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# $H^\infty$ -Optimal Fractional Delay Filters

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## Abstract

Fractional delay filters are digital filters to delay discrete-time signals by a fraction of the sampling period. Since the delay is fractional, the intersample behavior of the original analog signal becomes crucial. In contrast to the conventional designs based on the Shannon sampling theorem with the band-limiting hypothesis, the present paper proposes a new approach based on the modern sampled-data  $H^\infty$  optimization that aims at restoring the intersample behavior beyond the Nyquist frequency. By using the lifting transform or continuous-time blocking the design problem is equivalently reduced to a discrete-time  $H^\infty$  optimization, which can be effectively solved by numerical computation softwares. Moreover, a closed-form solution is obtained under an assumption on the original analog signals. Design examples are given to illustrate the advantage of the proposed method.

## Index Terms

Fractional delay filters, interpolation, sampled-data systems,  $H^\infty$  optimization, linear matrix inequality.

## I. INTRODUCTION

Fractional delay filters are digital filters to delay discrete-time signals by a fractional amount of the sampling period. Such filters have wide applications in signal processing, including sampling rate conversion [1], [2], [3], nonuniform sampling [4], [5], wavelet transform [6], [7], digital modeling of musical instruments [8], [9], to name a few. For more applications, see survey papers [10], [11], [12].

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Conventionally, fractional delay filters are designed based on the Shannon sampling theorem [13], [14] for strictly bandlimited analog signals. Based on this theory, the optimal filter coefficients are obtained by sampling a delayed sinc function. This ideal low-pass filter is however not realizable because of its non-causality and instability, and hence many studies have focused their attention on approximating the ideal filter by, for example, windowed sinc functions [15], [16], maximally-flat FIR approximation [17], [18], [19], [20], [21], all-pass approximation [22], [23], and minmax (Chebyshev) optimization [24].

In particular,  $H^2$  (or weighted least-squares) design has been prevalent in the literature [10], [25], [26], [21]. This method minimizes the  $H^2$  norm of the weighted difference between the ideal low-pass filter and a filter to be designed, and is based on the projection theorem in Hilbert space. There are, however, two major drawbacks in this conventional approach. One is that due to the averaging nature of the  $H^2$  criterion, the obtained frequency response can have a sharp peak at a certain frequency, thereby yielding a poor performance at that frequency, while still maintaining small  $H^2$  error in the overall frequency response. The other is that  $H^2$  criterion can yield a truncated frequency response as an optimal approximant of the ideal low-pass filter, which yields a distortion due to the Gibbs phenomenon in the time domain. Furthermore, such a design is mostly executed in the discrete-time domain, which yields poor intersample response.

In view of these problems we employ *sampled-data  $H^\infty$  optimization*, recently introduced for signal processing by [27]<sup>1</sup>. This is based on *sampled-data control theory* [29] which accounts for the mixed nature of *continuous- and discrete-time* thereby enabling optimization of the intersample signals via discrete-time controllers (filters). This also allows for optimization according to the  $H^\infty$  norm, namely *minimizing the maximum of the error frequency response*. This worst-case design is clearly desirable in that it does not have the drawback due to the averaging property of the  $H^2$  criterion. Due to the nature of the  $H^\infty$  norm, however, this optimization problem has been difficult to solve, but one can now utilize a standardized method to solve this class of problems [30], [29]. Furthermore, the obtained filter shows greater robustness against unknown disturbances due to the nature of the *uniform attenuation* of the error frequency response; see [27] for details. Based on this  $H^\infty$  optimization method, we formulate the design of fractional delay filters as a sampled-data  $H^\infty$  optimization problem<sup>2</sup>.

In order to optimize the intersample behavior, we must deal with both continuous- and discrete-time

<sup>1</sup>The approach dates back to [28], though.

<sup>2</sup>This method was first proposed in our conference articles [31], [32]. The present paper reorganizes these works with new results on the state-space formulation (Proposition 1, Appendix A). Simulation results in Section IV are also new.

signals, and hence the overall system is not time-invariant. The key to solving this problem is *lifting*, which is introduced in the early studies of modern sampled-data control theory [33], [34], [35], [36], [37]. Indeed, continuous-time lifting gives an exact, not approximated, time-invariant discrete-time model for a sampled-data system, albeit with infinite-dimensional input and output spaces. Hence the problem of the mixed time sets is circumvented without approximation.

Lifting can also be interpreted as a *continuous-time blocking or polyphase decomposition*. As in multi-rate signal processing [38], lifting makes it possible to capture continuous-time signals and systems in the discrete-time domain without approximation; see Section III-A for details. The remaining system becomes a *time-invariant discrete-time* system, albeit with infinite-dimensional input and output spaces. In view of this infinite-dimensionality, we retain the term lifting to avoid confusion. With such a representation, we show that our design problem is reducible to a finite-dimensional discrete-time  $H^\infty$  optimization without approximation. This type of  $H^\infty$  optimization is easily solvable by standard softwares such as MATLAB [39].

In some applications, digital filters with variable delay responses (*variable fractional delay filters* [40], [10], [25], [26]) are desired. In this case, a filter should have a tunable delay parameter, and hence a closed-form formula should be derived. In general,  $H^\infty$  optimal filters are difficult to solve analytically. However, we provide a closed-form formula of the optimal filter with the delay variable as a parameter under the assumption that the underlying frequency characteristic of continuous-time input signals is governed by a low-pass filter of first order. While this assumption may appear somewhat restrictive, it covers many typical cases and variations by some robustness properties [31].

The paper is organized as follows. Section II defines fractional delay filters, and reviews a standard  $H^2$  design method. We then reformulate our design problem as a sampled-data  $H^\infty$  optimization to overcome the difficulty due to the  $H^2$  design. Section III gives a procedure to solve the sampled-data  $H^\infty$  optimization problem based on the lifting transform. Section IV shows numerical examples to illustrate the superiority of the proposed method.

### Notation

Throughout this paper, we use the following notation. We denote by  $L^2[0, \infty)$  and  $L^2[0, T)$  the Lebesgue spaces consisting of all square integrable real functions on  $\mathbb{R}_+ := [0, \infty)$  and  $[0, T)$ , respectively.  $L^2[0, \infty)$  may be abbreviated as  $L^2$ . By  $\ell^2$  we denote the set of all real-valued square summable sequences on  $\mathbb{Z}_+ := \{0, 1, 2, \dots\}$ . For a normed space  $X$ , we denote by  $\ell^2(\mathbb{Z}_+, X)$  the set of all sequences on  $\mathbb{Z}_+$  taking values in  $X$  with squared norms being summable. For normed linear spaces  $X$  and  $Y$ , we denote

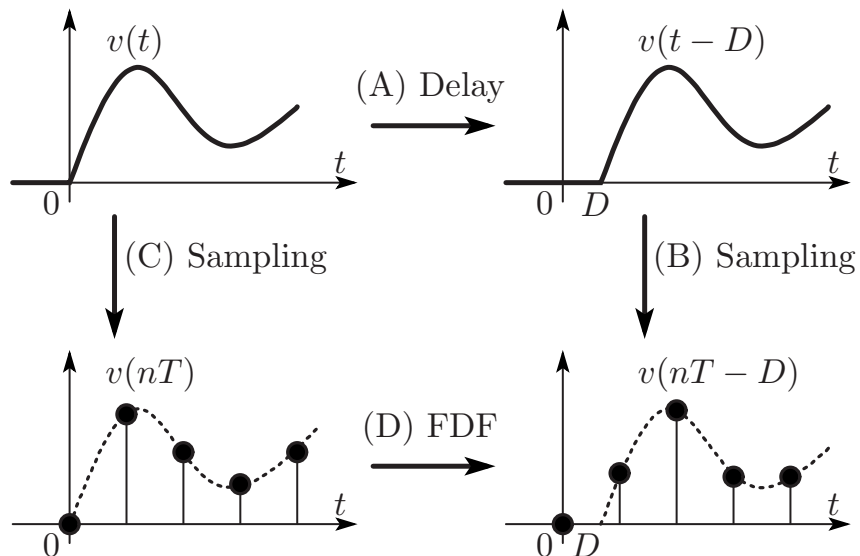


Fig. 1. Fractional delay process: (A) a continuous-time signal  $v(t)$  (top left) is delayed by  $D > 0$ . (B) the delayed signal  $v(t - D)$  is sampled at  $t = nT$ ,  $n = 0, 1, \dots$  (C) the signal  $v(t)$  is sampled at  $t = nT$ ,  $n = 0, 1, \dots$  (D) digital filtering (fractional delay filter, FDF) to produce (or estimate) the sequence  $\{v(nT - D)\}$  from the sampled-data  $\{v(nT)\}$ .

