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# Lifshitz tails estimate for the density of states of the Anderson model

By

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## Abstract

We prove an upper bound for the (differentiated) density of states of the Anderson model at the bottom of the spectrum. The density of states is shown to exhibit the same Lifshitz tails upper bound as the integrated density of states.

## § 1. Introduction and main result

We consider the Anderson model, the simplest random Schrödinger operator, given by the random Hamiltonian

$$(1.1) \quad H_{\omega} := -\Delta + V_{\omega} \quad \text{on} \quad \ell^2(\mathbb{Z}^d),$$

$-\Delta$  is the  $d$ -dimensional discrete Laplacian operator (i.e.  $-\Delta u(j) = \sum_{|k-j|=1} u(k)$ , where  $u \in \ell^2(\mathbb{Z}^d)$ ) and  $V_{\omega}$  is the random potential given by  $V_{\omega}(j) = \omega_j$  for  $j \in \mathbb{Z}^d$ , where  $\omega = \{\omega_j\}_{j \in \mathbb{Z}^d}$  is a family of independent identically distributed random variables whose common probability distribution  $\mu$  is non-degenerate, with support bounded from below, and has a bounded density  $\rho$ . (The requirement  $\inf \text{supp} \mu > -\infty$  is equivalent to requiring  $H_{\omega}$  to be bounded from below with probability one.) Note that

$$(1.2) \quad V_{\omega} = \sum_{j \in \mathbb{Z}^d} \omega_j \Pi_j,$$

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where  $\Pi_j$  is the orthogonal projection onto  $\delta_j$ , the delta function at site  $j$ .

The integrated density of states (IDS) of this celebrated model is known to exhibit Lifshitz tails behavior at the bottom of its spectrum (e.g., [CaL, K, PF]), a property that can be interpreted as a first signature of localization.

In the physics literature there is much interest on the density of the IDS, the density of states (DOS). There is an implicit assumption that the IDS  $N(E)$  is absolutely continuous, and hence has a density  $n(E)$  given by its almost everywhere derivative. Very few mathematical results are available for the DOS of random Schrödinger operators. For nice enough models, like the Anderson model as above, the existence of the DOS is a consequence of the celebrated Wegner estimate (see (2.4) below), which also shows the DOS to be bounded. We note that while the absolute continuity of the IDS has been known for a long time for the Anderson model [W, FS], this is a more recent result for models in the continuum [CoH, CoHK]. In dimension  $d = 1$  this DOS is known to be regular [ST, CK, VSW]. For arbitrary dimension  $d$ , the DOS of the Anderson model described above is known to be regular at high disorder, a result obtained using supersymmetric methods by [BoCKP]. Regularity of the DOS is an open question for models in the continuum.

In this article, we show that the DOS exhibits the same Lifshitz tails upper bound as the IDS. To our best knowledge, this is the first time that such a bound is proved for the DOS of a random Schrödinger operator.

There is another reason to study the DOS  $n(E)$ . As proved by Minami [M], in the localization region the properly rescaled eigenvalues of a discrete Anderson Hamiltonian are distributed as a Poisson point process with intensity  $n(E)$ . Thus our results (see (2.13)) estimate the intensity of these Poisson point processes.

An Anderson Hamiltonian  $H_\omega$  is a  $\mathbb{Z}^d$ -ergodic family of random self-adjoint operators. It follows from standard results (cf. [KM]) that there exists fixed subsets  $\Sigma$ ,  $\Sigma_{\text{pp}}$ ,  $\Sigma_{\text{ac}}$  and  $\Sigma_{\text{sc}}$  of  $\mathbb{R}$  so that the spectrum  $\sigma(H_\omega)$  of  $H_\omega$ , as well as its pure point, absolutely continuous, and singular continuous components, are equal to these fixed sets with probability one. This non-random spectrum is given by

$$(1.3) \quad \Sigma = \sigma(-\Delta) + \text{supp } \mu = [0, 4d] + \text{supp } \mu.$$

Note  $\inf \Sigma = \inf \text{supp } \mu > -\infty$ .

