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LEFSCHETZ FIBRATIONS WITH SMALL SLOPE

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We construct Lefschetz fibrations over S^2 which do not satisfy the slope inequality. This disproves a conjecture of Hain.

1. Introduction

Lefschetz fibrations have been an active area of research ever since the remarkable work in [Donaldson 1999] and [Gompf and Stipsicz 1999] revealed a close connection between them and symplectic 4-manifolds. In this paper, we consider the geography problem of Lefschetz fibrations over S^2, which derives from that of complex surfaces fibred over curves.

We are interested in two kinds of geography problems. Let σ and e be the signature and the Euler characteristic of a closed oriented smooth 4-manifold X, respectively. For an almost complex closed 4-manifold X, we set $K^2 := 3\sigma + 2e$ and $\chi_h := (\sigma + e)/4$ (the *holomorphic Euler characteristic*).

One is the geography problem for complex surfaces: the characterization of pairs (K^2, χ_h) corresponding to minimal complex surfaces. It is well known that any minimal complex surface of general type satisfies $K^2 > 0$, $\chi_h > 0$, the *Noether inequality* $2\chi_h - 6 \leq K^2$ and the *Bogomolov–Miyaoka–Yau inequality* $K^2 \leq 9\chi_h$ (see [Barth et al. 1984], for example). The above geography problem can be extended to the symplectic 4-manifolds. However, Fintushel and Stern [1998] constructed Lefschetz fibration which does not satisfy the Noether inequality. In particular, for most pairs (p, q) satisfying $p < 2q - 6$, there exists a minimal symplectic 4-manifold with $p = K^2$ and $q = \chi_h$ (see [Gompf and Stipsicz 1999]). On the other hand, no examples of a minimal symplectic 4-manifold with $K^2 > 9\chi_h$ have been found yet.

The other is the geography problem for complex surfaces fibred over curves. Hereafter, we assume $g \geq 2$. Let $f : S \to C$ be a relatively minimal holomorphic genus-g fibration, where S is a complex surface and C is a complex curve of genus k. We define relative numerical invariants $\chi_f := \chi_h - (g-1)(k-1)$ and $K_f^2 := K^2 - 8(g-1)(k-1)$ for $f : S \to C$. Then, we have two inequalities $\chi_f \geq 0$

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and $K_f^2 \geq 0$, known as *Beauville's inequality* (see [Beauville 1979]) and *Arakelov's inequality* (see [Arakelov 1971]), respectively. For $\chi_f \neq 0$, which is equivalent to the fact that f is not a holomorphic bundle, we define λ_f to be the quotient K_f^2/χ_f, called the slope of f. Xiao [1987] proved that $4-4/g \leq \lambda_f \leq 12$ (that is, $(4-4/g)\chi_f \leq K_f^2 \leq 12\chi_f$). The former inequality is called the *slope inequality*. For a relatively minimal genus-g Lefschetz fibration, χ_f, K_f^2 and the slope λ_f are defined in the same way as for complex surfaces fibred over curves. To the author's knowledge, the slope of all known Lefschetz fibrations over S^2 is greater than or equal to $4-4/g$.

Conjecture 1.1 (Hain; see [Amorós et al. 2000, Question 5.10; Endo and Nagami 2005, Conjecture 4.12]). *For every relatively minimal genus-g Lefschetz fibration $f : X \to S^2$, the slope inequality $\lambda_f \geq 4-4/g$ holds.*

In this paper, we give a negative answer to Conjecture 1.1.

Theorem 3.1. *For each $g \geq 3$, there exists a genus-g Lefschetz fibration over S^2 with slope $\lambda_f = 4-4/g-1/3g$ whose total space is simply connected.*

Moreover, by fiber sum operations, we have the following results:

Corollary 3.6. *For each $g \geq 3$, $m \geq 0$ and $l \geq 0$, there exists a genus-g Lefschetz fibration $f_{m,l} : X_{m,l} \to S^2$ with slope $\lambda_{f_{m,l}} = 4-4/g-1/(m+3)g$ such that $\pi_1(X_{m,l}) = 1$. Moreover, if $(m,l) \neq (0,0)$, then $X_{m,l}$ is a minimal symplectic 4-manifold.*

Corollary 3.7. *For each $g \geq 3$, $m \geq 1$ and $l \geq 0$, there exists a genus-g Lefschetz fibration $f'_{m,l} : Y_{m,l} \to S^2$ with slope $\lambda_{f'_{m,l}} = 4-4/g-1/2g+1/(2\cdot 3^{m-1}g)$ such that $\pi_1(Y_{m,l}) = 1$. Moreover, if $l \geq 1$, then $Y_{m,l}$ is a minimal symplectic 4-manifold.*

As a consequence, we have the following results.