by  $\mathbb{B}(X, Y)$  the set of all bounded linear operators of  $X$  into  $Y$ .  $\mathbb{R}^\nu$  and  $\mathbb{R}^{m \times n}$  denote respectively the set of real vectors of size  $\nu$  and real matrices of size  $m \times n$ . Finite-dimensional vectors and matrices are denoted by bold letters, such as  $\mathbf{x}$  or  $\mathbf{A}$ , and infinite-dimensional operators by calligraphic letters, such as  $\mathcal{B}$ . The transpose of a matrix  $\mathbf{A}$  is denoted by  $\mathbf{A}^\top$ . Symbols  $s$  and  $z$  are used for the variables of Laplace and  $Z$  transforms, respectively. For a linear system  $F$ , its transfer function is denoted by  $\hat{F}(z)$  (if  $F$  is discrete-time) or  $\hat{F}(s)$  (if  $F$  is continuous-time), and its impulse response by the lower-case letter,  $f[n]$  or  $f(t)$ . The imaginary unit  $\sqrt{-1}$  is denoted by  $j$ .

## II. FRACTIONAL DELAY FILTERS

In this section, we review fractional delay filters with conventional design methods based on the Shannon sampling theorem. Then, we reformulate the design problem as a sampled-data  $H^\infty$  optimization problem.

### A. Definition and standard design method

Consider a continuous-time signal  $v$  shown in Fig. 1 (top-left figure). Assume  $v(t) = 0$  for  $t < 0$  (i.e., it is a causal signal). Delaying this signal by  $D > 0$  gives the delayed continuous-time signal  $v(t - D)$

shown in Fig. 1 (top-right in Fig. 1). Then by sampling  $v(t - D)$  with sampling period  $T$ , we obtain the discrete-time signal  $\{v(nT - D)\}_{n \in \mathbb{Z}}$  as shown in Fig. 1 (bottom-right in Fig. 1).

Next, let us consider the directly sampled signal  $\{v(nT)\}_{n \in \mathbb{Z}}$  of the original analog signal  $v$  as shown in Fig. 1 (bottom-left in Fig. 1). The objective of fractional delay filters is to reconstruct or estimate the delayed sampled signal  $\{v(nT - D)\}_{n \in \mathbb{Z}}$  directly from the sampled data  $\{v(nT)\}_{n \in \mathbb{Z}}$  when  $D$  is not an integer multiple of  $T$ . We now define the *ideal* fractional delay filter.

*Definition 1:* The *ideal fractional delay filter*  $K_{\text{id}}$  with delay  $D > 0$  is the mapping that produces  $\{v(nT - D)\}_{n \in \mathbb{Z}}$  from  $\{v(nT)\}_{n \in \mathbb{Z}}$ , that is,

$$K_{\text{id}} : \{v(nT)\}_{n \in \mathbb{Z}} \mapsto \{v(nT - D)\}_{n \in \mathbb{Z}}.$$

Assume for the moment that the original analog signal  $v$  is fully band-limited below the Nyquist frequency  $\Omega_N = \pi/T$ , that is,

$$\hat{v}(j\omega) = 0, \quad |\omega| \geq \Omega_N, \quad (1)$$

where  $\hat{v}$  is the Fourier transform of  $v$ . Then the impulse response of the ideal fractional delay filter is obtained by [10]:

$$k_{\text{id}}[n] = \frac{\sin \pi(n - D/T)}{\pi(n - D/T)} = \text{sinc}(n - D/T), \quad (2)$$

$$n = 0, \pm 1, \pm 2, \dots, \quad \text{sinc}(t) := \frac{\sin(\pi t)}{\pi t}.$$

The frequency response of this ideal filter is given in the frequency domain as

$$\hat{K}_{\text{id}}(e^{j\omega T}) = e^{-j\omega D}, \quad \omega \leq \Omega_N. \quad (3)$$

Since the impulse response (2) does not vanish at  $n = -1, -2, \dots$  and is not absolutely summable, the ideal filter is noncausal and unstable, and hence the ideal filter is not physically realizable. Conventional designs thus aim at approximating the impulse response (2) or the frequency response (3) by a causal and stable filter. We here review in particular the  $H^2$  optimization, also known as weighted least squares [10].

Define the weighted approximation error by

$$E_2 := (K_{\text{id}} - K)W_d \quad (4)$$

where  $W_d$  is a weighting function and  $K$  is a filter to be designed, which is assumed to be FIR (finite impulse response). The  $H^2$  design aims at finding the FIR coefficients of the transfer function  $\hat{K}(z)$  of  $K$  that minimize the  $H^2$  norm of the weighted error system  $E_2$ :

$$\begin{aligned} \|E_2\|_2^2 &= \|(K_{\text{id}} - K)W_d\|_2^2 \\ &= \frac{1}{\Omega_N} \int_0^{\Omega_N} \left| \left[ \hat{K}_{\text{id}}(e^{j\omega T}) - \hat{K}(e^{j\omega T}) \right] \hat{W}_d(e^{j\omega T}) \right|^2 d\omega. \end{aligned} \quad (5)$$

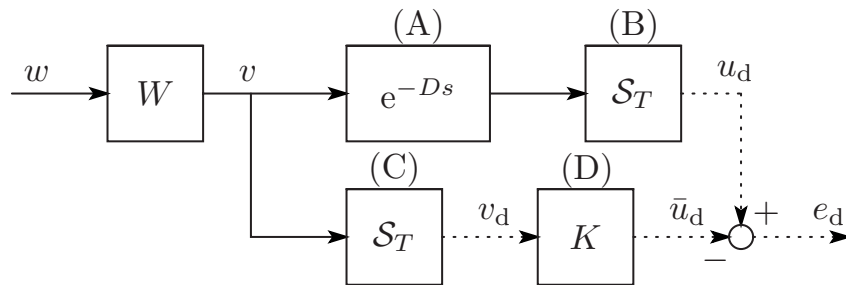


Fig. 2. Error system  $E$  for designing fractional delay filter  $K$ . (A)–(D) correspond to those in Fig. 1.

As pointed out in the Introduction, this  $H^2$  design has some drawbacks. One is that the designed filter  $K$  may yield a large peak in the error frequency response  $\hat{E}_2(e^{j\omega T})$  due to the averaging nature of the  $H^2$  norm (5). If an input signal has a frequency component at around such a peak of  $\hat{E}_2(e^{j\omega T})$ , the error will become very large. The second is that the perfect band-limiting assumption (1) implies that the  $H^2$  suboptimal filter is given as an approximant of the ideal low-pass filter [41], which induces large errors in the time domain [27]. Moreover, real analog signals always contain frequency components beyond the Nyquist frequency, and hence (1) never holds exactly for real signals.

### B. Reformulation of design problem

To simultaneously solve the two problems pointed out above, we introduce sampled-data  $H^\infty$  optimization [27]. This method has advantages as mentioned in Section I. To adapt sampled-data  $H^\infty$  optimization for the design of fractional delay filters, we reformulate the design problem, instead of mimicking the “ideal” filter given in (2) or (3).

