad+\left(\nabla B\cdot\overleftrightarrow{A}^1_{08a.11a.01}\right)\cdot\nabla p_a+\left(\nabla B\cdot\overleftrightarrow{A}^1_{08a.13a.01}\right)\cdot\nabla Q_{a\|}\\
&\quad+C^1_{08a.03}E_\zeta+C^1_{08a.04}\bar{E}_\chi+\sum_b C^1_{08a.08b}n_au_{b\|}+\sum_b C^1_{08a.13b}Q_{b\|}\\
&\quad+\left(\boldsymbol{C}^2_{08a.08a.01}\cdot\nabla B\right)n_au_{a\|}\\
&\quad+\left(\nabla B\cdot\overleftrightarrow{C}^3_{08a.09a.01.01}\cdot\nabla B\right)n_au_{a\zeta}+\left(\nabla B\cdot\overleftrightarrow{C}^3_{08a.10a.01.01}\cdot\nabla B\right)n_au_a^{\chi}\\
&\qquad+\left(\nabla B\cdot\overleftrightarrow{C}^3_{08a.14a.01.01}\cdot\nabla B\right)Q_{a\zeta}+\left(\nabla B\cdot\overleftrightarrow{C}^3_{08a.15a.01.01}\cdot\nabla B\right)Q_a^{\chi}\\
&=S_{08a.08a}
\end{aligned} \tag{A.83}$$

A.10 Equation for covariant toroidal particle flux

$$\left.\frac{\partial}{\partial t}\left(n_a u_{a\zeta}\right)\right|_{\boldsymbol{x}} + \frac{F^{\rm ine}_{a\zeta}}{m_a} + \frac{F^{\rm vis}_{a\zeta}}{m_a} = \frac{F^{\rm Lor}_{a\zeta}}{m_a} + \frac{F^{\rm fir}_{a\zeta}}{m_a} + \frac{S_{ma\zeta}}{m_a} \tag{A.84}$$

Inertial term

$$\begin{aligned}\left.\frac{\partial}{\partial t}\left(n_a u_{a\zeta}\right)\right|_{\boldsymbol{x}} + \frac{F^{\rm ine}_{a\zeta}}{m_a} &= \frac{1}{\sqrt{g}}\frac{\partial}{\partial t}\left(\sqrt{g} n_a u_{a\zeta}\right) - \nabla\cdot\left(\boldsymbol{u}_g n_a u_{a\zeta}\right) + \nabla\cdot\left(n_a u_{a\zeta}\boldsymbol{u}_a\right)\\ &= \frac{1}{\sqrt{g}}\frac{\partial}{\partial t}\left(\sqrt{g} M_{09a.09a} n_a u_{a\zeta}\right) + \nabla\cdot\left(\boldsymbol{V}^1_{09a.09a} n_a u_{a\zeta}\right),\end{aligned} \tag{A.85}$$

where

$$\sqrt{g} M_{09a.09a} = \sqrt{g} \tag{A.86}$$

$$\sqrt{g}\boldsymbol{V}^1_{09a.09a} = \begin{pmatrix} \sqrt{g}\left(u^\sigma_a - u^\sigma_g\right) & \sqrt{g}\left(u^\chi_a - u^\chi_g\right)\end{pmatrix} \tag{A.87}$$

Viscous force

$$\begin{aligned}\frac{F^{\rm vis}_{a\zeta}}{m_a} &= B\nabla_\parallel\left(\frac{I}{B^2}\frac{\pi_{\parallel a}}{m_a}\right)\\
&= -\frac{1}{\sqrt{g}}\frac{\partial}{\partial\chi}\left\{\sqrt{g}\frac{3\bar{\mu}_{01a}}{n_a m_a}\frac{B^\chi}{B}\frac{I}{B}\left[\frac{B^\chi}{B}\frac{\partial}{\partial\chi}\left(n_a u_{a\parallel}\right) - u_{a\parallel}\frac{B^\chi}{B}\frac{\partial n_a}{\partial\chi} + \frac{B_t^2}{B^3}\frac{B^\chi}{I}\frac{\partial B}{\partial\chi}n_a u_{a\zeta} - \frac{B_t^2}{B^3}\frac{\partial B}{\partial\chi}n_a u^\chi_a\right]\right\}\\
&\quad - \frac{1}{\sqrt{g}}\frac{\partial}{\partial\chi}\left\{\sqrt{g}\frac{3\bar{\mu}_{02a}}{p_a m_a}\frac{B^\chi}{B}\frac{I}{B}\left[\frac{B^\chi}{B}\frac{\partial Q_{a\parallel}}{\partial\chi} - w_{a\parallel}\frac{B^\chi}{B}\frac{\partial p_a}{\partial\chi} + \frac{B_t^2}{B^3}\frac{B^\chi}{I}\frac{\partial B}{\partial\chi}Q_{a\zeta} - \frac{B_t^2}{B^3}\frac{\partial B}{\partial\chi}Q^\chi_a\right]\right\}\\
&= -\frac{1}{\sqrt{g}}\frac{\partial}{\partial\chi}\left\{\sqrt{g}\left[-\frac{3\bar{\mu}_{01a}u_{a\parallel}}{n_a m_a}\frac{I}{B}\left(\frac{B^\chi}{B}\right)^2\right]\frac{\partial n_a}{\partial\chi}\right\}\\
&\quad - \frac{1}{\sqrt{g}}\frac{\partial}{\partial\chi}\left\{\sqrt{g}\left[\frac{3\bar{\mu}_{01a}}{n_a m_a}\left(\frac{B^\chi}{B}\right)^2\frac{I}{B}\right]\frac{\partial}{\partial\chi}\left(n_a u_{a\parallel}\right)\right\}\\
&\quad - \frac{1}{\sqrt{g}}\frac{\partial}{\partial\chi}\left\{\sqrt{g}\left[\frac{3\bar{\mu}_{01a}}{n_a m_a}\frac{B_t^2}{B^3}\left(\frac{B^\chi}{B}\right)^2\right]\frac{\partial B}{\partial\chi}n_a u_{a\zeta}\right\}\\
&\quad - \frac{1}{\sqrt{g}}\frac{\partial}{\partial\chi}\left\{\sqrt{g}\left[-\frac{3\bar{\mu}_{01a}}{n_a m_a}\frac{B_t^2}{B^3}\frac{B^\chi}{B}\frac{I}{B}\right]\frac{\partial B}{\partial\chi}n_a u^\chi_a\right\}\\
&\quad - \frac{1}{\sqrt{g}}\frac{\partial}{\partial\chi}\left\{\sqrt{g}\left[-\frac{3\bar{\mu}_{02a}w_{a\parallel}}{p_a m_a}\left(\frac{B^\chi}{B}\right)^2\frac{I}{B}\right]\frac{\partial p_a}{\partial\chi}\right\} - \frac{1}{\sqrt{g}}\frac{\partial}{\partial\chi}\left\{\sqrt{g}\left[\frac{3\bar{\mu}_{02a}}{p_a m_a}\left(\frac{B^\chi}{B}\right)^2\frac{I}{B}\right]\frac{\partial Q_{a\parallel}}{\partial\chi}\right\}\\
&\quad - \frac{1}{\sqrt{g}}\frac{\partial}{\partial\chi}\left\{\sqrt{g}\left[\frac{3\bar{\mu}_{02a}}{p_a m_a}\frac{B_t^2}{B^3}\left(\frac{B^\chi}{B}\right)^2\right]\frac{\partial B}{\partial\chi}Q_{a\zeta}\right\} - \frac{1}{\sqrt{g}}\frac{\partial}{\partial\chi}\left\{\sqrt{g}\left[-\frac{3\bar{\mu}_{02a}}{p_a m_a}\frac{B_t^2}{B^3}\frac{B^\chi}{B}\frac{I}{B}\right]\frac{\partial B}{\partial\chi}Q^\chi_a\right\}\end{aligned} \tag{A.88}$$

$$
\begin{aligned}
\frac{F_{a\zeta}^{\mathrm{vis}}}{m_a} = &-\nabla\cdot\left[\left(\overleftrightarrow{V}^2_{09a.09a.01}\cdot\nabla B\right)n_a u_{a\zeta}\right]-\nabla\cdot\left[\left(\overleftrightarrow{V}^2_{09a.10a.01}\cdot\nabla B\right)n_a u_a^{\chi}\right]\\
&-\nabla\cdot\left[\left(\overleftrightarrow{V}^2_{09a.14a.01}\cdot\nabla B\right)Q_{a\zeta}\right]-\nabla\cdot\left[\left(\overleftrightarrow{V}^2_{09a.15a.01}\cdot\nabla B\right)Q_a^{\chi}\right]\\
&-\nabla\cdot\left(\overleftrightarrow{D}_{09a.06a}\cdot\nabla n_a\right)-\nabla\cdot\left[\overleftrightarrow{D}_{09a.08a}\cdot\nabla\left(n_a u_{a\|}\right)\right]\\
&-\nabla\cdot\left(\overleftrightarrow{D}_{09a.11a}\cdot\nabla p_a\right)-\nabla\cdot\left(\overleftrightarrow{D}_{09a.13a}\cdot\nabla Q_{a\|}\right),
\end{aligned}
\tag{A.89}
$$

where

$$
\sqrt{g}\overleftrightarrow{V}^2_{09a.09a.01} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{g}\dfrac{3\bar{\mu}_{01a}}{n_a m_a}\dfrac{B_t^2}{B^3}\left(\dfrac{B^{\chi}}{B}\right)^2 \end{pmatrix} \tag{A.90}
$$

$$
\sqrt{g}\overleftrightarrow{V}^2_{09a.10a.01} = \begin{pmatrix} 0 & 0 \\ 0 & -\sqrt{g}\dfrac{3\bar{\mu}_{01a}}{n_a m_a}\dfrac{B_t^2}{B^3}\dfrac{B^{\chi}}{B}\dfrac{I}{B} \end{pmatrix} \tag{A.91}
$$

$$
\sqrt{g}\overleftrightarrow{V}^2_{09a.14a.01} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{g}\dfrac{3\bar{\mu}_{02a}}{p_a m_a}\dfrac{B_t^2}{B^3}\left(\dfrac{B^{\chi}}{B}\right)^2 \end{pmatrix} \tag{A.92}
$$

$$
\sqrt{g}\overleftrightarrow{V}^2_{09a.15a.01} = \begin{pmatrix} 0 & 0 \\ 0 & -\sqrt{g}\dfrac{3\bar{\mu}_{02a}}{p_a m_a}\dfrac{B_t^2}{B^3}\dfrac{B^{\chi}}{B}\dfrac{I}{B} \end{pmatrix} \tag{A.93}
$$

$$
\sqrt{g}\overleftrightarrow{D}_{09a.06a} = \begin{pmatrix} 0 & 0 \\ 0 & -\sqrt{g}\dfrac{3\bar{\mu}_{01a}u_{a\|}}{n_a m_a}\left(\dfrac{B^{\chi}}{B}\right)^2\dfrac{I}{B} \end{pmatrix} \tag{A.94}
$$

$$
\sqrt{g}\overleftrightarrow{D}_{09a.08a} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{g}\dfrac{3\bar{\mu}_{01a}}{n_a m_a}\left(\dfrac{B^{\chi}}{B}\right)^2\dfrac{I}{B} \end{pmatrix} \tag{A.95}
$$

$$
\sqrt{g}\overleftrightarrow{D}_{09a.11a} = \begin{pmatrix} 0 & 0 \\ 0 & -\sqrt{g}\dfrac{3\bar{\mu}_{02a}w_{a\|}}{p_a m_a}\left(\dfrac{B^{\chi}}{B}\right)^2\dfrac{I}{B} \end{pmatrix} \tag{A.96}
$$

$$
\sqrt{g}\overleftrightarrow{D}_{09a.13a} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{g}\dfrac{3\bar{\mu}_{02a}}{p_a m_a}\left(\dfrac{B^{\chi}}{B}\right)^2\dfrac{I}{B} \end{pmatrix} \tag{A.97}
$$

Lorentz force

$$
\begin{aligned}
\frac{F_{a\zeta}^{\mathrm{Lor}}}{m_a} &= \frac{e_a}{m_a}n_a E_\zeta + \frac{e_a}{m_a}\sqrt{g}B^{\chi}\sigma n_a \bar{u}_a^{\sigma}\\
&= -C^1_{09a.03}E_\zeta - C^1_{09a.07a}n_a\bar{u}_a^{\sigma},
\end{aligned}
\tag{A.98}
$$

where

$$\sqrt{g}C^1_{09a.03} = -\sqrt{g}\frac{e_a n_a}{m_a} \tag{A.99}$$

$$\sqrt{g}C^1_{09a.07a} = -\sqrt{g}^2\frac{e_a B^\chi \sigma}{m_a} \tag{A.100}$$

Friction force

$$\begin{aligned}\frac{F^{\text{fri}}_{a\zeta}}{m_a} &= \sum_b \left(\frac{\bar{l}_{01ab}}{m_a n_b} n_a u_{b\zeta} + \frac{\bar{l}_{02ab}}{m_a p_b} Q_{b\zeta}\right) \\ &= -\sum_b \left(C^1_{09a.09b} n_b u_{b\zeta} + C^1_{09a.13b} Q_{b\zeta}\right),\end{aligned} \tag{A.101}$$

where

$$\sqrt{g}C^1_{09a.09b} = -\sqrt{g}\frac{\bar{l}_{01ab}}{m_a n_b} \tag{A.102}$$

$$\sqrt{g}C^1_{09a.14b} = -\sqrt{g}\frac{\bar{l}_{02ab}}{m_a p_b} \tag{A.103}$$

Source term

$$\sqrt{g}S_{09a.09a} = \sqrt{g}\frac{S_{ma\zeta}}{m_a} \tag{A.104}$$

Equation for covariant toroidal particle flux in advection-diffusion form

$$\begin{aligned}&\frac{1}{\sqrt{g}}\frac{\partial}{\partial t}\left(\sqrt{g}M_{09a.09a} n_a u_{a\zeta}\right) + \nabla\cdot\left(\boldsymbol{V}^1_{09a.09a} n_a u_{a\zeta}\right) \\ &\quad - \nabla\cdot\left[\left(\overleftrightarrow{V}^2_{09a.09a.01}\cdot\nabla B\right) n_a u_{a\zeta}\right] - \nabla\cdot\left[\left(\overleftrightarrow{V}^2_{09a.10a.01}\cdot\nabla B\right) n_a u^\chi_a\right] \\ &\qquad - \nabla\cdot\left[\left(\overleftrightarrow{V}^2_{09a.14a.01}\cdot\nabla B\right) Q_{a\zeta}\right] - \nabla\cdot\left[\left(\overleftrightarrow{V}^2_{09a.15a.01}\cdot\nabla B\right) Q^\chi_a\right] \\ &\quad - \nabla\cdot\left(\overleftrightarrow{D}_{09a.06a}\cdot\nabla n_a\right) - \nabla\cdot\left[\overleftrightarrow{D}_{09a.08a}\cdot\nabla\left(n_a u_{a\parallel}\right)\right] \\ &\qquad - \nabla\cdot\left(\overleftrightarrow{D}_{09a.11a}\cdot\nabla p_a\right) - \nabla\cdot\left(\overleftrightarrow{D}_{09a.13a}\cdot\nabla Q_{a\parallel}\right) \\ &\quad + C^1_{09a.03}E_\zeta + C^1_{09a.07a} n_a \bar{u}^\sigma_a + \sum_b C^1_{09a.09b} n_a u_{b\zeta} + \sum_b C^1_{09a.14b} Q_{b\zeta} \\ &= S_{09a.09a}.\end{aligned} \tag{A.105}$$

A.11 Expression of contravariant poloidal particle flux

$$
\begin{aligned}
B^\chi g_{\chi\chi} n_a u_a^\chi - B n_a u_{a\parallel} + B^\zeta n_a u_{a\zeta} = 0 \\
C^1_{10a.10a} n_a u_a^\chi + C^1_{10a.08a} n_a u_{a\parallel} + C^1_{10a.09a} n_a u_{a\zeta} = 0
\end{aligned}
\tag{A.106}
$$

where

$$\sqrt{g} C^1_{10a.08a} = -\sqrt{g} B \tag{A.107}$$

$$\sqrt{g} C^1_{10a.09a} = \sqrt{g} B^\zeta \tag{A.108}$$

$$\sqrt{g} C^1_{10a.10a} = \sqrt{g} g_{\chi\chi} B^\chi \tag{A.109}$$

A.12 Equation for pressure

$$\frac{3}{2} \left.\frac{\partial p_a}{\partial t}\right|_{\boldsymbol{x}} + \nabla \cdot \left(\boldsymbol{Q}_a - \frac{1}{2} m_a n_a u_a^2 \boldsymbol{u}_a \right) = \boldsymbol{u}_a \cdot \nabla p_a + Q_a^{\text{vis}} + Q_{\Delta a} + S_{pa} \tag{A.110}$$

Inertial term

$$
\begin{aligned}
&\frac{3}{2} \left.\frac{\partial p_a}{\partial t}\right|_{\boldsymbol{x}} + \nabla \cdot \left(\boldsymbol{Q}_a - \frac{1}{2} m_a n_a u_a^2 \boldsymbol{u}_a \right) \\
&= \frac{1}{\sqrt{g}} \frac{\partial}{\partial t} \left(\sqrt{g} \frac{3}{2} p_a \right) - \nabla \cdot \left(\boldsymbol{u}_g \frac{3}{2} p_a \right) + \nabla \cdot \left[\frac{1}{p_a} \left(\boldsymbol{Q}_a - \frac{1}{2} m_a n_a u_a^2 \boldsymbol{u}_a \right) p_a \right] \\
&= \frac{1}{\sqrt{g}} \frac{\partial}{\partial t} \left(\sqrt{g} M_{11a.11a} p_a \right) + \nabla \cdot \left(\boldsymbol{V}^1_{11a.11a} p_a \right)
\end{aligned}
\tag{A.111}
$$

where

$$\sqrt{g} M_{11a.11a} = \frac{3}{2} \sqrt{g} \tag{A.112}$$

$$\sqrt{g} \boldsymbol{V}^1_{11a.11a} = \left(\sqrt{g} \left(\frac{Q_a^\sigma}{p_a} - \frac{1}{2} \frac{m_a n_a u_a^2}{p_a} u_a^\sigma - \frac{3}{2} u_g^\sigma \right) \quad \sqrt{g} \left(\frac{Q_a^\chi}{p_a} - \frac{1}{2} \frac{m_a n_a u_a^2}{p_a} u_a^\chi - \frac{3}{2} u_g^\chi \right) \right) \tag{A.113}$$

Pressure heating term

$$\boldsymbol{u}_a \cdot \nabla p_a = -\boldsymbol{A}^1_{11a.11a} \cdot \nabla p_a \tag{A.114}$$

where

$$\sqrt{g} \boldsymbol{A}^1_{11a.11a} = \left(-\sqrt{g} u_a^\sigma \quad -\sqrt{g} u_a^\chi \right) \tag{A.115}$$

