RIMS Kôkyûroku Bessatsu
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Wavelet characterization of weighted spaces

北海道大学大学院理学研究院　立澤一哉 (Kazuya Tachizawa)
Department of Mathematics, Hokkaido University

1 Wavelets

In this article we investigate a generalization of the Sobolev-Lieb-Thirring inequality or Lieb's inequality for Bessel potentials. First we recall the definition of Meyer's wavelet basis. Let θ be a function which satisfies the following conditions.

(i) θ is a real valued and even function in $C_0^\infty(\mathbb{R})$.

(ii) $0 \le \theta(\xi) \le 1$ and $\operatorname{supp}\theta \subset [-4\pi/3, 4\pi/3]$.

(iii) $\theta(\xi) = 1$ for all $\xi \in [-2\pi/3, 2\pi/3]$.

(iv) $\theta(\xi)^2 + \theta(2\pi - \xi)^2 = 1$ for all $\xi \in [0, 2\pi]$.

We define a function $\psi \in \mathcal{S}(\mathbb{R})$ by

$$\hat{\psi}(\xi) = \int_{\mathbb{R}} \psi(x) e^{-i\xi x}\, dx = \{\theta(\xi/2)^2 - \theta(\xi)^2\}^{1/2} e^{-i\xi/2}.$$

For integers j, k we set $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$. Then it turns out that $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{R})$ which we call Meyer's wavelet basis([8]).

We define n-dimensional Meyer's wavelet basis as follows. Let φ be a function in $\mathcal{S}(\mathbb{R})$ such that $\hat{\varphi}(\xi) = \theta(\xi)$. Set $E = \{0,1\}^n \setminus \{\mathbf{0}\}$, $\psi^0(x) = \varphi(x)$, and $\psi^1(x) = \psi(x)$. For $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n) \in E$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ we define

$$\psi^\varepsilon(x) = \psi^{\varepsilon_1}(x_1) \cdots \psi^{\varepsilon_n}(x_n).$$

Let $\Lambda = \{\, (\varepsilon, j, k) \,:\, \varepsilon \in E,\ j \in \mathbb{Z},\ k \in \mathbb{Z}^n \,\}$. For $\lambda = (\varepsilon, j, k) \in \Lambda, x \in \mathbb{R}^n$, set

$$\psi_\lambda(x) = 2^{nj/2}\psi^\varepsilon(2^j x - k).$$

Then $\{\psi_\lambda\}_{\lambda\in\Lambda}$ is an orthonormal basis of $L^2(\mathbb{R}^n)$ which we call n-dimensional Meyer's wavelet basis([8]).

We can construct another orthonormal basis by φ and ψ. Let

$$\Lambda_0 = \{\, (\varepsilon, j, k) \,:\, \varepsilon \in E,\ j \in \mathbb{Z},\ j \ge 0,\ k \in \mathbb{Z}^n \,\},$$
$$\Phi(x) = \varphi(x_1) \cdots \varphi(x_n), \quad \text{and} \quad \Phi_k(x) = \Phi(x - k) \quad (k \in \mathbb{Z}^n).$$

Then we can prove that

$$\{ \Phi_k,\ \psi_\lambda \ :\ \lambda \in \Lambda_0,\ k \in \mathbb{Z}^n \}$$

is an orthonormal basis of $L^2(\mathbb{R}^n)$([8]). The function Φ is called a scaling function.

2 Weighted spaces

We recall the definition of A_p-weights. By a cube in $\mathbb{R}^n$ we mean a cube which sides are parallel to coordinate axes. A locally integrable function $w > 0$ *a.e.* on $\mathbb{R}^n$ is an A_p-weight for some $p \in (1, \infty)$ if there exists a positive constant C such that

$$\frac{1}{|Q|}\int_Q w(x)\,dx \left(\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\right)^{p-1} \leq C$$

for all cubes $Q \subset \mathbb{R}^n$, where $|Q|$ is the volume of Q.

We say that w is an A_1-weight if there exists a positive constant C such that

$$\frac{1}{|Q|}\int_Q w(y)\,dy \leq Cw(x) \qquad a.e.\ x \in Q$$

for all cubes $Q \subset \mathbb{R}^n$.