Let us consider the error system shown in Fig. 2.  $W$  is a stable continuous-time system with strictly proper transfer function  $\hat{W}(s)$  that defines the frequency-domain characteristic of the original analog signal  $v$ . More precisely, we assume that the analog signal  $v$  is in the following subspace of  $L^2$ :

$$WL^2 := \{v \in L^2 : v = Ww, w \in L^2\}.$$

Note that the signal subspace  $WL^2$  is much wider than that of band-limited  $L^2$  signals [42].

The upper path of the diagram in Fig. 2 is the ideal process of the fractional delay filter (the process (A)  $\rightarrow$  (B) in Fig. 1); that is, the continuous-time signal  $v$  is delayed by the continuous-time delay denoted by  $e^{-Ds}$  (we use the notation  $e^{-Ds}$ , the transfer function of the  $D$ -delay system, as the system

itself), and then sampled by the ideal sampler denoted by  $\mathcal{S}_T$  with period  $T > 0$  to become an  $\ell^2$  signal<sup>3</sup>  
 $u_d := \mathcal{S}_T e^{-Ds} v$ , or

$$u_d[n] := (\mathcal{S}_T e^{-Ds} v)[n] = v(nT - D), \quad n \in \mathbb{Z}_+.$$

On the other hand, the lower path represents the real process ((C)  $\rightarrow$  (D) in Fig. 1); that is, the continuous-time signal  $v$  is directly sampled with the same period  $T$  to produce a discrete-time signal  $v_d \in \ell^2$  defined by

$$v_d[n] := (\mathcal{S}_T v)[n] = v(nT), \quad n \in \mathbb{Z}_+.$$

This signal is then filtered by a digital filter  $K$  to be designed, and we obtain an estimation signal  $\bar{u}_d = K \mathcal{S}_T v \in \ell^2$ .

Put  $e_d := u_d - \bar{u}_d$  (the difference between the ideal output  $u_d$  and the estimation  $\bar{u}_d$ ), and let  $E$  denote the error system from  $w \in L^2$  to  $e_d \in \ell^2$  (see Fig. 2). Symbolically,  $E$  is represented by (cf. (4))

$$E = (\mathcal{S}_T e^{-Ds} - K \mathcal{S}_T) W. \quad (6)$$

Then our problem is to find a digital filter  $K$  that minimizes the  $H^\infty$  norm of the error system  $E$ .

*Problem 1:* Given a stable, strictly proper  $W(s)$ , a delay time  $D > 0$ , and a sampling period  $T > 0$ , find the digital filter  $K$  that minimizes (cf. (5))

$$\begin{aligned} \|E\|_\infty &= \|(\mathcal{S}_T e^{-Ds} - K \mathcal{S}_T) W\|_\infty \\ &= \sup_{w \in L^2, \|w\|_2=1} \|(\mathcal{S}_T e^{-Ds} - K \mathcal{S}_T) W w\|_{\ell^2}. \end{aligned}$$

Note that  $W$ , or its transfer function  $\hat{W}(s)$ , can be interpreted as a frequency-domain weighting function for the optimization. This is comparable to  $\hat{W}_d(z)$  in the discrete-time  $H^2$  design minimizing (5). The point to use continuous-time  $\hat{W}(s)$  is that one can model the frequency characteristic of signals beyond the Nyquist frequency. Also, the advantage of using the sampled-data setup here is that we can minimize the norm of the overall transfer operator from continuous-time  $w$  to the error  $e_d$ . In the next section, we will show a procedure to solve Problem 1 based on sampled-data control theory.

### III. $H^\infty$ DESIGN OF FRACTIONAL DELAY FILTERS

The error system  $E$  in Fig. 2 contains both continuous- and discrete-time signals, and hence the system is not time-invariant; in fact, it is  $T$ -periodic [29]. In this section, we introduce the continuous-time lifting technique [37], [29] to derive a norm-preserving transformation from  $E$  to a time-invariant

<sup>3</sup>If  $\hat{W}(s)$  is stable and strictly proper, the discrete-time signal  $u_d = \mathcal{S}_T e^{-Ds} v$  belongs to  $\ell^2$ . Otherwise,  $\mathcal{S}_T$  is not a bounded operator on  $L^2$ ; see [29, Section 9.3].



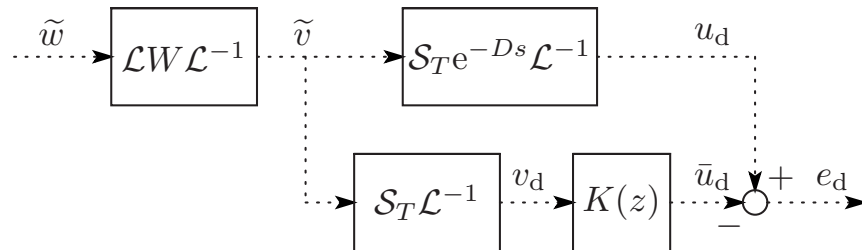


Fig. 3. Lifted error system  $\mathcal{E}$ .

finite-dimensional discrete-time system. After this, one can use a standard discrete-time  $H^\infty$  optimization implemented on a computer software such as MATLAB to obtain an optimal filter. We also give a closed-form solution of the optimization under an assumption on  $W(s)$ .

#### A. Lifted model of sampled-data error system

Let  $\{\mathbf{A}, \mathbf{B}, \mathbf{C}\}$  be a minimal realization [43] of  $\hat{W}(s)$ :

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}w(t), \quad v(t) = \mathbf{C}\mathbf{x}(t), \quad t \in \mathbb{R}_+, \quad (7)$$

where  $\mathbf{x}(t) \in \mathbb{R}^\nu$  is the state variable ( $\nu$  is a positive integer). We assume  $\mathbf{A} \in \mathbb{R}^{\nu \times \nu}$ ,  $\mathbf{B} \in \mathbb{R}^{\nu \times 1}$ ,  $\mathbf{C} \in \mathbb{R}^{1 \times \nu}$ , and  $\mathbf{x}(0) = \mathbf{0}$ . Let  $D = mT + d$  where  $m \in \mathbb{Z}_+$  and  $d$  is a real number such that  $0 \leq d < T$ . First, we introduce the lifting operator  $\mathcal{L}$  [37], [29] that transforms a continuous-time signal in  $L^2[0, \infty)$  to an  $\ell^2$  sequence of functions in  $L^2[0, T)$ . Apply  $\mathcal{L}$  to the continuous-time signals  $w$  and  $v$ , and put  $\tilde{w} := \mathcal{L}w$ ,  $\tilde{v} := \mathcal{L}v$ . By this, the error system in Fig. 2, is transformed into a time-invariant discrete-time system  $\mathcal{E}$  shown in Fig. 3. Since the operator  $\mathcal{L}$  gives an isometry between  $L^2[0, \infty)$  and  $\tilde{\ell}^2 := \ell^2(\mathbb{Z}_+, L^2[0, T))$ , we have

$$\|E\|_\infty = \|\mathcal{E}\|_\infty := \sup_{\tilde{w} \in \tilde{\ell}^2, \|\tilde{w}\|_{\tilde{\ell}^2} = 1} \|\mathcal{E}\tilde{w}\|_{\ell^2} \quad (8)$$

The following proposition is fundamental to the sampled-data optimization in (8).