Viscous heating term

$$
\begin{aligned}
Q_a^{\mathrm{vis}} &= B\nabla_\parallel \left(\frac{u_{a\parallel}\pi_{\parallel a}}{B}\right) - \pi_{\parallel a}\left(\nabla_\parallel u_{a\parallel} - u_{a\kappa}\right) - \frac{1}{3}\nabla\cdot\left(\boldsymbol{u}_a \pi_{\parallel a}\right) \\
&= -\nabla\cdot\left(\frac{\boldsymbol{u}_a^*}{3}\pi_{\parallel a}\right) - \left(\nabla_\parallel u_{a\parallel} - u_{a\kappa}\right)\pi_{\parallel a},
\end{aligned} \tag{A.116}
$$

where $\boldsymbol{u}_a^*$ and $u_a^{\dagger\chi}$ are defined by

$$
\boldsymbol{u}_a^* = \boldsymbol{u}_a - 3\frac{B^\chi}{B}u_{a\parallel}\boldsymbol{e}_\chi, \tag{A.117}
$$

$$
u_a^{\dagger\chi} = \frac{B^\chi}{I}u_{a\zeta} - u_a^\chi \tag{A.118}
$$

By the use of $\boldsymbol{u}_a^*$ and $u_a^{\dagger\chi}$, the viscous heating term can be written as

$$
\begin{aligned}
Q_a^{\mathrm{vis}} = &\sum_i \frac{1}{\sqrt{g}}\frac{\partial}{\partial\xi_i}\left\{\sqrt{g}\frac{\bar\mu_{01a}u_a^{*\xi_i}}{n_a}\left[\frac{B^\chi}{B}\frac{\partial}{\partial\chi}\left(n_a u_{a\parallel}\right) - u_{a\parallel}\frac{B^\chi}{B}\frac{\partial n_a}{\partial\chi} + \frac{B_t^2}{B^3}\frac{B^\chi}{I}\frac{\partial B}{\partial\chi}n_a u_{a\zeta} - \frac{B_t^2}{B^3}\frac{\partial B}{\partial\chi}n_a u_a^\chi\right]\right\} \\
&+ \sum_i \frac{1}{\sqrt{g}}\frac{\partial}{\partial\xi_i}\left\{\sqrt{g}\frac{\bar\mu_{02a}u_a^{*\xi_i}}{p_a}\left[\frac{B^\chi}{B}\frac{\partial Q_{a\parallel}}{\partial\chi} - w_{a\parallel}\frac{B^\chi}{B}\frac{\partial p_a}{\partial\chi} + \frac{B_t^2}{B^3}\frac{B^\chi}{I}\frac{\partial B}{\partial\chi}Q_{a\zeta} - \frac{B_t^2}{B^3}\frac{\partial B}{\partial\chi}Q_a^\chi\right]\right\} \\
&+ \frac{B^\chi}{B}\frac{\partial u_{a\parallel}}{\partial\chi}\frac{3\bar\mu_{01a}}{n_a}\left[\frac{B^\chi}{B}\frac{\partial}{\partial\chi}\left(n_a u_{a\parallel}\right) - u_{a\parallel}\frac{B^\chi}{B}\frac{\partial n_a}{\partial\chi} + \frac{B_t^2}{B^3}\frac{B^\chi}{I}\frac{\partial B}{\partial\chi}n_a u_{a\zeta} - \frac{B_t^2}{B^3}\frac{\partial B}{\partial\chi}n_a u_a^\chi\right] \\
&+ \frac{B^\chi}{B}\frac{\partial u_{a\parallel}}{\partial\chi}\frac{3\bar\mu_{02a}}{p_a}\left[\frac{B^\chi}{B}\frac{\partial Q_{a\parallel}}{\partial\chi} - w_{a\parallel}\frac{B^\chi}{B}\frac{\partial p_a}{\partial\chi} + \frac{B_t^2}{B^3}\frac{B^\chi}{I}\frac{\partial B}{\partial\chi}Q_{a\zeta} - \frac{B_t^2}{B^3}\frac{\partial B}{\partial\chi}Q_a^\chi\right] \\
&+ u_a^{\dagger\chi}\frac{B_t^2}{B^3}\frac{\partial B}{\partial\chi}\frac{3\bar\mu_{01a}}{n_a}\left[\frac{B^\chi}{B}\frac{\partial}{\partial\chi}\left(n_a u_{a\parallel}\right) - u_{a\parallel}\frac{B^\chi}{B}\frac{\partial n_a}{\partial\chi} + \frac{B_t^2}{B^3}\frac{B^\chi}{I}\frac{\partial B}{\partial\chi}n_a u_{a\zeta} - \frac{B_t^2}{B^3}\frac{\partial B}{\partial\chi}n_a u_a^\chi\right] \\
&+ u_a^{\dagger\chi}\frac{B_t^2}{B^3}\frac{\partial B}{\partial\chi}3\frac{\bar\mu_{02a}}{p_a}\left[\frac{B^\chi}{B}\frac{\partial Q_{a\parallel}}{\partial\chi} - w_{a\parallel}\frac{B^\chi}{B}\frac{\partial p_a}{\partial\chi} + \frac{B_t^2}{B^3}\frac{B^\chi}{I}\frac{\partial B}{\partial\chi}Q_{a\zeta} - \frac{B_t^2}{B^3}\frac{\partial B}{\partial\chi}Q_a^\chi\right] \\
= &\sum_i \frac{1}{\sqrt{g}}\frac{\partial}{\partial\xi_i}\left[\sqrt{g}\left(-\frac{\bar\mu_{01a}u_a^{*\xi_i}u_{a\parallel}}{n_a}\frac{B^\chi}{B}\right)\frac{\partial n_a}{\partial\chi}\right] + \sum_i \frac{1}{\sqrt{g}}\frac{\partial}{\partial\xi_i}\left[\sqrt{g}\left(\frac{\bar\mu_{01a}u_a^{*\xi_i}}{n_a}\frac{B^\chi}{B}\right)\frac{\partial}{\partial\chi}\left(n_a u_{a\parallel}\right)\right] \\
&+ \sum_i \frac{1}{\sqrt{g}}\frac{\partial}{\partial\xi_i}\left[\sqrt{g}\left(\frac{\bar\mu_{01a}u_a^{*\xi_i}}{n_a}\frac{B_t^2}{B^3}\frac{B^\chi}{I}\right)\frac{\partial B}{\partial\chi}n_a u_{a\zeta}\right] + \sum_i \frac{1}{\sqrt{g}}\frac{\partial}{\partial\xi_i}\left[\sqrt{g}\left(-\frac{\bar\mu_{01a}u_a^{*\xi_i}}{n_a}\frac{B_t^2}{B^3}\right)\frac{\partial B}{\partial\chi}n_a u_a^\chi\right] \\
&+ \sum_i \frac{1}{\sqrt{g}}\frac{\partial}{\partial\xi_i}\left[\sqrt{g}\left(-\frac{\bar\mu_{02a}u_a^{*\xi_i}w_{a\parallel}}{p_a}\frac{B^\chi}{B}\right)\frac{\partial p_a}{\partial\chi}\right] + \sum_i \frac{1}{\sqrt{g}}\frac{\partial}{\partial\xi_i}\left[\sqrt{g}\left(\frac{\bar\mu_{02a}u_a^{*\xi_i}}{p_a}\frac{B^\chi}{B}\right)\frac{\partial}{\partial\chi}\left(Q_{a\parallel}\right)\right] \\
&+ \sum_i \frac{1}{\sqrt{g}}\frac{\partial}{\partial\xi_i}\left[\sqrt{g}\left(\frac{\bar\mu_{02a}u_a^{*\xi_i}}{p_a}\frac{B_t^2}{B^3}\frac{B^\chi}{I}\right)\frac{\partial B}{\partial\chi}Q_{a\zeta}\right] + \sum_i \frac{1}{\sqrt{g}}\frac{\partial}{\partial\xi_i}\left[\sqrt{g}\left(-\frac{\bar\mu_{02a}u_a^{*\xi_i}}{p_a}\frac{B_t^2}{B^3}\right)\frac{\partial B}{\partial\chi}Q_a^\chi\right] \\
&- \frac{\partial u_{a\parallel}}{\partial\chi}\left[\frac{3\bar\mu_{01a}u_{a\parallel}}{n_a}\left(\frac{B^\chi}{B}\right)^2\right]\frac{\partial n_a}{\partial\chi} - \frac{\partial u_{a\parallel}}{\partial\chi}\left[-\frac{3\bar\mu_{01a}}{n_a}\left(\frac{B^\chi}{B}\right)^2\right]\frac{\partial}{\partial\chi}\left(n_a u_{a\parallel}\right) \\
&- \frac{\partial u_{a\parallel}}{\partial\chi}\left[-\frac{3\bar\mu_{01a}}{n_a I}\frac{B_t^2}{B^2}\left(\frac{B^\chi}{B}\right)^2\right]\frac{\partial B}{\partial\chi}n_a u_{a\zeta} - \frac{\partial u_{a\parallel}}{\partial\chi}\left[\frac{3\bar\mu_{01a}}{n_a B}\frac{B_t^2}{B^2}\frac{B^\chi}{B}\right]\frac{\partial B}{\partial\chi}n_a u_a^\chi \\
&- \frac{\partial u_{a\parallel}}{\partial\chi}\left[\frac{3\bar\mu_{02a}w_{a\parallel}}{p_a}\left(\frac{B^\chi}{B}\right)^2\right]\frac{\partial p_a}{\partial\chi} - \frac{\partial u_{a\parallel}}{\partial\chi}\left[-\frac{3\bar\mu_{02a}}{p_a}\left(\frac{B^\chi}{B}\right)^2\right]\frac{\partial Q_{a\parallel}}{\partial\chi} \\
&- \frac{\partial u_{a\parallel}}{\partial\chi}\left[-\frac{3\bar\mu_{02a}}{p_a I}\frac{B_t^2}{B^2}\left(\frac{B^\chi}{B}\right)^2\right]\frac{\partial B}{\partial\chi}Q_{a\zeta} - \frac{\partial u_{a\parallel}}{\partial\chi}\left[\frac{3\bar\mu_{02a}}{p_a B}\frac{B_t^2}{B^2}\frac{B^\chi}{B}\right]\frac{\partial B}{\partial\chi}Q_a^\chi \\
&- \frac{\partial B}{\partial\chi}\left[\frac{3\bar\mu_{01a}u_a^{\dagger\chi}u_{a\parallel}}{n_a}\frac{B_t^2}{B^3}\frac{B^\chi}{B}\right]\frac{\partial n_a}{\partial\chi} - \frac{\partial B}{\partial\chi}\left[-\frac{3\bar\mu_{01a}u_a^{\dagger\chi}}{n_a}\frac{B_t^2}{B^3}\frac{B^\chi}{B}\right]\frac{\partial}{\partial\chi}\left(n_a u_{a\parallel}\right) \\
&- \frac{\partial B}{\partial\chi}\left[-\frac{3\bar\mu_{01a}u_a^{\dagger\chi}}{n_a}\frac{B_t^4}{B^6}\frac{B^\chi}{I}\right]\frac{\partial B}{\partial\chi}n_a u_{a\zeta} - \frac{\partial B}{\partial\chi}\left[\frac{3\bar\mu_{01a}u_a^{\dagger\chi}}{n_a}\frac{B_t^4}{B^6}\right]\frac{\partial B}{\partial\chi}n_a u_a^\chi \\
&- \frac{\partial B}{\partial\chi}\left[\frac{3\bar\mu_{02a}u_a^{\dagger\chi}w_{a\parallel}}{p_a}\frac{B_t^2}{B^3}\frac{B^\chi}{B}\right]\frac{\partial p_a}{\partial\chi} - \frac{\partial B}{\partial\chi}\left[-\frac{3\bar\mu_{02a}u_a^{\dagger\chi}}{p_a}\frac{B_t^2}{B^3}\frac{B^\chi}{B}\right]\frac{\partial Q_{a\parallel}}{\partial\chi} \\
&- \frac{\partial B}{\partial\chi}\left[-\frac{3\bar\mu_{02a}u_a^{\dagger\chi}}{p_a}\frac{B_t^4}{B^6}\frac{B^\chi}{I}\right]\frac{\partial B}{\partial\chi}Q_{a\zeta} - \frac{\partial B}{\partial\chi}\left[\frac{3\bar\mu_{02a}u_a^{\dagger\chi}}{p_a}\frac{B_t^4}{B^6}\right]\frac{\partial B}{\partial\chi}Q_a^\chi
\end{aligned}
\tag{A.119}
$$

$$
\begin{aligned}
Q_a^{\mathrm{vis}} = {} & \nabla\cdot\left[\left(\overleftrightarrow{V}^2_{11a.09a.01}\cdot\nabla B\right)n_a u_{a\zeta}\right] + \nabla\cdot\left[\left(\overleftrightarrow{V}^2_{11a.10a.01}\cdot\nabla B\right)n_a u_a^{\chi}\right] \\
& + \nabla\cdot\left[\left(\overleftrightarrow{V}^2_{11a.14a.01}\cdot\nabla B\right)Q_{a\zeta}\right] + \nabla\cdot\left[\left(\overleftrightarrow{V}^2_{11a.15a.01}\cdot\nabla B\right)Q_a^{\chi}\right] \\
& + \nabla\cdot\left[\overleftrightarrow{D}_{11a.06a}\cdot\nabla n_a\right] + \nabla\cdot\left[\overleftrightarrow{D}_{11a.08a}\cdot\nabla\left(n_a u_{a\|}\right)\right] \\
& + \nabla\cdot\left[\overleftrightarrow{D}_{11a.11a}\cdot\nabla p_a\right] + \nabla\cdot\left[\overleftrightarrow{D}_{11a.13a}\cdot\nabla Q_{a\|}\right] \\
& - \left[\nabla B\cdot\overleftrightarrow{A}^2_{11a.06a.01}\right]\cdot\nabla n_a - \left[\nabla B\cdot\overleftrightarrow{A}^2_{11a.08a.01}\right]\cdot\nabla\left(n_a u_{a\|}\right) \\
& - \left[\nabla B\cdot\overleftrightarrow{A}^2_{11a.11a.01}\right]\cdot\nabla p_a - \left[\nabla B\cdot\overleftrightarrow{A}^2_{11a.13a.01}\right]\cdot\nabla Q_{a\|} \\
& - \left[\nabla u_{a\|}\cdot\overleftrightarrow{A}^2_{11a.06a.03a}\right]\cdot\nabla n_a - \left[\nabla u_{a\|}\cdot\overleftrightarrow{A}^2_{11a.08a.03a}\right]\cdot\nabla\left(n_a u_{a\|}\right) \\
& - \left[\nabla u_{a\|}\cdot\overleftrightarrow{A}^2_{11a.11a.03a}\right]\cdot\nabla p_a - \left[\nabla u_{a\|}\cdot\overleftrightarrow{A}^2_{11a.13a.03a}\right]\cdot\nabla Q_{a\|} \\
& - \left[\nabla B\cdot\overleftrightarrow{C}^3_{11a.09a.01.01}\cdot\nabla B\right]n_a u_{a\zeta} - \left[\nabla B\cdot\overleftrightarrow{C}^3_{11a.10a.01.01}\cdot\nabla B\right]n_a u_a^{\chi} \\
& - \left[\nabla B\cdot\overleftrightarrow{C}^3_{11a.14a.01.01}\cdot\nabla B\right]Q_{a\zeta} - \left[\nabla B\cdot\overleftrightarrow{C}^3_{11a.15a.01.01}\cdot\nabla B\right]Q_a^{\chi} \\
& - \left[\nabla u_{a\|}\cdot\overleftrightarrow{C}^3_{11a.09a.03a.01}\cdot\nabla B\right]n_a u_{a\zeta} - \left[\nabla u_{a\|}\cdot\overleftrightarrow{C}^3_{11a.10a.03a.01}\cdot\nabla B\right]n_a u_a^{\chi} \\
& - \left[\nabla u_{a\|}\cdot\overleftrightarrow{C}^3_{11a.14a.03a.01}\cdot\nabla B\right]Q_{a\zeta} - \left[\nabla u_{a\|}\cdot\overleftrightarrow{C}^3_{11a.15a.03a.01}\cdot\nabla B\right]Q_a^{\chi}
\end{aligned}
\tag{A.120}
$$

where

$$
\sqrt{g}\overleftrightarrow{V}^2_{11a.09a.01} = \begin{pmatrix} 0 & \sqrt{g}\dfrac{\bar{\mu}_{01a}u_a^{*\sigma}}{n_a}\dfrac{B_t^2}{B^3}\dfrac{B^{\chi}}{I} \\ 0 & \sqrt{g}\dfrac{\bar{\mu}_{01a}u_a^{*\chi}}{n_a}\dfrac{B_t^2}{B^3}\dfrac{B^{\chi}}{I} \end{pmatrix} \tag{A.121}
$$

$$
\sqrt{g}\overleftrightarrow{V}^2_{11a.10a.01} = \begin{pmatrix} 0 & -\sqrt{g}\dfrac{\bar{\mu}_{01a}u_a^{*\sigma}}{n_a}\dfrac{B_t^2}{B^3} \\ 0 & -\sqrt{g}\dfrac{\bar{\mu}_{01a}u_a^{*\chi}}{n_a}\dfrac{B_t^2}{B^3} \end{pmatrix} \tag{A.122}
$$

$$
\sqrt{g}\overleftrightarrow{V}^2_{11a.14a.01} = \begin{pmatrix} 0 & \sqrt{g}\dfrac{\bar{\mu}_{02a}u_a^{*\sigma}}{p_a}\dfrac{B_t^2}{B^3}\dfrac{B^{\chi}}{I} \\ 0 & \sqrt{g}\dfrac{\bar{\mu}_{02a}u_a^{*\chi}}{p_a}\dfrac{B_t^2}{B^3}\dfrac{B^{\chi}}{I} \end{pmatrix} \tag{A.123}
$$

$$
\sqrt{g}\overleftrightarrow{V}^2_{11a.15a.01} = \begin{pmatrix} 0 & -\sqrt{g}\dfrac{\bar{\mu}_{02a}u_a^{*\sigma}}{p_a}\dfrac{B_t^2}{B^3} \\ 0 & -\sqrt{g}\dfrac{\bar{\mu}_{02a}u_a^{*\chi}}{p_a}\dfrac{B_t^2}{B^3} \end{pmatrix} \tag{A.124}
$$

$$\sqrt{g}\overleftrightarrow{D}_{11a.06a} = \begin{pmatrix} 0 & -\sqrt{g}\dfrac{\bar{\mu}_{01a}u_a^{*\sigma}u_{a\|}}{n_a}\dfrac{B^\chi}{B} \\ 0 & -\sqrt{g}\dfrac{\bar{\mu}_{01a}u_a^{*\chi}u_{a\|}}{n_a}\dfrac{B^\chi}{B} \end{pmatrix} \tag{A.125}$$

$$\sqrt{g}\overleftrightarrow{D}_{11a.08a} = \begin{pmatrix} 0 & \sqrt{g}\dfrac{\bar{\mu}_{01a}u_a^{*\sigma}}{n_a}\dfrac{B^\chi}{B} \\ 0 & \sqrt{g}\dfrac{\bar{\mu}_{01a}u_a^{*\chi}}{n_a}\dfrac{B^\chi}{B} \end{pmatrix} \tag{A.126}$$

$$\sqrt{g}\overleftrightarrow{D}_{11a.11a} = \begin{pmatrix} 0 & -\sqrt{g}\dfrac{\bar{\mu}_{02a}u_a^{*\sigma}w_{a\|}}{p_a}\dfrac{B^\chi}{B} \\ 0 & -\sqrt{g}\dfrac{\bar{\mu}_{02a}u_a^{*\chi}w_{a\|}}{p_a}\dfrac{B^\chi}{B} \end{pmatrix} \tag{A.127}$$

$$\sqrt{g}\overleftrightarrow{D}_{11a.13a} = \begin{pmatrix} 0 & \sqrt{g}\dfrac{\bar{\mu}_{02a}u_a^{*\sigma}}{p_a}\dfrac{B^\chi}{B} \\ 0 & \sqrt{g}\dfrac{\bar{\mu}_{02a}u_a^{*\chi}}{p_a}\dfrac{B^\chi}{B} \end{pmatrix} \tag{A.128}$$

$$\sqrt{g}\overleftrightarrow{A}^2_{11a.06a.01} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{g}\dfrac{3\bar{\mu}_{01a}u_a^{\dagger\chi}u_{a\|}}{n_a}\dfrac{B_t^2}{B^3}\dfrac{B^\chi}{B} \end{pmatrix} \tag{A.129}$$

$$\sqrt{g}\overleftrightarrow{A}^2_{11a.08a.01} = \begin{pmatrix} 0 & 0 \\ 0 & -\sqrt{g}\dfrac{3\bar{\mu}_{01a}u_a^{\dagger\chi}}{n_a}\dfrac{B_t^2}{B^3}\dfrac{B^\chi}{B} \end{pmatrix} \tag{A.130}$$

$$\sqrt{g}\overleftrightarrow{A}^2_{11a.11a.01} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{g}\dfrac{3\bar{\mu}_{02a}u_a^{\dagger\chi}w_{a\|}}{p_a}\dfrac{B_t^2}{B^3}\dfrac{B^\chi}{B} \end{pmatrix} \tag{A.131}$$

$$\sqrt{g}\overleftrightarrow{A}^2_{11a.13a.01} = \begin{pmatrix} 0 & 0 \\ 0 & -\sqrt{g}\dfrac{3\bar{\mu}_{02a}u_a^{\dagger\chi}}{p_a}\dfrac{B_t^2}{B^3}\dfrac{B^\chi}{B} \end{pmatrix} \tag{A.132}$$

$$\sqrt{g}\overleftrightarrow{A}^2_{11a.06a.03a} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{g}\dfrac{3\bar{\mu}_{01a}u_{a\|}}{n_a}\left(\dfrac{B^\chi}{B}\right)^2 \end{pmatrix} \tag{A.133}$$

$$\sqrt{g}\overleftrightarrow{A}^2_{11a.08a.03a} = \begin{pmatrix} 0 & 0 \\ 0 & -\sqrt{g}\dfrac{3\bar{\mu}_{01a}}{n_a}\left(\dfrac{B^\chi}{B}\right)^2 \end{pmatrix} \tag{A.134}$$

$$\sqrt{g}\overleftrightarrow{A}^2_{11a.11a.03a} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{g}\dfrac{3\bar{\mu}_{02a}w_{a\|}}{p_a}\left(\dfrac{B^\chi}{B}\right)^2 \end{pmatrix} \tag{A.135}$$

$$\sqrt{g}\overleftrightarrow{A}^2_{11a.13a.03a} = \begin{pmatrix} 0 & 0 \\ 0 & -\sqrt{g}\dfrac{3\bar{\mu}_{02a}}{p_a}\left(\dfrac{B^\chi}{B}\right)^2 \end{pmatrix} \tag{A.136}$$

$$\sqrt{g}\overleftrightarrow{C}^3_{11a.09a.01.01} = \begin{pmatrix} 0 & 0 \\ 0 & -\sqrt{g}\frac{3\bar{\mu}_{01a}u_a^{\dagger\chi}}{n_a}\frac{B_t^4}{B^6}\frac{B^\chi}{I} \end{pmatrix} \tag{A.137}$$

$$\sqrt{g}\overleftrightarrow{C}^3_{11a.10a.01.01} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{g}\frac{3\bar{\mu}_{01a}u_a^{\dagger\chi}}{n_a}\frac{B_t^4}{B^6} \end{pmatrix} \tag{A.138}$$

$$\sqrt{g}\overleftrightarrow{C}^3_{11a.14a.01.01} = \begin{pmatrix} 0 & 0 \\ 0 & -\sqrt{g}\frac{3\bar{\mu}_{02a}u_a^{\dagger\chi}}{p_a}\frac{B_t^4}{B^6}\frac{B^\chi}{I} \end{pmatrix} \tag{A.139}$$

$$\sqrt{g}\overleftrightarrow{C}^3_{11a.15a.01.01} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{g}\frac{3\bar{\mu}_{02a}u_a^{\dagger\chi}}{p_a}\frac{B_t^4}{B^6} \end{pmatrix} \tag{A.140}$$

$$\sqrt{g}\overleftrightarrow{C}^3_{11a.09a.03a.01} = \begin{pmatrix} 0 & 0 \\ 0 & -\sqrt{g}\frac{3\bar{\mu}_{01a}}{n_a I}\frac{B_t^2}{B^2}\left(\frac{B^\chi}{B}\right)^2 \end{pmatrix} \tag{A.141}$$

$$\sqrt{g}\overleftrightarrow{C}^3_{11a.10a.03a.01} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{g}\frac{3\bar{\mu}_{01a}}{n_a B}\frac{B_t^2}{B^2}\frac{B^\chi}{B} \end{pmatrix} \tag{A.142}$$

$$\sqrt{g}\overleftrightarrow{C}^3_{11a.14a.03a.01} = \begin{pmatrix} 0 & 0 \\ 0 & -\sqrt{g}\frac{3\bar{\mu}_{02a}}{p_a I}\frac{B_t^2}{B^2}\left(\frac{B^\chi}{B}\right)^2 \end{pmatrix} \tag{A.143}$$

$$\sqrt{g}\overleftrightarrow{C}^3_{11a.15a.03a.01} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{g}\frac{3\bar{\mu}_{02a}}{p_a B}\frac{B_t^2}{B^2}\frac{B^\chi}{B} \end{pmatrix} \tag{A.144}$$

