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| Title | ON THE PERSISTENCE PROPERTIES OF SOLUTIONS OF NONLINEAR DISPERSIVE EQUATIONS IN WEIGHTED SOBOLEV SPACES (Harmonic Analysis and Nonlinear Partial Differential Equations) |
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| Citation | 数理解析研究所講究録別冊 = RIMS Kokyuroku Bessatsu (2011), B26: 23-36 |
| Issue Date | 2011-05 |
| URL | http://hdl.handle.net/2433/187885 |
| Right | |
| Type | Departmental Bulletin Paper |
| Textversion | publisher |

# ON THE PERSISTENCE PROPERTIES OF SOLUTIONS OF NONLINEAR DISPERSIVE EQUATIONS IN WEIGHTED SOBOLEV SPACES

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ABSTRACT. We study persistence properties of solutions to some canonical dispersive models, namely the semi-linear Schrödinger equation, the $k$-generalized Korteweg-de Vries equation and the Benjamin-Ono equation, in weighted Sobolev spaces $H^s(\mathbb{R}^n)\cap L^2(|x|^l dx),\ s,\, l>0$.

## 1. INTRODUCTION

This work is concerned with persistence properties of solutions to some nonlinear dispersive equations in weighted Sobolev spaces $H^s(\mathbb{R}^n)\cap L^2(|x|^l dx), s, l>0$. We shall consider the initial value problems (IVP) associated to the following dispersive models : the nonlinear Schrödinger (NLS) equation

$$i\partial_t u+\Delta u=\mu|u|^{a-1}u, \quad t\in\mathbb{R}, \quad x\in\mathbb{R}^n, \quad \mu=\pm1, \quad a>1, \tag{1.1}$$

the $k$-generalized Korteweg-de Vries ($k$-gKdV) equations

$$\partial_t u+\partial_x^3 u+u^k\partial_x u=0, \quad t,\, x\in\mathbb{R}, \ \ k\in\mathbb{Z}^+, \tag{1.2}$$

and the Benjamin-Ono (BO) equation

$$\partial_t u+\mathcal{H}\partial_x^2 u+u\partial_x u=0, \qquad t,\, x\in\mathbb{R}, \tag{1.3}$$

where $\mathcal{H}$ denotes the Hilbert transform

$$\mathcal{H}f(x)=\frac{1}{\pi}\lim_{\epsilon\downarrow 0}\int_{|y|\geq\epsilon}\frac{f(x-y)}{y}dy=-i\,(\mathrm{sgn}(\xi)\,\widehat{f}(\xi))^{\vee}(x). \tag{1.4}$$

These models have been widely studied in several contexts. For example, the KdV $k=1$ in (1.2) was first deduced as a model for long waves propagating in a channel. Subsequently the KdV and its modified form ($k=2$ in (1.2)) were found to be relevant in a number of different physical systems. Also they have been studied because of their relation to inverse scattering theory [20]. The NLS arises as a model in several different physical phenomena (see [61] and references therein). In the particular, case $n=1$ and $a=3$ it has been shown to be completely integrable [66]. The BO equation (1.3) was first deduced in [3] and [54] as a model for long internal gravity waves in deep stratified fluids. It was also shown that it is a completely integrable system (see [2], [12] and references therein).

We recall the notion of well posedness given in [34] : the IVP is said to be locally well posed (LWP) in the function space $X$ if for each $u_0\in X$ there exist $T>0$ and a unique solution $u\in C([-T,T]:X)\cap ....=Y_T$ of the equation, with the map data $\to$ solution being locally continuous from $X$ to $Y_T$. This notion of LWP includes the "persistent" property, i.e. the solution describes a continuous curve on $X$. In particular, it implies that the solution flow defines a dynamical system in $X$. When

$T$ can be taken arbitrarily large one says that the corresponding IVP is globally well posed (GWP) in $X$.

First, we shall study the Schrödinger equation (1.1).

## 2. The Schrödinger equation (1.1)

The results in [9], [10], [21], [35], and [65] yield the following LWP theory in the classical Sobolev spaces $H^s(\mathbb{R}^n)$ for the IVP associated to the NLS equation (1.1).

**Theorem A.** *Let $s_c = n/2 - 2/(a-1)$.*

(I) *If $s > s_c$, $s \geq 0$, with $[s] \leq a-1$ if $a$ is not an odd integer, then for each $u_0 \in H^s(\mathbb{R}^n)$ there exist $T = T(\|u_0\|_{s,2}) > 0$ and a unique solution $u = u(x,t)$ of the IVP associated to the NLS equation* (1.1) *with*

$$u \in C([-T,T] : H^s(\mathbb{R}^n)) \cap L^q([-T,T] : L^p_s(\mathbb{R}^n)) = Z^s_T. \tag{2.1}$$

*Moreover, the map data $\to$ solution is locally continuous from $H^s(\mathbb{R}^n)$ into $Z^s_T$.*