*Proposition 1:* A state-space realization of the lifted error system  $\mathcal{E}$  is given by

$$\begin{aligned} \boldsymbol{\xi}[n+1] &= \mathbf{A}_d \boldsymbol{\xi}[n] + \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix} \tilde{w}[n], \\ e_d[n] &= \mathbf{C}_1 \boldsymbol{\xi}[n] - \bar{u}_d[n], \quad v_d[n] = \mathbf{C}_2 \boldsymbol{\xi}[n], \\ \bar{u}_d[n] &= (k * v_d)[n], \end{aligned} \quad (9)$$

where  $*$  stands for convolution, and the pertinent operators  $\mathbf{A}_d$ ,  $\mathbf{B}$ ,  $\mathbf{C}_1$  and  $\mathbf{C}_2$  are given as follows:  
 First,  $\mathbf{B}$  is a linear (infinite-dimensional) operator defined by

$$\begin{aligned} \mathbf{B} : L^2[0, T] &\rightarrow \mathbb{R}^{\nu+1}, \\ \tilde{w} &\mapsto \mathbf{B}\tilde{w} = \begin{bmatrix} \mathbf{B}_1 \tilde{w} \\ \mathbf{B}_2 \tilde{w} \end{bmatrix} \\ &= \begin{bmatrix} \int_0^T e^{\mathbf{A}(T-\tau)} \mathbf{B} \tilde{w}(\tau) d\tau \\ \int_0^{T-d} \mathbf{C} e^{\mathbf{A}(T-d-\tau)} \mathbf{B} \tilde{w}(\tau) d\tau \end{bmatrix} \end{aligned} \quad (10)$$

The matrices  $\mathbf{A}_d$ ,  $\mathbf{C}_1$ , and  $\mathbf{C}_2$  in (9) are defined by

$$\begin{aligned} \mathbf{A}_d &:= \begin{bmatrix} e^{\mathbf{A}T} & 0 & 0 \\ \mathbf{C} e^{\mathbf{A}(T-d)} & 0 & 0 \\ 0 & \mathbf{B}_m & \mathbf{A}_m \end{bmatrix} \in \mathbb{R}^{(\nu+1+m) \times (\nu+1+m)}, \\ \mathbf{C}_1 &:= [0, 0, \mathbf{C}_m] \in \mathbb{R}^{1 \times (\nu+1+m)}, \\ \mathbf{C}_2 &:= [\mathbf{C}, 0, 0] \in \mathbb{R}^{1 \times (\nu+1+m)}, \end{aligned}$$

where  $\mathbf{A}_m$ ,  $\mathbf{B}_m$ , and  $\mathbf{C}_m$  are state-space realization matrices of the discrete-time delay  $z^{-m}$ .

*Proof:* See Appendix A. ■

The state-space representation (9) then gives the transfer function of the lifted system  $\mathcal{E}$  as

$$\hat{\mathcal{E}}(z) = \hat{\mathcal{G}}_1(z) - \hat{K}(z) \hat{\mathcal{G}}_2(z), \quad (11)$$

where

$$\hat{\mathcal{G}}_i(z) := \mathbf{C}_i (z\mathbf{I} - \mathbf{A}_d)^{-1} \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix}, \quad i = 1, 2.$$

Put

$$\hat{E}_0(z) := (\mathbf{C}_1 - \hat{K}(z) \mathbf{C}_2) (z\mathbf{I} - \mathbf{A}_d)^{-1}. \quad (12)$$

Note that  $E_0$  is a finite-dimensional discrete-time system. Then the lifted system  $\mathcal{E}(z)$  in (11) can be factorized (see Fig. 4) as

$$\hat{\mathcal{E}}(z) = \hat{E}_0(z) \begin{bmatrix} \mathbf{B} \\ 0 \end{bmatrix}. \quad (13)$$

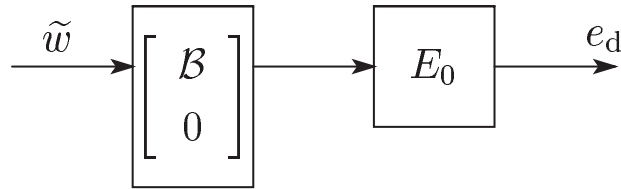


Fig. 4. Factorization of  $\mathcal{E}$ .

### B. Norm-equivalent finite-dimensional system

The lifted system  $\mathcal{E}$  given in (9), or its transfer function  $\hat{\mathcal{E}}$  in (11), involves an infinite-dimensional operator  $\mathcal{B} : L^2[0, T) \rightarrow \mathbb{R}^{\nu+1}$ . Introducing the dual operator [44]  $\mathcal{B}^* : \mathbb{R}^{\nu+1} \rightarrow L^2[0, T)$  of  $\mathcal{B}$ , and composing this with  $\mathcal{B}$ , we can obtain a norm-equivalent finite dimensional system of the infinite-dimensional system  $\mathcal{E}$ .

The dual operator  $\mathcal{B}^*$  of  $\mathcal{B}$  is given by [44]

$$\mathcal{B}^* = \begin{bmatrix} \mathcal{B}_1^* & \mathcal{B}_2^* \end{bmatrix}, \quad \mathcal{B}_1^*(\theta) := \mathbf{B}^\top e^{\mathbf{A}^\top(T-\theta)},$$

$$\mathcal{B}_2^*(\theta) := \mathbf{1}_{[0, T-d)}(\theta) \mathbf{B}^\top e^{\mathbf{A}^\top(T-d-\theta)} \mathbf{C}^\top, \quad \theta \in [0, T),$$

where  $\mathbf{1}_{[0, T-d)}$  is the characteristic function of the interval  $[0, T-d)$ , that is,

$$\mathbf{1}_{[0, T-d)}(\theta) := \begin{cases} 1, & \theta \in [0, T-d), \\ 0, & \text{otherwise.} \end{cases}$$

Then we have the following lemma:

*Lemma 1:* The operator  $\mathcal{B}\mathcal{B}^*$  is a positive semi-definite matrix given by

$$\begin{aligned} \mathcal{B}\mathcal{B}^* &= \begin{bmatrix} \mathcal{B}_1\mathcal{B}_1^* & \mathcal{B}_1\mathcal{B}_2^* \\ \mathcal{B}_2\mathcal{B}_1^* & \mathcal{B}_2\mathcal{B}_2^* \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{M}(T) & e^{\mathbf{A}d}\mathbf{M}(T-d)\mathbf{C}^\top \\ \mathbf{C}\mathbf{M}(T-d)e^{\mathbf{A}^\top d} & \mathbf{C}\mathbf{M}(T-d)\mathbf{C}^\top \end{bmatrix} \end{aligned} \tag{14}$$

where  $\mathbf{M}(\cdot)$  is defined by

$$\mathbf{M}(t) := \int_0^t e^{\mathbf{A}\theta} \mathbf{B}\mathbf{B}^\top e^{\mathbf{A}^\top\theta} d\theta \in \mathbb{R}^{\nu \times \nu}, \quad t \geq 0.$$

*Proof:* We first prove  $\mathcal{B}_1\mathcal{B}_2^* = e^{\mathbf{A}d}\mathbf{M}(T-d)\mathbf{C}^\top$ .

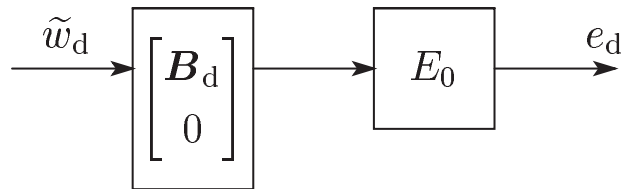


Fig. 5. Discrete-time system  $E_d$ .