Corollary 4.2. *The Lefschetz fibrations $f_{m,l}$ $(m \geq 0)$ and $f'_{m,l}$ $(m \geq 2)$ are non-holomorphic.*

Let $f : X \to S^2$ be a relatively minimal genus-g Lefschetz fibration with $n > 0$ singular fibers. From $e(X) = -4(g-1)+n$ and results from [Smith 1999; Stipsicz 1999; Ozbagci 2002], we have $\chi_f > 0$, $K_f^2 \geq 4g-4$ and $\lambda_f \leq 10$. Moreover, it is well known that any hyperelliptic Lefschetz fibration satisfies the slope inequality. This fact follows from the signature formula for genus-g hyperelliptic Lefschetz fibrations obtained by Matsumoto [1983; 1996] for $g = 1, 2$ and Endo [2000] for $g \geq 3$. Therefore, genus-2 Lefschetz fibrations satisfy the slope inequality. In particular, if f is a hyperelliptic Lefschetz fibration with only nonseparating vanishing cycles, then λ_f is equal to $4-4/g$. For Lefschetz fibrations with $b_2^+ = 1$, we prove the following result.

Theorem 5.1. *Let $g \geq 2$ and let $f : X \to S^2$ be a genus-g Lefschetz fibration with $b_2^+(X) = 1$.*

(1) *If X is not diffeomorphic to the blow-up of a ruled surface, then*

(i) $$4 - 4/g \leq \lambda_f \leq 8 \quad \textit{for } b_1(X) = 0,$$

(ii) $$4 \leq \lambda_f \leq 8 \quad \textit{for } b_1(X) = 2.$$

(2) *If X is diffeomorphic to the blow-up of an S^2-bundle over Σ_k, then*

$$4 + 4(k-1)/(g-k) \leq \lambda_f \leq 8,$$

and the lower bound is sharp.

The study of the slope of holomorphic fibrations was mainly motivated by Severi's inequality, which states that if S is a minimal surface of general type of maximal Albanese dimension, then $K^2 \geq 4\chi_h$. Equivalently, if $K^2 < 4\chi_h$, then S is a surface fibred over C of genus $b_1(S)/2$. Severi [1932] claimed it, but his proof was not correct (see [Catanese 1983]). The inequality was independently posed as a conjecture by Reid [1979] and by Catanese [1983]. Xiao [1987] proved the conjecture when S is a surface fibred over a curve of positive genus. He showed that if S admits a holomorphic genus-g fibration f over C of positive genus k with $K^2 < 4\chi_h + 4(g-1)(k-1)$ (that is, $\lambda_f < 4$), then $k = b_1(S)/2$. Konno [1996] proved it in the case of *even* surfaces. The conjecture was solved by Manetti [2003] when S has ample canonical bundle. Pardini [2005] proved the conjecture completely by using the slope inequality for holomorphic fibrations over $\mathbb{CP}^1$.

In Section 2, we review some standard facts on Lefschetz fibrations. Our main results are proved in Section 3. We give Lefschetz fibrations which violate the slope inequality. Consequently, we obtain examples of nonholomorphic Lefschetz fibrations in Section 4. In the last section, we investigate the slopes of Lefschetz fibrations with $b_2^+ = 1$.

Remark 1.2. The slope inequality of Conjecture 1.1 can be reformulated in terms of the Deligne–Mumford compactified moduli space of stable curves of genus g, denoted by $\overline{\mathcal{M}}_g$, as follows. For a relatively minimal genus-g Lefschetz fibration $f : X \to S^2$ with n singular fibers, we obtain a symplectic structure on X such that for all $x \in S^2$, $f^{-1}(x)$ is a pseudoholomorphic curve. Since a 2-dimensional almost-complex structure is integrable, $f^{-1}(x)$ determines a point in $\overline{\mathcal{M}}_g$. Thus, we obtain the moduli map $\phi_f : S^2 \to \overline{\mathcal{M}}_g$ which is defined by $\phi_f(x) = [f^{-1}(x)] \in \overline{\mathcal{M}}_g$ for $x \in S^2$. We denote by $\mathcal{H}_g$ the Hodge bundle on $\overline{\mathcal{M}}_g$ with fiber the determinant line $\bigwedge^g H^0(C; K_C)$, where C is the set of critical points of f. By Smith's signature formula [1999] and the slope inequality, we have the following inequality:

$$(8g+4)\langle c_1(\mathcal{H}_g), [\phi_f(S^2)]\rangle - g \cdot n \geq 0.$$

2. Preliminaries

In this section, we first recall the definition and basic properties of Lefschetz fibrations. More details can be found in [Gompf and Stipsicz 1999].

Let Σ_g be a closed oriented surface of genus $g \geq 2$ and let Γ_g be the *mapping class group* of Σ_g, which is the group of isotopy classes of orientation-preserving diffeomorphisms of Σ_g. We denote by t_c the right-handed *Dehn twist* about a simple closed curve c on an oriented surface. The notation $t_c t_d$ means that we first apply t_d then t_c.