(II) *If $s = s_c$ and $s \geq 0$, then part (I) holds with $T = T(u_0) > 0$.*

Notations : (a) for $1 < p < \infty$ and $s \in \mathbb{R}$

$$L^p_s(\mathbb{R}^n) \equiv (1-\Delta)^{-s/2} L^p(\mathbb{R}^n) = J^{-s/2} L^p(\mathbb{R}^n), \qquad \|\cdot\|_{s,p} \equiv \|(1-\Delta)^s \cdot \|_p, \tag{2.2}$$

with $L^2_s(\mathbb{R}^n) = H^s(\mathbb{R}^n)$,

(b) the pair of indices $(q,p)$ in (2.1) are given by the Strichartz estimates (see [60] and [21]):

$$\left(\int_{-\infty}^{\infty} \|e^{it\Delta} u_0\|_p^q dt\right)^{1/q} \leq c\|u_0\|_2, \tag{2.3}$$

where

$$\frac{n}{2} = \frac{2}{q} + \frac{n}{p}, \qquad 2 \leq p \leq \infty, \text{ if } n = 1, \quad 2 \leq p < 2n/(n-2), \quad \text{if } n \geq 2.$$

The value $s_c = n/2 - 2/(a-1)$ in Theorem A is determined by a scaling argument : if $u(x,t)$ is a solution of the IVP associated to the NLS equation (1.1), then $u_\lambda(x,t) = \lambda^{2/(a-1)} u(\lambda x, \lambda^2 t)$ satisfies the same equation with data $u_\lambda(x,0) = \lambda^{2/(a-1)} u_0(\lambda x)$. Hence, for $s \in \mathbb{R}$

$$\|D^s u_\lambda(x,0)\|_2 = c\||\xi|^s \widehat{u_\lambda}(\xi,0)\|_2 = c\lambda^{2/(a-1)+s-n/2}\|u_0\|_2, \tag{2.4}$$

is independent of $\lambda$ when $s = s_c$. In Theorem A the case (I) corresponds to the sub-critical case and (II) to the critical one. In the latter, one has that if $\|D^{s_c} u_0\|_2$ is sufficiently small, then the local solution extends globally in time.

For the optimality of the results in Theorem A see [4], [11], and [40].

Formally, solutions of the NLS equation (1.1) satisfies the following conservation laws:

$$\|u(\cdot,t)\|_2 = \|u_0\|_2,$$

and

$$E(t) = \int_{\mathbb{R}^n} (|\nabla_x u(x,t)|^2 + \frac{2\mu}{a+1}|u(x,t)|^{a+1}) dx = E(0).$$

Using these conservation laws one can extend the LWP results in Theorem A to a GWP one, for details we refer to [6], [64], and references therein.

Concerning the persistence properties in weighted Sobolev spaces of solutions of the IVP associated to the NLS equation (1.1) one has the following result established in [26], [27], and [28].

**Theorem B.** *In addition to the hypothesis in Theorem A assume* $u_0 \in L^2(|x|^{2m}dx)$, $m \in \mathbb{Z}^+$ *with* $m \le a-1$ *if* $a$ *is not an odd integer.*

(I) *If* $s \ge m$*, then*

$$u \in C([-T,T] : H^s \cap L^2(|x|^{2m}dx)) \cap L^q([-T,T] : L^p_s \cap L^p(|x|^{2m}dx) = Z_T^{s,m}. \tag{2.5}$$

(II) *If* $1 \le s < m$*, then* (2.5) *holds with* $[s]$ *instead of* $m$ *and*

$$\Gamma^\beta u = (x_j + 2it\partial_{x_j})^\beta u \in C([-T,T] : L^2) \cap L^q([-T,T] : L^p), \tag{2.6}$$

*for any* $\beta \in (\mathbb{Z}^+)^n$ *with* $|\beta| \le m$.

The proof of Theorem B (see [26], [27], [28]) combines the operators ("vector fields")

$$\Gamma_j = x_j + 2it\partial_{x_j} = e^{i|x|^2/4t} 2it\partial_{x_j}(e^{-i|x|^2/4t}\cdot) = e^{it\Delta}x_j e^{-it\Delta}\cdot, \quad j = 1,..,n, \tag{2.7}$$

their commutative relation

$$(i\partial_t + \Delta)\Gamma_j u = \Gamma_j(i\partial_t u + \Delta u), \qquad j = 1,..,n, \tag{2.8}$$

so that $e^{it\Delta}(x_j u_0) = \Gamma_j\, e^{it\Delta}u_0$, and the structure of the nonlinearity in (1.1).

It should be remarked that Theorem B shows that the amount of decay in $L^2(|x|^{2m}dx)$ preserved by the solution depends on the regularity in the Sobolev scale $H^s$, $s \ge 0$) of the data, and the non-preserved decay is transformed in "local regularity". In particular, (2.6) tells us that $t^\beta \partial_x^\beta u \in L^2_{loc}(\mathbb{R}^n)$, for $|\beta| \le m$ and $t \in [-T,T] - \{0\}$.

Also one notices that the power of the weight $m$ in Theorem B is assumed to be an integer. In [53] we were able to remove this restriction.

**Theorem 1.** *In addition to the hypothesis in Theorem A assume* $u_0 \in L^2(|x|^{2m}dx)$, $m > 0$ *with* $[m] \le a-1$ *if* $a$ *is not an odd integer.*

(I) *If* $s \ge m$*,*

$$u \in C([-T,T] : H^s \cap L^2(|x|^{2m}dx)) \cap L^q([-T,T] : L^p_s \cap L^p(|x|^{2m}dx) = Z_T^{s,m}. \tag{2.9}$$

(II) *If* $1 \le s < m$*, then* (2.9) *holds with* $[s]$ *instead of* $m$ *and*

$$\mathit{\Gamma}^b\, \Gamma^\beta u(\cdot,t) \in C([-T,T] : L^2) \cap L^q([-T,T] : L^p), \tag{2.10}$$

*where* $\mathit{\Gamma}^b = e^{i|x|^2/4t}2^b t^b D^b \left(e^{-i|x|^2/4t}\cdot\right)$ *with* $|\beta| = [m]$ *and* $b = m - [m]$.