For every  $u \in \mathbb{R}$ , we have

$$\begin{aligned}
 \mathcal{B}_1 \mathcal{B}_2^* u &= \int_0^T e^{\mathbf{A}(T-\theta)} \mathbf{B} (\mathcal{B}_2^*(\theta) u) d\theta \\
 &= \int_0^T e^{\mathbf{A}(T-\theta)} \mathbf{B} \left( \mathbf{1}_{[0, T-d)}(\theta) \mathbf{B}^\top e^{\mathbf{A}^\top (T-d-\theta)} \mathbf{C}^\top u \right) d\theta \\
 &= e^{\mathbf{A}d} \int_0^{T-d} e^{\mathbf{A}(T-d-\theta)} \mathbf{B} \mathbf{B}^\top e^{\mathbf{A}^\top (T-d-\theta)} d\theta \mathbf{C}^\top u \\
 &= e^{\mathbf{A}d} \mathbf{M}(T-d) \mathbf{C}^\top u.
 \end{aligned}$$

Similarly, we can prove the equalities  $\mathcal{B}_1 \mathcal{B}_1^* = \mathbf{M}(T)$  and  $\mathcal{B}_2 \mathcal{B}_2^* = \mathbf{C} \mathbf{M}(T-d) \mathbf{C}^\top$ . ■

*Remark 1:* The matrix  $\mathbf{M}(t)$  can be computed via the matrix exponential formula [45]:

$$\begin{aligned}
 \mathbf{M}(t) &= \mathbf{F}_{22}^\top(t) \mathbf{F}_{12}(t), \\
 \begin{bmatrix} \mathbf{F}_{11}(t) & \mathbf{F}_{12}(t) \\ 0 & \mathbf{F}_{22}(t) \end{bmatrix} &:= \exp \left\{ \begin{bmatrix} -\mathbf{A} & \mathbf{B} \mathbf{B}^\top \\ 0 & \mathbf{A}^\top \end{bmatrix} t \right\}.
 \end{aligned}$$

By this formula, we can easily compute the matrices  $\mathbf{M}(T)$  and  $\mathbf{M}(T-d)$  in (14) without performing a numerical integration.

From Lemma 1,  $\mathcal{B} \mathcal{B}^*$  is a positive semi-definite matrix and hence there exists a matrix  $\mathbf{B}_d$  such that  $\mathcal{B} \mathcal{B}^* = \mathbf{B}_d \mathbf{B}_d^\top$ . With matrix  $\mathbf{B}_d$  and discrete-time system  $E_0$  given in (12), define a finite-dimensional discrete-time system by

$$E_d := E_0 \begin{bmatrix} \mathbf{B}_d \\ 0 \end{bmatrix}.$$

See Fig. 5 for the block diagram of  $E_d$ . Then the discrete-time system  $E_d$  is equivalent to the original sampled-data error system  $E$  in Fig. 2 with respect to their  $H^\infty$  norm as described in the following theorem:

*Theorem 1:* Assume that the sampled-data error system  $E$  gives an operator belonging to  $\mathbb{B}(L^2, \ell^2)$ , the set of all bounded linear operators of  $L^2$  into  $\ell^2$ . Then the discrete-time system  $E_d$  belongs to  $\mathbb{B}(\ell^2, \ell^2)$  and equivalent to  $E$  with respect to their  $H^\infty$  norm, that is,  $\|E\|_\infty = \|E_d\|_\infty$ .

*Proof:* First, the equality in (8) and  $E \in \mathbb{B}(L^2, \ell^2)$  give  $\|E\|_\infty = \|\mathcal{E}\|_\infty < \infty$ . Using the factorization (13), we have

$$\begin{aligned} \|E\|_\infty^2 = \|\mathcal{E}\|_\infty^2 &= \left\| E_0 \begin{bmatrix} \mathcal{B} \\ 0 \end{bmatrix} \right\|_\infty^2 \\ &= \left\| E_0 \begin{bmatrix} \mathcal{B} \\ 0 \end{bmatrix} \begin{bmatrix} \mathcal{B} \\ 0 \end{bmatrix}^* E_0^* \right\|_\infty \\ &= \left\| E_0 \begin{bmatrix} \mathbf{B}_d \\ 0 \end{bmatrix} \begin{bmatrix} \mathbf{B}_d \\ 0 \end{bmatrix}^\top E_0^* \right\|_\infty = \|E_d\|_\infty^2. \end{aligned}$$

■

Thus the sampled-data  $H^\infty$  optimization (Problem 1) is equivalently transformed to discrete-time  $H^\infty$  optimization. A MATLAB code for the  $H^\infty$ -optimal fractional delay filter is available on the web at [46]. Moreover, if we assume that the filter  $K(z)$  is an FIR filter, the design is reduced to a convex optimization with a linear matrix inequality. See [32], [47] for details.

### C. Closed-form solution under a first-order assumption

Assume that the weighting function  $\hat{W}(s)$  is a first-order low-pass filter with cutoff frequency  $\omega_c > 0$ :

$$\hat{W}(s) = \frac{\omega_c}{s + \omega_c}. \quad (15)$$

Under this assumption, a closed-form solution for the optimal filter is obtained [31], [32]:

*Theorem 2:* Assume that  $\hat{W}(s)$  is given by (15). Then the optimal filter  $\hat{K}(z)$  is given by

$$\hat{K}(z) = a_0(d)z^{-m} + a_1(d)z^{-m-1}, \quad (16)$$

where

$$a_0(d) := \frac{\sinh(\omega_c(T-d))}{\sinh(\omega_c T)}, \quad a_1(d) := e^{-\omega_c T} (e^{\omega_c d} - a_0(d)).$$

Moreover, the optimal value of  $\|E\|_\infty$  is given by

$$\|E\|_\infty = \sqrt{\frac{\omega_c \sinh(\omega_c d) \sinh(\omega_c(T-d))}{\sinh(\omega_c T)}}. \quad (17)$$

Since the optimal filter  $\hat{K}(z)$  in (16) is a function of the fractional delay  $d$  and the integer delay  $m$ , the filter can be used as a *variable fractional delay filter* [10].

*Remark 2:* Fix  $d > 0$  and  $m \in \mathbb{Z}_+$  arbitrarily. By definition, we have  $T-d < T$ . It follows that as  $\omega_c \rightarrow \infty$ , we have  $a_0(d) \rightarrow 0$ ,  $a_1(d) \rightarrow 0$ , and  $\|E\|_\infty \rightarrow \infty$ . This means that if the original analog

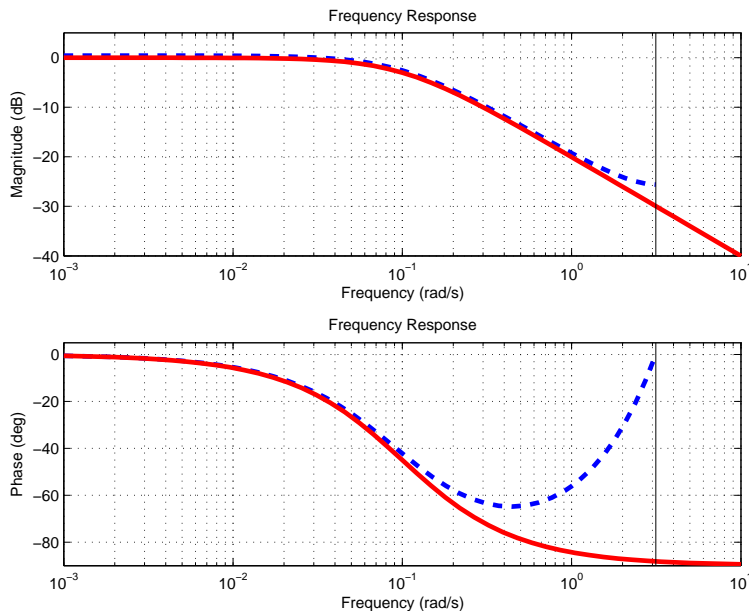


Fig. 6. Bode plot of  $\hat{W}(s)$  (solid) and its impulse-invariant discretization  $\hat{W}_d(z)$  (dash). The vertical line shows the Nyquist frequency.

signals contain higher frequency components (far beyond the Nyquist frequency), the worst-case input signal becomes more severe, and the  $H^\infty$ -optimal filter becomes closer to 0.

#### IV. DESIGN EXAMPLES

We here present design examples of fractional delay filters.