Energy equipartition

$$\begin{aligned} Q_{\Delta a} &= \sum_b \frac{3}{2} n_a \frac{T_b - T_a}{\tau_{ab}} \\ &= \sum_b \frac{3}{2}\frac{(T_b/T_a) - 1}{\tau_{ab}} p_a \\ &= -C^1_{10a.10a} p_a \end{aligned} \tag{A.145}$$

where

$$\sqrt{g}C^1_{10a.10a} = \sqrt{g}\frac{3}{2}\sum_b \frac{1 - (T_b/T_a)}{\tau_{ab}} \tag{A.146}$$

Source term

$$\sqrt{g}S_{11a.11a} = \sqrt{g}S_{pa} \tag{A.147}$$

Equation for pressure in advection-diffusion form

$$
\begin{aligned}
&\frac{1}{\sqrt{g}}\frac{\partial}{\partial t}\left(\sqrt{g}M_{11a.11a}p_a\right) + \nabla\cdot\left(\boldsymbol{V}^1_{11a.11a}p_a\right) \\
&\quad - \nabla\cdot\left[\left(\overleftrightarrow{V}^2_{11a.09a.01}\cdot\nabla B\right)n_a u_{a\zeta}\right] - \nabla\cdot\left[\left(\overleftrightarrow{V}^2_{11a.10a.01}\cdot\nabla B\right)n_a u_a^\chi\right] \\
&\qquad - \nabla\cdot\left[\left(\overleftrightarrow{V}^2_{11a.14a.01}\cdot\nabla B\right)Q_{a\zeta}\right] - \nabla\cdot\left[\left(\overleftrightarrow{V}^2_{11a.15a.01}\cdot\nabla B\right)Q_a^\chi\right] \\
&\quad - \nabla\cdot\left(\overleftrightarrow{D}_{11a.06a}\cdot\nabla n_a\right) - \nabla\cdot\left[\overleftrightarrow{D}_{11a.08a}\cdot\nabla\left(n_a u_{a\parallel}\right)\right] \\
&\qquad - \nabla\cdot\left(\overleftrightarrow{D}_{11a.11a}\cdot\nabla p_a\right) - \nabla\cdot\left(\overleftrightarrow{D}_{11a.13a}\cdot\nabla Q_{a\parallel}\right) \\
&\quad + \boldsymbol{A}^1_{11a.11a}\cdot\nabla p_a \\
&\quad + \left(\nabla B\cdot\overleftrightarrow{A}^2_{11a.06a.01}\right)\cdot\nabla n_a + \left(\nabla u_{a\parallel}\cdot\overleftrightarrow{A}^2_{11a.06a.03a}\right)\cdot\nabla n_a \\
&\qquad + \left(\nabla B\cdot\overleftrightarrow{A}^2_{11a.08a.01}\right)\cdot\nabla\left(n_a u_{a\parallel}\right) + \left(\nabla u_{a\parallel}\cdot\overleftrightarrow{A}^2_{11a.08a.03a}\right)\cdot\nabla\left(n_a u_{a\parallel}\right) \\
&\qquad + \left(\nabla B\cdot\overleftrightarrow{A}^2_{11a.11a.01}\right)\cdot\nabla p_a + \left(\nabla u_{a\parallel}\cdot\overleftrightarrow{A}^2_{11a.11a.03a}\right)\cdot\nabla p_a \\
&\qquad + \left(\nabla B\cdot\overleftrightarrow{A}^2_{11a.13a.01}\right)\cdot\nabla Q_{a\parallel} + \left(\nabla u_{a\parallel}\cdot\overleftrightarrow{A}^2_{11a.13a.03a}\right)\cdot\nabla Q_{a\parallel} \\
&\quad + C^1_{11a.11a}p_a \\
&\quad + \left(\nabla B\cdot\overleftrightarrow{C}^3_{11a.09a.01.01}\cdot\nabla B\right)n_a u_{a\zeta} + \left(\nabla u_{a\parallel}\cdot\overleftrightarrow{C}^3_{11a.10a.03a.01}\cdot\nabla B\right)n_a u_{a\zeta} \\
&\qquad + \left(\nabla B\cdot\overleftrightarrow{C}^3_{11a.10a.01.01}\cdot\nabla B\right)n_a u_a^\chi + \left(\nabla u_{a\parallel}\cdot\overleftrightarrow{C}^3_{10a.10a.03a.01}\cdot\nabla B\right)n_a u_a^\chi \\
&\qquad + \left(\nabla B\cdot\overleftrightarrow{C}^3_{10a.14a.01.01}\cdot\nabla B\right)Q_{a\zeta} + \left(\nabla u_{a\parallel}\cdot\overleftrightarrow{C}^3_{10a.14a.03a.01}\cdot\nabla B\right)Q_{a\zeta} \\
&\qquad + \left(\nabla B\cdot\overleftrightarrow{C}^3_{10a.15a.01.01}\cdot\nabla B\right)Q_a^\chi + \left(\nabla u_{a\parallel}\cdot\overleftrightarrow{C}^3_{10a.15a.03a.01}\cdot\nabla B\right)Q_a^\chi \\
&= S_{11a.11a}
\end{aligned}
\tag{A.148}
$$

A.13 Equation for contravariant radial total heat flux

$$\nabla\sigma\cdot\nabla\left(\frac{5}{2}\frac{T_a}{m_a}p_a\right)=\frac{5}{2}\frac{T_a}{m_a}e_an_aE^\sigma+\frac{e_a}{m_a}\frac{IB}{\psi'}Q_{a\|}-\frac{e_a}{m_a}\frac{B^2}{\psi'}Q_{a\zeta}$$
$$B^\chi\nabla\sigma\cdot\nabla\left(\frac{5}{2}\frac{T_a}{m_a}p_a\right)=\frac{5}{2}\frac{T_a}{m_a}e_an_aE^\sigma B^\chi+\frac{e_a}{m_a}\frac{IB}{\psi'}Q_{a\|}B^\chi-\frac{e_a}{m_a}\frac{B^2}{\psi'}Q_{a\zeta}B^\chi \tag{A.149}$$

EW pressure gradient

$$\begin{aligned}G_a^{\nabla p\ \sigma}&=B^\chi\frac{5}{2}\nabla\sigma\cdot\nabla\left(\frac{p_a^2}{m_an_a}\right)\\&=-\frac{5}{2}\frac{T_a^2}{m_a}B^\chi\nabla\sigma\cdot\nabla n_a+5\frac{T_a}{m_a}B^\chi\nabla\sigma\cdot\nabla p_a\\&=\boldsymbol{A}^1_{12a.06a}\cdot\nabla n_a+\boldsymbol{A}^1_{12a.11a}\cdot\nabla p_a,\end{aligned} \tag{A.150}$$

where

$$\sqrt{g}\boldsymbol{A}^1_{12a.06a}=\begin{pmatrix}-\sqrt{g}\frac{5}{2}\frac{T_a^2}{m_a}g^{\sigma\sigma}B^\chi & -\sqrt{g}\frac{5}{2}\frac{T_a^2}{m_a}g^{\sigma\chi}B^\chi\end{pmatrix} \tag{A.151}$$
$$\sqrt{g}\boldsymbol{A}^1_{12a.11a}=\begin{pmatrix}\sqrt{g}\frac{5T_a}{m_a}g^{\sigma\sigma}B^\chi & \sqrt{g}\frac{5T_a}{m_a}g^{\sigma\chi}B^\chi\end{pmatrix} \tag{A.152}$$

EW Lorentz force

$$\begin{aligned}G_a^{\mathrm{Lor}\ \sigma}&=\frac{5}{2}\frac{T_a}{m_a}e_an_aE^\sigma B^\chi+\frac{e_a}{m_a}\frac{IB}{\psi'}Q_{a\|}B^\chi-\frac{e_a}{m_a}\frac{B^2}{\psi'}Q_{a\zeta}B^\chi\\&=\frac{5}{2}\frac{e_a}{m_a}p_aB^\chi\left(g^{\sigma\chi}\sigma\bar{E}_\chi+g^{\sigma\sigma}E_\sigma\right)+\frac{e_a}{m_a}B^\chi\frac{IB}{\psi'}Q_{a\|}-\frac{e_a}{m_a}B^\chi\frac{B^2}{\psi'}Q_{a\zeta}\\&=-C^1_{12a.04}\bar{E}_\chi-C^1_{12a.05}E_\sigma-C^1_{12a.13a}Q_{a\|}-C^1_{12a.14a}Q_{a\zeta},\end{aligned} \tag{A.153}$$

where

$$\sqrt{g}C^1_{12a.04}=-\sqrt{g}\frac{5}{2}\frac{e_a}{m_a}p_ag^{\sigma\chi}B^\chi\sigma \tag{A.154}$$
$$\sqrt{g}C^1_{12a.05}=-\sqrt{g}\frac{5}{2}e_ap_ag^{\sigma\sigma}B^\chi \tag{A.155}$$
$$\sqrt{g}C^1_{12a.13a}=-\frac{e_a}{m_a}IB \tag{A.156}$$
$$\sqrt{g}C^1_{12a.14a}=\frac{e_a}{m_a}B^2 \tag{A.157}$$

Equation for contravariant radial total heat flux in advection-diffusion form

$$\boldsymbol{A}^1_{12.06a}\cdot\nabla n_a+\boldsymbol{A}^1_{12.11a}\cdot\nabla p_a+C^1_{12a.04}\bar{E}_\chi+C^1_{12a.05}E_\sigma+C^1_{12a.13a}Q_{a\|}+C^1_{12a.14a}Q_{a\zeta}=0 \tag{A.158}$$

A.14 Equation for parallel total heat flux

$$\frac{\partial}{\partial t}\left(Q_{a\|}B\right)\Big|_{\boldsymbol{x}} + G^{\text{ine}}_{a\|}B + G^{\nabla p}_{a\|}B + G^{\text{vis}}_{a\|}B = G^{\text{Lor}}_{a\|}B + G^{\text{fri}}_{a\|}B + S_{qa}B \tag{A.159}$$

Inertial term

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(Q_{a\|}B\right)\Big|_{\boldsymbol{x}} + G^{\text{ine}}_{a\|}B \\
&\quad = \frac{1}{\sqrt{g}}\frac{\partial}{\partial t}\left(\sqrt{g}Q_{a\|}B\right) - \nabla\cdot\left(\boldsymbol{u}_g Q_{a\|}B\right) + B\nabla_\|\left(Q_{a\|}u_{a\|} + u_{a\|}Q_{a\|} - \frac{3}{2}p_a u_{a\|}u_{a\|}\right) \\
&\qquad - \left(Q_{a\|}u_{a\|} + u_{a\|}Q_{a\|} - \frac{3}{2}p_a u_{a\|}u_{a\|}\right)\nabla_\| B \\
&\quad = \frac{1}{\sqrt{g}}\frac{\partial}{\partial t}\left(\sqrt{g}BQ_{a\|}\right) + \frac{1}{\sqrt{g}}\frac{\partial}{\partial\chi}\left[\sqrt{g}\frac{B^\chi}{n_a}\left(Q_{a\|} - \frac{3}{2}p_a u_{a\|}\right)n_a u_{a\|}\right] \\
&\qquad + \frac{1}{\sqrt{g}}\frac{\partial}{\partial\sigma}\left[\sqrt{g}\left(-u_g^\sigma B\right)Q_{a\|}\right] + \frac{1}{\sqrt{g}}\frac{\partial}{\partial\chi}\left[\sqrt{g}\left(u_{a\|}B^\chi - u_g^\chi B\right)Q_{a\|}\right] \\
&\qquad + \left\{\frac{\partial B}{\partial\chi}\left[-\frac{B^\chi}{B}\frac{1}{n_a}\left(Q_{a\|} - \frac{3}{2}p_a u_{a\|}\right)\right]\right\}n_a u_{a\|} + \left[\frac{\partial B}{\partial\chi}\left(-\frac{B^\chi}{B}u_{a\|}\right)\right]Q_{a\|} \\
&= \frac{1}{\sqrt{g}}\frac{\partial}{\partial t}\left(\sqrt{g}M_{13a.13a}Q_{a\|}\right) + \nabla\cdot\left(\boldsymbol{V}^1_{13a.08a}n_a u_{a\|}\right) + \nabla\cdot\left(\boldsymbol{V}^1_{13a.13a}Q_{a\|}\right) \\
&\qquad + \left(\nabla B\cdot\boldsymbol{C}^2_{13a.08a.01}\right)n_a u_{a\|} + \left(\nabla B\cdot\boldsymbol{C}^2_{13a.13a.01}\right)Q_{a\|},
\end{aligned} \tag{A.160}$$

where

$$\sqrt{g}M_{13a.13a} = \sqrt{g}B \tag{A.161}$$

$$\sqrt{g}\boldsymbol{V}^1_{13a.08a} = \begin{pmatrix} 0 & \dfrac{\sqrt{g}}{n_a}\left(Q_{a\|} - \dfrac{3}{2}p_a u_{a\|}\right)B^\chi \end{pmatrix} \tag{A.162}$$

$$\sqrt{g}\boldsymbol{V}^1_{13a.13a} = \begin{pmatrix} -\sqrt{g}u_g^\sigma B & \sqrt{g}\left(u_{a\|}B^\chi - u_g^\chi B\right) \end{pmatrix} \tag{A.163}$$

$$\sqrt{g}\boldsymbol{C}^2_{13a.08a.01} = \begin{pmatrix} 0 & -\dfrac{\sqrt{g}}{n_a}\left(Q_{a\|} - \dfrac{3}{2}p_a u_{a\|}\right)\dfrac{B^\chi}{B} \end{pmatrix} \tag{A.164}$$

$$\sqrt{g}\boldsymbol{C}^2_{13a.13a.01} = \begin{pmatrix} 0 & -\sqrt{g}u_{a\|}\dfrac{B^\chi}{B} \end{pmatrix} \tag{A.165}$$

EW pressure gradient term

$$G^{\nabla p}_{a\|} = B\nabla_\|\left(\frac{5T_a}{2m_a}p_a\right) = \frac{1}{\sqrt{g}}\frac{\partial}{\partial\chi}\left[\sqrt{g}\left(B^\chi\frac{5T_a}{2m_a}\right)p_a\right] = \nabla\cdot\left(\boldsymbol{V}^1_{13a.11a}p_a\right) \tag{A.166}$$

where

$$\sqrt{g}\boldsymbol{V}^1_{13a.11a} = \begin{pmatrix} 0 & \sqrt{g}\dfrac{5}{2}\dfrac{T_a}{m_a}B^\chi \end{pmatrix} \tag{A.167}$$

EW viscous force

$$
\begin{aligned}
G_{a\parallel}^{\mathrm{vis}} B &= \left(-r_{\parallel a} \nabla_\parallel B + \frac{2}{3} B \nabla_\parallel r_{\parallel a} \right) \\
&= \frac{\partial B}{\partial \chi} \left[-\frac{3\bar{\mu}_{03a} u_{a\parallel}}{n_a} \frac{T_a}{m_a} \left(\frac{B^\chi}{B} \right)^2 \right] \frac{\partial n_a}{\partial \chi} + \frac{\partial B}{\partial \chi} \left[\frac{3\bar{\mu}_{03a}}{n_a} \frac{T_a}{m_a} \left(\frac{B^\chi}{B} \right)^2 \right] \frac{\partial}{\partial \chi} \left(n_a u_{a\parallel} \right) \\
&\quad + \frac{\partial B}{\partial \chi} \left[\frac{3\bar{\mu}_{03a}}{n_a} \frac{T_a}{m_a} \frac{B_t^2}{B^3} \frac{B^\chi}{I} \frac{B^\chi}{B} \right] \frac{\partial B}{\partial \chi} n_a u_{a\zeta} + \frac{\partial B}{\partial \chi} \left[-\frac{3\bar{\mu}_{03a}}{n_a} \frac{T_a}{m_a} \frac{B_t^2}{B^3} \frac{B^\chi}{B} \right] \frac{\partial B}{\partial \chi} n_a u_a^\chi \\
&\quad + \frac{\partial B}{\partial \chi} \left[-\frac{3\bar{\mu}_{04a} w_{a\parallel}}{p_a} \frac{T_a}{m_a} \left(\frac{B^\chi}{B} \right)^2 \right] \frac{\partial p_a}{\partial \chi} + \frac{\partial B}{\partial \chi} \left[\frac{3\bar{\mu}_{04a}}{p_a} \frac{T_a}{m_a} \left(\frac{B^\chi}{B} \right)^2 \right] \frac{\partial Q_{a\parallel}}{\partial \chi} \\
&\quad + \frac{\partial B}{\partial \chi} \left[\frac{3\bar{\mu}_{04a}}{p_a} \frac{T_a}{m_a} \frac{B_t^2}{B^3} \frac{B^\chi}{I} \frac{B^\chi}{B} \right] \frac{\partial B}{\partial \chi} Q_{a\zeta} + \frac{\partial B}{\partial \chi} \left[-\frac{3\bar{\mu}_{04a}}{p_a} \frac{T_a}{m_a} \frac{B_t^2}{B^3} \frac{B^\chi}{B} \right] \frac{\partial B}{\partial \chi} Q_a^\chi \\
&\quad - \frac{1}{\sqrt{g}} \frac{\partial}{\partial \chi} \left[\sqrt{g} \left\{ -\frac{2\bar{\mu}_{03a} u_{a\parallel} B}{n_a} \frac{T_a}{m_a} \left(\frac{B^\chi}{B} \right)^2 \right\} \frac{\partial n_a}{\partial \chi} \right] \\
&\quad - \frac{1}{\sqrt{g}} \frac{\partial}{\partial \chi} \left[\sqrt{g} \left\{ \frac{2\bar{\mu}_{03a} B}{n_a} \frac{T_a}{m_a} \left(\frac{B^\chi}{B} \right)^2 \right\} \frac{\partial}{\partial \chi} \left(n_a u_{a\parallel} \right) \right] \\
&\quad - \frac{1}{\sqrt{g}} \frac{\partial}{\partial \chi} \left[\sqrt{g} \left(\frac{2\bar{\mu}_{03a}}{n_a} \frac{T_a}{m_a} \frac{B_t^2}{B^2} \frac{B^\chi}{B} \frac{B^\chi}{I} \right) \frac{\partial B}{\partial \chi} n_a u_{a\zeta} \right] \\
&\quad - \frac{1}{\sqrt{g}} \frac{\partial}{\partial \chi} \left[\sqrt{g} \left(-\frac{2\bar{\mu}_{03a}}{n_a} \frac{T_a}{m_a} \frac{B_t^2}{B^2} \frac{B^\chi}{B} \right) \frac{\partial B}{\partial \chi} n_a u_a^\chi \right] \\
&\quad - \frac{1}{\sqrt{g}} \frac{\partial}{\partial \chi} \left[\sqrt{g} \left\{ -\frac{2\bar{\mu}_{04a} w_{a\parallel} B}{p_a} \frac{T_a}{m_a} \left(\frac{B^\chi}{B} \right)^2 \right\} \frac{\partial p_a}{\partial \chi} \right] \\
&\quad - \frac{1}{\sqrt{g}} \frac{\partial}{\partial \chi} \left[\sqrt{g} \left\{ \frac{2\bar{\mu}_{04a} B}{p_a} \frac{T_a}{m_a} \left(\frac{B^\chi}{B} \right)^2 \right\} \frac{\partial Q_{a\parallel}}{\partial \chi} \right] \\
&\quad - \frac{1}{\sqrt{g}} \frac{\partial}{\partial \chi} \left[\sqrt{g} \left(\frac{2\bar{\mu}_{04a}}{p_a} \frac{T_a}{m_a} \frac{B_t^2}{B^2} \frac{B^\chi}{B} \frac{B^\chi}{I} \right) \frac{\partial B}{\partial \chi} Q_{a\zeta} \right] \\
&\quad - \frac{1}{\sqrt{g}} \frac{\partial}{\partial \chi} \left[\sqrt{g} \left(-\frac{2\bar{\mu}_{04a}}{p_a} \frac{T_a}{m_a} \frac{B_t^2}{B^2} \frac{B^\chi}{B} \right) \frac{\partial B}{\partial \chi} Q_a^\chi \right]
\end{aligned}
\tag{A.168}
$$