*In particular,*

$$t^m\, \partial_x^\beta\, D^b u(\cdot,t) \in L^2_{loc}(\mathbb{R}^n), \quad |\beta| = [m], \;\; b = m-[m], \;\; t \in (-T,T) - \{0\}. \tag{2.11}$$

As an application of this result we also prove that the persistence property in these weighted spaces can only hold for regular enough solutions. More precisely:

**Lemma 1.** *Let* $u$ *be a solution of the IVP associated to the NLS equation* (1.1) *provided by Theorem A. If there exist two times* $t_1, t_2 \in [0,T]$, $t_1 \ne t_2$ *such that*

$$|x|^m u(t_1), \quad |x|^m u(t_2) \in L^2(\mathbb{R}^n), \qquad m > s, \tag{2.12}$$

*$m \le a-1$ if $a$ is not an odd integer, then*

$$u \in C([-T,T]:H^m \cap L^2(|x|^{2m}dx)) \cap L^q([-T,T]:L^p_m \cap L^p(|x|^{2m}dx).$$

*Moreover, if $a$ is an odd integer and (2.12) holds for all $m \in \mathbb{Z}^+$, then*

$$u \in C([-T,T]:\mathbb{S}(\mathbb{R}^n)). \tag{2.13}$$

A key ingredient in our proof was an appropriate version of the Leibnitz rule for homogeneous fractional derivatives of order $b \in \mathbb{R}$

$$D^b f(x) \equiv ((2\pi|\xi|)^b \hat{f})^\vee(x) \tag{2.14}$$

deduced as a direct consequence of the characterization of the $L^p_s(\mathbb{R}^n)$ spaces (see (2.2)) given in [58].

**Theorem D.** *Let $b \in (0,1)$ and $2n/(n+2b) \le p < \infty$. Then $f \in L^p_b(\mathbb{R}^n)$ if and only if*

$$\begin{aligned} &(a)\ \ f \in L^p(\mathbb{R}^n), \\ &(b)\quad \mathcal{D}^b f(x) = (\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^2}{|x-y|^{n+2b}} dy)^{1/2} \in L^p(\mathbb{R}^n), \end{aligned} \tag{2.15}$$

*with*

$$\|f\|_{b,p} = \|(1-\Delta)^{b/2} f\|_p \simeq \|f\|_p + \|D^b f\|_p \simeq \|f\|_p + \|\mathcal{D}^b f\|_p. \tag{2.16}$$

For the proof of Theorem D we refer to [58], where the optimality of the lower bound $2n/(n+2b)$ was also established. The case $p = 2n/(n+2b)$ was proven in [18]. For a detailed discussion on the different characterizations of the $L^p_s(\mathbb{R}^n)$ spaces we refer to [58] and [59].

It is easy to see that for $p=2$ and $b \in (0,1)$ one has

$$\|\mathcal{D}^b f\|_2 \simeq \|D^b f\|_2, \tag{2.17}$$

$$\|\mathcal{D}^b (fg)\|_2 \le c(\|f\,\mathcal{D}^b g\|_2 + \|g\,\mathcal{D}^b f\|_2), \tag{2.18}$$

and for $p > 2n/(n+2b)$

$$\mathcal{D}^b(fg)(x) \le \|f\|_\infty \,\mathcal{D}^b g(x) + |g(x)|\,\mathcal{D}^b f(x). \tag{2.19}$$

We observe that in (2.18) both terms on the right hand side are estimates on the product of functions. We do not know whether or not (2.18) still holds with $D^b$ instead of $\mathcal{D}^b$ ,or for $p \neq 2$ .

Theorem D (i.e. the estimates (2.18)-(2.17)) allows us to get the following inequalities:

–(i) Let $b \in (0,1)$. For any $t > 0$

$$\mathcal{D}^b(e^{it|x|^2}) \le c(t^{b/2} + t^b|x|^b). \tag{2.20}$$

–(ii) Let $b \in (0,1)$. Then there exists $c = c(b) > 0$ such that for any $t \in \mathbb{R}$

$$\||x|^b\, e^{it\Delta} f\|_2 \le c(t^{b/2}\|f\|_2 + t^b\|D^b f\|_2 + \||x|^b f\|_2). \tag{2.21}$$

–(iii) Defining the operator $\varGamma^b$ for $b > 0$ as in Theorem 1 (see (2.10))

$$\varGamma^b \equiv \varGamma^b(t) = e^{i|x|^2/4t} 2^b t^b D^b \left( e^{-i|x|^2/4t} \cdot \right), \tag{2.22}$$

one has for $b>0$ and $t\in\mathbb{R}$ that

$$\varGamma^b(t)e^{it\Delta}f=e^{it\Delta}(|x|^bf), \tag{2.23}$$

and consequently

$$\varGamma^b(t)f=e^{it\Delta}(|x|^be^{-it\Delta}f). \tag{2.24}$$

In addition to the estimates (2.20)-(2.24) the following two lemmas were essential in the proof of Theorem 1 given in [53]. The first is a version of the Gagliardo-Nirenberg inequality for fractional derivatives.