The design parameters are as follows: the sampling period  $T = 1$  (sec), the delay  $D = 5.5$  (sec), that is,  $m = 5$  and  $d = 0.5$ . The frequency-domain characteristic of analog signals to be sampled is modeled by

$$\hat{W}(s) = \frac{\omega_c}{s + \omega_c}, \quad \omega_c = 0.1.$$

Note that  $\hat{W}(s)$  has the cutoff frequency  $\omega_c = 0.1$  (rad/sec)  $\approx 0.016$  (Hz), which is below the Nyquist frequency  $\pi$  (rad/sec) = 0.5 (Hz).

We compare the sampled-data  $H^\infty$  optimal filter obtained by Theorem 2 with conventional FIR filters designed by discrete-time  $H^2$  optimization [10], which minimizes the cost function (5). The weighting function  $\hat{W}_d(z)$  in (4) or (5) is chosen as the impulse-invariant discretization [48] of  $\hat{W}(s)$ . Fig. 6 shows the Bode plots of  $\hat{W}(s)$  and  $\hat{W}_d(z)$ .

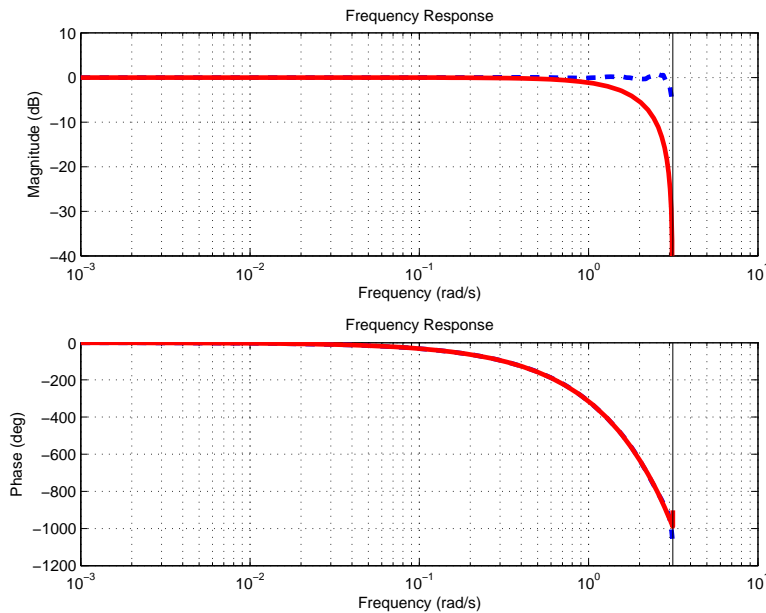


Fig. 7. Bode plot of filters: sampled-data  $H^\infty$  design (solid), and discrete-time  $H^2$  design (dash).

The transfer function of the proposed filter is given by

$$\hat{K}(z) = z^{-5} (0.4994 + 0.4994z^{-1}).$$

Fig. 7 shows the Bode plots of the designed filters. As illustrated in Fig. 7, the  $H^2$  optimal filter is closer to the ideal filter (3) as expected, so that it appears better in the context of the conventional design methodology.

However, the  $H^2$  optimal filter exhibits *much larger errors* in the high-frequency domain as shown in Fig. 8 that shows the frequency response gain of the sampled-data error system  $E$  shown in Fig. 2. This is because the conventional designs cannot take into account the frequency response of the source analog signals while the present method does.

To see the difference between the present filter and the conventional one, we show the time response against a piecewise regular signal produced by the `MakeSignal` function of WaveLab [49] in Fig. 9. The present method is superior to the conventional one that shows much ringing at edges of the wave. To see the difference more finely, we show the reconstruction error in Fig. 10. The  $H^2$ -optimal filter has much larger errors around edges of the signal than the proposed  $H^\infty$ -optimal one. In fact, the  $L^2$  norm of the error is  $1.34 \times 10^{-2}$  for  $H^\infty$  design and  $2.07 \times 10^{-2}$  for  $H^2$  design. This illustrates the effectiveness of our method.

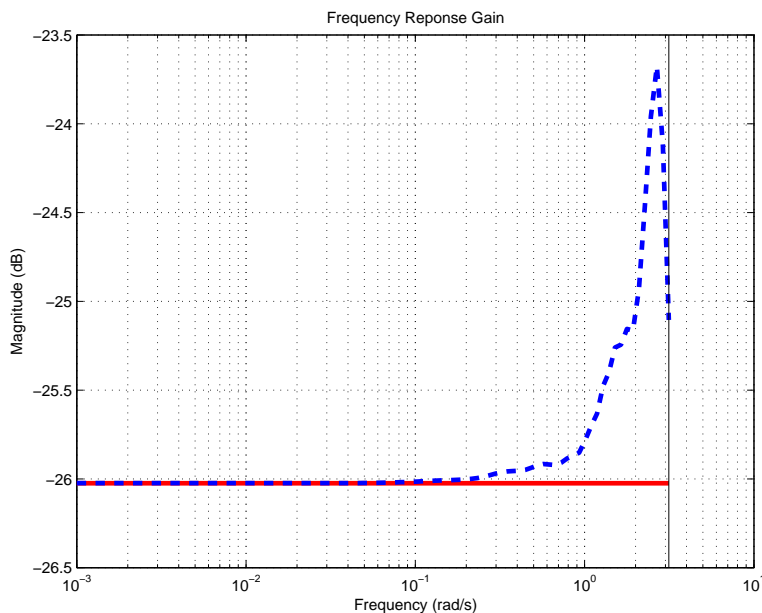


Fig. 8. Frequency response gain of error system  $E$  in Fig. 2: sampled-data  $H^\infty$  design (solid), and discrete-time  $H^2$  design (dash).

## V. CONCLUSION

We have presented a new method of designing fractional delay filters via sampled-data  $H^\infty$  optimization. An advantage here is that an *optimal analog performance* can be attained. The optimal design problem can be equivalently transformed to discrete-time  $H^\infty$  optimization, which is easily executed by standard numerical optimization toolboxes. A closed-form solution is given when the frequency distribution of the input analog signal is modeled as a first-order low-pass filter. Design examples show that the  $H^\infty$ -optimal filter exhibits a much more satisfactory performance than the conventional  $H^2$ -optimal filter.

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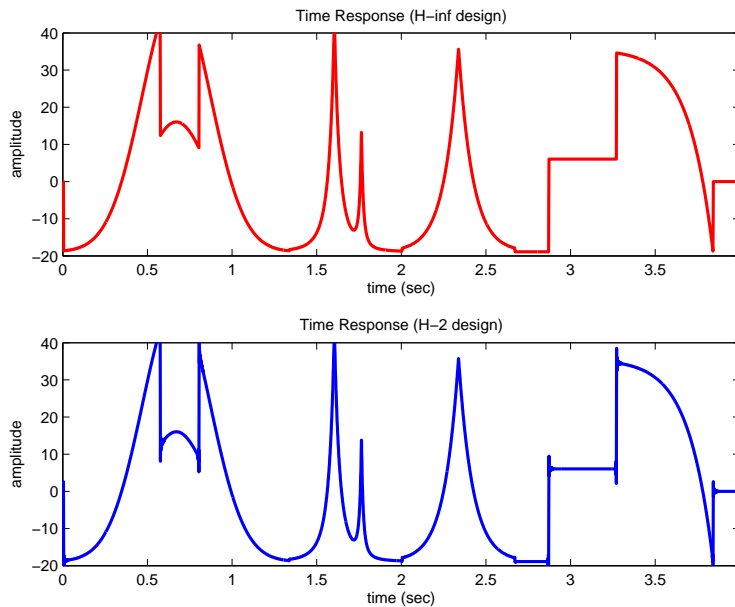


Fig. 9. Time response: sampled-data  $H^\infty$  design (above) and discrete-time  $H^2$  design (below) with sampling frequency 1000 (Hz).