If we set  $\omega'_j = \omega_j - \inf \text{supp } \mu$ , the new common probability distribution  $\mu'$  satisfies  $\inf \text{supp } \mu' = 0$ , and we have  $H_\omega = H_{\omega'} + \inf \text{supp } \mu$ . Thus, without loss of generality we assume from now on that

$$(1.4) \quad \inf \Sigma = \inf \text{supp } \mu = 0.$$

Lifshitz tails and localization are known to hold at the bottom of the spectrum [FMSS, DK, K]. If the support of  $\mu$  is also bounded from above, then so is  $\Sigma$ , and

Lifshitz tails and localization also hold at the top, i.e., upper edge, of the spectrum. In this case the results of this paper also hold at the top of the spectrum.

The integrated density of states (IDS),  $N(E)$ , can be written as (e.g., [K])

$$(1.5) \quad N(E) = \mathbb{E} \left\{ \text{tr} \left( \Pi_0 \chi_{]-\infty, E]}(H_\omega) \Pi_0 \right) \right\} = \mathbb{E} \left\{ \text{tr} \left( \Pi_0 \chi_{[0, E]}(H_\omega) \Pi_0 \right) \right\}.$$

The Anderson model is known to satisfy the following Lifshitz tails estimate, which asserts that the IDS has an exponential fall off as one approaches the edges of  $\Sigma$ . At the bottom of the spectrum, i.e., at energy  $E = 0$ , the IDS satisfies (e.g., [K])

$$(1.6) \quad \lim_{E \downarrow 0} \frac{\log |\log N(E)|}{\log E} \leq -\frac{d}{2}.$$

Equality is actually known to hold in (1.6). Since  $N(E)$  is an increasing function, it has a derivative  $n(E) := N'(E)$  almost everywhere, which is the density of states.

We prove

**Theorem 1.1.** *Let  $H_\omega$  be an Anderson Hamiltonian. Then there exists a Borel set  $\mathcal{N} \subset [0, 1]$  of zero Lebesgue measure, such that*

$$(1.7) \quad \lim_{E \downarrow 0; E \notin \mathcal{N}} \frac{\log |\log n(E)|}{\log E} \leq -\frac{d}{2}.$$

*Remark 1.2.* The same Lifshitz tails estimate holds for models in the continuum; see [CoGK3].

The proof of Theorem 1.1 takes advantage of a new double averaging procedure introduced in [CoGK2, Theorem 2.2] to extract better control on the constant in the Wegner estimate. Theorem 1.1 will be an immediate consequence of Theorem 2.1.

## § 2. Finite volume operators, the integrated density of states, and the Wegner estimate

Finite volume operators will be defined for finite boxes

$$(2.1) \quad \Lambda = \Lambda_L(j) := j + \left[-\frac{L}{2}, \frac{L}{2}\right]^d,$$

where  $j \in \mathbb{Z}^d$  and  $L \in 2\mathbb{N}$ ,  $L > 1$ . Given such  $\Lambda$ , we will consider the random Schrödinger operator  $H_\omega^{(\Lambda)}$  on  $\ell^2(\Lambda)$  given by the restriction of the Anderson Hamiltonian  $H_\omega$  to  $\Lambda$  with periodic boundary condition. To do so, we identify  $\Lambda$  with the torus  $\mathbb{Z}^d/L\mathbb{Z}^d$  in the usual way, and define finite volume operators

$$(2.2) \quad H_\omega^{(\Lambda)} := -\Delta^{(\Lambda)} + V_\omega^{(\Lambda)} \quad \text{on} \quad \ell^2(\Lambda),$$

where  $\Delta^{(\Lambda)}$  is the Laplacian on  $\Lambda$  with periodic boundary condition, and the random potential  $V_{\omega}^{(\Lambda)}$  is the restriction of  $V_{\omega^{(\Lambda)}}$  to  $\Lambda$ , where, given  $\omega = \{\omega_i\}_{i \in \mathbb{Z}^d}$ , we define  $\omega^{(\Lambda)} = \{\omega_i^{(\Lambda)}\}_{i \in \mathbb{Z}^d}$  by

$$(2.3) \quad \begin{aligned} \omega_i^{(\Lambda)} &= \omega_i \quad \text{if } i \in \Lambda, \\ \omega_i^{(\Lambda)} &= \omega_k^{(\Lambda)} \quad \text{if } k - i \in L\mathbb{Z}^d. \end{aligned}$$

The finite volume random operator  $H_{\omega}^{(\Lambda)}$  is covariant with respect to translations in the torus. If  $B \subset \mathbb{R}$  is a Borel set, we write  $P_{\omega}^{(\Lambda)}(B) := \chi_B \left( H_{\omega}^{(\Lambda)} \right)$  and  $P_{\omega}(B) := \chi_B(H_{\omega})$  for the spectral projections.