Definition 2.1. Let X be a closed, oriented smooth 4-manifold. A smooth map $f : X \to S^2$ is a genus-g *Lefschetz fibration* if it satisfies the following conditions:

(i) f has finitely many critical values $b_1, \dots, b_n \in S^2$, and f is a smooth Σ_g-bundle over $S^2 - \{b_1, \dots, b_n\}$.

(ii) For each i $(i = 1, \dots, n)$, there exists a unique critical point p_i in the *singular fiber* $f^{-1}(b_i)$ such that about each p_i and b_i there are local complex coordinate charts agreeing with the orientations of X and S^2 on which f is of the form $f(z_1, z_2) = z_1^2 + z_2^2$.

(iii) f is relatively minimal (no fiber contains a (-1)-sphere).

Each singular fiber is obtained by collapsing a simple closed curve (the *vanishing cycle*) in the regular fiber. The monodromy of the fibration around a singular fiber is given by a right-handed Dehn twist along the corresponding vanishing cycle. A Lefschetz fibration $f : X \to S^2$ is *holomorphic* if there are complex structures on both X and S^2 with holomorphic projection f.

Once we fix an identification of Σ_g with the fiber over a base point of S^2, we can characterize the Lefschetz fibration $f : X \to S^2$ by its *monodromy representation* $\pi_1(S^2 - \{b_1, \dots, b_n\}) \to \Gamma_g$. This map is really an antihomomorphism, since elements of $\pi_1(S^2 - \{b_1, \dots, b_n\})$ are written left-to-right and elements of Γ_g are written right-to-left. Let $\gamma_1, \dots, \gamma_n$ be an ordered system of generating loops for $\pi_1(S^2 - \{b_1, \dots, b_n\})$, such that each γ_i encircles only b_i and $\prod \gamma_i$ is homotopically trivial. Thus, the monodromy of f comprises a factorization

$$t_{v_n} \dots t_{v_2} t_{v_1} = 1 \in \Gamma_g,$$

where v_i are vanishing cycles of the singular fibers. This factorization is called the *positive relator*.

According to theorems of Kas [1980] and Matsumoto [1996], if $g \geq 2$, then the isomorphism class of a Lefschetz fibration is determined by a positive relator modulo simultaneous conjugations

$$t_{v_n} \dots t_{v_2} t_{v_1} \sim t_{\phi(v_n)} \dots t_{\phi(v_2)} t_{\phi(v_1)} \quad \text{for all } \phi \in \Gamma_g$$

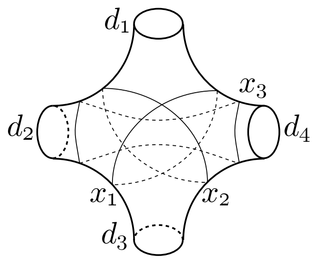

Figure 1. The curves $d_1, d_2, d_3, d_4, x_1, x_2, x_3$.

and elementary transformations

$$t_{v_n} \dots t_{v_{i+2}} t_{v_{i+1}} t_{v_i} t_{v_{i-1}} t_{v_{i-2}} \dots t_{v_1} \sim t_{v_n} \dots t_{v_{i+2}} t_{v_i} t_{t_{v_i}^{-1}(v_{i+1})} t_{v_{i-1}} t_{v_{i-2}} \dots t_{v_1},$$

$$t_{v_n} \dots t_{v_{i+2}} t_{v_{i+1}} t_{v_i} t_{v_{i-1}} t_{v_{i-2}} \dots t_{v_1} \sim t_{v_n} \dots t_{v_{i+2}} t_{v_{i+1}} t_{t_{v_i}(v_{i-1})} t_{v_i} t_{v_{i-2}} \dots t_{v_1}.$$

Note that $\phi t_{v_i} \phi^{-1} = t_{\phi(v_i)}$. For all $\phi \in \Gamma_g$, let $\phi(\varrho)$ be the positive relator which is obtained by applying simultaneous conjugations by ϕ to a positive relator ϱ. We denote a Lefschetz fibration associated to a positive relator $\varrho \in \Gamma_g$ by $f_\varrho : X_\varrho \to S^2$. Clearly, if $\varrho_1 \sim \varrho_2$ in Γ_g (that is, ϱ_2 is obtained by applying elementary transformations or simultaneous conjugations to ϱ_1), then

$$\chi_{f_{\varrho_1}} = \chi_{f_{\varrho_2}} \quad \text{and} \quad K^2_{f_{\varrho_1}} = K^2_{f_{\varrho_2}}.$$