$$
\begin{aligned}
G^{\mathrm{vis}}_{a\|}B = &-\nabla\cdot\left[\left(\nabla B\cdot\overleftrightarrow{V}^2_{13a.09a.01}\right)n_a u_{a\zeta}\right] - \nabla\cdot\left[\left(\nabla B\cdot\overleftrightarrow{V}^2_{13a.10a.01}\right)n_a u^{\chi}_a\right] \\
&- \nabla\cdot\left[\left(\nabla B\cdot\overleftrightarrow{V}^2_{13a.14a.01}\right)Q_{a\zeta}\right] - \nabla\cdot\left[\left(\nabla B\cdot\overleftrightarrow{V}^2_{13a.15a.01}\right)Q^{\chi}_a\right] \\
&- \nabla\cdot\left(\overleftrightarrow{D}_{13a.06a}\cdot\nabla n_a\right) - \nabla\cdot\left[\overleftrightarrow{D}_{13a.08a}\cdot\nabla\left(n_a u_{a\|}\right)\right] \\
&- \nabla\cdot\left(\overleftrightarrow{D}_{13a.11a}\cdot\nabla p_a\right) - \nabla\cdot\left(\overleftrightarrow{D}_{13a.13a}\cdot\nabla Q_{a\|}\right) \\
&+ \left(\nabla B\cdot\overleftrightarrow{A}^2_{13a.06a.01}\right)\cdot\nabla n_a + \left(\nabla B\cdot\overleftrightarrow{A}^2_{13a.08a.01}\right)\cdot\nabla\left(n_a u_{a\|}\right) \\
&+ \left(\nabla B\cdot\overleftrightarrow{A}^2_{13a.11a.01}\right)\cdot\nabla p_a + \left(\nabla B\cdot\overleftrightarrow{A}^2_{13a.13a.01}\right)\cdot\nabla Q_{a\|} \\
&+ \left(\nabla B\cdot\overleftrightarrow{C}^3_{13a.09a.01.01}\cdot\nabla B\right)n_a u_{a\zeta} + \left(\nabla B\cdot\overleftrightarrow{C}^3_{13a.10a.01.01}\cdot\nabla B\right)n_a u^{\chi}_a \\
&+ \left(\nabla B\cdot\overleftrightarrow{C}^3_{13a.14a.01.01}\cdot\nabla B\right)Q_{a\zeta} + \left(\nabla B\cdot\overleftrightarrow{C}^3_{13a.15a.01.01}\cdot\nabla B\right)Q^{\chi}_a
\end{aligned}
\tag{A.169}
$$

where

$$
\sqrt{g}\overleftrightarrow{V}^2_{13a.09a.01} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{g}\dfrac{2\bar{\mu}_{03a}}{n_a}\dfrac{T_a}{m_a}\dfrac{B_t^2}{B^2}\dfrac{B^{\chi}}{B}\dfrac{B^{\chi}}{I} \end{pmatrix} \tag{A.170}
$$

$$
\sqrt{g}\overleftrightarrow{V}^2_{13a.10a.01} = \begin{pmatrix} 0 & 0 \\ 0 & -\sqrt{g}\dfrac{2\bar{\mu}_{03a}}{n_a}\dfrac{T_a}{m_a}\dfrac{B_t^2}{B^2}\dfrac{B^{\chi}}{B} \end{pmatrix} \tag{A.171}
$$

$$
\sqrt{g}\overleftrightarrow{V}^2_{13a.14a.01} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{g}\dfrac{2\bar{\mu}_{04a}}{p_a}\dfrac{T_a}{m_a}\dfrac{B_t^2}{B^2}\dfrac{B^{\chi}}{B}\dfrac{B^{\chi}}{I} \end{pmatrix} \tag{A.172}
$$

$$
\sqrt{g}\overleftrightarrow{V}^2_{13a.15a.01} = \begin{pmatrix} 0 & 0 \\ 0 & -\sqrt{g}\dfrac{2\bar{\mu}_{04a}}{p_a}\dfrac{T_a}{m_a}\dfrac{B_t^2}{B^2}\dfrac{B^{\chi}}{B} \end{pmatrix} \tag{A.173}
$$

$$
\sqrt{g}\overleftrightarrow{D}_{13a.06a} = \begin{pmatrix} 0 & 0 \\ 0 & -\sqrt{g}\dfrac{2\bar{\mu}_{03a}u_{a\|}B}{n_a}\dfrac{T_a}{m_a}\left(\dfrac{B^{\chi}}{B}\right)^2 \end{pmatrix} \tag{A.174}
$$

$$
\sqrt{g}\overleftrightarrow{D}_{13a.08a} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{g}\dfrac{2\bar{\mu}_{03a}B}{n_a}\dfrac{T_a}{m_a}\left(\dfrac{B^{\chi}}{B}\right)^2 \end{pmatrix} \tag{A.175}
$$

$$
\sqrt{g}\overleftrightarrow{D}_{13a.11a} = \begin{pmatrix} 0 & 0 \\ 0 & -\sqrt{g}\dfrac{2\bar{\mu}_{04a}w_{a\|}B}{p_a}\dfrac{T_a}{m_a}\left(\dfrac{B^{\chi}}{B}\right)^2 \end{pmatrix} \tag{A.176}
$$

$$
\sqrt{g}\overleftrightarrow{D}_{13a.13a} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{g}\dfrac{2\bar{\mu}_{04a}B}{p_a}\dfrac{T_a}{m_a}\left(\dfrac{B^{\chi}}{B}\right)^2 \end{pmatrix} \tag{A.177}
$$

$$\sqrt{g}\overleftrightarrow{A}^{2}_{13a.06a.01} = \begin{pmatrix} 0 & 0 \\ 0 & -\sqrt{g}\frac{3\bar{\mu}_{03a}u_{a\|}}{n_a}\frac{T_a}{m_a}\left(\frac{B^\chi}{B}\right)^2 \end{pmatrix} \tag{A.178}$$

$$\sqrt{g}\overleftrightarrow{A}^{2}_{13a.08a.01} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{g}\frac{3\bar{\mu}_{03a}}{n_a}\frac{T_a}{m_a}\left(\frac{B^\chi}{B}\right)^2 \end{pmatrix} \tag{A.179}$$

$$\sqrt{g}\overleftrightarrow{A}^{2}_{13a.11a.01} = \begin{pmatrix} 0 & 0 \\ 0 & -\sqrt{g}\frac{3\bar{\mu}_{04a}w_{a\|}}{p_a}\frac{T_a}{m_a}\left(\frac{B^\chi}{B}\right)^2 \end{pmatrix} \tag{A.180}$$

$$\sqrt{g}\overleftrightarrow{A}^{2}_{13a.13a.01} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{g}\frac{3\bar{\mu}_{04a}}{p_a}\frac{T_a}{m_a}\left(\frac{B^\chi}{B}\right)^2 \end{pmatrix} \tag{A.181}$$

$$\sqrt{g}\overleftrightarrow{C}^{3}_{13a.09a.01.01} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{g}\frac{3\bar{\mu}_{03a}}{n_a}\frac{T_a}{m_a}\frac{B_t^2}{B^3}\frac{B^\chi}{I}\frac{B^\chi}{B} \end{pmatrix} \tag{A.182}$$

$$\sqrt{g}\overleftrightarrow{C}^{3}_{13a.10a.01.01} = \begin{pmatrix} 0 & 0 \\ 0 & -\sqrt{g}\frac{3\bar{\mu}_{03a}}{n_a}\frac{T_a}{m_a}\frac{B_t^2}{B^3}\frac{B^\chi}{B} \end{pmatrix} \tag{A.183}$$

$$\sqrt{g}\overleftrightarrow{C}^{3}_{13a.14a.01.01} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{g}\frac{3\bar{\mu}_{04a}}{p_a}\frac{T_a}{m_a}\frac{B_t^2}{B^3}\frac{B^\chi}{I}\frac{B^\chi}{B} \end{pmatrix} \tag{A.184}$$

$$\sqrt{g}\overleftrightarrow{C}^{3}_{13a.15a.01.01} = \begin{pmatrix} 0 & 0 \\ 0 & -\sqrt{g}\frac{3\bar{\mu}_{04a}}{p_a}\frac{T_a}{m_a}\frac{B_t^2}{B^3}\frac{B^\chi}{B} \end{pmatrix} \tag{A.185}$$

EW Lorentz force

$$
\begin{aligned}
G^{\mathrm{Lor}}_{a\|} B = {} & \frac{5e_a}{2m_a} p_a \left(B^\zeta E_\zeta + B^\chi E_\chi\right) \\
& - \frac{3\bar{\mu}_{01a} e_a}{m_a} \left[\frac{B^\chi}{B}\frac{\partial u_{a\|}}{\partial \chi} + u_a^{\dagger\chi} \frac{B_t^2}{B^3}\frac{\partial B}{\partial \chi}\right] \left(B^\zeta E_\zeta + B^\chi E_\chi\right) \\
& - \frac{3\bar{\mu}_{02a} e_a}{m_a} \left[\frac{B^\chi}{B}\frac{\partial w_{a\|}}{\partial \chi} + w_a^{\dagger\chi} \frac{B_t^2}{B^3}\frac{\partial B}{\partial \chi}\right] \left(B^\zeta E_\zeta + B^\chi E_\chi\right) \\
= {} & - \left[-\frac{5e_a}{2m_a} p_a B^\zeta\right] E_\zeta - \left[-\frac{5e_a}{2m_a} p_a B^\chi \sigma\right] \bar{E}_\chi \\
& - \frac{\partial B}{\partial \chi}\left[\frac{3e_a}{m_a}\left(\bar{\mu}_{01a} u_a^{\dagger\chi} + \bar{\mu}_{02a} w_a^{\dagger\chi}\right)\frac{B_t^2}{B^3} B^\zeta\right] E_\zeta \\
& - \frac{\partial u_{a\|}}{\partial \chi}\left[\frac{3\bar{\mu}_{01a} e_a}{m_a}\frac{B^\chi}{B} B^\zeta\right] E_\zeta - \frac{\partial w_{a\|}}{\partial \chi}\left[\frac{3\bar{\mu}_{02a} e_a}{m_a}\frac{B^\chi}{B} B^\zeta\right] E_\zeta \\
& - \frac{\partial B}{\partial \chi}\left[\frac{3e_a}{m_a}\left(\bar{\mu}_{01a} u_a^{\dagger\chi} + \bar{\mu}_{02a} w_a^{\dagger\chi}\right)\frac{B_t^2}{B^3} B^\chi \sigma\right] \bar{E}_\chi \\
& - \frac{\partial u_{a\|}}{\partial \chi}\left[\frac{3\bar{\mu}_{01a} e_a}{m_a}\frac{B^\chi}{B} B^\chi \sigma\right] \bar{E}_\chi - \frac{\partial w_{a\|}}{\partial \chi}\left[\frac{3\bar{\mu}_{02a} e_a}{m_a}\frac{B^\chi}{B} B^\chi \sigma\right] \bar{E}_\chi \\
= {} & -C^1_{13a.03} E_\zeta - C^1_{13a.04} \bar{E}_\chi \\
& - \left[\nabla B \cdot \boldsymbol{C}^2_{13a.03.01}\right] E_\zeta - \left[\nabla u_{a\|} \cdot \boldsymbol{C}^2_{13a.03.03a}\right] E_\zeta - \left[\nabla w_{a\|} \cdot \boldsymbol{C}^2_{13a.03.04a}\right] E_\zeta \\
& - \left[\nabla B \cdot \boldsymbol{C}^2_{13a.04.01}\right] \bar{E}_\chi - \left[\nabla u_{a\|} \cdot \boldsymbol{C}^2_{13a.04.03a}\right] \bar{E}_\chi - \left[\nabla w_{a\|} \cdot \boldsymbol{C}^2_{13a.04.04a}\right] \bar{E}_\chi
\end{aligned}
\tag{A.186}
$$

where

$$\sqrt{g} C^1_{13a.03} = -\sqrt{g}\frac{5 p_a e_a}{2m_a} B^\zeta \tag{A.187}$$

$$\sqrt{g} C^1_{13a.04} = -\sqrt{g}\frac{5 p_a e_a}{2m_a} B^\chi \tag{A.188}$$

$$\sqrt{g} \boldsymbol{C}^1_{13a.03.01} = \begin{pmatrix} 0 & \sqrt{g}\frac{3e_a}{m_a}\frac{B_t^2}{B^3} B^\zeta \left(\bar{\mu}_{01a} u_a^{\dagger\chi} + \bar{\mu}_{02a} w_a^{\dagger\chi}\right) \end{pmatrix} \tag{A.189}$$

$$\sqrt{g} \boldsymbol{C}^1_{13a.03.03a} = \begin{pmatrix} 0 & \sqrt{g}\frac{3e_a}{m_a}\frac{B^\chi}{B} B^\zeta \bar{\mu}_{01a} \end{pmatrix} \tag{A.190}$$

$$\sqrt{g} \boldsymbol{C}^1_{13a.03.04a} = \begin{pmatrix} 0 & \sqrt{g}\frac{3e_a}{m_a}\frac{B^\chi}{B} B^\zeta \bar{\mu}_{02a} \end{pmatrix} \tag{A.191}$$

$$\sqrt{g} \boldsymbol{C}^1_{13a.04.01} = \begin{pmatrix} 0 & \sqrt{g}\frac{3e_a}{m_a}\frac{B_t^2}{B^3} B^\chi \sigma \left(\bar{\mu}_{01a} u_a^{\dagger\chi} + \bar{\mu}_{02a} w_a^{\dagger\chi}\right) \end{pmatrix} \tag{A.192}$$

$$\sqrt{g} \boldsymbol{C}^1_{13a.04.03a} = \begin{pmatrix} 0 & \sqrt{g}\frac{3e_a}{m_a}\frac{B^\chi}{B} B^\chi \sigma \bar{\mu}_{01a} \end{pmatrix} \tag{A.193}$$

$$\sqrt{g} \boldsymbol{C}^1_{13a.04.04a} = \begin{pmatrix} 0 & \sqrt{g}\frac{3e_a}{m_a}\frac{B^\chi}{B} B^\chi \sigma \bar{\mu}_{02a} \end{pmatrix} \tag{A.194}$$

EW friction force

$$
\begin{aligned}
G^{\text{fri}}_{a\|}B &= \sum_b \frac{T_a}{m_a}\frac{\bar{l}_{03ab}}{n_b} n_b u_{b\|} B + \sum_b \frac{T_a}{m_a}\frac{\bar{l}_{04ab}}{p_b} Q_{b\|} B \\
&= -\sum_b \left[-\frac{T_a}{m_a}\frac{\bar{l}_{03ab}B}{n_b}\right] n_b u_{b\|} - \sum_b \left[-\frac{T_a}{m_a}\frac{\bar{l}_{04ab}B}{p_b}\right] Q_{b\|} \\
&= -\sum_b C^1_{13a.08b} n_b u_{b\|} - \sum_b C^1_{13a.13b} Q_{b\|},
\end{aligned} \tag{A.195}
$$

where

$$
\sqrt{g}C^1_{13a.08b} = -\sqrt{g}\frac{T_a}{m_a}\frac{\bar{l}_{03ab}B}{n_b} \tag{A.196}
$$

$$
\sqrt{g}C^1_{13a.13b} = -\sqrt{g}\frac{T_a}{m_a}\frac{\bar{l}_{04ab}B}{p_b} \tag{A.197}
$$

Source term

$$
\sqrt{g}S_{13a.13a} = \sqrt{g}S_{qa\|}B \tag{A.198}
$$

Equation for parallel total heat flux in advection-diffusion form

$$
\begin{aligned}
&\frac{1}{\sqrt{g}}\frac{\partial}{\partial t}\left(\sqrt{g}M_{13a.13a}Q_{a\|}\right) + \nabla\cdot\left(\boldsymbol{V}^1_{13a.08a}n_a u_{a\|}\right) + \nabla\cdot\left(\boldsymbol{V}^1_{13a.11a}p_a\right) + \nabla\cdot\left(\boldsymbol{V}^1_{13a.13a}Q_{a\|}\right) \\
&\quad - \nabla\cdot\left[\left(\nabla B\cdot \overleftrightarrow{V}^2_{13a.09a.01}\right) n_a u_{a\zeta}\right] - \nabla\cdot\left[\left(\nabla B\cdot \overleftrightarrow{V}^2_{13a.10a.01}\right) n_a u^\chi_a\right] \\
&\qquad - \nabla\cdot\left[\left(\nabla B\cdot \overleftrightarrow{V}^2_{13a.14a.01}\right) Q_{a\zeta}\right] - \nabla\cdot\left[\left(\nabla B\cdot \overleftrightarrow{V}^2_{13a.15a.01}\right) Q^\chi_a\right] \\
&\quad - \nabla\cdot\left(\overleftrightarrow{D}_{13a.06a}\cdot\nabla n_a\right) - \nabla\cdot\left[\overleftrightarrow{D}_{13a.08a}\cdot\nabla\left(n_a u_{a\|}\right)\right] \\
&\qquad - \nabla\cdot\left(\overleftrightarrow{D}_{13a.11a}\cdot\nabla p_a\right) - \nabla\cdot\left(\overleftrightarrow{D}_{13a.13a}\cdot\nabla Q_{a\|}\right) \\
&\quad + \left(\nabla B\cdot\overleftrightarrow{A}^2_{13a.06a.01}\right)\cdot\nabla n_a + \left(\nabla B\cdot\overleftrightarrow{A}^2_{13a.08a.01}\right)\cdot\nabla\left(n_a u_{a\|}\right) \\
&\qquad + \left(\nabla B\cdot\overleftrightarrow{A}^2_{13a.11a.01}\right)\cdot\nabla p_a + \left(\nabla B\cdot\overleftrightarrow{A}^2_{13a.13a.01}\right)\cdot\nabla Q_{a\|} \\
&\quad + C^1_{13a.03}E_\zeta + C^1_{13a.04}\bar{E}_\chi + \sum_b C^1_{13a.08b}n_b u_{b\|} + \sum_b C^1_{13a.13b}Q_{b\|}, \\
&\quad + \left(\nabla B\cdot\boldsymbol{C}^2_{13a.03.01}\right)E_\zeta + \left(\nabla u_{a\|}\cdot\boldsymbol{C}^2_{13a.03.03a}\right)E_\zeta + \left(\nabla w_{a\|}\cdot\boldsymbol{C}^2_{13a.03.04a}\right)E_\zeta \\
&\qquad + \left(\nabla B\cdot\boldsymbol{C}^2_{13a.04.01}\right)\bar{E}_\chi + \left(\nabla u_{a\|}\cdot\boldsymbol{C}^2_{13a.04.03a}\right)\bar{E}_\chi + \left(\nabla w_{a\|}\cdot\boldsymbol{C}^2_{13a.04.04a}\right)\bar{E}_\chi \\
&\quad + \left(\nabla B\cdot\boldsymbol{C}^2_{13a.08a.01}\right)n_a u_{a\|} + \left(\nabla B\cdot\boldsymbol{C}^2_{13a.13a.01}\right)Q_{a\|} \\
&\quad + \left(\nabla B\cdot\overleftrightarrow{C}^3_{13a.09a.01.01}\cdot\nabla B\right)n_a u_{a\zeta} + \left(\nabla B\cdot\overleftrightarrow{C}^3_{13a.10a.01.01}\cdot\nabla B\right)n_a u^\chi_a \\
&\qquad + \left(\nabla B\cdot\overleftrightarrow{C}^3_{13a.14a.01.01}\cdot\nabla B\right)Q_{a\zeta} + \left(\nabla B\cdot\overleftrightarrow{C}^3_{13a.15a.01.01}\cdot\nabla B\right)Q^\chi_a \\
&= S_{13a.13a}
\end{aligned} \tag{A.199}
$$