We write A_p for the class of A_p-weights. An example of A_p-weight for $1 < p < \infty$ is given by $w(x) = |x|^\alpha \in A_p$ where $x \in \mathbb{R}^n$ and $-n < \alpha < p(n-1)$. The inclusion $A_p \subset A_q$ holds for $p < q$.

For $w \in A_p$ we set

$$L^p(w) = \{f\ :\ \text{measurable},\ \|f\|_{L^p(w)} = \left\{\int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx\right\}^{1/p} < \infty\}.$$

For $\lambda = (\varepsilon, j, k) \in \Lambda$ set

$$Q(\lambda) = \{(x_1, \ldots, x_n)\ :\ k_i \leq 2^j x_i < k_i + 1, i = 1, \ldots, n\}$$

and

$$\tilde{\chi}_\lambda(x) = |Q(\lambda)|^{-1/2}\chi_{Q(\lambda)}(x),$$

where $\chi_{Q(\lambda)}(x)$ is the characteristic function of $Q(\lambda)$. The cube as above is called a dyadic cube.

Now we give the definition of an unconditional basis in a Banach space B over $\mathbb{C}$. Let $\{e_i\}_{i=1}^\infty$ be a family of elements in B. We say $\{e_i\}_{i=1}^\infty$ is a Schauder basis of B if every $f \in B$ can be written

$$f = \alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_k e_k + \cdots \tag{1}$$

where the $\alpha_1, \alpha_2, \ldots, \alpha_k, \ldots$ are uniquely determined coefficients in $\mathbb{C}$ and the convergence in (1) is defined by

$$\lim_{N\to\infty} \|f - \alpha_1 e_1 - \alpha_2 e_2 - \cdots - \alpha_N e_N\|_B = 0.$$

A Schauder basis $\{e_i\}_{i=1}^{\infty}$ in B is an unconditional basis if the following property is satisfied: for all $f \in B$ $f = \sum_{i=1}^{\infty} \alpha_{\sigma(i)} e_{\sigma(i)}$ in B for any permutation σ of $\mathbb{N}$, where α_i are coefficients given by (1).

The following theorem is a simple modification of results by Lemarié and Meyer([4],[5],[8]), where we use the notation

$$(f, g) = \int_{\mathbb{R}^n} f(x)\overline{g(x)}\, dx.$$

Theorem 2.1. *Let*

$$1 < p < \infty \quad \text{and} \quad w \in A_p.$$

Then $\{\psi_\lambda\}_{\lambda\in\Lambda}$ *is an unconditional basis of* $L^p(w)$. *Furthermore for* $f \in L^p(w)$ *we have*

$f = \sum_{\lambda\in\Lambda} (f, \psi_\lambda)\psi_\lambda$ *in* $L^p(w)$ *and*

$$\|f\|_{L^p(w)} \approx \left\| \left(\sum_{\lambda\in\Lambda} (|(f, \psi_\lambda)|\tilde{\chi}_\lambda)^2 \right)^{1/2} \right\|_{L^p(w)}.$$

Moreover

$$\{\ \Phi_k,\ \psi_\lambda\ :\ \lambda \in \Lambda_0,\ k \in \mathbb{Z}^n\ \}$$

is an unconditional basis of $L^p(w)$. *For* $f \in L^p(w)$ *we have*

$$f = \sum_{k\in\mathbb{Z}^n} (f, \Phi_k)\Phi_k + \sum_{\lambda\in\Lambda_0} (f, \psi_\lambda)\psi_\lambda$$

in $L^p(w)$ *and*

$$\|f\|_{L^p(w)} \approx \left(\sum_k |(f, \Phi_k)|^p w(Q_k) \right)^{1/p} + \left\| \left(\sum_{\lambda\in\Lambda_0} (|(f, \psi_\lambda)|\tilde{\chi}_\lambda(x))^2 \right)^{1/2} \right\|_{L^p(w)},$$

where

$$Q_k = \{(x_1, \ldots, x_n)\ :\ k_i \le x_i < k_i + 1, i = 1, \ldots, n\},$$

and

$$w(Q_k) = \int_{Q_k} w\, dx.$$

We will use this result in the proofs of Theorem 3.2 in Section 3 and Theorem 6.2 in Section 6.

