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Existence and stability of periodic solutions in the isosceles three-body problem

By

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Abstract

The isosceles three-body problem is a special case of the three-body problem, which is a sufficiently difficult and interesting problem. In the problem there is a family of well-known periodic solutions called Euler solutions. We investigate the Birkhoff normal form around the circular Euler solution and check the twist condition to prove the KAM-stability of the Euler solutions with small eccentricity. Next by using the variational method we prove the existence of new periodic and quasi-periodic solutions which are bifurcated from the circular Euler solutions.

§ 1. Introduction and Main Theorems

The three-body problem is given by the following set of ODEs:

$$\frac{d^2\mathbf{q}_i}{dt^2} = -\sum_{j\neq i} \frac{m_j(\mathbf{q}_i - \mathbf{q}_j)}{|\mathbf{q}_i - \mathbf{q}_j|^3}, \quad \mathbf{q}_i \in \mathbb{R}^3, m_i \geqq 0, i = 1, 2, 3.$$

In this paper we deal with the isosceles three-body problem; assume $m_1 = m_2$, and consider motions for which m_3 remains on the z-axis in $\mathbb{R}^3$, while m_1 and m_2 remain symmetric with respect to this axis (see figure 1). The Lagrangian of the three-body problem is

$$L = \frac{1}{2}\sum_{k=1}^{3} m_k|\dot{\mathbf{q}}_k|^2 + \sum_{j<k} \frac{m_j m_k}{|\mathbf{q}_j - \mathbf{q}_k|}.$$

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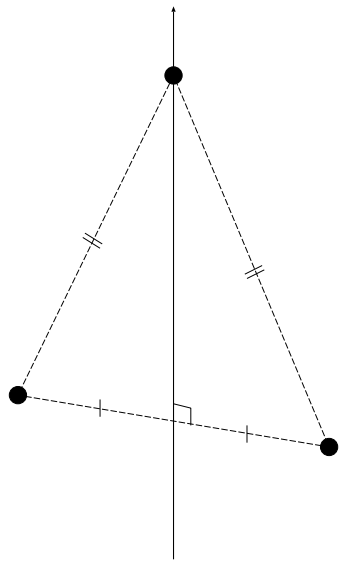


Figure 1. Spatial isosceles three-body problem

Let α be a mass ratio m_3/m_1 and use $(x, y, z) \in \mathbb{R}^3$ as coordinates on the configuration space where

$$\mathbf{q}_1 = \left(x, y, \frac{\alpha}{\alpha+2}z\right), \quad \mathbf{q}_2 = \left(-x, -y, \frac{\alpha}{\alpha+2}z\right), \quad \mathbf{q}_3 = \left(0, 0, -\frac{2}{\alpha+2}z\right).$$

The Lagrangian becomes

$$L = m_1\left(\dot{x}^2 + \dot{y}^2 + \frac{\alpha}{\alpha+2}\dot{z}^2\right) + m_1 m_3\left(\frac{1}{2\alpha\sqrt{x^2+y^2}} + \frac{2}{\sqrt{x^2+y^2+z^2}}\right).$$

It is possible to alter the above Lagrangian to

$$L = \dot{x}^2 + \dot{y}^2 + \frac{\alpha}{\alpha+2}\dot{z}^2 + \frac{1}{2\alpha\sqrt{x^2+y^2}} + \frac{2}{\sqrt{x^2+y^2+z^2}}$$

by rescaling the variables.

The isosceles three-body problem has an invariant manifold $\{z \equiv \dot{z} \equiv 0\}$ on which the differential equations are reduced to Kepler problem and on which all solutions are periodic solutions, so-called Euler solutions. With respect to the Euler solutions m_1 and m_2 move on ellipses and m_3 always stays at the origin.