**Lemma 2.** *Let $1<q,p,r<\infty$ and $0<\alpha<\beta$. Then*

$$\|D^\alpha f\|_p\le c\|f\|_r^{1-\theta}\,\|D^\beta f\|_q^\theta, \tag{2.25}$$

*with*

$$\frac{1}{p}-\frac{\alpha}{n}=(1-\theta)\frac{1}{r}+\theta\left(\frac{1}{q}-\frac{\beta}{n}\right),\qquad \theta\in[\alpha/\beta,1]. \tag{2.26}$$

The second is an interpolation estimate, which as Lemma 2, is a consequence of the three line theorem.

**Lemma 3.** *Let $a$, $b>0$. Assume that $J^af=(1-\Delta)^{a/2}f\in L^2(\mathbb{R})$ and $\langle x\rangle^bf=(1+|x|^2)^{b/2}f\in L^2(\mathbb{R})$. Then for any $\theta\in(0,1)$*

$$\|J^{\theta a}(\langle x\rangle^{(1-\theta)b}f)\|_2\le c\|\langle x\rangle^bf\|_2^{1-\theta}\,\|J^af\|_2^\theta. \tag{2.27}$$

For the study of persistence properties of the solution to the IVP associated to the NLS equation (1.1) in exponential weighted spaces we refer to [16], [17], and references therein.

Next, we shall consider the $k$-gKdV equation (1.2).

## 3. The $k$-generalized Korteweg-de Vries equation (1.2)

The following theorem describes the LWP theory in the classical Sobolev spaces $H^s(\mathbb{R})$ for the IVP associated to the $k$gKdV equation (1.2).

**Theorem E.**
(I) *The IVP associated to the equation (1.2) with $k=1$ is LWP in $H^s(\mathbb{R})$ for $s\ge s_1^*=-3/4$.*
(II) *The IVP associated to the equation (1.2) with $k=2$ is LWP in $H^s(\mathbb{R})$ for $s\ge s_2^*=1/4$.*
(III) *The IVP associated to the equation (1.2) with $k=3$ is LWP in $H^s(\mathbb{R})$ for $s\ge s_3^*=-1/6$.*
(IV) *The IVP associated to the equation (1.2) with $k\ge4$ is LWP in $H^s(\mathbb{R})$ for $s\ge s_k^*=(k-4)/2k$.*

The result $s>-3/4$ for the case $k=1$ was established in [39]. The limiting value $s=-3/4$ was obtained in [11], [24], and [42]. The result for the case $k=2$ was proven in [38]. The result $s>-1/6$ for the case $k=3$ was given in [22]. The limiting value $s=-1/6$ was obtained in [63]. The proof of the cases $k\ge4$ was given in [38].

The above local results apply to both real and complex valued functions.

The scaling argument described in (2.4) affirms that LWP should hold for $s\ge s_k=(k-4)/2k$. As Theorem E shows this is the case for $k\ge3$ (where for $s_k=s_k^*$

one has $T = T(u_0)$). However, in the cases $k = 1$ and $k = 2$ the values suggested by the scaling do not seem to be reachable in the Sobolev scale, see [40], and [11]. For the sharpness of these results we refer to [4], [40], and [11].

Real valued solutions of the $k$-gKdV equation (1.2) formally satisfy at least three conservation laws:

$$I_1(u) = \int_{-\infty}^{\infty} u(x,t)dx, \qquad I_2(u) = \int_{-\infty}^{\infty} (u(x,t))^2 dx,$$
$$I_3(u) = \int_{-\infty}^{\infty} ((\partial_x u(x,t))^2 - \frac{2}{(k+1)(k+2)} u(x,t)^{k+2})dx.$$

It was proven in [13] that for $k = 1$ and $k = 2$ one has global well posedness for $s > -3/4$ and $s > 1/4$, respectively. The global cases for $k = 1,\ s = -3/4$ and $k = 2,\ s = 1/4$ were proven in [24] and [42]. For the case $k = 3$ the global well posedness is known for $s > -1/42$, see [23].

For $k = 4$ blow up of "large" enough solutions was proven in [48]. Similar results for $k \geq 5$ remain an open problem.

Concerning the persistence of these solutions in weighted Sobolev spaces one has the following result found in [34].

**Theorem F.** *Let $m \in \mathbb{Z}^+$. Let $u \in C([-T,T] : H^s(\mathbb{R})) \cap .....$ with $s \geq 2m$ be the solution of the IVP associated to the equation* (1.2) *provided by Theorem E. If $u(x,0) = u_0(x) \in L^2(|x|^{2m}dx)$, then*

$$u \in C([-T,T] : H^s(\mathbb{R}) \cap L^2(|x|^{2m}dx)).$$

We recall that if for a solution $u \in C([0,T] : H^s(\mathbb{R}))$ of (1.2) one has that $\exists\, t_0 \in [0,T]$ such that $u(\cdot,t_0) \in H^{s'}(\mathbb{R})$, $s' > s$, then $u \in C([0,T] : H^{s'}(\mathbb{R}))$. So we shall mainly consider the most interesting case $s = 2m$ in Theorem F.