## APPENDIX A

### PROOF OF PROPOSITION 1

From the relation (6), the lifted system  $\mathcal{E}$  is described as (see also Fig. 3)

$$\begin{aligned} \mathcal{E} &= \mathcal{S}_T e^{-Ds} W \mathcal{L}^{-1} - K \mathcal{S}_T W \mathcal{L}^{-1} \\ &= \mathcal{S}_T e^{-mTs} e^{-ds} W \mathcal{L}^{-1} - K \mathcal{S}_T W \mathcal{L}^{-1} \\ &= z^{-m} \mathcal{S}_T e^{-ds} W \mathcal{L}^{-1} - K \mathcal{S}_T W \mathcal{L}^{-1} \\ &= z^{-m} y_d - K v_d, \end{aligned}$$

where  $y_d := \mathcal{S}_T e^{-ds} W \mathcal{L}^{-1} \tilde{w}$  and  $v_d := \mathcal{S}_T W \mathcal{L}^{-1} \tilde{w}$ . From the state-space representation of  $W$  in (7), for any  $t_1$  and  $t_2$  such that  $0 \leq t_1 \leq t_2 < \infty$ , we have

$$\mathbf{x}(t_2) = e^{\mathbf{A}(t_2-t_1)} \mathbf{x}(t_1) + \int_{t_1}^{t_2} e^{\mathbf{A}(t_2-t)} \mathbf{B} w(t) dt.$$

Putting  $t_1 := nT$  and  $t_2 := (n+1)T$  for  $n \in \mathbb{Z}_+$  gives

$$\mathbf{x}(nT+T) = e^{\mathbf{A}T} \mathbf{x}(nT) + \int_0^T e^{\mathbf{A}(T-\tau)} \mathbf{B} w(nT+\tau) d\tau.$$

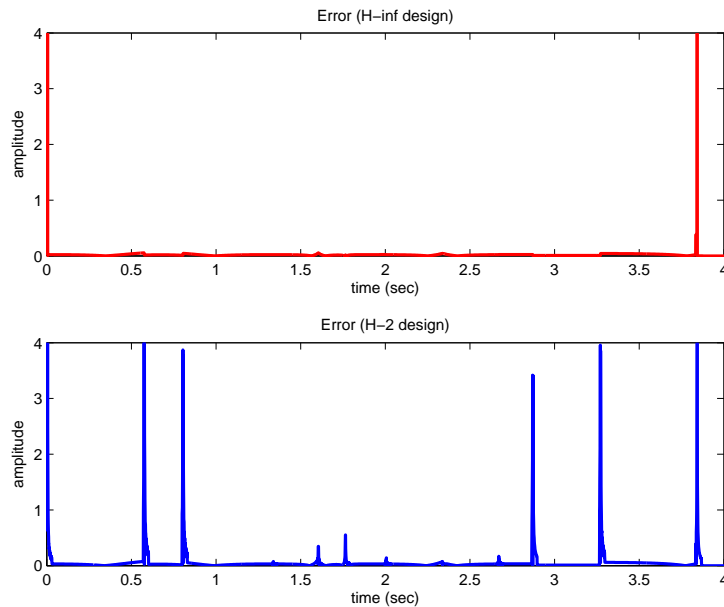


Fig. 10. Absolute value of reconstruction error: sampled-data  $H^\infty$  design (above) and discrete-time  $H^2$  design (below).

Define  $\mathbf{x}_1[n] := \mathbf{x}(nT)$  and  $\tilde{w}[n] := (\mathcal{L}w)[n]$ . Then we have

$$\mathbf{x}_1[n+1] = e^{AT} \mathbf{x}_1[n] + \mathcal{B}_1 \tilde{w}[n], \quad (18)$$

where  $\mathcal{B}_1$  is defined in (10). On the other hand, from (7), we have  $v(t) = \mathbf{C}\mathbf{x}(t)$  for  $t \in \mathbb{R}_+$ . Putting  $t_1 := nT$  and  $t_2 := nT + \theta$  for  $n \in \mathbb{Z}_+$  and  $\theta \in [0, T)$ , we have

$$\begin{aligned} v(nT + \theta) &= \mathbf{C}\mathbf{x}(nT + \theta) \\ &= \mathbf{C}e^{A\theta} \mathbf{x}(nT) + \int_0^\theta \mathbf{C}e^{A(\theta-\tau)} \mathbf{B}w(nT + \tau) d\tau. \end{aligned} \quad (19)$$

By this, we have

$$v_d[n] = v(nT) = \mathbf{C}\mathbf{x}_1[n]. \quad (20)$$

Next, from (19), we have

$$\begin{aligned} y_d[n] &= v(nT - d) = v(nT - T + T - d) \\ &= \mathbf{C}e^{A(T-d)} \mathbf{x}_1[n-1] \\ &\quad + \int_0^{T-d} \mathbf{C}e^{A(\theta-\tau)} \mathbf{B}\tilde{w}[n-1](\tau) d\tau. \end{aligned}$$

Put  $x_2[n] := y_d[n]$ . Then we have

$$\begin{aligned} x_2[n+1] &= \mathbf{C}e^{\mathbf{A}(T-d)}\mathbf{x}_1[n] + \mathbf{B}_2\tilde{w}[n], \\ y_d[n] &= x_2[n], \end{aligned} \tag{21}$$

where  $\mathbf{B}_2$  is defined in (10). By the relation

$$u_d[n] = y_d[n-m] = z^{-m}y_d[n],$$

and the state-space matrices  $\mathbf{A}_m$ ,  $\mathbf{B}_m$ , and  $\mathbf{C}_m$  for  $m$ -step delay  $z^{-m}$ , we have

$$\begin{aligned} \mathbf{x}_3[n+1] &= \mathbf{A}_m\mathbf{x}_3[n] + \mathbf{B}_m y_d[n], \\ u_d[n] &= \mathbf{C}_m\mathbf{x}_3[n]. \end{aligned} \tag{22}$$

Combining (18), (20), (21), and (22) all together gives the state-space representation (9) with  $\boldsymbol{\xi}^\top := [\mathbf{x}_1^\top, x_2, \mathbf{x}_3^\top]^\top$ . ■

## REFERENCES

- [1] T. A. Ramstad, "Digital methods for conversion between arbitrary sampling frequencies," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 32, no. 3, pp. 577–591, 1984.
- [2] J. O. Smith and P. Gossett, "A flexible sampling-rate conversion method," in *IEEE ICASSP'84*, 1984, pp. 19.4.1–19.4.4.
- [3] S. Park, G. Hillman, and R. Robles, "A novel structure for real-time digital sample-rate converters with finite precision error analysis," in *IEEE ICASSP'91*, 1991, pp. 3613–3616.
- [4] H. Johansson and P. Löwenborg, "Reconstruction of nonuniformly sampled bandlimited signals by means of digital fractional delay filters," *IEEE Trans. Signal Processing*, vol. 50, no. 11, pp. 2757 – 2767, Nov. 2002.
- [5] R. Prendergast, B. Levy, and P. Hurst, "Reconstruction of band-limited periodic nonuniformly sampled signals through multirate filter banks," *IEEE Trans. Circuits Syst. I*, vol. 51, no. 8, pp. 1612–1622, Aug. 2004.
- [6] R. Yu, "Characterization and sampled-data design of dual-tree filter banks for Hilbert transform pairs of wavelet bases," *IEEE Trans. Signal Processing*, vol. 55, no. 6, pp. 2458 –2471, Jun. 2007.
- [7] B. Dumitrescu, "SDP approximation of a fractional delay and the design of dual-tree complex wavelet transform," *IEEE Trans. Signal Processing*, vol. 56, no. 9, pp. 4255 –4262, Sep. 2008.
- [8] V. V. H.-M. Lehtonen and T. I. Laakso, "Musical signal analysis using fractional-delay inverse comb filters," in *Proc. of the 10th Int. Conf. on Digital Audio Effects*, Sep. 2007, pp. 261–268.
- [9] V. Välimäki, J. Pakarinen, C. Erkut, and M. Karjalainen, "Discrete-time modelling of musical instruments," *Rep. Prog. Phys.*, vol. 69, no. 1, pp. 1–78, 2006.
- [10] T. I. Laakso, V. Välimäki, M. Karjalainen, and U. K. Laine, "Splitting the unit delay," *IEEE Signal Processing Mag.*, vol. 13, pp. 30–60, 1996.
- [11] V. Välimäki and T. I. Laakso, "Principles of fractional delay filters," in *IEEE ICASSP'00*, 2000, pp. 3870–3873.
- [12] V. Välimäki and T. I. Laakso, "Fractional delay filters — design and applications," in *Nonuniform Sampling*. New York: Kluwer Academic/Plenum Publishers, 2001, pp. 835–885.
- [13] C. E. Shannon, "Communication in the presence of noise," *Proc. IRE*, vol. 37, no. 1, pp. 10–21, 1949.