3 The Sobolev-Lieb-Thirring inequality

In 1976 Lieb and Thirring proved the following inequality([7]).

Theorem 3.1 (The Sobolev-Lieb-Thirring inequality). *Suppose that* $n \in \mathbb{N}$, $f_i \in H^1(\mathbb{R}^n)$ $(i = 1, \dots, N)$, *and that* $\{f_i\}_{i=1}^N$ *is an orthonormal family in* $L^2(\mathbb{R}^n)$. *Then we have*

$$\int_{\mathbb{R}^n} \rho^{1+2/n} dx \leq c_n \sum_{i=1}^N \int_{\mathbb{R}^n} |\nabla f_i|^2 dx,$$

where

$$\rho(x) = \sum_{i=1}^N |f_i(x)|^2.$$

In the statement of the Sobolev-Lieb-Thirring inequality $H^1(\mathbb{R}^n)$ denotes the Sobolev space of order one. The Sobolev-Lieb-Thirring inequality has important applications such as the stability of matter or the estimates of the dimension of attractors of nonlinear equations([7]).

In this section we give a weighted version of the Sobolev-Lieb-Thirring inequality. Let $w \in A_2$ and $\mathcal{H}^1(w)$ be the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$\|f\|_{\mathcal{H}^1(w)} = \left\{ \int_{\mathbb{R}^n} |\nabla f(x)|^2 w(x)\, dx + \|f\|^2 \right\}^{1/2},$$

where $\|\cdot\|$ denotes the norm in $L^2(\mathbb{R}^n)$. We have the following generalization of the Sobolev-Lieb-Thirring inequality for $n \geq 3$(c.f.[9]).

Theorem 3.2. *Let* $n \in \mathbb{N}$, $n \geq 3$, $w \in A_2$ *and* $w^{-n/2} \in A_{n/2}$. *Suppose that* $f_i \in \mathcal{H}^1(w)$ $(i = 1, \dots, N)$, *and* $\{f_i\}_{i=1}^N$ *is orthonormal in* $L^2(\mathbb{R}^n)$. *Then we have*

$$\int_{\mathbb{R}^n} \rho(x)^{1+2/n} w(x)\, dx \leq c \sum_{i=1}^N \int_{\mathbb{R}^n} |\nabla f_i(x)|^2 w(x) dx,$$

where

$$\rho(x) = \sum_{i=1}^N |f_i(x)|^2$$

and c *is a positive constant depending only on* n *and* w.

An example of w which satisfies the conditions in Theorem 3.2 is given by $w(x) = |x|^\alpha$ for $-n + 2 < \alpha < 2$.

We explain about the outline of a proof of Theorem 3.2 in the next section. We use the estimates of some weighted integrals by means of wavelets. These estimates enable us to prove a weighted version of the Sobolev-Lieb-Thirring inequality.

4 Proof of Theorem 3.2

For $f \in L^1_{loc}(\mathbb{R}^n)$, we define the Hardy-Littlewood maximal operator as

$$M(f)(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q |f(y)|\, dy,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ such that $x \in Q$.

The proof of the following proposition is in [3].

Proposition 4.1. *(i) Let $1 < p < \infty$ and $w \in A_p$. Then M is bounded on $L^p(w)$.*

(ii) Let $0 < \tau < 1$, $f \in L^1_{loc}(\mathbb{R}^n)$, and $M(f)(x) < \infty$ a.e.. Then $(M(f)(x))^\tau \in A_1$.

(iii) Let $1 < p < \infty$ and $w_1, w_2 \in A_1$. Then $w_1 w_2^{1-p} \in A_p$.