For positive relators ϱ_1 and ϱ_2 in Γ_g, the genus-g Lefschetz fibration

$$f_{\varrho_1\varrho_2} : X_{\varrho_1\varrho_2} \to S^2$$

is the (trivial) fiber sum of f_{ϱ_1} and f_{ϱ_2}. Since $\sigma(X_{\varrho_1\varrho_2}) = \sigma(X_{\varrho_1}) + \sigma(X_{\varrho_2})$ and $e(X_{\varrho_1\varrho_2}) = e(X_{\varrho_1}) + e(X_{\varrho_2}) + 4(g-1)$, we see that $\chi_{f_{\varrho_1\varrho_2}} = \chi_{f_{\varrho_1}} + \chi_{f_{\varrho_2}}$ and $K^2_{f_{\varrho_1\varrho_2}} = K^2_{f_{\varrho_1}} + K^2_{f_{\varrho_2}}$. In particular, if $\varrho_1 \sim \varrho_2$, then

$$\chi_{f_{\varrho_1\varrho_2}} = 2\chi_{f_{\varrho_1}} = 2\chi_{f_{\varrho_2}} \quad \text{and} \quad K^2_{f_{\varrho_1\varrho_2}} = 2K^2_{f_{\varrho_1}} = 2K^2_{f_{\varrho_2}}.$$

We next begin with a definition of the lantern relation (see [Dehn 1938; Johnson 1979]).

Definition 2.2. Let Σ_0^4 denote a sphere with 4 boundary components. Let d_1, d_2, d_3, d_4 be the 4 boundary curves of Σ_0^4 and let x_1, x_2, x_3 be the interior curves as shown in Figure 1. Then, we have the *lantern relation*

$$t_{d_1} t_{d_2} t_{d_3} t_{d_4} = t_{x_1} t_{x_2} t_{x_3}.$$

Let ϱ be a positive relator of Γ_g. Let d_1, d_2, d_3, d_4, x_1, x_2, x_3 be curves as in Definition 2.2. Suppose that ϱ includes $t_{d_1} t_{d_2} t_{d_3} t_{d_4}$ as a subword:

$$\varrho = U \cdot t_{d_1} t_{d_2} t_{d_3} t_{d_4} \cdot V,$$

where U and V are products of right-handed Dehn twists. Then, by the lantern relation, the product of right-handed Dehn twists

$$\varrho' = U \cdot t_{x_1} t_{x_2} t_{x_3} \cdot V$$

is also a positive relator of Γ_g.

This operation is one of substitution techniques introduced by Fuller.

Definition 2.3. We say that ϱ' is obtained by applying an *L-substitution* to ϱ. Conversely, ϱ is said to be obtained by applying an L^{-1}-*substitution* to ϱ'. We also call these two kinds of operations *lantern substitutions*.

Proposition 2.4 [Endo and Nagami 2005, Theorem 4.3 and Proposition 3.12]. *Let ϱ, ϱ' be positive relators of Γ_g and let X_ϱ, $X_{\varrho'}$ be the corresponding Lefschetz fibrations over S^2, respectively. Suppose that ϱ is obtained by applying an L^{-1}-substitution to ϱ'. Then, $\sigma(X_\varrho) = \sigma(X_{\varrho'}) - 1$ and $e(X_\varrho) = e(X_{\varrho'}) + 1$. Therefore,*

$$\chi_{f_\varrho} = \chi_{f_{\varrho'}} \quad \textit{and} \quad K^2_{f_\varrho} = K^2_{f_{\varrho'}} - 1.$$

Remark 2.5. Endo and Gurtas [2010] showed that $X_{\varrho'}$ is a rational blowdown of X_ϱ introduced by Fintushel and Stern [1997]. Such relations were also generalized by Endo, Mark, and Van Horn-Morris [Endo et al. 2011].

3. Main results

In this section, we give a negative answer to Conjecture 1.1.

Theorem 3.1. *For each $g \geq 3$, there exists a genus-g Lefschetz fibration over S^2 with slope $\lambda_f = 4 - 4/g - 1/3g$ whose total space is simply connected.*

In order to prove Theorem 3.1, we recall some standard facts on hyperelliptic Lefschetz fibrations. Let Δ_g be the *hyperelliptic mapping class group* of genus g, that is, the subgroup of Γ_g which consists of all isotopy classes of orientation-preserving diffeomorphisms of Σ_g commuting with the isotopy class of ι, called the hyperelliptic involution. Note that $\Delta_g = \Gamma_g$ for $g = 1, 2$.

Definition 3.2. Let $\varrho = t_{a_1} \dots t_{a_n}$ be a positive relator in Γ_g. A genus-g Lefschetz fibration $f_\varrho : X_\varrho \to S^2$ is called *hyperelliptic* if for each $k \in \{1, \dots, n\}$, t_{a_k} is in Δ_g. Equivalently, $\iota(a_k) = a_k$ for each k.

The following theorem was established in [Matsumoto 1983; 1996] for $g = 1, 2$ and in [Endo 2000] for $g \geq 3$.