- [14] M. Unser, "Sampling — 50 years after Shannon," *Proc. IEEE*, vol. 88, no. 4, 2000.
- [15] G. D. Cain, A. Yardim, and P. Henry, "Offset windowing for FIR fractional-sample delay," in *IEEE ICASSP'95*, 1995, pp. 1276–1279.
- [16] J. Selva, "An efficient structure for the design of variable fractional delay filters based on the windowing method," *IEEE Trans. Signal Processing*, vol. 56, no. 8, pp. 3770–3775, Aug. 2008.
- [17] E. Hermanowicz, "Explicit formulas for weighting coefficients of maximally flat tunable FIR delayers," *Electronics Letters*, vol. 28, pp. 1936–1937, 1992.
- [18] S.-C. Pei and P.-H. Wang, "Closed-form design of maximally flat FIR Hilbert transformers, differentiators, and fractional delayers by power series expansion," *IEEE Trans. Circuits Syst. I*, vol. 48, pp. 389–398, 2001.
- [19] S. Samadi, O. Ahmad, and M. N. S. Swamy, "Results on maximally flat fractional-delay systems," *IEEE Trans. Circuits Syst. I*, vol. 51, no. 11, pp. 2271–2286, 2004.
- [20] H. Hachabiboglu, B. Gunel, and A. Kondo, "Analysis of root displacement interpolation method for tunable allpass fractional-delay filters," *IEEE Trans. Signal Processing*, vol. 55, no. 10, pp. 4896–4906, Oct. 2007.
- [21] J.-J. Shyu and S.-C. Pei, "A generalized approach to the design of variable fractional-delay FIR digital filters," *Signal Processing*, vol. 88, no. 6, pp. 1428–1435, Jun. 2008.
- [22] Z. Jing, "A new method for digital all-pass filter design," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. 35, pp. 1557–1564, 1987.
- [23] S.-C. Pei and P.-H. Wang, "Closed-form design of all-pass fractional delay filters," *IEEE Signal Processing Lett.*, vol. 11, pp. 788–791, 2004.
- [24] W. Putnam and J. Smith, "Design of fractional delay filters using convex optimization," in *Applications of Signal Processing to Audio and Acoustics, 1997 IEEE ASSP Workshop on*, Oct 1997.
- [25] A. Tarczynski, G. D. Cain, E. Hermanowicz, and M. Rojewski, "WLS design of variable frequency response FIR filters," in *Proc. IEEE Int. Symp. Circuits Syst.*, 1997, pp. 2244–2247.
- [26] T.-B. Deng and Y. Nakagawa, "SVD-based design and new structures for variable fractional-delay digital filters," *IEEE Trans. Signal Processing*, vol. 52, pp. 2513–2527, 1991.
- [27] Y. Yamamoto, M. Nagahara, and P. P. Khargonekar, "Signal reconstruction via  $H^\infty$  sampled-data control theory — Beyond the Shannon paradigm," *IEEE Trans. Signal Processing*, vol. 60, no. 2, pp. 613–625, 2012.
- [28] P. P. Khargonekar and Y. Yamamoto, "Delayed signal reconstruction using sampled-data control," in *Proc. 35th IEEE CDC*, 1996, pp. 1259–1263.
- [29] T. Chen and B. A. Francis, *Optimal Sampled-data Control Systems*. Springer, 1995.
- [30] J. C. Doyle, K. Glover, P. P. Khargonekar, and B. A. Francis, "State-space solutions to standard  $H_\infty$  and  $H_2$  control problems," *IEEE Trans. Automat. Contr.*, vol. 34, pp. 831–847, 1989.
- [31] M. Nagahara and Y. Yamamoto, "Optimal design of fractional delay filters," in *Proc. of 35th Conf. on Decision and Control*, 2003, pp. 6539–6544.
- [32] M. Nagahara and Y. Yamamoto, "Optimal design of fractional delay FIR filters without band-limiting assumption," in *IEEE ICASSP'05*, 2005, pp. 221–224.
- [33] Y. Yamamoto, "New approach to sampled-data systems: a function space method," in *Proc. 29th Conf. on Decision and Control*, 1990, pp. 1881–1887.
- [34] B. Bamieh, J. B. Pearson, B. A. Francis, and A. Tannenbaum, "A lifting technique for linear periodic systems with applications to sampled-data control systems," *Syst. Control Lett.*, vol. 17, pp. 79–88, 1991.

- [35] H. T. Toivonen, "Sampled-data control of continuous-time systems with an  $H_\infty$  optimality criterion," *Automatica*, vol. 28, pp. 45–54, 1992.
- [36] B. Bamieh and J. B. Pearson, "A general framework for linear periodic systems with applications to  $H_\infty$  sampled-data control," *IEEE Trans. Automat. Contr.*, vol. 37, pp. 418–435, 1992.
- [37] Y. Yamamoto, "A function space approach to sampled-data control systems and tracking problems," *IEEE Trans. Automat. Contr.*, vol. 39, pp. 703–712, 1994.
- [38] P. P. Vaidyanathan, *Multirate Systems and Filter Banks*. Prentice Hall, 1993.
- [39] G. Balas, R. Chiang, A. Packard, and M. Safonov, *Robust Control Toolbox, Version 3*. The Math Works, 2005.
- [40] C. W. Farrow, "A continuously variable digital delay element," in *Proc. IEEE Int. Symp. Circuits Syst.*, vol. 3, 1988, pp. 2641–2645.
- [41] N. J. Fliege, *Multirate Digital Signal Processing*. New York: John Wiley, 1994.
- [42] M. Nagahara, M. Ogura, and Y. Yamamoto, " $H^\infty$  design of periodically nonuniform interpolation and decimation for non-band-limited signals," *SICE Journal of Control, Measurement, and System Integration*, vol. 4, no. 5, pp. 341–348, 2011.
- [43] W. J. Rugh, *Linear Systems Theory*. Prentice Hall, 1996.
- [44] Y. Yamamoto, "On the state space and frequency domain characterization of  $H^\infty$ -norm of sampled-data systems," *Syst. Control Lett.*, vol. 21, pp. 163–172, 1993.
- [45] C. F. V. Loan, "Computing integrals involving the matrix exponential," *IEEE Trans. Automat. Contr.*, vol. 23, pp. 395–404, 1994.
- [46] <http://www-ics.acs.i.kyoto-u.ac.jp/mat/dfd/>.
- [47] M. Nagahara, "Min-max design of FIR digital filters by semidefinite programming," in *Applications of Digital Signal Processing*. InTech, Nov. 2011.
- [48] L. Jackson, "A correction to impulse invariance," *IEEE Signal Processing Lett.*, vol. 7, no. 10, pp. 273–275, Oct. 2000.
- [49] D. Donoho, A. Maleki, and M. Shaharam, "Wavelab 850," 2006.



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