The finite volume operator  $H_{\omega}^{(\Lambda)}$  is a finite dimensional operator, and hence its ( $\omega$ -dependent) spectrum consists of a finite number of isolated eigenvalues with finite multiplicity. These finite volume operators satisfy a Wegner estimate [W, FS] (see also [CoGK1]), which provides informations on the number of eigenvalues in a given energy interval: given  $E_0 > 0$ , there exists a constant  $K_W(E_0)$ , independent of  $\Lambda$ , such that for all intervals  $I \subset [0, E_0]$  we have

$$(2.4) \quad \mathbb{E} \left\{ \text{tr } P_{\omega}^{(\Lambda)}(I) \right\} \leq K_W(E_0) \|\rho\|_{\infty} |I| |\Lambda|.$$

The Wegner estimate is an immediate consequence of the following spectral averaging property (e.g., [CoHK, CoGK1]) : for any  $k \in \Lambda$ ,

$$(2.5) \quad \mathbb{E}_{\omega_k} \left\{ \langle \delta_k, P_{\omega}^{(\Lambda)}(I) \delta_k \rangle \right\} = \mathbb{E}_{\omega_k} \left\{ \text{tr } \Pi_k P_{\omega}^{(\Lambda)}(I) \Pi_k \right\} \leq \|\rho\|_{\infty} |I|,$$

where  $\mathbb{E}_{\omega_k}$  denotes integration over the variable  $\omega_k$ , and  $(\delta_k)_{k \in \mathbb{Z}^d}$  stands for the canonical basis (so  $\Pi_k = |\delta_k\rangle\langle\delta_k|$ ). It follows that for the Anderson model (2.4) holds with

$$(2.6) \quad K_W(E_0) \leq 1 \quad \text{for all } E_0 > 0.$$

We shall prove that at the bottom of the spectrum, in the Lifshitz tails region, the constant  $K_W(E_0)$  falls off in the same way as the IDS.

We set

$$(2.7) \quad N_{\omega}^{(\Lambda)}(E) := |\Lambda|^{-1} \text{tr } \chi_{]-\infty, E]} \left( H_{\omega}^{(\Lambda)} \right),$$

and recall that (e.g., [CaL, K, PF]) for  $\mathbb{P}$ -a.e.  $\omega$  we have, using the fact that  $N(E)$  is a continuous function by (2.4),

$$(2.8) \quad N(E) = \lim_{L \rightarrow \infty} N_{\omega}^{(\Lambda L(0))}(E) \quad \text{for all } E \in \mathbb{R},$$

where  $N(E)$  is the integrated density of states (IDS) given in (1.5)). Setting

$$(2.9) \quad N^{(\Lambda)}(E) := \mathbb{E} \left\{ N_{\omega}^{(\Lambda)}(E) \right\} = \mathbb{E} \left\{ \text{tr} \left( \Pi_j \chi_{]-\infty, E]} \left( H_{\omega}^{(\Lambda)} \right) \Pi_j \right) \right\} \quad \text{for } j \in \Lambda,$$

the last equality holding in view of the periodic boundary condition, it follows that

$$(2.10) \quad N(E) = \lim_{L \rightarrow \infty} N^{(\Lambda_L(0))}(E) = \mathbb{E} \left\{ \text{tr} \left( \Pi_0 \chi_{]-\infty, E]}(H_\omega) \Pi_0 \right) \right\}$$

for all  $E \in \mathbb{R}$ .