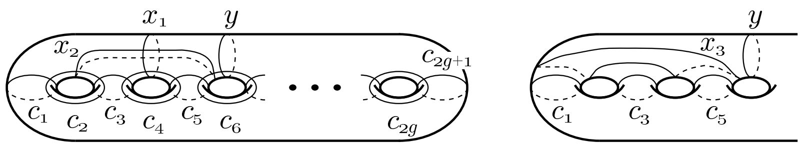

Figure 2. The curves $c_1, \dots, c_{2g+1}, x_1, x_2, x_3, y$.

Theorem 3.3 (Matsumoto, Endo). *Let $f_\varrho : X_\varrho \to S^2$ be a genus-g hyperelliptic Lefschetz fibration with m nonseparating and*

$$s = \sum_{h=1}^{[g/2]} s_h$$

separating vanishing cycles, where s_h denotes the number of separating vanishing cycles that separate Σ_g into two surfaces, one of which has genus h. Then, we have

$$\sigma(X_\varrho) = -\frac{g+1}{2g+1} m + \sum_{h=1}^{[g/2]} \left(\frac{4h(g-h)}{2g+1} - 1 \right) s_h.$$

We need the following positive relator to prove Theorem 3.1. As shown in Figure 2, let $c_1, c_2, \dots, c_{2g+1}$ be the curves in Σ_g. We denote by h_g ($\in \Gamma_g$) the product of $8g+4$ right-handed Dehn twists

$$h_g := (t_{c_1} t_{c_2} \dots t_{c_{2g+1}}^2 \dots t_{c_2} t_{c_1})^2.$$

It is well known that h_g is a positive relator in Δ_g and that $\sigma(X_{h_g}) = -4(g+1)$, by Theorem 3.3 and $e(X_{h_g}) = 4(g+2)$. This gives $\chi_{f_{h_g}} = g$, $K^2_{f_{h_g}} = 4g-4$ and $\lambda_{f_{h_g}} = 4 - 4/g$ (that is, f_{h_g} is lying on the slope line).

Proof of Theorem 3.1. Suppose $g \geq 3$. Let x_1, x_2, x_3, y be the curves as shown in Figure 2. Since c_1, x_i are nonseparating curves, there exists a diffeomorphism ϕ_i such that $\phi_i(c_1) = x_i$. Hence, we have the following positive relator r_i ($i = 1, 2, 3$):

$$\begin{aligned} r_i = \phi_i h_g \phi_i^{-1} &= \phi_i (t_{c_1} t_{c_2} \dots t_{c_{2g+1}}^2 \dots t_{c_2} t_{c_1})^2 \phi_i^{-1} \\ &= (t_{\phi_i(c_1)} t_{\phi_i(c_2)} \dots t_{\phi_i(c_{2g+1})}^2 \dots t_{\phi_i(c_2)} t_{\phi_i(c_1)})^2 \\ &= (t_{x_i} t_{\phi_i(c_2)} \dots t_{\phi_i(c_{2g+1})}^2 \dots t_{\phi_i(c_2)} t_{\phi_i(c_1)})^2 \\ &= 1 \in \Gamma_g. \end{aligned}$$

Let $r'_g = r_1 r_2 r_3 = (t_{x_1} \dots t_{\phi_1(c_1)})^2 (t_{x_2} \dots)^2 (t_{x_3} \dots)^2$. Since $f_{r'_g}$ is the fiber sum of f_{r_1}, f_{r_2} and f_{r_3} which are obtained by applying simultaneous conjugations to h_g, we have

$$\chi_{f_{r'_g}} = 3\chi_{f_{h_g}} = 3g \quad \text{and} \quad K^2_{f_{r'_g}} = 3K^2_{f_{h_g}} = 3(4g-4).$$

We apply elementary transformations to r'_g as follows:

$$\begin{aligned}
r'_g &= r_1 r_2 r_3 \\
&= t_{x_1} t_{\phi_1(c_2)} \cdots t_{\phi_1(c_2)} \underline{t_{\phi_1(c_1)} \cdot t_{x_2}} t_{\phi_2(c_2)} \cdots t_{\phi_2(c_1)} \cdot t_{x_3} t_{\phi_3(c_2)} \cdots t_{\phi_3(c_1)} \\
&\sim t_{x_1} t_{\phi_1(c_2)} \cdots t_{\phi_1(c_2)} \underline{t_{x_2} t_{t_{x_2}^{-1}(\phi_1(c_1))}} t_{\phi_2(c_2)} \cdots t_{\phi_2(c_1)} \cdot t_{x_3} t_{\phi_3(c_2)} \cdots t_{\phi_3(c_1)} \\
&\quad \vdots \qquad\qquad\qquad \vdots \\
&\sim t_{x_1} t_{x_2} t_{t_{x_2}^{-1}(\phi_1(c_2))} \cdots t_{t_{x_2}^{-1}(\phi_1(c_2))} t_{t_{x_2}^{-1}(\phi_1(c_1))} t_{\phi_2(c_2)} \cdots \underline{t_{\phi_2(c_1)} \cdot t_{x_3}} t_{\phi_3(c_2)} \cdots t_{\phi_3(c_1)} \\
&\sim t_{x_1} t_{x_2} t_{t_{x_2}^{-1}(\phi_1(c_2))} \cdots t_{t_{x_2}^{-1}(\phi_1(c_2))} t_{t_{x_2}^{-1}(\phi_1(c_1))} t_{\phi_2(c_2)} \cdots \underline{t_{x_3} t_{t_{x_3}^{-1}(\phi_2(c_1))}} t_{\phi_3(c_2)} \cdots t_{\phi_3(c_1)} \\
&\quad \vdots \qquad\qquad\qquad \vdots \\
&\sim (t_{x_1} t_{x_2} t_{x_3}) W,
\end{aligned}$$