A.15 Equation for covariant toroidal total heat flux

$$\left.\frac{\partial Q_{a\zeta}}{\partial t}\right|_{\boldsymbol{x}} + G_{a\zeta}^{\rm ine} + G_{a\zeta}^{\rm vis} = G_{a\zeta}^{\rm Lor} + G_{a\zeta}^{\rm fri} + S_{qa\zeta} \tag{A.200}$$

EW inertial term

$$\begin{aligned}\left.\frac{\partial Q_{a\zeta}}{\partial t}\right|_{\boldsymbol{x}} + G_{a\zeta}^{\rm fri} &= \frac{1}{\sqrt{g}}\frac{\partial}{\partial t}\left(\sqrt{g}Q_{a\zeta}\right) - \nabla\cdot\left(\boldsymbol{u}_g Q_{a\zeta}\right) + \nabla\cdot\left(Q_\zeta \boldsymbol{u}_a + u_{a\zeta}\boldsymbol{Q}_a - \frac{3}{2}p_a u_{a\zeta}\boldsymbol{u}_a\right)\\ &= \frac{1}{\sqrt{g}}\frac{\partial}{\partial t}\left(\sqrt{g}Q_{a\zeta}\right) + \nabla\cdot\left[\frac{1}{n_a}\left(\boldsymbol{Q}_a - \frac{3}{2}p_a n_a \boldsymbol{u}_a\right) n_a u_{a\zeta}\right] + \nabla\cdot\left[\left(\boldsymbol{u}_a - \boldsymbol{u}_g\right) Q_{a\zeta}\right]\\ &= \frac{1}{\sqrt{g}}\frac{\partial}{\partial t}\left(\sqrt{g}M_{14a.14a}Q_{a\zeta}\right) + \nabla\cdot\left(\boldsymbol{V}^1_{14a.09a} n_a u_{a\zeta}\right) + \nabla\cdot\left(\boldsymbol{V}^1_{14a.14a} Q_{a\zeta}\right)\end{aligned} \tag{A.201}$$

where

$$\sqrt{g}M_{14a.14a} = \sqrt{g} \tag{A.202}$$

$$\sqrt{g}\boldsymbol{V}^1_{14a.09a} = \begin{pmatrix} \dfrac{\sqrt{g}}{n_a}\left(Q_a^\sigma - \dfrac{3}{2}p_a u_a^\sigma\right) & \dfrac{\sqrt{g}}{n_a}\left(Q_a^\chi - \dfrac{3}{2}p_a u_a^\chi\right)\end{pmatrix} \tag{A.203}$$

$$\sqrt{g}\boldsymbol{V}^1_{14a.14a} = \begin{pmatrix} \sqrt{g}\left(u_a^\sigma - u_g^\sigma\right) & \sqrt{g}\left(u_a^\chi - u_g^\chi\right)\end{pmatrix} \tag{A.204}$$

EW viscous force

$$\begin{aligned}G_{a\zeta}^{\rm vis} = B\nabla_\parallel\left(\frac{I}{B^2}r_{\parallel a}\right) = &-\frac{1}{\sqrt{g}}\frac{\partial}{\partial\chi}\left\{\sqrt{g}\left[-\frac{3\bar{\mu}_{03a}u_{a\parallel}}{n_a}\frac{T_a}{m_a}\frac{I}{B}\left(\frac{B^\chi}{B}\right)^2\right]\frac{\partial n_a}{\partial\chi}\right\}\\ &-\frac{1}{\sqrt{g}}\frac{\partial}{\partial\chi}\left\{\sqrt{g}\left[\frac{3\bar{\mu}_{03a}}{n_a}\frac{T_a}{m_a}\left(\frac{B^\chi}{B}\right)^2\frac{I}{B}\right]\frac{\partial}{\partial\chi}\left(n_a u_{a\parallel}\right)\right\}\\ &-\frac{1}{\sqrt{g}}\frac{\partial}{\partial\chi}\left\{\sqrt{g}\left[\frac{3\bar{\mu}_{03a}}{n_a}\frac{T_a}{m_a}\frac{B_t^2}{B^3}\left(\frac{B^\chi}{B}\right)^2\right]\frac{\partial B}{\partial\chi}n_a u_{a\zeta}\right\}\\ &-\frac{1}{\sqrt{g}}\frac{\partial}{\partial\chi}\left\{\sqrt{g}\left[-\frac{3\bar{\mu}_{03a}}{n_a}\frac{T_a}{m_a}\frac{B_t^2}{B^3}\frac{B^\chi}{B}\frac{I}{B}\right]\frac{\partial B}{\partial\chi}n_a u_a^\chi\right\}\\ &-\frac{1}{\sqrt{g}}\frac{\partial}{\partial\chi}\left\{\sqrt{g}\left[-\frac{3\bar{\mu}_{04a}w_{a\parallel}}{p_a}\frac{T_a}{m_a}\left(\frac{B^\chi}{B}\right)^2\frac{I}{B}\right]\frac{\partial p_a}{\partial\chi}\right\}\\ &-\frac{1}{\sqrt{g}}\frac{\partial}{\partial\chi}\left\{\sqrt{g}\left[\frac{3\bar{\mu}_{04a}}{p_a}\frac{T_a}{m_a}\left(\frac{B^\chi}{B}\right)^2\frac{I}{B}\right]\frac{\partial Q_{a\parallel}}{\partial\chi}\right\}\\ &-\frac{1}{\sqrt{g}}\frac{\partial}{\partial\chi}\left\{\sqrt{g}\left[\frac{3\bar{\mu}_{04a}}{p_a}\frac{T_a}{m_a}\frac{B_t^2}{B^3}\left(\frac{B^\chi}{B}\right)^2\right]\frac{\partial B}{\partial\chi}Q_{a\zeta}\right\}\\ &-\frac{1}{\sqrt{g}}\frac{\partial}{\partial\chi}\left\{\sqrt{g}\left[-\frac{3\bar{\mu}_{04a}}{p_a}\frac{T_a}{m_a}\frac{B_t^2}{B^3}\frac{B^\chi}{B}\frac{I}{B}\right]\frac{\partial B}{\partial\chi}Q_a^\chi\right\}\end{aligned} \tag{A.205}$$

$$
\begin{aligned}
G_{a\zeta}^{\text{vis}} = &-\nabla\cdot\left[\left(\overleftrightarrow{V}^{2}_{14a.09a.01}\cdot\nabla B\right)n_a u_{a\zeta}\right]-\nabla\cdot\left[\left(\overleftrightarrow{V}^{2}_{14a.10a.01}\cdot\nabla B\right)n_a u_a^{\chi}\right]\\
&-\nabla\cdot\left[\left(\overleftrightarrow{V}^{2}_{14a.14a.01}\cdot\nabla B\right)Q_{a\zeta}\right]-\nabla\cdot\left[\left(\overleftrightarrow{V}^{2}_{14a.15a.01}\cdot\nabla B\right)Q_a^{\chi}\right]\\
&-\nabla\cdot\left(\overleftrightarrow{D}_{13a.06a}\cdot\nabla n_a\right)-\nabla\cdot\left[\overleftrightarrow{D}_{13a.08a}\cdot\nabla\left(n_a u_{a\|}\right)\right]\\
&-\nabla\cdot\left(\overleftrightarrow{D}_{13a.10a}\cdot\nabla p_a\right)-\nabla\cdot\left(\overleftrightarrow{D}_{13a.12a}\cdot\nabla Q_{a\|}\right),
\end{aligned}
\tag{A.206}
$$

where

$$
\sqrt{g}\overleftrightarrow{V}^{2}_{14a.09a.01} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{g}\dfrac{3\bar{\mu}_{03a}}{n_a}\dfrac{T_a}{m_a}\dfrac{B_t^2}{B^3}\left(\dfrac{B^{\chi}}{B}\right)^2 \end{pmatrix} \tag{A.207}
$$

$$
\sqrt{g}\overleftrightarrow{V}^{2}_{14a.10a.01} = \begin{pmatrix} 0 & 0 \\ 0 & -\sqrt{g}\dfrac{3\bar{\mu}_{03a}}{n_a}\dfrac{T_a}{m_a}\dfrac{B_t^2}{B^3}\dfrac{B^{\chi}}{B}\dfrac{I}{B} \end{pmatrix} \tag{A.208}
$$

$$
\sqrt{g}\overleftrightarrow{V}^{2}_{14a.14a.01} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{g}\dfrac{3\bar{\mu}_{04a}}{p_a}\dfrac{T_a}{m_a}\dfrac{B_t^2}{B^3}\left(\dfrac{B^{\chi}}{B}\right)^2 \end{pmatrix} \tag{A.209}
$$

$$
\sqrt{g}\overleftrightarrow{V}^{2}_{14a.15a.01} = \begin{pmatrix} 0 & 0 \\ 0 & -\sqrt{g}\dfrac{3\bar{\mu}_{04a}}{p_a}\dfrac{T_a}{m_a}\dfrac{B_t^2}{B^3}\dfrac{B^{\chi}}{B}\dfrac{I}{B} \end{pmatrix} \tag{A.210}
$$

$$
\sqrt{g}\overleftrightarrow{D}_{14a.06a} = \begin{pmatrix} 0 & 0 \\ 0 & -\sqrt{g}\dfrac{3\bar{\mu}_{03a}u_{a\|}}{n_a}\dfrac{T_a}{m_a}\left(\dfrac{B^{\chi}}{B}\right)^2\dfrac{I}{B} \end{pmatrix} \tag{A.211}
$$

$$
\sqrt{g}\overleftrightarrow{D}_{14a.08a} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{g}\dfrac{3\bar{\mu}_{03a}}{n_a}\dfrac{T_a}{m_a}\left(\dfrac{B^{\chi}}{B}\right)^2\dfrac{I}{B} \end{pmatrix} \tag{A.212}
$$

$$
\sqrt{g}\overleftrightarrow{D}_{14a.11a} = \begin{pmatrix} 0 & 0 \\ 0 & -\sqrt{g}\dfrac{3\bar{\mu}_{04a}w_{a\|}}{p_a}\dfrac{T_a}{m_a}\left(\dfrac{B^{\chi}}{B}\right)^2\dfrac{I}{B} \end{pmatrix} \tag{A.213}
$$

$$
\sqrt{g}\overleftrightarrow{D}_{14a.13a} = \begin{pmatrix} 0 & 0 \\ 0 & \sqrt{g}\dfrac{3\bar{\mu}_{04a}}{p_a}\dfrac{T_a}{m_a}\left(\dfrac{B^{\chi}}{B}\right)^2\dfrac{I}{B} \end{pmatrix} \tag{A.214}
$$

EW Lorentz force

$$
\begin{aligned}
G_{a\zeta}^{\mathrm{Lor}} &= \frac{e_a}{m_a}\left[\frac{5}{2}p_a E_\zeta + \left(\frac{B_t^2}{B^2}-\frac{1}{3}\right)\pi_{\|a}E_\zeta + \frac{I}{B}\frac{B^\chi}{B}\pi_{\|a}E_\chi + \psi' Q_a^\sigma\right] \\
&= \frac{5e_a}{2m_a}p_a E_\zeta + \frac{e_a}{m_a}\sqrt{g}B^\chi\sigma\bar{Q}_a^\sigma \\
&\quad - \frac{3e_a}{m_a}\left(\frac{B_t^2}{B^2}-\frac{1}{3}\right)\left[\bar{\mu}_{01a}\left(\frac{B^\chi}{B}\frac{\partial u_{a\|}}{\partial\chi} + u_a^{\dagger\chi}\frac{B_t^2}{B^3}\frac{\partial B}{\partial\chi}\right) + \bar{\mu}_{02a}\left(\frac{B^\chi}{B}\frac{\partial w_{a\|}}{\partial\chi} + w_a^{\dagger\chi}\frac{B_t^2}{B^3}\frac{\partial B}{\partial\chi}\right)\right]E_\zeta \\
&\quad - \frac{3e_a}{m_a}\frac{I}{B}\frac{B^\chi}{B}\left[\bar{\mu}_{01a}\left(\frac{B^\chi}{B}\frac{\partial u_{a\|}}{\partial\chi} + u_a^{\dagger\chi}\frac{B_t^2}{B^3}\frac{\partial B}{\partial\chi}\right) + \bar{\mu}_{02a}\left(\frac{B^\chi}{B}\frac{\partial w_{a\|}}{\partial\chi} + w_a^{\dagger\chi}\frac{B_t^2}{B^3}\frac{\partial B}{\partial\chi}\right)\right]\sigma\bar{E}_\chi \\
&= -\left[-\frac{5e_a}{2m_a}p_a\right]E_\zeta - \left[-\frac{e_a}{m_a}\sqrt{g}B^\chi\sigma\right]\bar{Q}_a^\sigma \\
&\quad - \frac{\partial B}{\partial\chi}\left[\frac{3e_a}{m_a}\left(\frac{B_t^2}{B^2}-\frac{1}{3}\right)\frac{B_t^2}{B^3}\left(\bar{\mu}_{01a}u_a^{\dagger\chi} + \bar{\mu}_{02a}w_a^{\dagger\chi}\right)\right]E_\zeta \\
&\quad - \frac{\partial u_{a\|}}{\partial\chi}\left[\frac{3\bar{\mu}_{01a}e_a}{m_a}\left(\frac{B_t^2}{B^2}-\frac{1}{3}\right)\frac{B^\chi}{B}\right]E_\zeta - \frac{\partial w_{a\|}}{\partial\chi}\left[\frac{3\bar{\mu}_{02a}e_a}{m_a}\left(\frac{B_t^2}{B^2}-\frac{1}{3}\right)\frac{B^\chi}{B}\right]E_\zeta \\
&\quad - \frac{\partial B}{\partial\chi}\left[\frac{3e_a}{m_a}\frac{I}{B}\frac{B^\chi}{B}\frac{B_t^2}{B^3}\left(\bar{\mu}_{01a}u_a^{\dagger\chi} + \bar{\mu}_{02a}w_a^{\dagger\chi}\right)\sigma\right]\bar{E}_\chi \\
&\quad - \frac{\partial u_{a\|}}{\partial\chi}\left[\frac{3\bar{\mu}_{01a}e_a}{m_a}\frac{I}{B}\frac{B^\chi}{B}\frac{B^\chi}{B}\sigma\right]\bar{E}_\chi - \frac{\partial w_{a\|}}{\partial\chi}\left[\frac{3\bar{\mu}_{02a}e_a}{m_a}\frac{I}{B}\frac{B^\chi}{B}\frac{B^\chi}{B}\sigma\right]\bar{E}_\chi \\
&= -C_{14a.03}^1 E_\zeta - C_{14a.12a}^1\bar{Q}_a^\sigma \\
&\quad - \left[\nabla B\cdot\boldsymbol{C}_{14a.03.01}^2\right]E_\zeta - \left[\nabla u_{a\|}\cdot\boldsymbol{C}_{14a.03.03a}^2\right]E_\zeta - \left[\nabla w_{a\|}\cdot\boldsymbol{C}_{14a.03.04a}^2\right]E_\zeta \\
&\quad - \left[\nabla B\cdot\boldsymbol{C}_{14a.04.01}^2\right]\bar{E}_\chi - \left[\nabla u_{a\|}\cdot\boldsymbol{C}_{14a.04.03a}^2\right]\bar{E}_\chi - \left[\nabla w_{a\|}\cdot\boldsymbol{C}_{14a.04.04a}^2\right]\bar{E}_\chi
\end{aligned} \tag{A.215}
$$

where

$$\sqrt{g}C_{14a.03}^1 = -\sqrt{g}\frac{5e_a}{2m_a}p_a \tag{A.216}$$

$$\sqrt{g}C_{14a.12a}^1 = -\sqrt{g}\frac{e_a}{m_a}\sqrt{g}B^\chi\sigma \tag{A.217}$$

$$\sqrt{g}\boldsymbol{C}_{14a.03.01}^2 = \begin{pmatrix} 0 & \sqrt{g}\frac{3e_a}{m_a}\left(\frac{B_t^2}{B^2}-\frac{1}{3}\right)\frac{B_t^2}{B^3}\left(\bar{\mu}_{01a}u_a^{\dagger\chi} + \bar{\mu}_{02a}w_a^{\dagger\chi}\right)\end{pmatrix} \tag{A.218}$$

$$\sqrt{g}\boldsymbol{C}_{14a.03.03a}^2 = \begin{pmatrix} 0 & \sqrt{g}\frac{3e_a}{m_a}\left(\frac{B_t^2}{B^2}-\frac{1}{3}\right)\frac{B^\chi}{B}\bar{\mu}_{01a}\end{pmatrix} \tag{A.219}$$

$$\sqrt{g}\boldsymbol{C}_{14a.03.04a}^2 = \begin{pmatrix} 0 & \sqrt{g}\frac{3e_a}{m_a}\left(\frac{B_t^2}{B^2}-\frac{1}{3}\right)\frac{B^\chi}{B}\bar{\mu}_{02a}\end{pmatrix} \tag{A.220}$$

$$\sqrt{g}\boldsymbol{C}_{14a.04.01}^2 = \begin{pmatrix} 0 & \sqrt{g}\frac{3e_a}{m_a}\frac{I}{B}\frac{B^\chi}{B}\frac{B_t^2}{B^3}\sigma\left(\bar{\mu}_{01a}u_a^{\dagger\chi} + \bar{\mu}_{02a}w_a^{\dagger\chi}\right)\end{pmatrix} \tag{A.221}$$

$$\sqrt{g}\boldsymbol{C}_{14a.04.03a}^2 = \begin{pmatrix} 0 & \sqrt{g}\frac{3e_a}{m_a}\frac{I}{B}\frac{B^\chi}{B}\frac{B^\chi}{B}\sigma\bar{\mu}_{01a}\end{pmatrix} \tag{A.222}$$

$$\sqrt{g}\boldsymbol{C}_{14a.03.04a}^2 = \begin{pmatrix} 0 & \sqrt{g}\frac{3e_a}{m_a}\frac{I}{B}\frac{B^\chi}{B}\frac{B^\chi}{B}\sigma\bar{\mu}_{02a}\end{pmatrix} \tag{A.223}$$

EW friction force

$$
\begin{aligned}
G_{a\zeta}^{\mathrm{fri}} &= \sum_b \frac{T_a}{m_a}\frac{\bar{l}_{03ab}}{n_b} n_a u_{b\zeta} + \sum_b \frac{T_a}{m_a}\frac{\bar{l}_{04ab}}{p_b} Q_{b\zeta} \\
&= -\sum_b \left[-\frac{T_a}{m_a}\frac{\bar{l}_{03ab}}{n_b}\right] n_a u_{b\zeta} - \sum_b \left[-\frac{T_a}{m_a}\frac{\bar{l}_{04ab}}{p_b}\right] Q_{b\zeta} \\
&= -\sum_b C^1_{14a.09b} n_b u_{b\zeta} - \sum_b C^1_{14a.14b} Q_{b\zeta} \qquad \text{(A.224)}
\end{aligned}
$$

where

$$
\sqrt{g} C^1_{14a.09b} = -\sqrt{g}\frac{T_a}{m_a}\frac{\bar{l}_{03ab}}{n_b} \qquad \text{(A.225)}
$$

$$
\sqrt{g} C^1_{14a.14b} = -\sqrt{g}\frac{T_a}{m_a}\frac{\bar{l}_{04ab}}{p_b} \qquad \text{(A.226)}
$$