Combining (2.4) and (2.10), we conclude that, if  $N(E)$  is differentiable at  $E$ , which is true for a.e.  $E$ , we have

$$(2.11) \quad n(E) := N'(E) \leq K_W(E) \|\rho\|_\infty.$$

Moreover, to obtain (2.11), it suffices to have the Wegner estimate (2.4) for boxes  $\Lambda_{L_n}(0)$  with  $L_n \rightarrow \infty$ . Thus Theorem 1.1 follows immediately from the following result.

**Theorem 2.1.** *Let  $H_\omega$  be an Anderson Hamiltonian. Then there is an energy  $E_0 > 0$ , and for a.e.  $E \in ]0, E_0]$  and all  $\varepsilon \in ]0, \frac{d}{2}[$  there exists a scale  $L(E, \varepsilon) < \infty$ , such that given  $L \in 4\mathbb{N}$  with  $L \geq L(E, \varepsilon)$ , the Wegner estimate (2.4) holds in all boxes  $\Lambda = \Lambda_L$  for all intervals  $I \subset [0, E]$  with a constant  $K_W(E)$  satisfying*

$$(2.12) \quad K_W(E) \leq C_{d,\varepsilon} e^{-E^{-\frac{d}{2}+\varepsilon}},$$

for some constant  $C_{d,\varepsilon} < \infty$ . As a consequence, we have

$$(2.13) \quad n(E) \leq C_{d,\varepsilon} \|\rho\|_\infty e^{-E^{-\frac{d}{2}+\varepsilon}} \quad \text{for a.e. } E \in ]0, E_0].$$

### § 3. The proof of Theorem 2.1

We borrow from [CoGK3] the following observation.

**Lemma 3.1** ([CoGK3]). *Let  $H = H_0 + W$ , where  $H, H_0$  are semi-bounded self-adjoint operators, say  $H, H_0 \geq -\Theta$  for some  $\Theta > 0$ , such that  $(H + \Theta + 1)^{-p}$  is a trace class operator for some  $p > 0$ , and  $W$  is a bounded self-adjoint operator. Given  $E_0 \in \mathbb{R}$ , let  $f, g$  be bounded Borel measurable nonnegative functions such that  $f = \chi_{(-\infty, E_0]} f$  and  $\chi_{(-\infty, E_0]} \leq g \leq 1$ . Then  $f(H)W$  and  $f(H)Wg(H_0)$  are trace class operators, and*

$$(3.1) \quad \text{tr} f(H)W \leq \text{tr} f(H)Wg(H_0).$$

The proof is elementary. It consists in proving that  $\text{tr} f(H)W(1 - g(H_0)) \leq 0$ , using  $W = H - H_0$ .

*Proof of Theorem 2.1.* All the operators in this proofs will be finite volume operators on a box  $\Lambda = \Lambda_L(0)$  as defined in (2.2)-(2.3). We will require  $L \in 4\mathbb{N}$ .

We set  $\omega_0^\perp = \omega \setminus \{\omega_0\}$  and  $H_{\omega_0^\perp} = H_\omega - \omega_0 \Pi_0$ . Given  $E > 0$ , we fix a  $C^\infty$  real-valued non-increasing function  $f_E$  on  $\mathbb{R}$ , such that  $f_E(t) = 1$  for  $t \leq E$ ,  $f_E(t) = 0$  for

$t \geq 2E$ , and  $\left|f_E^{(j)}(t)\right| \leq CE^{-j}$  for all  $t \in \mathbb{R}$  and  $j = 1, 2, \dots, 2d + 3$ , where  $C$  is a constant independent of  $E$ . We let  $\tilde{P}_0 = \tilde{P}_{\omega_0^\perp, E}^{(\Lambda)} = f_E(H_{\omega_0^\perp}^{(\Lambda)})$ .