where W is a product of $24g+9$ right-handed Dehn twists. By the lantern relation, we get the following positive relator r_g:

$$r_g := (t_{c_1} t_{c_3} t_{c_5} t_y) W.$$

Since r_g is obtained by applying an L^{-1}-substitution to r'_g, by Proposition 2.4

$$\chi_{f_{r_g}} = 3g \quad \text{and} \quad K^2_{f_{r_g}} = 3(4g-4) - 1.$$

Then, the slope of f_{r_g} is equal to $4 - 4/g - 1/3g$.

Since it is easy to check that r_g includes the Dehn twist about a curve $\phi_3(c_i)$ for $1 \leq i \leq 2g+1$, $\pi_1(X_{r_g}) = 1$. This follows from [Gompf and Stipsicz 1999] and the fact that f_{r_g} has a section. This completes the proof of Theorem 3.1. □

Remark 3.4. Since r_g is obtained by applying an L^{-1}-substitution to r'_g, X_{r_g} is a *rational blow-up* of $X_{r'_g}$. By applying elementary transformations to a relator corresponding to a Lefschetz fibration which is obtained by taking a twisted fiber sum with sufficiently many Lefschetz fibrations, we obtain a positive relator such that we can apply a monodromy substitution, which corresponds to the operation of rational blowdown (resp. rational blow-up) in [Endo et al. 2011], to it.

Remark 3.5. Miyachi and Shiga [2011] produced genus-g Lefschetz fibrations over Σ_{2m} which do not satisfy the slope inequality.

Moreover, by fiber sum operations, we have the following results:

Corollary 3.6. *For each $g \geq 3$, $m \geq 0$ and $l \geq 0$, there exists a genus-g Lefschetz fibration $f_{m,l} : X_{m,l} \to S^2$ with slope $\lambda_{f_{m,l}} = 4 - 4/g - 1/(m+3)g$ such that $\pi_1(X_{m,l}) = 1$. Moreover, if $(m,l) \neq (0,0)$, then $X_{m,l}$ is a minimal symplectic 4-manifold.*

Proof of Corollary 3.6. For any $m \geq 0$, we consider the Lefschetz fibration $f_{r_g h_g^m}$: $X_{r_g h_g^m} \to S^2$ which is the fiber sum of f_{r_g} and m copies of f_{h_g}. Then,

$$\chi_{f_{r_g h_g^m}} = \chi_{f_{r_g}} + m\chi_{f_{h_g}} = (3+m)g,$$
$$K^2_{f_{r_g h_g^m}} = K^2_{f_{r_g}} + mK^2_{f_{h_g}} = (3+m)(4g-4)-1.$$

Therefore, we obtain

$$\lambda_{f_{r_g h_g^m}} = 4 - 4/g - 1/(m+3)g. \tag{1}$$

Let $f_{m,l} : X_{m,l} \to S^2$ be the fiber sum of l copies of $f_{r_g h_g^m}$ (that is, $f_{m,l} = f_{(r_g h_g^m)^l}$). By Using (1) and an argument similar to the proof of Theorem 3.1, we have $\lambda_{f_{m,l}} = 4 - 4/g - 1/(m+3)g$ and $\pi_1(X_{m,l}) = 1$. By the result of Usher [2006], $X_{m,l}$ is minimal for $(m,l) \neq (0,0)$. This completes the proof. □

Corollary 3.7. *For each $g \geq 3$, $m \geq 1$ and $l \geq 0$, there exists a genus-g Lefschetz fibration $f'_{m,l} : Y_{m,l} \to S^2$ with slope $\lambda_{f'_{m,l}} = 4 - 4/g - 1/2g + 1/(2 \cdot 3^{m-1} g)$ such that $\pi_1(Y_{m,l}) = 1$. Moreover, if $l \geq 1$, then $Y_{m,l}$ is a minimal symplectic 4-manifold.*