By using the cylindrical coordinates: $(x, y, z) = (r\cos\theta, r\sin\theta, z)$, the Lagrangian is

$$L = \dot{r}^2 + r^2\dot{\theta}^2 + \frac{\alpha}{\alpha+2}\dot{z}^2 + \frac{1}{2\alpha r} + \frac{2}{\sqrt{r^2+z^2}}.$$

By the Legendre transformation, we obtain the Hamiltonian:

$$H = \frac{1}{4}p_r^2 + \frac{1}{4r^2}p_\theta^2 + \frac{\alpha+2}{4\alpha}p_z^2 - \frac{1}{2\alpha r} - \frac{2}{\sqrt{r^2+z^2}}. \tag{1.1}$$

The variable p_θ is the angular momentum and constant along solutions. By fixing $p_\theta = \omega > 0$ and ignoring θ, we can reduce it to a system with two degrees of freedom. Under the reduction we prove the stability of the Euler solutions with small eccentricities:

Theorem 1.1. *For $0 < \alpha < \infty$, the circular Euler solution is stable. In particular there are "many" KAM tori around the solution. Furthermore if $\alpha \neq \frac{1}{3}$, the elliptic Euler solutions with small eccentricity are stable and there are "many" KAM tori around the solution on the energy surface.*

KAM tori stand for quasi-periodic solutions around the Euler solutions. The word "many" KAM tori means that the smaller neighborhood of the Euler solution is chosen, the nearer to the full measure the measure of KAM tori on the neighborhood is.

Next we prove the existence of symmetric periodic solutions bifurcated from the circular Euler solution. Define two functions f and g by

$$f(\alpha) = \frac{\pi}{2}\sqrt{\frac{4\alpha+1}{\alpha+2}}, \quad g(\alpha) = \frac{\pi(2\alpha+1)}{4\alpha+1}\sqrt{\frac{2(2\alpha+1)}{\alpha+2}}.$$

We use the coordinates (x, y, z) here.

Theorem 1.2. *For any $T > 0$ and $c \in (f(\alpha), g(\alpha))$, there is a solution $(x(t), y(t), z(t))$ such that*

$$\begin{pmatrix} x(-t) \\ y(-t) \\ z(-t) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix},$$

$$\begin{pmatrix} x(t+2T) \\ y(t+2T) \\ z(t+2T) \end{pmatrix} = \begin{pmatrix} \cos 2c & -\sin 2c & 0 \\ \sin 2c & \cos 2c & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

and $z(t) \not\equiv 0$.

It turns out that if $c/\pi \in \mathbb{Q}$, the obtained solution is a periodic solution and if $c/\pi \notin \mathbb{Q}$, quasi-periodic solution. Note that $z(t) \not\equiv 0$ means that the obtained solutions are different from the Euler solutions.

We prove Theorem 1.1 in section 2. Since in the case of $\alpha \neq 1/3$ the frequency ratio is non-resonant, Birkhoff normal form can be constructed and hence the stability follows from KAM theory by checking the twist condition. In the case of $\alpha = 1/3$, the 1:2 resonance occurs. But fortunately Birkhoff normal form can be constructed in such a way that the resonance terms vanish. For the computations the computer algebra system Maple is effectively used.

Theorem 1.2 can be proved by using a variational method. A difficulty is to prove that the minimzer has no collision. The symmetry the desired solutions have is too