Source term

$$
\sqrt{g} S_{14a.14a} = \sqrt{g} S_{qa\zeta} \qquad \text{(A.227)}
$$

Equation for covariant toroidal total heat flux in advection-diffusion form

$$
\begin{aligned}
&\frac{1}{\sqrt{g}}\frac{\partial}{\partial t}\left(\sqrt{g} M_{14a.14a} n_a u_{a\zeta}\right) + \nabla\cdot\left(\boldsymbol{V}^1_{14a.09a} n_a u_{a\zeta}\right) + \nabla\cdot\left(\boldsymbol{V}^1_{14a.14a} Q_{a\zeta}\right) \\
&\quad - \nabla\cdot\left[\left(\overleftrightarrow{V}^2_{14a.09a.01}\cdot\nabla B\right) n_a u_{a\zeta}\right] - \nabla\cdot\left[\left(\overleftrightarrow{V}^2_{14a.10a.01}\cdot\nabla B\right) n_a u_a^\chi\right] \\
&\qquad - \nabla\cdot\left[\left(\overleftrightarrow{V}^2_{14a.14a.01}\cdot\nabla B\right) Q_{a\zeta}\right] - \nabla\cdot\left[\left(\overleftrightarrow{V}^2_{14a.15a.01}\cdot\nabla B\right) Q_a^\chi\right] \\
&\quad - \nabla\cdot\left(\overleftrightarrow{D}_{14a.06a}\cdot\nabla n_a\right) - \nabla\cdot\left[\overleftrightarrow{D}_{14a.08a}\cdot\nabla\left(n_a u_{a\parallel}\right)\right] \\
&\qquad - \nabla\cdot\left(\overleftrightarrow{D}_{14a.11a}\cdot\nabla p_a\right) - \nabla\cdot\left(\overleftrightarrow{D}_{14a.13a}\cdot\nabla Q_{a\parallel}\right) \\
&\quad + C^1_{14a.03} E_\zeta + C^1_{14a.12a}\bar{Q}_a^\sigma + \sum_b C^1_{14a.09b} n_b u_{b\zeta} + \sum_b C^1_{14a.14b} Q_{b\zeta} \\
&\quad + \left(\nabla B\cdot\boldsymbol{C}^2_{14a.03.01}\right) E_\zeta + \left(\nabla u_{a\parallel}\cdot\boldsymbol{C}^2_{14a.03.03a}\right) E_\zeta + \left(\nabla w_{a\parallel}\cdot\boldsymbol{C}^2_{14a.03.04a}\right) E_\zeta \\
&\quad + \left(\nabla B\cdot\boldsymbol{C}^2_{14a.04.01}\right) \bar{E}_\chi + \left(\nabla u_{a\parallel}\cdot\boldsymbol{C}^2_{14a.04.03a}\right) \bar{E}_\chi + \left(\nabla w_{a\parallel}\cdot\boldsymbol{C}^2_{14a.04.04a}\right) \bar{E}_\chi \qquad \text{(A.228)}
\end{aligned}
$$

A.16 Expression of contravariant poloidal total heat flux

$$
\begin{aligned}
B^\chi g_{\chi\chi} Q_a^\chi - B Q_{a\parallel} + B^\zeta Q_{a\zeta} &= 0 \\
C^1_{15a.15a} Q_a^\chi + C^1_{15a.13a} Q_{a\parallel} + C^1_{15a.14a} Q_{a\zeta} &= 0 \qquad \text{(A.229)}
\end{aligned}
$$

where

$$\sqrt{g}C^1_{15a.13a} = -\sqrt{g}B \tag{A.230}$$

$$\sqrt{g}C^1_{15a.14a} = \sqrt{g}B^\zeta \tag{A.231}$$

$$\sqrt{g}C^1_{15a.15a} = \sqrt{g}g_{\chi\chi}B^\chi \tag{A.232}$$

Appendix B

Coefficient matrices of advection-diffusion equations

Mass submatrix: $\mathsf{M}_{ab}^{\text{elm}}$

$$\begin{aligned}
&\iint w_a^{\text{elm}} \frac{\partial}{\partial t}\left(\sqrt{g} M_{ab} f_b^{\text{elm}}\right) d\rho d\chi \\
&= \sum_{IJK}^{n_{\text{node}}} \iint \{w_a\}_I^{\text{elm}} \phi_I^{\text{elm}} \frac{\partial}{\partial t}\left(\{\sqrt{g} M_{ab}\}_K^{\text{elm}} \phi_K^{\text{elm}} \{f_b\}_J^{\text{elm}} \phi_J^{\text{elm}}\right) J d\eta_1 d\eta_2 \\
&= \sum_{IJK}^{n_{\text{node}}} \{w_a\}_I^{\text{elm}} \frac{\partial}{\partial t}\left(\iint \phi_I^{\text{elm}} \phi_J^{\text{elm}} \phi_K^{\text{elm}} J d\eta_1 d\eta_2 \{\sqrt{g} M_{ab}\}_K^{\text{elm}} \{f_b\}_J^{\text{elm}}\right) \\
&= \mathsf{w}_a^{\text{elm}\mathsf{T}} \frac{\partial}{\partial t}\left(\mathsf{M}_{ab}^{\text{elm}} \mathsf{f}_b^{\text{elm}}\right),
\end{aligned} \tag{B.1}$$

where

$$\mathsf{M}_{abIJ}^{\text{elm}} = \sum_{K}^{n_{\text{node}}} \iint \phi_I^{\text{elm}} \phi_J^{\text{elm}} \phi_K^{\text{elm}} J d\eta_1 d\eta_2 \{\sqrt{g} M_{ab}\}_K^{\text{elm}} \tag{B.2}$$

Advection martix (1): $\mathsf{V}_{ab}^{1\text{elm}}$

$$\begin{aligned}
&\iint w_a^{\text{elm}} \frac{\partial}{\partial \xi_i}\left[\left(\sqrt{g} V_{ab}^{\xi_i}\right)^{\text{elm}} f_b^{\text{elm}}\right] d\rho d\chi \\
&= \iint \{w_a\}_I^{\text{elm}} \phi_I^{\text{elm}} J_{ij}^* \frac{\partial}{\partial \eta_j}\left(\left\{\sqrt{g} V_{ab}^{\xi_i}\right\}_K^{\text{elm}} \phi_K^{\text{elm}} \{f_b\}_J^{\text{elm}} \phi_J^{\text{elm}}\right) J d\eta_1 d\eta_2 \\
&= \sum_{IJK}^{n_{\text{node}}} \sum_{ij}^{2} \{w_a\}_I^{\text{elm}} \left[\iint \phi_I^{\text{elm}} \frac{\partial}{\partial \eta_j}\left(\phi_J^{\text{elm}} \phi_K^{\text{elm}}\right) J J_{ij}^* d\eta_1 d\eta_2 \left\{\sqrt{g} V_{ab}^{\xi_i}\right\}_K^{\text{elm}}\right] \{f_b\}_J^{\text{elm}} \\
&= \mathsf{w}_a^{\text{elm}\mathsf{T}} \mathsf{V}_{ab}^{1\text{elm}} \mathsf{f}_b^{\text{elm}},
\end{aligned} \tag{B.3}$$

where

$$\mathsf{V}_{abIJ}^{1\text{elm}} = \sum_{K}^{n_{\text{node}}} \sum_{ij}^{2} \iint \phi_I^{\text{elm}} \frac{\partial}{\partial \eta_j}\left(\phi_J^{\text{elm}} \phi_K^{\text{elm}}\right) J J_{ij}^* d\eta_1 d\eta_2 \left\{\sqrt{g} V_{ab}^{\xi_i}\right\}_K^{\text{elm}} \tag{B.4}$$

Advection martix (2): $\mathsf{V}_{ab}^{2\mathrm{elm}}$

$$
\begin{aligned}
&\sum_{x}\sum_{ij}^{2}\iint w_a^{\mathrm{elm}}\frac{\partial}{\partial \xi_i}\left[\left(\sqrt{g}V_{abx}^{\xi_i\xi_j}\right)^{\mathrm{elm}}\frac{\partial g_x^{\mathrm{elm}}}{\partial \xi_j}f_b^{\mathrm{elm}}\right]d\rho d\chi \\
&\quad=\sum_{x}\sum_{ij}^{2}\int\left[w_a^{\mathrm{elm}}\left(\sqrt{g}V_{abx}^{\xi_i\xi_j}\right)^{\mathrm{elm}}\frac{\partial g_x^{\mathrm{elm}}}{\partial \xi_j}f_b^{\mathrm{elm}}\right]d\Gamma_{2,i} \\
&\qquad-\sum_{x}\sum_{ij}^{2}\iint\frac{\partial w_a^{\mathrm{elm}}}{\partial \xi_i}\left(\sqrt{g}V_{abx}^{\xi_i\xi_j}\right)^{\mathrm{elm}}\frac{\partial f_x^{\mathrm{elm}}}{\partial \xi_j}f_b^{\mathrm{elm}}d\rho d\chi \\
&\quad=-\sum_{x}\sum_{ijkl}^{2}\sum_{IJKL}^{n_{\mathrm{node}}}\iint J_{ik}^{*}\frac{\partial}{\partial \eta_k}\left(\{w_a\}_I^{\mathrm{elm}}\phi_I^{\mathrm{elm}}\right)\left\{\sqrt{g}V_{abx}^{\xi_i\xi_j}\right\}_L^{\mathrm{elm}}\phi_L^{\mathrm{elm}} \\
&\qquad\qquad\qquad\qquad\times J_{jl}^{*}\frac{\partial}{\partial \eta_l}\left(\{f_x\}_K^{\mathrm{elm}}\phi_K^{\mathrm{elm}}\right)\{f_b\}_J^{\mathrm{elm}}\phi_J^{\mathrm{elm}}Jd\eta_1 d\eta_2 \\
&=-\sum_{x}\sum_{ijkl}^{2}\sum_{IJKL}^{n_{\mathrm{node}}}\{w_a\}_I^{\mathrm{elm}}\left[\iint\frac{\partial \phi_I^{\mathrm{elm}}}{\partial \eta_k}\phi_J^{\mathrm{elm}}\frac{\partial \phi_K^{\mathrm{elm}}}{\partial \eta_l}\phi_L^{\mathrm{elm}}J_{ik}^{*}J_{jl}^{*}Jd\eta_1 d\eta_2\left\{\sqrt{g}V_{abx}^{\xi_i\xi_j}\right\}_L^{\mathrm{elm}}\{f_x\}_K^{\mathrm{elm}}\right]\{f_b\}_J^{\mathrm{elm}} \\
&=-\mathsf{w}_a^{\mathrm{elm}\mathsf{T}}\mathsf{V}_{ab}^{2\mathrm{elm}}\mathsf{f}_b^{\mathrm{elm}},
\end{aligned}
\tag{B.5}
$$

where

$$
\mathsf{V}_{abIJ}^{2\mathrm{elm}}=\sum_{x}\sum_{ijkl}^{2}\sum_{KL}^{n_{\mathrm{node}}}\iint\frac{\partial \phi_I^{\mathrm{elm}}}{\partial \eta_k}\phi_J^{\mathrm{elm}}\frac{\partial \phi_K^{\mathrm{elm}}}{\partial \eta_l}\phi_L^{\mathrm{elm}}J_{ik}^{*}J_{jl}^{*}Jd\eta_1 d\eta_2\left\{\sqrt{g}V_{abx}^{\xi_i\xi_j}\right\}_L^{\mathrm{elm}}\{f_x\}_K^{\mathrm{elm}} \tag{B.6}
$$

Diffusion submatrix: $\mathsf{D}_{ab}^{\mathrm{elm}}$ **and flux subvector** $\mathsf{F}_{ab}^{\mathrm{elm}}$

$$
\begin{aligned}
&\sum_{ij}^{2} \iint w_a^{\mathrm{elm}} \frac{\partial}{\partial \xi_i} \left[\left(\sqrt{g} D_{ab}^{\xi_i \xi_j} \right)^{\mathrm{elm}} \frac{\partial f_b^{\mathrm{elm}}}{\partial \xi_j} \right] d\rho d\chi \\
&= \sum_{ij}^{2} \int \left[w_a^{\mathrm{elm}} \left(\sqrt{g} D_{ab}^{\xi_i \xi_j} \right)^{\mathrm{elm}} q_{b,j}^{\mathrm{elm}} \right] d\xi_i^* - \sum_{ij}^{2} \iint \frac{\partial w_a^{\mathrm{elm}}}{\partial \xi_i} \left(\sqrt{g} D_{ab}^{\xi_i \xi_j} \right)^{\mathrm{elm}} \frac{\partial f_b^{\mathrm{elm}}}{\partial \xi_j} d\rho d\chi \\
&= \sum_{ij}^{2} \sum_{IJ}^{n_{\mathrm{node}}} \int \left[\{w_a\}_I^{\mathrm{elm}} \phi_I^{\mathrm{elm}} \left\{ \sqrt{g} D_{ab}^{\xi_i \xi_j} q_{b,j} \right\}_J^{\mathrm{elm}} \phi_J^{\mathrm{elm}} \right] h_i^* d\eta_i^* \\
&\quad - \sum_{ijkl}^{2} \sum_{IJK}^{n_{\mathrm{node}}} \iint J_{ik}^* \frac{\partial}{\partial \eta_k} \left(\{w_a\}_I^{\mathrm{elm}} \phi_I^{\mathrm{elm}} \right) \left\{ \sqrt{g} D_{ab}^{\xi_i \xi_j} \right\}_K^{\mathrm{elm}} \phi_K^{\mathrm{elm}} J_{jl}^* \frac{\partial}{\partial \eta_l} \left(\{f_b\}_J^{\mathrm{elm}} \phi_J^{\mathrm{elm}} \right) J d\eta_1 d\eta_2 \\
&= \sum_{ij}^{2} \sum_{IJ}^{n_{\mathrm{node}}} \{w_a\}_I^{\mathrm{elm}} \left[\int \phi_I^{\mathrm{elm}} \phi_J^{\mathrm{elm}} h_i^* d\eta_i^* \left\{ \sqrt{g} D_{ab}^{\xi_i \xi_j} q_{b,j} \right\}_J^{\mathrm{elm}} \right] \\
&\quad - \sum_{ijkl}^{2} \sum_{IJK}^{n_{\mathrm{node}}} \{w_a\}_I^{\mathrm{elm}} \left[\iint \frac{\partial \phi_I^{\mathrm{elm}}}{\partial \eta_k} \frac{\partial \phi_J^{\mathrm{elm}}}{\partial \eta_l} \phi_K^{\mathrm{elm}} J_{ik}^* J_{jl}^* J d\eta_1 d\eta_2 \left\{ \sqrt{g} D_{ab}^{\xi_i \xi_j} \right\}_K^{\mathrm{elm}} \right] \{f_b\}_J^{\mathrm{elm}} \\
&= \mathsf{w}_a^{\mathrm{elm}\mathsf{T}} \mathsf{F}_{ab}^{\mathrm{elm}} - \mathsf{w}_a^{\mathrm{elm}\mathsf{T}} \mathsf{D}_{ab}^{\mathrm{elm}} \mathsf{f}_b^{\mathrm{elm}},
\end{aligned} \tag{B.7}
$$

where

$$
\mathsf{F}_{abI}^{\mathrm{elm}} = \sum_{ij}^{2} \sum_{J}^{n_{\mathrm{node}}} \int \phi_I^{\mathrm{elm}} \phi_J^{\mathrm{elm}} h_i^* d\eta_i^* \left\{ \sqrt{g} D_{ab}^{\xi_i \xi_j} q_{b,j} \right\}_J^{\mathrm{elm}} \tag{B.8}
$$

$$
\mathsf{D}_{abIJ}^{\mathrm{elm}} = \sum_{ijkl}^{2} \sum_{K}^{n_{\mathrm{node}}} \iint \frac{\partial \phi_I^{\mathrm{elm}}}{\partial \eta_k} \frac{\partial \phi_J^{\mathrm{elm}}}{\partial \eta_l} \phi_K^{\mathrm{elm}} J_{ik}^* J_{jl}^* J d\eta_1 d\eta_2 \left\{ \sqrt{g} D_{ab}^{\xi_i \xi_j} \right\}_K^{\mathrm{elm}} \tag{B.9}
$$

Gradient submatrix (1): $\mathsf{A}_{ab}^{1\mathrm{elm}}$

$$
\begin{aligned}
&\sum_{i}^{2} \iint w_a^{\mathrm{elm}} \left(\sqrt{g} A_{ab}^{\xi_i} \right)^{\mathrm{elm}} \frac{\partial f_b^{\mathrm{elm}}}{\partial \xi_i} d\rho d\chi \\
&\quad = \sum_{ij}^{2} \iint \{w_a\}_I^{\mathrm{elm}} \phi_I^{\mathrm{elm}} \left\{ \sqrt{g} A_{ab}^{\xi_i} \right\}_K^{\mathrm{elm}} \phi_K^{\mathrm{elm}} J_{ij}^* \frac{\partial}{\partial \eta_j} \left(\{f_b\}_J^{\mathrm{elm}} \phi_J^{\mathrm{elm}} \right) J d\eta_1 d\eta_2 \\
&\quad = \sum_{ij}^{2} \sum_{IJK}^{n_{\mathrm{node}}} \{w_a\}_I^{\mathrm{elm}} \left[\iint \phi_I^{\mathrm{elm}} \frac{\partial \phi_J^{\mathrm{elm}}}{\partial \eta_j} \phi_K^{\mathrm{elm}} J_{ij}^* J d\eta_1 d\eta_2 \left\{ \sqrt{g} A_{ab}^{\xi_i} \right\}_K^{\mathrm{elm}} \right] \{f_b\}_J^{\mathrm{elm}} \\
&= \mathsf{w}_a^{\mathrm{elm}\mathsf{T}} \mathsf{A}_{abIJ}^{1\mathrm{elm}} \mathsf{f}_b^{\mathrm{elm}},
\end{aligned} \tag{B.10}
$$

where

$$
\mathsf{A}_{abIJ}^{1\mathrm{elm}} = \sum_{ij}^{2} \sum_{K}^{n_{\mathrm{node}}} \iint \phi_I^{\mathrm{elm}} \frac{\partial \phi_J^{\mathrm{elm}}}{\partial \eta_j} \phi_K^{\mathrm{elm}} J_{ij}^* J d\eta_1 d\eta_2 \left\{ \sqrt{g} A_{ab}^{\xi_i} \right\}_K^{\mathrm{elm}} \tag{B.11}
$$

Gradient submatrix (2): $\mathsf{A}_{ab}^{2\text{elm}}$

$$
\begin{aligned}
&\sum_x \sum_{ij}^{2} \iint w_a^{\text{elm}} \frac{\partial g_x}{\partial \xi_i} \left(\sqrt{g} A_{abx}^{\xi_i \xi_j}\right)^{\text{elm}} \frac{\partial f_b^{\text{elm}}}{\partial \xi_j} d\rho d\chi \\
&\quad = \sum_x \sum_{ij}^{2} \sum_{IJKL}^{n_{\text{node}}} \iint \{w_a\}_I^{\text{elm}} \phi_I^{\text{elm}} J_{ik}^* \frac{\partial}{\partial \eta_k} \left(\{g_x\}_K^{\text{elm}} \phi_K^{\text{elm}}\right) \\
&\qquad\qquad\qquad \times \left\{\sqrt{g} A_{abx}^{\xi_i \xi_j}\right\}_L^{\text{elm}} \phi_L^{\text{elm}} J_{jl}^* \frac{\partial}{\partial \eta_l} \left(\{f_b\}_J^{\text{elm}} \phi_J^{\text{elm}}\right) J d\eta_1 d\eta_2 \\
&\quad = \sum_x \sum_{ijkl}^{2} \sum_{IJKL}^{n_{\text{node}}} \{w_a\}_I^{\text{elm}} \left[\iint \phi_I^{\text{elm}} \frac{\partial \phi_J^{\text{elm}}}{\partial \eta_l} \frac{\partial \phi_K^{\text{elm}}}{\partial \eta_k} \phi_L^{\text{elm}} J_{ik}^* J_{jl}^* J d\eta_1 d\eta_2 \left\{\sqrt{g} A_{abx}^{\xi_i \xi_j}\right\}_K^{\text{elm}}\right] \{f_b\}_J^{\text{elm}} \\
&= \mathsf{w}_a^{\text{elm}\mathsf{T}} \mathsf{A}_{ab}^{2\text{elm}} \mathsf{f}_b^{\text{elm}},
\end{aligned}
\tag{B.12}
$$

where

$$
\mathsf{A}_{ab}^{2\text{elm}} = \sum_x \sum_{ijkl}^{2} \sum_{KL}^{n_{\text{node}}} \iint \phi_I^{\text{elm}} \frac{\partial \phi_J^{\text{elm}}}{\partial \eta_l} \frac{\partial \phi_K^{\text{elm}}}{\partial \eta_k} \phi_L^{\text{elm}} J_{ik}^* J_{jl}^* J d\eta_1 d\eta_2 \left\{\sqrt{g} A_{abx}^{\xi_i \xi_j}\right\}_K^{\text{elm}} \tag{B.13}
$$