Given an interval  $I \subset [0, E]$ , it follows from Lemma 3.1 that

$$\begin{aligned}
\text{tr } P_\omega^{(\Lambda)}(I)\Pi_0 &\leq \text{tr } P_\omega^{(\Lambda)}(I)\Pi_0\tilde{P}_0 = \sum_{k \in \Lambda} \text{tr } P_\omega^{(\Lambda)}(I)\Pi_0\tilde{P}_0\Pi_k \\
(3.2) \quad &\leq \sum_{k \in \Lambda} \|P_\omega^{(\Lambda)}(I)\Pi_0\|_2 \|P_\omega^{(\Lambda)}(I)\Pi_k\|_2 \|\Pi_0\tilde{P}_0\Pi_k\| \\
&\leq \frac{1}{2} \sum_{k \in \Lambda} \left\{ \text{tr}(\Pi_0 P_\omega^{(\Lambda)}(I)\Pi_0) + \text{tr}(\Pi_k P_\omega^{(\Lambda)}(I)\Pi_k) \right\} \|\Pi_0\tilde{P}_0\Pi_k\|.
\end{aligned}$$

Next, for any given  $k$  (including  $k = 0$ ) there exists  $k_0 \in \mathbb{Z}^d$ , so that, defining the sublattice  $\Gamma_k = k_0 + (4\mathbb{Z})^d \subset \mathbb{Z}^d$ , we have  $0, k \notin \Gamma_k$ . Since we required  $L \in 4\mathbb{N}$ , we have  $|\Gamma_k \cap \Lambda| = 4^{-d}|\Lambda|$ . We define  $\omega_{\Gamma_k} = \{\omega_\gamma\}_{\gamma \in \Gamma_k}$ . We have, using the fast decay of the kernel of smooth functions of Schrödinger operators (e.g. [GK]), using the notation  $\langle k \rangle = \sqrt{1 + |k|^2}$ , and letting  $r$  stand for either 0 or  $k$ ,

$$\begin{aligned}
\|\Pi_0\tilde{P}_0\Pi_k\| &\leq \|\Pi_0\tilde{P}_0\Pi_k\|^{\frac{1}{2}} \|\Pi_r\tilde{P}_0\|^{\frac{1}{2}} \leq \|\Pi_0\tilde{P}_0\Pi_k\|^{\frac{1}{2}} \|\Pi_r\tilde{P}_0\Pi_r\|^{\frac{1}{4}} \\
(3.3) \quad &\leq \frac{C_d}{E^{d+2} \langle k \rangle^{d+1}} \left( \text{tr } \Pi_r\tilde{P}_0\Pi_r \right)^{\frac{1}{4}} \\
&\leq \frac{C_d}{E^{d+2} \langle k \rangle^{d+1}} \left( e^{2tE} \text{tr } \Pi_r e^{-tH_{\omega_0^\perp}^{(\Lambda)}} \Pi_r \right)^{\frac{1}{4}} \\
&\leq \frac{C_d}{E^{d+2} \langle k \rangle^{d+1}} \left( e^{2tE} \text{tr } \Pi_r e^{-tH_{\omega_{\Gamma_k}^{(\Lambda)}}} \Pi_r \right)^{\frac{1}{4}},
\end{aligned}$$

for all  $t > 0$ , where we used  $r \notin \Gamma_k$  and a positivity preserving argument like [BGKS, Lemma 2.2] to get the last inequality. Hence, again letting  $r$  stand for either 0 or  $k$ , so  $r \notin \Gamma_k$ , using the spectral averaging (2.5) with the bound (2.6) and (3.3) leads to

$$\begin{aligned}
&\mathbb{E} \left( \text{tr} \left\{ \Pi_r P_\omega^{(\Lambda)}(I)\Pi_r \right\} \|\Pi_0\tilde{P}_0\Pi_k\| \right) \\
&\leq \frac{C_d}{E^{d+2} \langle k \rangle^{d+1}} \mathbb{E}_{\omega_r^\perp} \left\{ \left( e^{2tE} \text{tr } \Pi_r e^{-tH_{\omega_{\Gamma_k}^{(\Lambda)}}} \Pi_r \right)^{\frac{1}{4}} \mathbb{E}_{\omega_r} \left\{ \text{tr } \Pi_r P_\omega^{(\Lambda)}(I)\Pi_r \right\} \right\} \\
&\leq \frac{C_d}{E^{d+2} \langle k \rangle^{d+1}} \|\rho\|_\infty |I| e^{\frac{t}{2}E} \left( \mathbb{E}_{\omega_{\Gamma_k}} \left\{ \text{tr } \Pi_r e^{-tH_{\omega_{\Gamma_k}^{(\Lambda)}}} \Pi_r \right\}^{\frac{1}{4}} \right) \\
(3.4) \quad &\leq \frac{C_d}{E^{d+2} \langle k \rangle^{d+1}} \|\rho\|_\infty |I| e^{\frac{t}{2}E} \left( \mathbb{E}_{\omega_{\Gamma_k}} \left\{ \text{tr } \Pi_r e^{-tH_{\omega_{\Gamma_k}^{(\Lambda)}}} \Pi_r \right\} \right)^{\frac{1}{4}}.
\end{aligned}$$