The proof of Theorem F combines the operator

$$\Gamma = x + 3t\partial_x^2,$$

and its commutative relation with the linear part $L = \partial_t + \partial_x^3$ of the equation (1.2) i.e.

$$\Gamma(\partial_t + \partial_x^3)v = (\partial_t + \partial_x^3)\Gamma v.$$

As in the case of the NLS equation (1.1) we would like to extend Theorem F where $m \in \mathbb{Z}^+$ to the case $m \in \mathbb{R}$, $m > 0$. Our first result in this direction is the following:

**Theorem 2.** *Let $m \geq 0$. Let $u \in C([-T,T] : H^m(\mathbb{R})) \cap .....$ with $m \geq \max\{s_k^*; 0\}$ be the solution of the IVP associated to the equation* (1.2) *provided by Theorem E. If $u(x,0) = u_0(x) \in L^2(|x|^m dx)$, then*

(I) *If $m < 1$, then for any $\epsilon > 0$*

$$u \in C([-T,T] : H^m(\mathbb{R}) \cap L^2(|x|^{m-\epsilon}dx)).$$

(II) *If $m \geq 1$, then*

$$u \in C([-T,T] : H^m(\mathbb{R}) \cap L^2(|x|^m dx)).$$

In [51] and [52] the loss of power $\epsilon > 0$ in the weight when $m < 1$ was removed for the equation (1.2) with non-linearity $k = 2, 4, 5, .....$ More precisely, the following optimal result was established in [52]:

**Theorem 3.** *Let $m \geq max\{s_k^*; 0\}$ with $k = 2, 4, 5, \ldots.$ Let $u \in C([-T, T] : H^m(\mathbb{R})) \cap \ldots..$ be the solution of the IVP associated to the equation (1.2) provided by Theorem E. If $u(x, 0) = u_0(x) \in L^2(|x|^m dx)$, then*

$$u \in C([-T, T] : H^m(\mathbb{R}) \cap L^2(|x|^m dx)).$$

It should be remarked that in the cases $k = 1$ and $k = 3$ the proof of the local theory in Theorem E is based on the spaces $X_{s,b}$ introduced in the context of dispersive equations in [5]. For all the other powers $k$ one has a local existence theory based on a contraction principle in a spaces defined by mixed norms of the type $L^p(\mathbb{R} : L^q([0, T]))$ or $L^q([0, T] : L^p(\mathbb{R}))$ (see [38]). This is the main difficulty in extending the optimal result in Theorem 3 to the powers $k = 1$ and $k = 3$ in (1.2).

Proof of Theorem 2

We shall sketch the ideas in the proof of Theorem 2 and refer to [51] and [52] for the justification of the argument and further details.

Following Kato's idea in [34] to establish the local smoothing effect (i.e. multiplying the equation (1.2) by $u(x, t)\phi(x)$, integrating the result, and using integration by parts) one formally gets the identity

$$\frac{d}{dt}\int u^2\phi dx + 3\int(\partial_x u)^2\phi' dx - \int u^2\phi^{(3)} dx - \frac{2}{k+2}\int u^{k+2}\phi' dx = 0. \tag{3.1}$$

Let us consider first the case $max\{s_k^*; 0\} \leq m < 1$.

From the local theory one has the following estimates for the solution $u = u(x, t)$

$$\sup_{x\in\mathbb{R}}\Big(\int_0^T |\partial_x D_x^m u(x, t)|^2 dt\Big)^{1/2} < c_T\|J^m u_0\|_2 = c_T\|u_0\|_{m,2}, \tag{3.2}$$

(the sharp form of the local smoothing effect found in [37]-[38]), and

$$\begin{aligned} \|D_x^m u\|_{L_x^2 L_T^2} &= \Big(\int_{-\infty}^{\infty}\int_0^T |D_x^m u(x, t)|^2 dt dx\Big)^{1/2} \\ &\leq T^{1/2}\sup_{t\in[0,T]}\|D_x^m u(t)\|_2 < c_T\|D^m u_0\|_2 \leq c_T\|u_0\|_{m,2}. \end{aligned} \tag{3.3}$$

Now, we consider the extensions of the estimates in (3.2)-(3.3) to the operators $D_x^{1+m+iy}$ and $D_x^{m+iy}$, $y \in \mathbb{R}$ respectively. First, in the linear case one has the estimates

$$\begin{aligned} &\|D_x^{m+1+iy} v\|_{L_x^\infty L_T^2} \leq c_T\|D^m v_0\|_2, \\ &\|D_x^{m+iy} v\|_{L_x^2 L_T^2} \leq c_T\|D^m v_0\|_2, \end{aligned} \tag{3.4}$$

for

$$v(x, t) = U(t)v_0(x) = c\int_{-\infty}^{\infty} e^{ix\xi}e^{it\xi^3}\widehat{v}_0(\xi)d\xi. \tag{3.5}$$