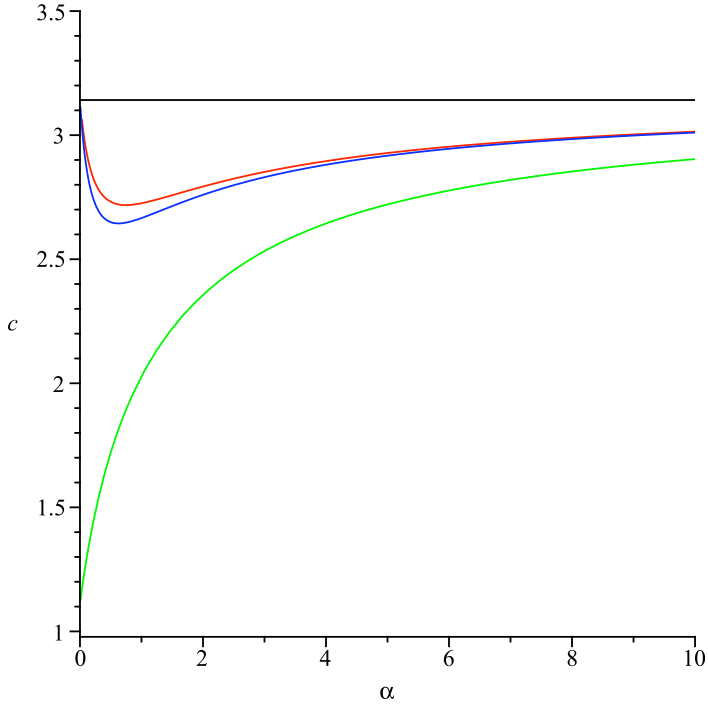


Figure 2. Graphs of f, g, h and $\pi(f(\alpha) < g(\alpha) < h(\alpha) < \pi)$

strong to satisfy "the rotating circle property", that is a known criterion for the elimination of collisions([2]). But a little less symmetric constraint satisfies the rotating circle property, and we can partly apply [2] and then we show that the minimizer has no collision in the interior $(0, T)$. So we prove that the minimizer does not have collision $t = 0$ and $t = T$ by comparing the values of the action functional between the circular motion and curves with collision at $t = 0$ or $t = T$. By using another test path instead of the circular motion, we find a better function $h(\alpha)$ than $g(\alpha)$ $(h(\alpha) > g(\alpha))$ such that Theorem 1.2 holds for $f(\alpha) < c < h(\alpha)$. But the expression of h is too complicated to denote explicitly. We just show the graph in figure 2.

The paper is organized as follows. In the next section we prove theorem 1.1. In subsection 2.1 we write down the Taylor expansion of the Hamiltonian at the circular Euler solution. In subsection 2.2 we use the Lie transforms to construct the Birkhoff normal form from the Taylor expansion. Then by checking the twist condition we prove the KAM-stability of the Euler solutions. Section 3 is devoted to the proof of theorem 2 which is based on the variational method. We express the spatial isosceles three-body problem by the Lagrangian formulation in subsection 3.1 and show the existence of a minimizer in subsection 3.2. In subsection 3.3, we show that the minimizer has no collision. So in subsection 3.4 we will prove z-component of the obtained minimizer is

not identically zero. In Appendix, we give a better estimate for collisionlessness.

§ 2. Stability of Euler solution

§ 2.1. Taylor expansion of Euler solution

We consider the Hamiltonian system with Hamiltonian (1.1). By fixing $p_\theta \equiv \omega$ and ignoring θ, the system is reduced to the one with Hamiltonian

$$H(p_r, p_z, r, z) = \frac{1}{4}p_r^2 + \frac{1}{4r^2}\omega^2 + \frac{\alpha+2}{4\alpha}p_z^2 - \frac{1}{2\alpha r} - \frac{2}{\sqrt{r^2+z^2}}.$$

The point $(p_r, p_z, r, z) = (0, 0, \frac{\omega^2\alpha}{4\alpha+1}, 0)$ is an equilibrium point corresponding to the circular Euler solution. By the linear canonical transformation

$$\begin{aligned}
q_1 &= \frac{4\alpha+1}{\omega^{3/2}\alpha}\left(r - \frac{\omega^2\alpha}{4\alpha+1}\right)\\
p_1 &= \frac{\omega^{3/2}\alpha}{4\alpha+1}p_r\\
q_2 &= \frac{2^{1/2}(4\alpha+1)^{3/4}}{\omega^{3/2}\alpha^{1/2}(\alpha+2)^{1/4}}z\\
p_2 &= \frac{\omega^{3/2}\alpha^{1/2}(\alpha+2)^{1/4}}{2^{1/2}(4\alpha+1)^{3/4}}p_z
\end{aligned}$$

the quadratic terms of H are taken into normal form. The Taylor expansion of H is

$$\begin{aligned}
H = {} & \frac{(4\alpha+1)^2}{4\omega^3\alpha^2}(p_1^2+q_1^2) + \frac{(4\alpha+1)^{3/2}(\alpha+2)^{1/2}}{2\omega^3\alpha^2}(p_2^2+q_2^2)\\
& - \frac{(4\alpha+1)^2}{2\omega^{7/2}\alpha^2}q_1^3 - \frac{9(4\alpha+1)^{3/2}(\alpha+2)^{1/2}}{2\omega^{7/2}\alpha^2}q_1q_2^2\\
& + \frac{3(4\alpha+1)^2}{4\omega^4\alpha^2}q_1^4 + \frac{18(4\alpha+1)^{2/3}(\alpha+2)^{1/2}}{\omega^4\alpha^2}q_1^2q_2^2 - \frac{3(4\alpha+1)^2(\alpha+2)}{16\omega^4\alpha^2}q_2^4 + O(|(p,q)|^5).
\end{aligned}$$

The frequency ratio is

$$\lambda = 2\sqrt{\frac{\alpha+2}{4\alpha+1}}$$

and the resonance of no more than degree four occurs in the case of $\alpha = 1/3$ (1:2 resonace). We will use λ as the parameter instead of α.