Proof of Corollary 3.7. When we apply the argument of Theorem 3.1 again, with $\varrho_1 = h_g$ replaced by $\varrho_2 = r_g$, we obtain a genus-g Lefschetz fibration $f_{\varrho_3} : X_{\varrho_3} \to S^2$ with

$$\chi_{f_{\varrho_3}} = 3\chi_{f_{\varrho_2}} = 3 \cdot 3\chi_{f_{\varrho_1}}$$
$$K^2_{f_{\varrho_3}} = 3K^2_{f_{\varrho_2}} - 1 = 3(3K^2_{f_{\varrho_1}} - 1) - 1.$$

By repeating this argument, we get a genus-g Lefschetz fibration f_{ϱ_m} $(m \geq 1)$ with

$$\chi_{f_{\varrho_m}} = 3^{m-1}\chi_{f_{\varrho_1}} = 3^{m-1}g,$$
$$K^2_{f_{\varrho_m}} = 3\big(\dots(3(3K^2_{f_{\varrho_1}} - 1) - 1)\dots\big) - 1 = 3^{m-1}K^2_{f_{\varrho_1}} - 3^{m-2} - \cdots - 3 - 1$$
$$= 3^{m-1}(4g-4) - (3^{m-1}-1)/2.$$

Therefore, $\lambda_{f_{\varrho_m}} = 4 - 4/g - 1/2g + 1/(2 \cdot 3^{m-1} g)$.

Let $f'_{m,l} : Y_{m,l} \to S^2$ be the fiber sum of l copies of f_{ϱ_m}, and so $\lambda_{f'_{m,l}} = 4 - 4/g - 1/2g + 1/(2 \cdot 3^{m-1} g)$. Similar to the proof of Corollary 3.6, we see that $\pi_1(Y_{m,l}) = 1$ and that $Y_{m,l}$ is minimal for $l \geq 1$. □

4. Nonholomorphic Lefschetz fibrations

There are various kinds of nonholomorphic Lefschetz fibrations. By fiber summing two copies of genus-2 Lefschetz fibration due to Matsumoto [1996], Ozbagci and Stipsicz [2000] constructed nonholomorphic genus-2 Lefschetz fibrations whose total space does not admit a complex structure. Korkmaz [2001] generalized their examples to $g \geq 3$. The above mentioned examples of Fintushel and Stern are also nonholomorphic Lefschetz fibrations. From the study of divisors in moduli space, Smith [2001] showed that a genus-3 Lefschetz fibration over S^2 which was produced by Fuller is nonholomorphic. Endo and Nagami [2005] constructed some examples of nonholomorphic Lefschetz fibrations which violate lower bounds of the slope for nonhyperelliptic fibrations of genus 3, 4 and 5 from the results of Konno [1991; 1993] and Chen [1993]. Hirose [2010] also gave some examples of $g = 3, 4$. In this section, we give new examples of nonholomorphic Lefschetz fibrations.

From the slope inequality for holomorphic fibrations, we have the following necessary condition for a Lefschetz fibration to be holomorphic:

Proposition 4.1 [Xiao 1987]. *If a Lefschetz fibration f is holomorphic, then the slope inequality $\lambda_f \geq 4 - 4/g$ holds.*

As a consequence, we have the following results.

Corollary 4.2. *The Lefschetz fibrations $f_{m,l}$ $(m \geq 0)$ and $f'_{m,l}$ $(m \geq 2)$ are nonholomorphic.*

Remark 4.3. The above mentioned examples of Fintushel and Stern satisfy the slope inequality but violate the Noether inequality. On the other hand, $f_{m,l}$ and $f'_{m,l}$ satisfy the Noether inequality but violate the slope inequality. Therefore, these two necessary conditions for a Lefschetz fibration to be holomorphic are independent in the sense that neither one implies the other.

5. The slopes of Lefschetz fibrations with $b_2^+ = 1$

We have the following natural question: Which Lefschetz fibrations satisfy the slope inequality? By Proposition 4.1, holomorphic Lefschetz fibrations satisfy the slope inequality. If a Lefschetz fibration is hyperelliptic, then $\lambda_f \geq 4 - 4/g$. This fact can proved as follows. In the notation of Theorem 3.3, we have $e(X_\varrho) = -4(g-1) + (m+s)$. Then, since $h \in \{1, \dots, [g/2]\}$ and $g \geq 2$, we have

$$K_{f_\varrho}^2 - (4 - 4/g)\chi_{f_\varrho} = \sum_{h=1}^{[g/2]} \frac{4h(g-h)-g}{g} s_h \geq 0.$$

In particular, this means that for any hyperelliptic Lefschetz fibrations with only nonseparating vanishing cycles, $\lambda_f = 4 - 4/g$.