Thanks to (3.2) and (3.4), it now suffices to bound the quantity

$$(3.5) \quad \mathbb{E}_{\omega_{\Gamma_k}} \left\{ \text{tr } \Pi_r e^{-tH_{\omega_{\Gamma_k}}^{(\Lambda)}} \Pi_r \right\} \quad \text{with } r = 0, k.$$

To alleviate the notation, we write from now on  $\Gamma = \Gamma_k = k_0 + (4\mathbb{Z})^d$ . For all  $t > 0$  we have

$$(3.6) \quad \begin{aligned} \text{tr } \Pi_r e^{-tH_{\omega_{\Gamma}}^{(\Lambda)}} \Pi_r &\leq \text{tr } \Pi_r \chi_{]-\infty, 4E]}(H_{\omega_{\Gamma}}^{(\Lambda)}) \Pi_r + e^{-4tE} \text{tr } \Pi_r \\ &= \text{tr } \Pi_r \chi_{]-\infty, 4E]}(H_{\omega_{\Gamma}}^{(\Lambda)}) \Pi_r + e^{-4tE}. \end{aligned}$$

By the  $(4\mathbb{Z})^d$ -ergodicity of  $H_{\omega_{\Gamma}}^{(\Lambda)}$ , and taking advantage of the periodic boundary condition, we have

$$(3.7) \quad \begin{aligned} \mathbb{E} \left\{ \text{tr } \Pi_r \chi_{]-\infty, 4E]}(H_{\omega_{\Gamma}}^{(\Lambda)}) \Pi_r \right\} &= \frac{1}{|\Gamma \cap \Lambda|} \sum_{\gamma \in (r + (4\mathbb{Z})^d) \cap \Lambda} \mathbb{E}_{\omega_{\Gamma}} \left\{ \text{tr } \Pi_{\gamma} \chi_{]-\infty, 4E]}(H_{\omega_{\Gamma}}^{(\Lambda)}) \Pi_{\gamma} \right\} \\ &\leq \frac{4^d}{|\Lambda|} \mathbb{E}_{\omega_{\Gamma}} \left\{ \text{tr } \chi_{]-\infty, 4E]}(H_{\omega_{\Gamma}}^{(\Lambda)}) \right\}. \end{aligned}$$

We now use the Lifshitz tails estimate for  $H_{\omega_{\Gamma}}$  to bound  $\mathbb{E}_{\omega_{\Gamma}} \left\{ \text{tr } \chi_{]-\infty, 4E]}(H_{\omega_{\Gamma}}^{(\Lambda)}) \right\}$ . Note that  $\Gamma$  is a strict sublattice of  $\mathbb{Z}^d$ , so we lack the so called covering condition. The Lifshitz tails estimate (1.6) is nevertheless valid for  $H_{\omega_{\Gamma}}$  (e.g. [K]), and it implies that for all  $\varepsilon \in ]0, \frac{d}{2}[$  there is an energy  $E_{\varepsilon} > 0$  such that (on  $\mathbb{Z}^d$ )

$$(3.8) \quad N_{\Gamma}(E) = \mathbb{E}_{\omega_{\Gamma}} \left\{ \text{tr } \Pi_{k_0} \chi_{]-\infty, E]}(H_{\omega_{\Gamma}}) \Pi_{k_0} \right\} \leq e^{-E^{-\frac{d}{2} + \varepsilon}} \quad \text{for all } E \leq E_{\varepsilon}.$$