We will carry out the Birkhoff normalization of H so that those terms up to order four depend only on

$$\tau_k = \frac{1}{2}(p_k^2 + q_k^2) \qquad (k = 1, 2).$$

Define the canonical transformation

$$p_k = \frac{1}{\sqrt{2}}(x_k - iy_k)$$
$$q_k = \frac{1}{\sqrt{2}}(-ix_k + y_k) \qquad (k = 1, 2).$$

Then we have $\tau_k = -ix_k y_k$. It will be useful for further calculation. The Hamiltonian in this coordinates is

$$\begin{aligned} H = &-\frac{392i(x_1y_1 + \lambda x_2y_2)}{\omega^3(8-\lambda^2)^2} + \frac{49\sqrt{2}}{\omega^{7/2}(8-\lambda^2)^2}\{-2i(x_1^3 - iy_1^3) + 6(x_1^2y_1 + ix_1y_1^2) \\ &+ 6\lambda(x_1x_2y_2 + ix_2y_1y_2) - 3i\lambda(x_1x_2^2 + ix_2^2y_1 - x_1y_2^2 - iy_1y_2^2)\} \\ &+ \frac{147}{2(\lambda^2-8)^3\,\omega^4}\{-24(\lambda^2-8){x_1}^2{y_1}^2 - 32\lambda(\lambda^2-8)x_1y_1x_2y_2 - 42\lambda^2{x_2}^2{y_2}^2 \\ &+ 4(\lambda^2-8)(x_1^4 + y_1^4) + 16i(\lambda^2-8)(x_1^3y_1 - x_1y_1^3) + 7\lambda^2(x_2^4 + y_2^4) \\ &+ 8\lambda(\lambda^2-8)({x_1}^2{x_2}^2 - x_1^2y_2^2 - y_1^2x_2^2 + y_1^2y_2^2) + 16i\lambda(\lambda^2-8)(x_1y_1x_2^2 - x_1y_1y_2^2) \\ &+ 16i\lambda(\lambda^2-8)(x_1^2x_2y_2 - y_1^2x_2y_2) + 28i\lambda^2({x_2}^3y_2 - x_2{y_2}^3)\} + \dots \end{aligned}$$

§ 2.2. Lie transforms

We denote

$$H(x_1, x_2, y_1, y_2) = \sum_{i=0}^{\infty} \frac{1}{i!} H_i^0(x_1, x_2, y_1, y_2),$$

where H_i^0 is a homogeneous polynomial of degree $i+2$. Consider an analytic function

$$W(x_1, x_2, y_1, y_2) = \sum_{i=0}^{\infty} \frac{1}{i!} W_{i+1}(x_1, x_2, y_1, y_2),$$

where W_i is a homogeneous polynomial of degree $i+2$. The time 1 map of the Hamiltonian system with the Hamiltonian W is a canonical transformation. We denote the Hamiltonian H transformed by the time 1 map by G:

$$G(x_1, x_2, y_1, y_2) = \sum_{i=0}^{\infty} \frac{1}{i!} H_0^i(x_1, x_2, y_1, y_2),$$

where H_0^i is a homogeneous polynomial of degree $i+2$. We can get H_0^i by a formula

$$H_j^i = H_{j+1}^{i-1} + \sum_{k=0}^{j} \binom{j}{k} \{H_{j-k}^{i-1}, W_{k+1}\} \tag{2.1}$$

(see [4]).

If $W' = \rho \mathbf{x}^{\mathbf{a}}\mathbf{y}^{\mathbf{b}} = \rho x_1^{a_1} x_2^{a_2} y_1^{b_1} y_2^{b_2}$,

$$\{H_0^0, W'\} = \rho b((1,\lambda)\cdot(\mathbf{b}-\mathbf{a}))\mathbf{x}^{\mathbf{a}}\mathbf{y}^{\mathbf{b}}$$

where $b = -\frac{392i}{\omega^3(8-\lambda^2)^2}$. Hence we can eliminate the terms except resonance terms $\mathbf{x}^{\mathbf{a}}\mathbf{y}^{\mathbf{b}}$ with