To apply the three line theorem we consider the function $F(z)$ defined on $\mathcal{S} = \{z \in \mathbb{C} : \Re(z) \in [0, 1]\}$

$$F(z) = \int_{-\infty}^{\infty}\int_0^T D_x^{s(z)} v(x, t)\,\phi(x, z)\,f(t)\,dtdx,$$

where

$$s(z)=(1-z)(1+m)+zm,\quad 1/q(z)=(1-z)+z/2,\quad q=2/(2-m),$$

$$\phi(x,z)=|g(x)|^{q/q(z)}\,\frac{g(x)}{|g(x)|},\qquad \text{with}\quad \|g\|_{L_x^{2/(2-m)}}=\|f\|_{L^2([0,T])}=1,$$

which is analytic on the interior of $\mathcal{S}$. So using that

$$\|\phi(\cdot,0+iy)\|_1=\|\phi(\cdot,1+iy)\|_2=1,$$

one gets that

$$\begin{aligned}\|\partial_x v\|_{L_x^{2/m}L_T^2}&\le c\|D_x v\|_{L_x^{2/m}L_T^2}\\ &\le c\sup_{y\in\mathbb{R}}\|D_x^{1+m+iy}v\|_{L_x^\infty L_T^2}^{1-m}\sup_{y\in\mathbb{R}}\|D_x^{m+iy}v\|_{L_x^2L_T^2}^m\le c_T\|D^m v_0\|_2.\end{aligned}\tag{3.6}$$

Inserting the estimate (3.6) in the proof of the local well posedness one obtains that

$$\|\partial_x u\|_{L_x^{2/m}L_T^2}\le c_T\|u_0\|_{m,2},\tag{3.7}$$

for $u=u(x,t)$ solution of the $k$-gKdV equation (1.2).

Now taking $\phi(x)=\langle x\rangle^{m-\epsilon}$, $\epsilon>0$ sufficiently small in (3.1), (we recall that $m<1$) and integrating in the time interval $[0,T]$ one finds that

$$\begin{aligned}&\int_0^T\int_{-\infty}^\infty(\partial_x u(x,t))^2\phi'(x)dxdt=c\|\partial_x u\,\langle x\rangle^{\frac{m}{2}-\frac12-\frac{\epsilon}{2}}\|_{L_x^2L_T^2}^2\\ &\le c\|\langle x\rangle^{m/2-1/2-\epsilon/2}\|_{L_x^{2/(1-m)}}\|\partial_x u\|_{L_x^{2/m}L_T^2}\le c_{m,\epsilon}\|\partial_x u\|_{L_x^{2/m}L_T^2},\end{aligned}\tag{3.8}$$

which combined with (3.6) and (3.1) shows that $\langle x\rangle^{m/2-\epsilon/2}u(\cdot,t)\in L^2(\mathbb{R})$ for $t\in[0,T]$. This basically completes the proof of the case $m<1$.

Next, we shall consider the case $m\ge 1$.

We take in (3.1) $\phi(x)=\langle x\rangle^m$ in (3.1), so we need to estimate the term

$$\int_{-\infty}^\infty|\partial_x u(x,t)|^2\langle x\rangle^{m-1}dx=\|\partial_x u(\cdot,t)\langle\cdot\rangle^{(m-1)/2}\|_{L_x^2}^2.$$

Thus, combining Lemma 3 in the previous section, the preservation of the $L^2$-norm of the solution, and Lemma 3 it follows that

$$\begin{aligned}&\|\partial_x u(\cdot,t)\langle\cdot\rangle^{(m-1)/2}\|_2\\ &\le\|\partial_x(u(\cdot,t)\langle\cdot\rangle^{(m-1)/2})\|_2+c\|u(\cdot,t)\langle\cdot\rangle^{(m-3)/2}\|_2\\ &\le\|\partial_x J^{-1}J(u(\cdot,t)\langle\cdot\rangle^{(m-1)/2})\|_2+c\|u(\cdot,t)\langle\cdot\rangle^{m/2}\|_2\\ &\le c\|J(u(\cdot,t)\langle\cdot\rangle^{(m-1)/2})\|_2+c\|u(\cdot,t)\langle\cdot\rangle^{m/2}\|_2\\ &\le c\|J^m u(\cdot,t)\|_2^{1/m}\|u(\cdot,t)\langle\cdot\rangle^{m/2}\|_2^{1-1/m}+c\|u(\cdot,t)\langle\cdot\rangle^{m/2}\|_2.\end{aligned}\tag{3.9}$$

Hence, inserting (3.9) in (3.1), using Young and Gronwall inequalities, the hypothesis $m\ge1$, and the fact that the $H^m$-norm of the solution is bounded in the time interval $[0,T]$ one obtains the desired result

$$\sup_{t\in[0,T]}\|\langle x\rangle^{m/2}u(\cdot,t)\|_{L^2}<\infty.$$

This completes the sketch of the proof of Theorem 2.

To finish this section concerning the $k$-gKdV equation (1.2) we will make some comments concerning the proof of Theorem 3 given in [51] and [52]. One of the key element in that proof is the following commutator estimate:

**Lemma 4.** *Let $0 < \alpha < 1$ and $1 < p < \infty$. Then for functions $f, g : \mathbb{R} \to \mathbb{C}$ one has that*

$$\|D^\alpha(fg) - fD^\alpha g\|_p \leq c\|Q_N(D^\alpha f)\|_{L^\infty l^1_N} \|g\|_2, \tag{3.10}$$

where

$$\|Q_N(f)\|_{L^\infty l^1_N} \equiv \| \sum_{N\in\mathbb{Z}} |Q_N(f)| \,\|_{L^\infty},$$

and

$$Q_N(f)(x) = ((\eta\left(\frac{\xi}{2^N}\right) + \eta\left(-\frac{\xi}{2^N}\right))\widehat{f}(\xi))^\vee(x),$$

where $\eta \in C_0^\infty(\mathbb{R})$ with $supp(\eta) \subseteq [1, 2, 2]$ so that

$$\sum_{N\in\mathbb{Z}} (\eta\left(\frac{x}{2^N}\right) + \eta\left(-\frac{x}{2^N}\right)) = 1, \quad \text{for } x \neq 0.$$

In the proof of Theorem 3 for the case $k = 2$ and $m = 1/4$ (extremal case) given in [51] Lemma 4 was combined with the inequality

$$\|D_\xi^{1/8} Q_N\left(\frac{e^{it\xi^3}}{(1+\xi^2)^{1/8}}\right)\|_{L^\infty_\xi l^1_N} < \infty,$$

to establish the main estimate in the proof.