We may assume $f_i \in C_0^\infty(\mathbb{R}^n)$ for $i = 1, \ldots, N$. Let $V(x) = \delta \rho(x)^{2/n} w(x)$ where δ is a positive constant. Then we get $\int_{\mathbb{R}^n} V^{1+n/2} w^{-n/2}\, dx < \infty$ and $w^{-n/2} \in A_{(1+n/2)/\kappa} = A_{n/2}$ for $\kappa = 1 + 2/n$. Set $v(x) = M(V^\kappa)(x)^{1/\kappa}$. Then (i) of Proposition 4.1 leads to

$$\int_{\mathbb{R}^n} v^{1+n/2} w^{-n/2}\, dx = \int_{\mathbb{R}^n} M(V^\kappa)^{(1+n/2)/\kappa} w^{-n/2}\, dx \le c_1 \int_{\mathbb{R}^n} V^{1+n/2} w^{-n/2}\, dx < \infty.$$

Furthermore we have $v \in A_2$ and $V \le v$ a.e..

The following lemma is essentially proved by Frazier and Jawerth (c.f.[9]).

Lemma 4.1. *Let $w \in A_2$. Then there exists a $\alpha > 0$ such that*

$$\alpha \sum_{\lambda \in \Lambda} |Q(\lambda)|^{-2/n} |(f, \psi_\lambda)|^2 \frac{1}{|Q(\lambda)|} \int_{Q(\lambda)} w\, dx \le \int_{\mathbb{R}^n} |\nabla f|^2 w\, dx$$

for all $f \in C_0^\infty(\mathbb{R}^n)$.

The next lemma is a corollary of Theorem 2.1.

Lemma 4.2. *Let $v \in A_2$. Then there exists a $\beta > 0$ such that*

$$\int_{\mathbb{R}^n} |f|^2 v\, dx \le \beta \sum_{\lambda \in \Lambda} |(f, \psi_\lambda)|^2 \frac{1}{|Q(\lambda)|} \int_{Q(\lambda)} v\, dx$$

for all $f \in C_0^\infty(\mathbb{R}^n)$.

By Lemmas 4.1 and 4.2 we have for $f \in C_0^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} |\nabla f|^2 w\, dx - \int_{\mathbb{R}^n} V |f|^2\, dx$$

$$\ge \alpha \sum_{\lambda \in \Lambda} |Q(\lambda)|^{-2/n} |(f, \psi_\lambda)|^2 \frac{1}{|Q(\lambda)|} \int_{Q(\lambda)} w\, dx - \beta \sum_{\lambda \in \Lambda} |(f, \psi_\lambda)|^2 \frac{1}{|Q(\lambda)|} \int_{Q(\lambda)} v\, dx.$$

Let

$$\mathcal{I} = \{\lambda \in \Lambda : \beta \int_{Q(\lambda)} v\,dx > \alpha |Q(\lambda)|^{-2/n} \int_{Q(\lambda)} w\,dx\}$$

and $\{\mu_k\}_{1 \le k}$ be the non-decreasing rearrangement of

$$\left\{ \alpha |Q(\lambda)|^{-2/n-1} \int_{Q(\lambda)} w\,dx - \beta |Q(\lambda)|^{-1} \int_{Q(\lambda)} v\,dx \right\}_{\lambda \in \mathcal{I}}.$$

When

$$\mu_k = \alpha |Q(\lambda)|^{-2/n-1} \int_{Q(\lambda)} w\,dx - \beta |Q(\lambda)|^{-1} \int_{Q(\lambda)} v\,dx,$$

we define $\Psi_k = \psi_\lambda$. Then we get

$$\begin{aligned}
&\sum_{i=1}^{N} \int_{\mathbb{R}^n} |\nabla f_i|^2 w\,dx - \sum_{i=1}^{N} \int_{\mathbb{R}^n} V |f_i|^2\,dx \\
&\ge \sum_{i=1}^{N} \sum_{\lambda \in \Lambda} |(f_i, \psi_\lambda)|^2 \left\{ \alpha |Q(\lambda)|^{-2/n-1} \int_{Q(\lambda)} w\,dx - \beta |Q(\lambda)|^{-1} \int_{Q(\lambda)} v\,dx \right\} \\
&\ge \sum_{i=1}^{N} \sum_{k} \mu_k |(f_i, \Psi_k)|^2 = \sum_{k} \mu_k \sum_{i=1}^{N} |(f_i, \Psi_k)|^2 \\
&\ge -c \sum_{k} |\mu_k|.
\end{aligned}$$

Now we use the following lemma in [9].