In this section, we investigate the slopes of Lefschetz fibrations with $b_2^+ = 1$. By combining the results of [Stipsicz 1999; 2002] and [Li 2000], we can show that Lefschetz fibrations with $b_2^+ = 1$ satisfy the slope inequality. Stipsicz showed that if $X \to S^2$ is a genus-g Lefschetz fibration over S^2 with $b_2^+(X) = 1$ and X is not diffeomorphic to the blow-up of a ruled surface (that is, diffeomorphic to an S^2-bundle over Σ_k), then $b_1(X) \in \{0, 2\}$ and $e \geq 0$ (see [Stipsicz 1999, Corollary 3.3

and 3.5]). In particular, if X is the blow-up of an S^2-bundle over Σ_k, then $k \leq g/2$ (see [Li 2000, Proposition 4.4]). Then, we obtain the following result.

Theorem 5.1. *Let $g \geq 2$ and let $f : X \to S^2$ be a genus-g Lefschetz fibration with $b_2^+(X) = 1$.*

(1) *If X is not diffeomorphic to the blow-up of a ruled surface, then*

$$\text{(i)} \qquad 4 - 4/g \leq \lambda_f \leq 8 \quad \text{for } b_1(X) = 0,$$

$$\text{(ii)} \qquad 4 \leq \lambda_f \leq 8 \qquad \text{for } b_1(X) = 2.$$

(2) *If X is diffeomorphic to the blow-up of an S^2-bundle over Σ_k, then*

$$4 + 4(k-1)/(g-k) \leq \lambda_f \leq 8,$$

and the lower bound is sharp.

An improvement of the previous result was suggested by the referee.

Proof of Theorem 5.1. For a genus-g Lefschetz fibration, a regular fiber has zero self-intersection. Since the intersection form is nondegenerate, it follows that $b_2^\pm \geq 1$. Let $f : X \to S^2$ be a nontrivial genus-g Lefschetz fibration with $b_2^+(X) = 1$. Note that $-4(g-1) \leq K^2$, and so $4(g-1) \leq K_f^2$ (see [Stipsicz 1999, Lemma 3.2]). Suppose that X is not diffeomorphic to the blow-up of a ruled surface.

First, suppose that $b_1 = 0$. Since $b_2^+ = 1$ and $\chi_f = (\sigma + e)/4 + (g-1) = (b_2^+ - b_1 + 1)/2 + (g-1) = g$, we have $4(g-1)/g \leq K_f^2/\chi_f = \lambda_f$. On the other hand, since $K^2 = 3\sigma + 2e = 5b_2^+ - b_2^- + 4 - 4b_1 = 9 - b_2^-$, by $b_2^- \geq 1$, we have $\lambda_f = K_f^2/\chi_f = \{9 - b_2^- + 8(g-1)\}/g \leq 8$.

Next, suppose $b_1 = 2$. Then, $\chi_f = g - 1$. Therefore, by $4(g-1) \leq K_f^2$, we have $4 \leq \lambda_f$. Since $0 \leq e = 2 - 2b_1 + b_2^+ + b_2^- = 2 - 4 + 1 + b_2^- = -1 + b_2^-$, we obtain $\lambda_f = \{1 - b_2^- + 8(g-1)\}/(g-1) \leq 8$.

Finally, suppose that X is the m-fold blow-up of an S^2-bundle over Σ_k. Let Y be the S^2-bundle over Σ_k. Then, since $b_1(Y) = 2k$, $b_2^\pm(Y) = 1$ and $X = Y \# m\overline{\mathbb{CP}^2}$, we have $b_1(X) = 2k$, $b_2^+(X) = 1$, $b_2^-(X) = m+1$, $e(X) = 4 - 4k + m$ and $\sigma(X) = -m$. Hence, we have $\lambda_f = 8 - m/(g-k)$. From $m \geq 0$, $\lambda_f \leq 8$. We will give lower bounds for λ_f. By Lemma 3.2 in [Stipsicz 2002], $4(2k-g) + m \leq 4$. We have $\lambda_f \geq 4 + 4(k-1)/(g-k)$ from $\lambda_f = 8 - m/(g-k)$, $4(2k-g) + m \leq 4$ and $0 \leq k \leq g/2$ [Li 2000, Proposition 4.4]. Fintushel and Stern [2004] showed that $(\Sigma_k \times S^2) \# 4m\overline{\mathbb{CP}^2}$ admits a genus-$(2k+m-1)$ Lefschetz fibration f_{FS} over S^2. When $m = g - 2k + 1$, we find $b_2^+ = 1$ and that $\lambda_{f_{FS}} = 4 + 4(k-1)/(g-k)$. □

Remark 5.2. If two Lefschetz fibrations f_1 and f_2 satisfy $\lambda_{f_1}, \lambda_{f_2} \geq 4 - 4/g$, then the (twisted) fiber sum f_3 of f_1 and f_2 satisfies $\lambda_{f_3} \geq 4 - 4/g$.

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