For the study of persistence properties of the solution to the IVP associated to the $k$-gKdV equation (1.2) in exponential weighted spaces we refer to [41] and [15] and references therein.

Finally, we shall consider the BO equation (1.3).

## 4. The Benjamin-Ono equation (1.3)

The LWP in the Sobolev spaces $H^s(\mathbb{R})$ of the IVP associated to the BO equation (1.3) has been largely considered : in [1] and [32] LWP was established for $s > 3/2$, in [56] for $s \geq 3/2$, in [44] for $s > 5/4$, in [36] for $s > 9/8$, in [62] for $s \geq 1$, in [7] for $s > 1/4$, and in [31] LWP was proven in $H^s(\mathbb{R})$ for $s \geq 0$.

Real valued solutions of the IVP (1.3) satisfy infinitely many conservation laws (time invariant quantities), the first three are the following:

$$\begin{aligned} &I_1(u) = \int_{-\infty}^{\infty} u(x,t)dx, \quad I_2(u) = \int_{-\infty}^{\infty} u^2(x,t)dx,\\ &I_3(u) = \int_{-\infty}^{\infty} (|D_x^{1/2}u|^2 - \frac{u^3}{3})(x,t)dx, \end{aligned} \tag{4.1}$$

where $D_x = \mathcal{H}\,\partial_x$.

The $k$-conservation law $I_k$ provides an *a priori* estimate of the $L^2$-norm of the derivatives of order $(k-2)/2$, $k > 2$ of the solution, i.e. $\|D_x^{(k-2)/2}u(t)\|_2$. This allows one to deduce GWP from LWP results.

In the BO equation the dispersive effect is described by a non-local operator and is significantly weaker than that exhibited by the Korteweg-de Vries (KdV) equation, i.e. $k=1$ in (1.2). Indeed, it was proven in [49] that for any $s\in\mathbb{R}$ the map data-solution from $H^s(\mathbb{R})$ to $C([0,T]:H^s(\mathbb{R}))$ is not locally $C^2$, and in [45] that it is not locally uniformly continuous. In particular, this implies that no LWP results can be obtained by an argument based only on a contraction method.

Consider the weighted Sobolev spaces

$$Z_{s,r}=H^s(\mathbb{R})\cap L^2(|x|^{2r}dx),\ \text{ and }\ \dot{Z}_{s,r}=\{f\in Z_{s,r}\,:\,\widehat{f}(0)=0\}\quad s,\,r\in\mathbb{R}. \tag{4.2}$$

In [32] the following results were obtained:

**Theorem G.** (I) *The IVP associated to the BO equation* (1.3) *is GWP in* $Z_{2,2}$.
(II) *If* $\widehat{u}_0(0)=0$*, then the IVP associated to the BO equation* (1.3) *is GWP in* $\dot{Z}_{3,3}$.
(III) *If* $u(x,t)$ *is a solution of the IVP associated to the BO equation*(1.3) *such that* $u\in C([0,T]:Z_{4,4})$ *for arbitrary* $T>0$*, then* $u(x,t)\equiv 0$.

We observe that the linear part of the equation in (1.3) $L=\partial_t+\mathcal{H}\partial_x^2$ commutes with the operator $\Gamma=x-2t\mathcal{H}\partial_x$, i.e.

$$[L;\Gamma]=L\Gamma-\Gamma L=0.$$

Also, the solution $v(x,t)$ of the associated IVP

$$v(x,t)=U(t)v_0(x)=e^{-it\mathcal{H}\partial_x^2}v_0(x)=\ (e^{-it\xi|\xi|}\,\widehat{v}_0)^{\vee}(x), \tag{4.3}$$

satisfies that $v(\cdot,t)\in L^2(|x|^{2k}dx)$, $t\in[0,T]$, when $v_0\in Z_{k,k},\ k\in\mathbb{Z}^+$ for $k=1,2,......$ and

$$\int_{-\infty}^{\infty}x^j\,v_0(x)dx=0,\quad j=0,1,...,k-3,\quad\text{if}\quad k\geq 3.$$

In [33] the unique continuation result in $Z_{4,4}$ in Theorem G was improved:

**Theorem I.** *Let* $u\in C([0,T]:H^2(\mathbb{R}))$ *be a solution of the IVP* (1.3)*. If there exist three different times* $t_1,t_2,t_3\in[0,T]$ *such that*

$$u(\cdot,t_j)\in Z_{4,4},\quad j=1,2,3,\quad \textit{then}\quad u(x,t)\equiv 0. \tag{4.4}$$

As in the previous cases, the goal was to extend the results in Theorem G and Theorem I from integer values to the continuum optimal range of indices $(s,r)$. In this direction one finds the following results established in [19]:

**Theorem 4.**
(I) *Let* $s\geq 1,\ r\in[0,s]$*, and* $r<5/2$*. If* $u_0\in Z_{s,r}$*, then the solution* $u(x,t)$ *of the IVP associated to the BO equation* (1.3) *satisfies that* $u\in C([0,\infty):Z_{s,r})$.
(II) *For* $s>9/8$ *(*$s\geq 3/2$*),* $r\in[0,s]$*, and* $r<5/2$ *the IVP associated to the BO equation*(1.3) *is LWP (GWP resp.) in* $Z_{s,r}$.
(III) *If* $r\in[5/2,7/2)$ *and* $r\leq s$*, then the IVP associated to the BO equation* (1.3) *is GWP in* $\dot{Z}_{s,r}$.

**Theorem 5.** *Let* $u\in C([0,T]:Z_{2,2})$ *be a solution of the IVP associated to the BO equation* (1.3)*. If there exist two different times* $t_1,t_2\in[0,T]$ *such that*

$$u(\cdot,t_j)\in Z_{5/2,5/2},\ \ j=1,2,\ \ \textit{then}\ \ \widehat{u}_0(0)=0\,,\ \ (\textit{so}\ \ u(\cdot,t)\in\dot{Z}_{5/2,5/2}). \tag{4.5}$$

**Theorem 6.** *Let $u \in C([0,T] : \dot{Z}_{3,3})$ be a solution of the IVP (1.3). If there exist three different times $t_1, t_2, t_3 \in [0,T]$ such that*

$$u(\cdot, t_j) \in Z_{7/2,7/2}, \quad j = 1,2,3, \quad \textit{then} \quad u(x,t) \equiv 0. \tag{4.6}$$

We also refer readers to the related works [47], [25], and [43].

Remarks : Theorem 5 and Theorem 6 show that the upper values of $r$ for the persisitence properties in $Z_{s,r}$ and $\dot{Z}_{s,k}$ in Theorem 4 are optimal. We recall that if $u \in C([0,T] : H^s(\mathbb{R}))$ is a solution of the BO equation (1.3) such that $\exists\, t_0 \in [0,T]$ for which $u(x,t_0) \in H^{s'}(\mathbb{R})$, $s' > s$, then $u \in C([0,T] : H^{s'}(\mathbb{R}))$. So it suffices to consider the most interesting case $s = r$ in (4.2).

The proof of Theorems 6 is based on weighted energy estimates and involves several inequalities concerning the Hilbert transform $\mathcal{H}$.

Among them one finds the $A_p$ condition introduced in [50].

**Definition 1.** A non-negative function $w \in L^1_{loc}(\mathbb{R})$ satisfies the $A_p$ inequality with $1 < p < \infty$ if

$$\sup_{Q \text{ interval}} \left(\frac{1}{|Q|}\int_Q w\right)\left(\frac{1}{|Q|}\int_Q w^{1-p'}\right)^{p-1} = c(w) < \infty, \tag{4.7}$$

where $1/p + 1/p' = 1$.

It was proven in [30] that this is a necessary and sufficient condition for the Hilbert transform $\mathcal{H}$ to be bounded in $L^p(w(x)dx)$ (see [30], ), i.e. $w \in A_p$, $1 < p < \infty$ if and only if

$$\left(\int_{-\infty}^{\infty} |\mathcal{H}f|^p w(x)dx\right)^{1/p} \le c^* \left(\int_{-\infty}^{\infty} |f|^p w(x)dx\right)^{1/p}, \tag{4.8}$$

In the case $p = 2$, a previous characterization of $w$ in (4.7) was found in [29]. However, even though the main case is for $p = 2$, the characterization (4.7) will be the one used in the proof. In particular, one has that in $\mathbb{R}$

$$|x|^\alpha \in A_p \quad \Leftrightarrow \quad \alpha \in (-1, p-1). \tag{4.9}$$

In order to justify some of the arguments in the proofs one need some further continuity properties of the Hilbert transform. More precisely, the proof requires the constant $c^*$ in (4.8) to depend only on $c(w)$ the constant describing the $A_p$ condition (see (4.7)) and on $p$. In [55] precise bounds for the constant $c^*$ in (4.7) were given which are sharp in the case $p = 2$ and sufficient for the purpose in [19].

It will be essential in the arguments in [19] that some commutator operators involving the Hilbert transform $\mathcal{H}$ are of "order zero". More precisely, one shall use the following estimate: $\forall\, p \in (1,\infty)$, $l$, $m \in \mathbb{Z}^+ \cup \{0\}$, $l + m \ge 1$ $\exists\, c = c(p;l;m) > 0$ such that

$$\|\partial_x^l [\mathcal{H};\, a] \partial_x^m f\|_p \le c \|\partial_x^{l+m} a\|_\infty \|f\|_p. \tag{4.10}$$

In the case $l + m = 1$, (4.10) is Calderón's first commutator estimate [8]. The case $l + m \ge 2$ of the estimate (4.10) was proved in [14].

ACKNOWLEDGMENT: J. N. was supported by the EAPSI NSF and JSPS program. G.P. was supported by NSF grant DMS-0800967. Part of this work was done while J. N. was visiting Prof. Y. Tsutsumi at the Department of Mathematics at Kyoto University whose hospitality he gratefully acknowledges.

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