Lemma 4.3. *There exists a positive constant c such that*

$$\sum_{k} |\mu_k| \le c \int_{\mathbb{R}^n} v^{1+n/2} w^{-n/2}\,dx,$$

where c depends only on n and w.

Hence by Lemma 4.3 we have

$$\begin{aligned}
&\sum_{i=1}^{N} \int_{\mathbb{R}^n} |\nabla f_i|^2 w\,dx - \sum_{i=1}^{N} \int_{\mathbb{R}^n} V |f_i|^2\,dx \\
&\ge -c \int_{\mathbb{R}^n} V^{1+n/2} w^{-n/2}\,dx = -c\delta^{1+n/2} \int_{\mathbb{R}^n} \rho^{1+2/n} w\,dx.
\end{aligned}$$

Therefore

$$\begin{aligned}
\sum_{i=1}^{N} \int_{\mathbb{R}^n} |\nabla f_i|^2 w\,dx &\ge \delta \int_{\mathbb{R}^n} \rho^{1+2/n} w\,dx - c\delta^{1+n/2} \int_{\mathbb{R}^n} \rho^{1+2/n} w\,dx \\
&= \{\delta - c\delta^{1+n/2}\} \int_{\mathbb{R}^n} \rho^{1+2/n} w\,dx.
\end{aligned}$$

If we take δ small enough, then we get the inequality in Theorem 3.2.

5 L^p Sobolev-Lieb-Thirring inequality

By Theorem 3.2 we are able to prove the following L^p version of the Sobolev-Lieb-Thirring inequality.

Theorem 5.1 ([10]). *Let $n \in \mathbb{N}$, $n \geq 3$ and $2n/(n+2) < p < n$. Then there exists a positive constant c such that for every family $\{f_i\}_{i=1}^N$ in $L^2(\mathbb{R}^n)$ which is orthonormal and $|\nabla f_i(x)| \in L^p(\mathbb{R}^n)$, $(i = 1, \dots, N)$, we have*

$$\int_{\mathbb{R}^n} \rho(x)^{(1+2/n)p/2}\, dx \leq c \int_{\mathbb{R}^n} \left(\sum_{i=1}^N |\nabla f_i(x)|^2 \right)^{p/2} dx,$$

where

$$\rho(x) = \sum_{i=1}^N |f_i(x)|^2$$

and c depends only on n and p.

Proof

Our proof is very similar to that of the extrapolation theorem in harmonic analysis(c.f.[2, Theorem 7.8]). Let $2 < p < n$ and $2/p + 1/q = 1$. Let $u \in L^q$, $u \geq 0$ and $\|u\|_{L^q} = 1$. We take a γ such that $n/(n-2) < \gamma < q$. Then we have $u \leq M(u^\gamma)^{1/\gamma}$ a.e and $M(u^\gamma)^{1/\gamma} \in A_1$. Furthermore let $\alpha = \frac{n}{(n-2)\gamma}$. Then $0 < \alpha < 1$ and

$$M(u^\gamma)^{-n/(2\gamma)} = \{M(u^\gamma)^\alpha\}^{1-n/2} \in A_{n/2},$$

where we used $M(u^\gamma)^\alpha \in A_1$ and (iii) of Proposition 4.1. Therefore we have

$$\begin{aligned} \int \rho^{1+2/n} u\, dx &\leq \int \rho^{1+2/n} M(u^\gamma)^{1/\gamma}\, dx \leq c \int \left(\sum_{i=1}^N |\nabla f_i|^2 \right) M(u^\gamma)^{1/\gamma}\, dx \\ &\leq c \left(\int \left(\sum_{i=1}^N |\nabla f_i|^2 \right)^{p/2} dx \right)^{2/p} \left(\int M(u^\gamma)^{q/\gamma} dx \right)^{1/q} \\ &\leq c \left(\int \left(\sum_{i=1}^N |\nabla f_i|^2 \right)^{p/2} dx \right)^{2/p}, \end{aligned}$$

where we used Theorem 3.2 and the inequality

$$\int M(u^\gamma)^{q/\gamma}\, dx \leq c \int u^q dx = c.$$

If we take the supremum for all $u \in L^q$, $u \geq 0$ and $\|u\|_{L^q} = 1$, then we get

$$\left(\int \rho^{(1+2/n)p/2}\, dx \right)^{2/p} \leq c \left(\int \left(\sum_{i=1}^N |\nabla f_i|^2 \right)^{p/2} dx \right)^{2/p}.$$