Excitation submatrix (1): $\mathsf{C}_{ab}^{1\text{elm}}$

$$
\begin{aligned}
&\iint w_a^{\text{elm}} \left(\sqrt{g} C_{ab}\right)^{\text{elm}} f_b^{\text{elm}} d\rho d\chi \\
&\quad = \sum_{IJK}^{n_{\text{node}}} \iint \{w_a\}_I^{\text{elm}} \phi_I^{\text{elm}} \{\sqrt{g} C_{ab}\}_K^{\text{elm}} \phi_K^{\text{elm}} \{f_b\}_J^{\text{elm}} \phi_J^{\text{elm}} J d\eta_1 d\eta_2 \\
&\quad = \sum_{IJK}^{n_{\text{node}}} \{w_a\}_I^{\text{elm}} \left[\iint \phi_I^{\text{elm}} \phi_J^{\text{elm}} \phi_K^{\text{elm}} J d\eta_1 d\eta_2 \{\sqrt{g} C_{ab}\}_K^{\text{elm}}\right] \{f_b\}_J^{\text{elm}} \\
&\quad = \mathsf{w}_a^{\text{elm}\mathsf{T}} \mathsf{C}_{ab}^{1\text{elm}} \mathsf{f}_b^{\text{elm}},
\end{aligned}
\tag{B.14}
$$

where

$$
\mathsf{C}_{ab}^{1\text{elm}} = \sum_{K}^{n_{\text{node}}} \iint \phi_I^{\text{elm}} \phi_J^{\text{elm}} \phi_K^{\text{elm}} J d\eta_1 d\eta_2 \{\sqrt{g} C_{ab}\}_K^{\text{elm}} \tag{B.15}
$$

Excitation submatrix (2): C_{ab}^{2}

$$
\begin{aligned}
&\sum_x \sum_i^2 \iint w_a^{\mathrm{elm}} \frac{\partial g_x^{\mathrm{elm}}}{\partial \xi_i} \left(\sqrt{g} C_{abx}^{\xi_i}\right)^{\mathrm{elm}} f_b^{\mathrm{elm}} d\rho d\chi \\
&= \sum_x \sum_{ij}^2 \sum_{IJKL}^{n_{\mathrm{node}}} \iint \{w_a\}_I^{\mathrm{elm}} \phi_I^{\mathrm{elm}} J_{ij}^* \frac{\partial}{\partial \eta_j} \left(\{g_x\}_K^{\mathrm{elm}} \phi_K^{\mathrm{elm}}\right) \left\{\sqrt{g} C_{abx}^{\xi_i}\right\}_L^{\mathrm{elm}} \phi_L^{\mathrm{elm}} \{f_b\}_J^{\mathrm{elm}} \phi_J^{\mathrm{elm}} J d\eta_1 d\eta_2 \\
&= \sum_x \sum_{ij}^2 \sum_{IJKL}^{n_{\mathrm{node}}} \{w_a\}_I^{\mathrm{elm}} \left[\iint \phi_I^{\mathrm{elm}} \phi_J^{\mathrm{elm}} \frac{\partial \phi_K^{\mathrm{elm}}}{\partial \eta_j} \phi_L^{\mathrm{elm}} J_{ij}^* J d\eta_1 d\eta_2 \{g_x\}_K^{\mathrm{elm}} \left\{\sqrt{g} C_{abx}^{\xi_i}\right\}_L^{\mathrm{elm}}\right] \{f_b\}_J^{\mathrm{elm}} \\
&= \mathsf{w}_a^{\mathrm{elm}\mathsf{T}} \mathsf{C}_{ab}^{2\mathrm{elm}} \mathsf{f}_b^{\mathrm{elm}},
\end{aligned}
\tag{B.16}
$$

where

$$
\mathsf{C}_{ab}^{2\mathrm{elm}} = \sum_x \sum_{ij}^2 \sum_{KL}^{n_{\mathrm{node}}} \iint \phi_I^{\mathrm{elm}} \phi_J^{\mathrm{elm}} \frac{\partial \phi_K^{\mathrm{elm}}}{\partial \eta_j} \phi_L^{\mathrm{elm}} J_{ij}^* J d\eta_1 d\eta_2 \{g_x\}_K^{\mathrm{elm}} \left\{\sqrt{g} C_{abx}^{\xi_i}\right\}_L^{\mathrm{elm}} \tag{B.17}
$$

Excitation submatrix (3): $\mathsf{C}_{ab}^{3\mathrm{elm}}$

$$
\begin{aligned}
&\sum_{xy} \sum_{ij}^2 \iint w_a^{\mathrm{elm}} \frac{\partial g_x^{\mathrm{elm}}}{\partial \xi_i} \left(\sqrt{g} C_{abxy}\right)^{\mathrm{elm}} \frac{\partial g_y^{\mathrm{elm}}}{\partial \xi_j} f_b^{\mathrm{elm}} d\rho d\chi \\
&= \sum_{xy} \sum_{ijkl}^2 \sum_{IJKLM}^{n_{\mathrm{node}}} \iint \{w_a\}_I^{\mathrm{elm}} \phi_I^{\mathrm{elm}} J_{ik}^* \frac{\partial}{\partial \eta_k} \left(\{g_x\}_K^{\mathrm{elm}} \phi_K^{\mathrm{elm}}\right) \\
&\qquad \times \left\{\sqrt{g} C_{abxy}^{\xi_i \xi_j}\right\}_M^{\mathrm{elm}} \phi_M^{\mathrm{elm}} J_{jl}^* \frac{\partial}{\partial \eta_l} \left(\{g_y\}_L^{\mathrm{elm}} \phi_L^{\mathrm{elm}}\right) \{f_b\}_J^{\mathrm{elm}} \phi_J^{\mathrm{elm}} J d\eta_1 d\eta_2 \\
&= \sum_{xy} \sum_{ijkl}^2 \sum_{IJKLM}^{n_{\mathrm{node}}} \{w_a\}_I^{\mathrm{elm}} \left[\iint \phi_I^{\mathrm{elm}} \phi_J^{\mathrm{elm}} \frac{\partial \phi_K^{\mathrm{elm}}}{\partial \eta_k} \frac{\partial \phi_L^{\mathrm{elm}}}{\partial \eta_l} \phi_M^{\mathrm{elm}} J_{ik}^* J_{jl}^* J d\eta_1 d\eta_2 \right. \\
&\qquad \left. \times \{g_x\}_K^{\mathrm{elm}} \left\{\sqrt{g} C_{abxy}^{\xi_i \xi_j}\right\}_M^{\mathrm{elm}} \{g_y\}_L^{\mathrm{elm}}\right] \{f_b\}_J^{\mathrm{elm}} \\
&= \mathsf{w}_a^{\mathrm{elm}\mathsf{T}} \mathsf{C}_{ab}^{3\mathrm{elm}} \mathsf{f}_b^{\mathrm{elm}},
\end{aligned}
\tag{B.18}
$$

where

$$
\mathsf{C}_{abIJ}^{3\mathrm{elm}} = \sum_{xy} \sum_{ijkl}^2 \sum_{KLM}^{n_{\mathrm{node}}} \iint \phi_I^{\mathrm{elm}} \phi_J^{\mathrm{elm}} \frac{\partial \phi_K^{\mathrm{elm}}}{\partial \eta_k} \frac{\partial \phi_L^{\mathrm{elm}}}{\partial \eta_l} \phi_M^{\mathrm{elm}} J_{ik}^* J_{jl}^* J d\eta_1 d\eta_2 \{g_x\}_K^{\mathrm{elm}} \left\{\sqrt{g} C_{abxy}^{\xi_i \xi_j}\right\}_M^{\mathrm{elm}} \{g_y\}_L^{\mathrm{elm}} \tag{B.19}
$$

Source subvector: $\mathsf{S}_a^{\text{elm}}$

$$
\begin{aligned}
&\iint w_a^{\text{elm}} \left(\sqrt{g}S_a\right)^{\text{elm}} d\rho d\chi \\
&\quad = \sum_{IJ}^{n_{\text{node}}} \iint \{w_a\}_I^{\text{elm}} \phi_I^{\text{elm}} \{\sqrt{g}S_a\}_J^{\text{elm}} J d\eta_1 d\eta_2 \\
&\quad = \sum_{IJ}^{n_{\text{node}}} \{w_a\}_I^{\text{elm}} \left[\iint \phi_I^{\text{elm}}\phi_J^{\text{elm}} J d\eta_1 d\eta_2 \{\sqrt{g}S_a\}_J^{\text{elm}}\right] \\
&= \mathsf{w}_a^{\text{elm}\mathsf{T}} \mathsf{S}_a^{\text{elm}},
\end{aligned} \tag{B.20}
$$

where

$$
\mathsf{S}_{aI}^{\text{elm}} = \sum_{J}^{n_{\text{node}}} \iint \phi_I^{\text{elm}}\phi_J^{\text{elm}} J d\eta_1 d\eta_2 \{\sqrt{g}S_a\}_J^{\text{elm}} \tag{B.21}
$$

SUPG mass submatrix: $\mathsf{M}_{ab}^{\text{selm}}$

$$
\begin{aligned}
&\sum_{i}^{2} \iint \tau_a^{\text{elm}} u_a^{\xi_i \text{elm}} \frac{\partial w_a^{\text{elm}}}{\partial \xi_i} \frac{\partial}{\partial t}\left[\left(\sqrt{g}M_{ab}\right)^{\text{elm}} f_b^{\text{elm}}\right] d\rho d\chi \\
&\quad = \sum_{ij}^{2} \sum_{IJKL}^{n_{\text{node}}} \iint \tau_a^{\text{elm}} \left\{u_a^{\xi_i}\right\}_L^{\text{elm}} \phi_L^{\text{elm}} J_{ij}^{*} \frac{\partial}{\partial \eta_j}\left(\left\{w_a^{\text{elm}}\right\}_I^{\text{elm}} \phi_I^{\text{elm}}\right) \\
&\quad \times \frac{\partial}{\partial t}\left(\{\sqrt{g}M_{ab}\}_K^{\text{elm}} \phi_K^{\text{elm}} \{f_b\}_J^{\text{elm}} \phi_J^{\text{elm}}\right) J d\eta_1 d\eta_2 \\
&\quad = \sum_{ij}^{2} \sum_{IJKL}^{n_{\text{node}}} \{w_a\}_I^{\text{elm}} \\
&\quad \times \frac{\partial}{\partial t}\left(\left[\tau_a^{\text{elm}} \iint J_{ij}^{*} \frac{\partial \phi_I^{\text{elm}}}{\partial \eta_j} \phi_J^{\text{elm}} \phi_K^{\text{elm}} \phi_L^{\text{elm}} J d\eta_1 d\eta_2 \{\sqrt{g}M_{ab}\}_K^{\text{elm}} \left\{u_a^{\xi_i}\right\}_L^{\text{elm}}\right] \{f_b\}_J^{\text{elm}}\right) \\
&\quad = \mathsf{w}_a^{\text{elm}\mathsf{T}} \mathsf{M}_{ab}^{\text{selm}} \mathsf{f}_b^{\text{elm}},
\end{aligned} \tag{B.22}
$$

where

$$
\mathsf{M}_{abIJ}^{\text{selm}} = \sum_{ij}^{2} \sum_{KL}^{n_{\text{node}}} \tau_a^{\text{elm}} \iint \frac{\partial \phi_I^{\text{elm}}}{\partial \eta_j} \phi_J^{\text{elm}} \phi_K^{\text{elm}} \phi_L^{\text{elm}} J_{ij}^{*} J d\eta_1 d\eta_2 \{\sqrt{g}M_{ab}\}_K^{\text{elm}} \left\{u_a^{\xi_i}\right\}_L^{\text{elm}} \tag{B.23}
$$

SUPG advection submatrix (1): $\mathsf{V}_{ab}^{1s\mathrm{elm}}$

$$\begin{aligned}
&\sum_{ij}^{2} \iint \tau_a^{\mathrm{elm}} u_a^{\xi_i \mathrm{elm}} \frac{\partial w_a^{\mathrm{elm}}}{\partial \xi_i} \frac{\partial}{\partial \xi_j} \left[\left(\sqrt{g} V_{ab}^{\xi_j} \right)^{\mathrm{elm}} f_b^{\mathrm{elm}} \right] d\rho d\chi \\
&= \sum_{ijkl}^{2} \sum_{IJKL}^{n_{\mathrm{node}}} \iint \tau_a^{\mathrm{elm}} \left\{ u_a^{\xi_i} \right\}_L^{\mathrm{elm}} \phi_L^{\mathrm{elm}} J_{ik}^* \frac{\partial}{\partial \eta_k} \left(\left\{ w_a^{\mathrm{elm}} \right\}_I^{\mathrm{elm}} \phi_I^{\mathrm{elm}} \right) \\
&\qquad \times J_{jl}^* \frac{\partial}{\partial \eta_l} \left(\left\{ \sqrt{g} V_{ab}^{\xi_j} \right\}_K^{\mathrm{elm}} \phi_K^{\mathrm{elm}} \left\{ f_b \right\}_J^{\mathrm{elm}} \phi_J^{\mathrm{elm}} \right) J d\eta_1 d\eta_2 \\
&= \sum_{ijkl}^{2} \sum_{IJK}^{n_{\mathrm{node}}} \left\{ w_a^{\mathrm{elm}} \right\}_I^{\mathrm{elm}} \\
&\quad \times \left[\tau_a^{\mathrm{elm}} \iint \frac{\partial \phi_I^{\mathrm{elm}}}{\partial \eta_k} \frac{\partial}{\partial \eta_l} \left(\phi_J^{\mathrm{elm}} \phi_K^{\mathrm{elm}} \right) \phi_L^{\mathrm{elm}} J_{ik}^* J_{jl}^* J d\eta_1 d\eta_2 \left\{ \sqrt{g} V_{ab}^{\xi_j} \right\}_K^{\mathrm{elm}} \left\{ u_a^{\xi_i} \right\}_L^{\mathrm{elm}} \right] \left\{ f_b \right\}_J^{\mathrm{elm}} \\
&= \mathsf{w}_a^{\mathrm{elm}\mathsf{T}} \mathsf{V}_{ab}^{1s\mathrm{elm}} \mathsf{f}_b^{\mathrm{elm}}, \qquad \text{(B.24)}
\end{aligned}$$

where

$$\mathsf{V}_{abIJ}^{1\mathrm{elm}} = \sum_{ijkl}^{2} \sum_{KL}^{n_{\mathrm{node}}} \tau_a^{\mathrm{elm}} \iint \frac{\partial \phi_I^{\mathrm{elm}}}{\partial \eta_k} \frac{\partial}{\partial \eta_l} \left(\phi_J^{\mathrm{elm}} \phi_K^{\mathrm{elm}} \right) \phi_L^{\mathrm{elm}} J_{ik}^* J_{jl}^* J d\eta_1 d\eta_2 \left\{ \sqrt{g} V_{ab}^{\xi_j} \right\}_K^{\mathrm{elm}} \left\{ u_a^{\xi_i} \right\}_L^{\mathrm{elm}} \qquad \text{(B.25)}$$

SUPG advection submatrix (2): $\mathsf{V}_{ab}^{2s\mathrm{elm}}$

$$\begin{aligned}
&\sum_{x} \sum_{ijk}^{2} \iint \tau_a^{\mathrm{elm}} u_a^{\xi_i \mathrm{elm}} \frac{\partial w_a^{\mathrm{elm}}}{\partial \xi_i} \frac{\partial}{\partial \xi_j} \left[\left(\sqrt{g} V_{abx}^{\xi_j \xi_k} \right)^{\mathrm{elm}} \frac{\partial g_x^{\mathrm{elm}}}{\partial \xi_k} f_b^{\mathrm{elm}} \right] d\rho d\chi \\
&= \sum_{x} \sum_{ijklmn}^{2} \sum_{IJKLM}^{n_{\mathrm{node}}} \iint \tau_a^{\mathrm{elm}} \left\{ u_a^{\xi_i} \right\}_M^{\mathrm{elm}} \phi_M^{\mathrm{elm}} J_{il}^* \frac{\partial}{\partial \eta_l} \left(\left\{ w_a^{\mathrm{elm}} \right\}_I^{\mathrm{elm}} \phi_I^{\mathrm{elm}} \right) \\
&\quad \times J_{jm}^* \frac{\partial}{\partial \eta_m} \left[\left\{ \sqrt{g} V_{abx}^{\xi_j \xi_k} \right\}_L^{\mathrm{elm}} \phi_L^{\mathrm{elm}} J_{kn}^* \frac{\partial}{\partial \eta_n} \left(\left\{ g_x \right\}_K^{\mathrm{elm}} \phi_K^{\mathrm{elm}} \right) \left\{ f_b \right\}_J^{\mathrm{elm}} \phi_J^{\mathrm{elm}} \right] J d\eta_1 d\eta_2 \\
&= \sum_{x} \sum_{ijklmn}^{2} \sum_{IJKLM}^{n_{\mathrm{node}}} \left\{ w_a^{\mathrm{elm}} \right\}_I^{\mathrm{elm}} \left[\tau_a^{\mathrm{elm}} \iint \frac{\partial \phi_I^{\mathrm{elm}}}{\partial \eta_l} \frac{\partial}{\partial \eta_m} \left(\phi_J^{\mathrm{elm}} \frac{\partial \phi_K^{\mathrm{elm}}}{\partial \eta_n} \phi_L^{\mathrm{elm}} J_{kn}^* \right) \phi_M^{\mathrm{elm}} J_{il}^* J_{jm}^* J d\eta_1 d\eta_2 \right. \\
&\qquad \left. \times \left\{ g_x \right\}_K^{\mathrm{elm}} \left\{ \sqrt{g} V_{abx}^{\xi_j \xi_k} \right\}_L^{\mathrm{elm}} \left\{ u_a^{\xi_i} \right\}_M^{\mathrm{elm}} \right] \left\{ f_b \right\}_J^{\mathrm{elm}} \\
&= \mathsf{w}_a^{\mathrm{elm}\mathsf{T}} \mathsf{V}_{ab}^{2s\mathrm{elm}} \mathsf{f}_b^{\mathrm{elm}}, \qquad \text{(B.26)}
\end{aligned}$$

where

$$\begin{aligned}
\mathsf{V}_{abIJ}^{2s\mathrm{elm}} = \sum_{x} \sum_{ijklmn}^{2} \sum_{KLM}^{n_{\mathrm{node}}} \tau_a^{\mathrm{elm}} \iint \frac{\partial \phi_I^{\mathrm{elm}}}{\partial \eta_l} \frac{\partial}{\partial \eta_m} \left(\phi_J^{\mathrm{elm}} \frac{\partial \phi_K^{\mathrm{elm}}}{\partial \eta_n} \phi_L^{\mathrm{elm}} J_{kn}^* \right) \phi_M^{\mathrm{elm}} J_{il}^* J_{jm}^* J d\eta_1 d\eta_2 \\
\times \left\{ g_x \right\}_K^{\mathrm{elm}} \left\{ \sqrt{g} V_{abx}^{\xi_j \xi_k} \right\}_L^{\mathrm{elm}} \left\{ u_a^{\xi_i} \right\}_M^{\mathrm{elm}} \qquad \text{(B.27)}
\end{aligned}$$

SUPG diffusion submatrix: $\mathsf{D}_{ab}^{s\mathrm{elm}}$

$$
\begin{aligned}
&\sum_{ijk}^{2} \iint \tau_a^{\mathrm{elm}} u_a^{\xi_i \mathrm{elm}} \frac{\partial w_a^{\mathrm{elm}}}{\partial \xi_i} \frac{\partial}{\partial \xi_j} \left[\left(\sqrt{g} D_{ab}^{\xi_j \xi_k} \right)^{\mathrm{elm}} \frac{\partial f_b^{\mathrm{elm}}}{\partial \xi_k} \right] d\rho d\chi \\
&\quad = \sum_{ijklmn}^{2} \sum_{IJKL}^{n_{\mathrm{node}}} \iint \tau_a^{\mathrm{elm}} \left\{ u_a^{\xi_i} \right\}_L^{\mathrm{elm}} \phi_L^{\mathrm{elm}} J_{il}^{*} \frac{\partial}{\partial \eta_l} \left(\left\{ w_a \right\}_I^{\mathrm{elm}} \phi_I^{\mathrm{elm}} \right) \\
&\qquad\qquad \times J_{jm}^{*} \frac{\partial}{\partial \eta_m} \left[\left\{ \sqrt{g} D_{ab}^{\xi_j \xi_k} \right\}_K^{\mathrm{elm}} \phi_K^{\mathrm{elm}} J_{kn}^{*} \frac{\partial}{\partial \eta_n} \left(\left\{ f_b \right\}_J^{\mathrm{elm}} \phi_J^{\mathrm{elm}} \right) \right] J d\eta_1 d\eta_2 \\
&\quad = \sum_{ijklmn}^{2} \sum_{IJKL}^{n_{\mathrm{node}}} \left\{ w_a \right\}_I^{\mathrm{elm}} \left[\tau_a^{\mathrm{elm}} \iint \frac{\partial \phi_I^{\mathrm{elm}}}{\partial \eta_l} \frac{\partial}{\partial \eta_m} \left(\frac{\partial \phi_J^{\mathrm{elm}}}{\partial \eta_n} \phi_K^{\mathrm{elm}} J_{kn}^{*} \right) \phi_L^{\mathrm{elm}} J_{jm}^{*} J_{il}^{*} J d\eta_1 d\eta_2 \right. \\
&\qquad\qquad \left. \times \left\{ \sqrt{g} D_{ab}^{\xi_j \xi_k} \right\}_K^{\mathrm{elm}} \left\{ u_a^{\xi_i} \right\}_L^{\mathrm{elm}} \right] \left\{ f_b \right\}_J^{\mathrm{elm}} \\
&\quad = \mathsf{w}_a^{\mathrm{elm}\mathsf{T}} \mathsf{D}_{ab}^{s\mathrm{elm}} \mathsf{f}_b^{\mathrm{elm}},
\end{aligned}
\tag{B.28}
$$

where

$$
\begin{aligned}
\mathsf{D}_{abIJ}^{s\mathrm{elm}} = \sum_{ijklmn}^{2} \sum_{KL}^{n_{\mathrm{node}}} \tau_a^{\mathrm{elm}} \iint \frac{\partial \phi_I^{\mathrm{elm}}}{\partial \eta_l} \frac{\partial}{\partial \eta_m} \left(\frac{\partial \phi_J^{\mathrm{elm}}}{\partial \eta_n} \phi_K^{\mathrm{elm}} J_{kn}^{*} \right) \phi_L^{\mathrm{elm}} J_{jm}^{*} J_{il}^{*} J d\eta_1 d\eta_2 \\
\times \left\{ \sqrt{g} D_{ab}^{\xi_j \xi_k} \right\}_K^{\mathrm{elm}} \left\{ u_a^{\xi_i} \right\}_L^{\mathrm{elm}}
\end{aligned}
\tag{B.29}
$$