Given a box  $\Lambda_L = \Lambda_L(0)$ , we set, similarly to (2.7) and (2.9),

$$(3.9) \quad N_{\omega_{\Gamma}}^{(\Lambda_L)}(E) = |\Lambda_L|^{-1} \text{tr } \chi_{]-\infty, E]}(H_{\omega_{\Gamma}}^{(\Lambda_L)}), \quad N_{\Gamma}^{(\Lambda_L)}(E) = \mathbb{E}_{\omega_{\Gamma}} \left\{ N_{\omega_{\Gamma}}^{(\Lambda_L)}(E) \right\}.$$

Since  $H_{\omega_{\Gamma}}$  is  $(4\mathbb{Z})^d$ -ergodic, for  $\mathbb{P}$ -a.e.  $\omega_{\Gamma}$  we have [CaL, PF]

$$(3.10) \quad N_{\Gamma}(E) = \lim_{L \rightarrow \infty} N_{\omega_{\Gamma}}^{(\Lambda_L)}(E) \quad \text{for a.e. } E \in \mathbb{R},$$

so we conclude

$$(3.11) \quad N_{\Gamma}(E) = \lim_{L \rightarrow \infty} N_{\Gamma}^{(\Lambda_L)}(E) \quad \text{for a.e. } E \in \mathbb{R}.$$

(Compare with (2.8) and (2.10), where convergence holds for all  $E$ . The difference is the lack of a Wegner estimate for  $H_{\omega_{\Gamma}}$ .) It follows that for a.e.  $E \in ]0, E_{\varepsilon}]$  there exists  $L(E, \varepsilon) < \infty$  such that for all  $L \geq L(E, \varepsilon)$  we have

$$(3.12) \quad N_{\Gamma}^{(\Lambda_L)}(E) \leq 2N_{\Gamma}(E) \leq 2e^{-E^{-\frac{d}{2} + \varepsilon}}.$$



Thus, for a.e.  $E \in ]0, \frac{1}{4}E_\varepsilon]$  and  $L \geq L(E, \varepsilon)$ , combining (3.6), (3.7), and (3.12), we get

$$(3.13) \quad \mathbb{E} \left\{ \text{tr} \Pi_r e^{-tH\omega_\Gamma} \Pi_r \right\} \leq 2 \cdot 4^d e^{-(4E)^{-\frac{d}{2}+\varepsilon}} + e^{-4tE}.$$

Choosing  $t = t_E$  by

$$(3.14) \quad e^{-4t_E E} = 2 \cdot 4^d e^{-(4E)^{-\frac{d}{2}+\varepsilon}}, \quad \text{i.e.,} \quad t_E = (4E)^{-\frac{d}{2}-1+\varepsilon} - (1+2d)(\log 2)(4E)^{-1},$$

and letting  $E'_\varepsilon = \max \{E \leq \frac{1}{4}E_\varepsilon; t_E > 0\}$ , we conclude that for a.e.  $E \in ]0, E'_\varepsilon]$  and  $L \geq L(E, \varepsilon)$  we have

$$(3.15) \quad \mathbb{E} \left\{ \text{tr} \Pi_r e^{-t_E H\omega_\Gamma} \Pi_r \right\} \leq 4^{d+1} e^{-(4E)^{-\frac{d}{2}+\varepsilon}} = 2e^{-4t_E E}.$$

Combining (3.2), (3.4), and (3.15), we conclude that, for a.e.  $E \in ]0, E'_\varepsilon]$  and  $L \geq L(E, \varepsilon)$  we have, for all intervals  $I \subset [0, E]$ ,

$$(3.16) \quad \begin{aligned} \mathbb{E} \left\{ \text{tr} P_\omega^{(\Lambda_L)}(I) \Pi_0 \right\} &\leq \sum_{k \in \Lambda} \frac{C_d}{E^{d+2} \langle k \rangle^{d+1}} \|\rho\|_\infty |I| e^{\frac{1}{2}t_E E} (2e^{-4t_E E})^{\frac{1}{4}} \\ &\leq C'_d E^{-d-2} e^{-\frac{1}{2}t_E E} \|\rho\|_\infty |I| = C''_d E^{-d-2} e^{-\frac{1}{8}(4E)^{-\frac{d}{2}+\varepsilon}} \|\rho\|_\infty |I|. \end{aligned}$$

Theorem 2.1 follows. □

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