Next we consider the case $2n/(n+2) < p < 2$. Let

$$f = \left(\sum_{i=1}^{N} |\nabla f_i|^2 \right)^{1/2}.$$

We can take γ such that $(2-p)n/2 < \gamma < p$. Then we have

$$M(f^\gamma)^{-(2-p)/\gamma} \in A_2$$

because

$$M(f^\gamma)^{(2-p)/\gamma} \in A_1$$

by (ii) of Proposition 4.1. Furthermore we have

$$\{M(f^\gamma)^{-(2-p)/\gamma}\}^{-n/2} = M(f^\gamma)^{(2-p)n/(2\gamma)} \in A_1 \subset A_{n/2}.$$

Therefore

$$\begin{aligned}
\int \rho^{(1+2/n)p/2}\, dx &= \int \rho^{(1+2/n)p/2} M(f^\gamma)^{-(2-p)p/(2\gamma)} M(f^\gamma)^{(2-p)p/(2\gamma)}\, dx \\
&\le \left(\int \rho^{1+2/n} M(f^\gamma)^{-(2-p)/\gamma}\, dx \right)^{p/2} \left(\int M(f^\gamma)^{p/\gamma}\, dx \right)^{1-p/2} \\
&\le c \left(\int f^2 M(f^\gamma)^{-(2-p)/\gamma}\, dx \right)^{p/2} \left(\int f^p\, dx \right)^{1-p/2} \\
&\le c \left(\int M(f^\gamma)^{2/\gamma} M(f^\gamma)^{-(2-p)/\gamma}\, dx \right)^{p/2} \left(\int f^p\, dx \right)^{1-p/2} \\
&\le c \left(\int M(f^\gamma)^{p/\gamma}\, dx \right)^{p/2} \left(\int f^p\, dx \right)^{1-p/2} \le c \int f^p\, dx,
\end{aligned}$$

where we used Theorem 3.2 in the second inequality.

6 Lieb's inequality for Bessel potentials

Lieb proved the following inequality in [6].

Theorem 6.1. *Let $n \in \mathbb{N}$, $s > 0$, $n > 2s$ and $m \ge 0$. Let $f_1, \dots, f_N$ be orthonormal in $L^2(\mathbb{R}^n)$ and*

$$u_i = (-\Delta + m^2)^{-s/2} f_i.$$

Then

$$\int_{\mathbb{R}^n} \left(\sum_{i=1}^{N} |u_i(x)|^2 \right)^{n/(n-2s)} dx \le C_{n,s} N.$$

Battle and Federbush([1]) proved this inequality for $n = 3$ and $s = 1$ in 1982. They applied it to the quantum field theory. Lieb proved the case $n \geq 4$ and $s > 0$.

We can prove the following generalization of Lieb's inequality by means of Theorem 2.1.

Theorem 6.2 (Tachizawa, 2007). *Let $n \in \mathbb{N}$, $s > 0$, $n > 2s$ and $m \geq 0$. Let $w \in A_{n/(n-2s)} \cap A_2$ and $w^{-n/(2s)} \in A_{n/(2s)}$. Let $f_1, \ldots, f_N$ be orthonormal in $L^2(\mathbb{R}^n)$, $f_i \in L^2(w)$, and*

$$u_i = (-\Delta + m^2)^{-s/2} f_i.$$

Then

$$\int_{\mathbb{R}^n} \left(\sum_{i=1}^{N} |u_i(x)|^2 \right)^{n/(n-2s)} w(x)dx \leq C \sum_{i=1}^{N} \int_{\mathbb{R}^n} |f_i(x)|^2 w(x)\, dx,$$

where the constant C depends only on $n, s,$ and w.

The proof of Theorem 6.2 is given by a similar argument to that of Theorem 3.2. We use the characterization of weighted spaces by means of wavelets and scaling function. The detail will appear elsewhere.

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