SUPG gradient submatrix (1): $\mathsf{A}_{ab}^{1s\mathrm{elm}}$

$$
\begin{aligned}
&\sum_{ij}^{2} \iint \tau_a^{\mathrm{elm}} u_a^{\xi_i \mathrm{elm}} \frac{\partial w_a^{\mathrm{elm}}}{\partial \xi_i} \left(\sqrt{g} A_{ab}^{\xi_j} \right)^{\mathrm{elm}} \frac{\partial f_b^{\mathrm{elm}}}{\partial \xi_j} d\rho d\chi \\
&\quad = \sum_{ijkl}^{2} \sum_{IJKL}^{n_{\mathrm{node}}} \iint \tau_a^{\mathrm{elm}} \left\{ u_a^{\xi_i} \right\}_L^{\mathrm{elm}} \phi_L^{\mathrm{elm}} J_{ik}^{*} \frac{\partial}{\partial \eta_k} \left(\left\{ w_a \right\}_I^{\mathrm{elm}} \phi_I^{\mathrm{elm}} \right) \\
&\qquad\qquad \times \left\{ \sqrt{g} A_{ab}^{\xi_j} \right\}_K^{\mathrm{elm}} \phi_K^{\mathrm{elm}} J_{jl}^{*} \frac{\partial}{\partial \eta_l} \left(\left\{ f_b \right\}_J^{\mathrm{elm}} \phi_J^{\mathrm{elm}} \right) J d\eta_1 d\eta_2 \\
&\quad = \sum_{ijkl}^{2} \sum_{IJKL}^{n_{\mathrm{node}}} \left\{ w_a \right\}_I^{\mathrm{elm}} \\
&\quad \times \left[\tau_a^{\mathrm{elm}} \iint \frac{\partial \phi_I^{\mathrm{elm}}}{\partial \eta_k} \frac{\partial \phi_J^{\mathrm{elm}}}{\partial \eta_l} \phi_K^{\mathrm{elm}} \phi_L^{\mathrm{elm}} J_{ik}^{*} J_{jl}^{*} J d\eta_1 d\eta_2 \left\{ \sqrt{g} A_{ab}^{\xi_j} \right\}_K^{\mathrm{elm}} \left\{ u_a^{\xi_i} \right\}_L^{\mathrm{elm}} \right] \left\{ f_b \right\}_J^{\mathrm{elm}} \\
&\quad = \mathsf{w}_a^{\mathrm{elm}\mathsf{T}} \mathsf{A}_{ab}^{1s\mathrm{elm}} \mathsf{f}_b^{\mathrm{elm}},
\end{aligned}
\tag{B.30}
$$

where

$$
\mathsf{A}_{abIJ}^{1s\mathrm{elm}} = \sum_{ijkl}^{2} \sum_{KL}^{n_{\mathrm{node}}} \tau_a^{\mathrm{elm}} \iint \frac{\partial \phi_I^{\mathrm{elm}}}{\partial \eta_k} \frac{\partial \phi_J^{\mathrm{elm}}}{\partial \eta_l} \phi_K^{\mathrm{elm}} \phi_L^{\mathrm{elm}} J_{ik}^{*} J_{jl}^{*} J d\eta_1 d\eta_2 \left\{ \sqrt{g} A_{ab}^{\xi_j} \right\}_K^{\mathrm{elm}} \left\{ u_a^{\xi_i} \right\}_L^{\mathrm{elm}} \tag{B.31}
$$

SUPG gradient submatrix (2): $\mathsf{A}_{ab}^{2s\mathrm{elm}}$

$$\begin{aligned}
&\sum_x \sum_{ijk}^{2} \iint \tau_a^{\mathrm{elm}} u_a^{\xi_i \mathrm{elm}} \frac{\partial w_a^{\mathrm{elm}}}{\partial \xi_i} \frac{\partial g_x^{\mathrm{elm}}}{\partial \xi_j} \left(\sqrt{g} A_{abx}^{\xi_j \xi_k}\right)^{\mathrm{elm}} \frac{\partial f_b^{\mathrm{elm}}}{\partial \xi_k} d\rho d\chi \\
&\quad = \sum_x \sum_{ijklmn}^{2} \sum_{IJKLM}^{n_{\mathrm{node}}} \iint \tau_a^{\mathrm{elm}} \left\{u_a^{\xi_i}\right\}_M^{\mathrm{elm}} \phi_M^{\mathrm{elm}} J_{il}^{*} \frac{\partial}{\partial \eta_l} \left(\{w_a\}_I^{\mathrm{elm}} \phi_I^{\mathrm{elm}}\right) J_{jm}^{*} \frac{\partial}{\partial \eta_m} \left(\{g_x\}_K^{\mathrm{elm}} \phi_K^{\mathrm{elm}}\right) \\
&\qquad\qquad \times \left\{\sqrt{g} A_{abx}^{\xi_j \xi_k}\right\}_L^{\mathrm{elm}} \phi_L^{\mathrm{elm}} J_{kn}^{*} \frac{\partial}{\partial \eta_n} \left(\{f_b\}_J^{\mathrm{elm}} \phi_J^{\mathrm{elm}}\right) J d\eta_1 d\eta_2 \\
&\quad = \sum_x \sum_{ijklmn}^{2} \sum_{IJKLM}^{n_{\mathrm{node}}} \{w_a\}_I^{\mathrm{elm}} \left[\tau_a^{\mathrm{elm}} \iint \frac{\partial \phi_I^{\mathrm{elm}}}{\partial \eta_l} \frac{\partial \phi_J^{\mathrm{elm}}}{\partial \eta_n} \frac{\partial \phi_K^{\mathrm{elm}}}{\partial \eta_m} \phi_L^{\mathrm{elm}} \phi_M^{\mathrm{elm}} J_{il}^{*} J_{jm}^{*} J_{kn}^{*} J d\eta_1 d\eta_2 \right. \\
&\qquad\qquad \left. \times \{g_x\}_K^{\mathrm{elm}} \left\{\sqrt{g} A_{abx}^{\xi_j \xi_k}\right\}_L^{\mathrm{elm}} \left\{u_a^{\xi_i}\right\}_M^{\mathrm{elm}}\right] \{f_b\}_J^{\mathrm{elm}} \\
&\quad = \mathsf{w}_a^{\mathrm{elm}\mathsf{T}} \mathsf{A}_{ab}^{2s\mathrm{elm}} \mathsf{f}_b^{\mathrm{elm}},
\end{aligned} \tag{B.32}$$

where

$$\begin{aligned}
\mathsf{A}_{abIJ}^{2s\mathrm{elm}} = \sum_x \sum_{ijklmn}^{2} \sum_{KLM}^{n_{\mathrm{node}}} \tau_a^{\mathrm{elm}} \iint \frac{\partial \phi_I^{\mathrm{elm}}}{\partial \eta_l} \frac{\partial \phi_J^{\mathrm{elm}}}{\partial \eta_n} \frac{\partial \phi_K^{\mathrm{elm}}}{\partial \eta_m} \phi_L^{\mathrm{elm}} \phi_M^{\mathrm{elm}} J_{il}^{*} J_{jm}^{*} J_{kn}^{*} J d\eta_1 d\eta_2 \\
\times \{g_x\}_K^{\mathrm{elm}} \left\{\sqrt{g} A_{abx}^{\xi_j \xi_k}\right\}_L^{\mathrm{elm}} \left\{u_a^{\xi_i}\right\}_M^{\mathrm{elm}}
\end{aligned} \tag{B.33}$$

SUPG excitation submatrix (1): $\mathsf{C}_{ab}^{1s\mathrm{elm}}$

$$\begin{aligned}
&\sum_i^{2} \iint \tau_a^{\mathrm{elm}} u_a^{\xi_i \mathrm{elm}} \frac{\partial w_a^{\mathrm{elm}}}{\partial \xi_i} \left(\sqrt{g} C_{ab}\right)^{\mathrm{elm}} f_b^{\mathrm{elm}} d\rho d\chi \\
&\quad = \sum_{ij}^{2} \sum_{IJKL}^{n_{\mathrm{node}}} \iint \tau_a^{\mathrm{elm}} \left\{u_a^{\xi_i}\right\}_L^{\mathrm{elm}} \phi_L^{\mathrm{elm}} J_{ij}^{*} \frac{\partial}{\partial \eta_j} \left(\{w_a\}_I^{\mathrm{elm}} \phi_I^{\mathrm{elm}}\right) \\
&\qquad\qquad \times \left\{\sqrt{g} C_{ab}\right\}_K^{\mathrm{elm}} \phi_K^{\mathrm{elm}} \{f_b\}_J^{\mathrm{elm}} \phi_J^{\mathrm{elm}} J d\eta_1 d\eta_2 \\
&\quad = \sum_{ij}^{2} \sum_{IJKL}^{n_{\mathrm{node}}} \{w_a\}_I^{\mathrm{elm}} \left[\tau_a^{\mathrm{elm}} \iint \frac{\partial \phi_I^{\mathrm{elm}}}{\partial \eta_j} \phi_J^{\mathrm{elm}} \phi_K^{\mathrm{elm}} \phi_L^{\mathrm{elm}} J_{ij}^{*} J d\eta_1 d\eta_2 \left\{\sqrt{g} C_{ab}\right\}_K^{\mathrm{elm}} \left\{u_a^{\xi_i}\right\}_L^{\mathrm{elm}}\right] \{f_b\}_J^{\mathrm{elm}} \\
&\quad = \mathsf{w}_a^{\mathrm{elm}\mathsf{T}} \mathsf{C}_{ab}^{1s\mathrm{elm}} \mathsf{f}_b^{\mathrm{elm}},
\end{aligned} \tag{B.34}$$

where

$$\mathsf{C}_{abIJ}^{1s\mathrm{elm}} = \sum_{ij}^{2} \sum_{KL}^{n_{\mathrm{node}}} \tau_a^{\mathrm{elm}} \iint \frac{\partial \phi_I^{\mathrm{elm}}}{\partial \eta_j} \phi_J^{\mathrm{elm}} \phi_K^{\mathrm{elm}} \phi_L^{\mathrm{elm}} J_{ij}^{*} J d\eta_1 d\eta_2 \left\{\sqrt{g} C_{ab}\right\}_K^{\mathrm{elm}} \left\{u_a^{\xi_i}\right\}_L^{\mathrm{elm}} \tag{B.35}$$

SUPG excitation submatrix (2): $\mathsf{C}_{ab}^{2s\mathrm{elm}}$

$$
\begin{aligned}
&\sum_{x}\sum_{ij}^{2}\iint \tau_a^{\mathrm{elm}} u_a^{\xi_i \mathrm{elm}} \frac{\partial w_a^{\mathrm{elm}}}{\partial \xi_i}\frac{\partial g_x^{\mathrm{elm}}}{\partial \xi_j}\left(\sqrt{g}C_{abx}^{\xi_j}\right)^{\mathrm{elm}} f_b^{\mathrm{elm}} d\rho d\chi \\
&= \sum_{x}\sum_{ijkl}^{2}\sum_{IJKLM}^{n_{\mathrm{node}}}\iint \tau_a^{\mathrm{elm}}\left\{u_a^{\xi_i}\right\}_M^{\mathrm{elm}}\phi_M^{\mathrm{elm}} J_{ik}^{*}\frac{\partial}{\partial \eta_k}\left(\{w_a\}_I^{\mathrm{elm}}\phi_I^{\mathrm{elm}}\right) \\
&\qquad\qquad \times J_{jl}^{*}\frac{\partial}{\partial \eta_l}\left(\{g_x\}_K^{\mathrm{elm}}\phi_K^{\mathrm{elm}}\right)\left\{\sqrt{g}C_{abx}^{\xi_j}\right\}_L^{\mathrm{elm}}\phi_L^{\mathrm{elm}}\{f_b\}_J^{\mathrm{elm}}\phi_J^{\mathrm{elm}} J d\eta_1 d\eta_2 \\
&= \sum_{x}\sum_{ijkl}^{2}\sum_{IJKLM}^{n_{\mathrm{node}}}\{w_a\}_I^{\mathrm{elm}}\left[\tau_a^{\mathrm{elm}}\iint \frac{\partial \phi_I^{\mathrm{elm}}}{\partial \eta_k}\phi_J^{\mathrm{elm}}\frac{\partial \phi_K^{\mathrm{elm}}}{\partial \eta_l}\phi_L^{\mathrm{elm}}\phi_M^{\mathrm{elm}} J_{ik}^{*}J_{jl}^{*} J d\eta_1 d\eta_2\right. \\
&\qquad\qquad \left.\times \{g_x\}_K^{\mathrm{elm}}\left\{\sqrt{g}C_{abx}^{\xi_j}\right\}_L^{\mathrm{elm}}\left\{u_a^{\xi_i}\right\}_M^{\mathrm{elm}}\right]\{f_b\}_J^{\mathrm{elm}} \\
&= \mathsf{w}_a^{\mathrm{elm}\mathsf{T}}\mathsf{C}_{ab}^{2s\mathrm{elm}}\mathsf{f}_b^{\mathrm{elm}},
\end{aligned}
\tag{B.36}
$$

where

$$
\begin{aligned}
\mathsf{C}_{ab}^{2s\mathrm{elm}} = \sum_{x}\sum_{ijkl}^{2}\sum_{KLM}^{n_{\mathrm{node}}}\tau_a^{\mathrm{elm}}\iint \frac{\partial \phi_I^{\mathrm{elm}}}{\partial \eta_k}\phi_J^{\mathrm{elm}}\frac{\partial \phi_K^{\mathrm{elm}}}{\partial \eta_l}\phi_L^{\mathrm{elm}}\phi_M^{\mathrm{elm}} J_{ik}^{*}J_{jl}^{*} J d\eta_1 d\eta_2 \\
\times \{g_x\}_K^{\mathrm{elm}}\left\{\sqrt{g}C_{abx}^{\xi_j}\right\}_L^{\mathrm{elm}}\left\{u_a^{\xi_i}\right\}_M^{\mathrm{elm}}
\end{aligned}
\tag{B.37}
$$

SUPG excitation submatrix (3): $\mathsf{C}_{ab}^{3s\mathrm{elm}}$

$$
\begin{aligned}
&\sum_{xy}\sum_{ijk}^{2}\iint \tau_a^{\mathrm{elm}} u_a^{\xi_i \mathrm{elm}} \frac{\partial w_a^{\mathrm{elm}}}{\partial \xi_i}\frac{\partial g_x^{\mathrm{elm}}}{\partial \xi_j}\left(\sqrt{g}C_{abxy}^{\xi_j\xi_k}\right)^{\mathrm{elm}}\frac{\partial g_y^{\mathrm{elm}}}{\partial \xi_k} f_b^{\mathrm{elm}} d\rho d\chi \\
&= \sum_{xy}\sum_{ijklmn}^{2}\sum_{IJKLMN}^{n_{\mathrm{node}}}\{w_a\}_I^{\mathrm{elm}}\left[\tau_a^{\mathrm{elm}}\iint \frac{\partial \phi_I^{\mathrm{elm}}}{\partial \eta_l}\phi_J^{\mathrm{elm}}\frac{\partial \phi_K^{\mathrm{elm}}}{\partial \eta_m}\frac{\partial \phi_L^{\mathrm{elm}}}{\partial \eta_n}\phi_M^{\mathrm{elm}}\phi_N^{\mathrm{elm}} J_{il}^{*}J_{jm}^{*}J_{kn}^{*} J d\eta_1 d\eta_2\right. \\
&\qquad\qquad \left.\times \{g_x\}_K^{\mathrm{elm}}\{g_y\}_L^{\mathrm{elm}}\left\{\sqrt{g}C_{abxy}^{\xi_j\xi_k}\right\}_M^{\mathrm{elm}}\left\{u_a^{\xi_i}\right\}_N^{\mathrm{elm}}\right]\{f_b\}_J^{\mathrm{elm}} \\
&= \mathsf{w}_a^{\mathrm{elm}\mathsf{T}}\mathsf{C}_{ab}^{3s\mathrm{elm}}\mathsf{f}_b^{\mathrm{elm}},
\end{aligned}
\tag{B.38}
$$

where

$$
\begin{aligned}
\mathsf{C}_{abIJ}^{3s\mathrm{elm}} = \sum_{xy}\sum_{ijklmn}^{2}\sum_{KLMN}^{n_{\mathrm{node}}}\tau_a^{\mathrm{elm}}\iint \frac{\partial \phi_I^{\mathrm{elm}}}{\partial \eta_l}\phi_J^{\mathrm{elm}}\frac{\partial \phi_K^{\mathrm{elm}}}{\partial \eta_m}\frac{\partial \phi_L^{\mathrm{elm}}}{\partial \eta_n}\phi_M^{\mathrm{elm}}\phi_N^{\mathrm{elm}} J_{il}^{*}J_{jm}^{*}J_{kn}^{*} J d\eta_1 d\eta_2 \\
\times \{g_x\}_K^{\mathrm{elm}}\{g_y\}_L^{\mathrm{elm}}\left\{\sqrt{g}C_{abxy}^{\xi_j\xi_k}\right\}_M^{\mathrm{elm}}\left\{u_a^{\xi_i}\right\}_N^{\mathrm{elm}}
\end{aligned}
\tag{B.39}
$$

SUPG source subvector: $\mathsf{S}_a^{s\mathrm{elm}}$

$$\begin{aligned}
&\sum_i^2 \iint \tau_a^{\mathrm{elm}} u_a^{\xi_i \mathrm{elm}} \frac{\partial w_a^{\mathrm{elm}}}{\partial \xi_i} \left(\sqrt{g} S_a\right)^{\mathrm{elm}} d\rho d\chi \\
&= \sum_{ij}^2 \sum_{IJK}^{n_{\mathrm{node}}} \iint \tau_a^{\mathrm{elm}} \left\{u_a^{\xi_i}\right\}_K^{\mathrm{elm}} \phi_K^{\mathrm{elm}} J_{ij}^* \frac{\partial}{\partial \eta_j} \left(\left\{w_a\right\}_I^{\mathrm{elm}} \phi_I^{\mathrm{elm}}\right) \left\{\sqrt{g} S_a\right\}_J^{\mathrm{elm}} \phi_J^{\mathrm{elm}} J d\eta_1 d\eta_2 \\
&= \sum_{ij}^2 \sum_{IJK}^{n_{\mathrm{node}}} \left\{w_a\right\}_I^{\mathrm{elm}} \left[\tau_a^{\mathrm{elm}} \iint \frac{\partial \phi_I^{\mathrm{elm}}}{\partial \eta_j} \phi_J^{\mathrm{elm}} \phi_K^{\mathrm{elm}} J_{ij}^* J d\eta_1 d\eta_2 \left\{\sqrt{g} S_a\right\}_J^{\mathrm{elm}} \left\{u_a^{\xi_i}\right\}_K^{\mathrm{elm}}\right] \\
&= \mathsf{w}_a^{\mathrm{elm}\mathsf{T}} \mathsf{S}_a^{s\mathrm{elm}},
\end{aligned} \tag{B.40}$$

where

$$\mathsf{S}_{aI}^{\mathrm{elm}} = \sum_{ij}^2 \sum_{JK}^{n_{\mathrm{node}}} \tau_a^{\mathrm{elm}} \iint \frac{\partial \phi_I^{\mathrm{elm}}}{\partial \eta_j} \phi_J^{\mathrm{elm}} \phi_K^{\mathrm{elm}} J_{ij}^* J d\eta_1 d\eta_2 \left\{\sqrt{g} S_a\right\}_J^{\mathrm{elm}} \left\{u_a^{\xi_i}\right\}_K^{\mathrm{elm}} \tag{B.41}$$