$$(1,\lambda)\cdot(\mathbf{b}-\mathbf{a}) = 0. \tag{2.2}$$

In the case of $i = 1$, the formula (2.1) is

$$H_0^1 = H_1^0 + \{H_0^0, W_1\}.$$

The resonance occur in the case of $\lambda = 2$ ($\alpha = 1/3$). The resonance terms satisfies $|\mathbf{a}| + |\mathbf{b}| = 3$ and (2.2), then $\mathbf{a} = (2,0), \mathbf{b} = (0,1)$ and $\mathbf{a} = (0,1), \mathbf{b} = (2,0)$. Hence the resonance terms of degree three are $x_1^2 y_2$ and $x_2 y_1^2$. But H_1^0 dose not have these terms, because the Hamiltonian (1.1) is invariant under the transformations:

$$p_r \to -p_r, \qquad p_z \to -p_z, \qquad z \to -z,$$

which correspond to

$$(x_1, y_1) \to (iy_1, -ix_1), \qquad (x_2, y_2) \to (iy_2, -ix_2), \qquad (x_2, y_2) \to (-iy_2, ix_2)$$

respectively. Therefore we can eliminate all terms of degree three.

For $i = 2$, the formula (2.1) is

$$\begin{aligned} H_1^1 &= H_2^0 + \{H_1^0, W_1\} + \{H_0^0, W_2\} \\ H_0^2 &= H_1^1 + \{H_0^1, W_1\}, \end{aligned}$$

hence

$$H_0^2 = H_2^0 + \{H_1^0 + H_0^1, W_1\} + \{H_0^0, W_2\}.$$

The part $H_2^0 + \{H_1^0 + H_0^1, W_1\}$ is

$$\begin{aligned} &\frac{147}{(\lambda^2-8)^2\omega^4}\Bigg\{8{x_1}^2{y_1}^2 + \frac{8\lambda(7\lambda^2-1)}{4\lambda^2-1}x_1y_1x_2y_2 + \frac{3\lambda^2(8\lambda^4-95\lambda^2+31)}{(\lambda^2-8)(4\lambda^2-1)}{x_2}^2{y_2}^2 \\ &+ 4({x_1}^4 + y_1^4) + \frac{31\lambda^2(\lambda^2-1)}{2(\lambda^2-8)(4\lambda^2-1)}({x_2}^4 + y_2^4) \\ &- \frac{2i\lambda^2(6\lambda^4-79\lambda^2+31)}{(\lambda^2-8)(4\lambda^2-1)}({x_2}^3y_2 - x_2y_2^3) \\ &- \frac{4i\lambda(\lambda^2-1)}{4\lambda^2-1}(x_1y_1{x_2}^2 - x_1y_1y_2^2 - {x_1}^2x_2y_2 + y_1^2x_2y_2) \\ &+ \frac{2\lambda(\lambda+1)(3\lambda-1)}{2\lambda-1}({x_1}^2{x_2}^2 + y_1^2y_2^2) + \frac{2\lambda(\lambda-1)(3\lambda+1)}{2\lambda+1}({x_1}^2{y_2}^2 + x_2^2y_1^2)\Bigg\}. \end{aligned}$$

Consequently for $1 < \lambda < 2\sqrt{2}$ we obtain Birkhoff normal form

(2.3)
$$G = \frac{392(\tau_1 + \lambda\tau_2)}{\omega^3(8-\lambda^2)^2} - \frac{588}{\omega^4(8-\lambda^2)^2}\tau_1^2 - \frac{588\lambda(1-7\lambda^2)}{\omega^4(8-\lambda^2)^2(1-4\lambda^2)}\tau_1\tau_2 - \frac{441\lambda^2(8\lambda^4-95\lambda^2+31)}{2\omega^4(8-\lambda^2)^3(1-4\lambda^2)}\tau_2^2 + \dots.$$

We denote
$$G = a_1\tau_1 + a_2\tau_2 + a_{11}\tau_1^2 + 2a_{12}\tau_1\tau_2 + a_{22}\tau_2^2 + \dots.$$

The twist condition is
$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix} = -\frac{43218\lambda^2(\lambda^2-1)(2\lambda^4+354\lambda^2-77)}{\omega^8(1-4\lambda^2)^2(\lambda^2-8)^5} \neq 0.$$

This holds for all $1 < \lambda < 2\sqrt{2}(0 < \alpha < \infty)$. The isoenergetic twist condition is
$$\det\begin{pmatrix} a_{11} & a_{12} & a_1 \\ a_{12} & a_{22} & a_2 \\ a_1 & a_2 & 0 \end{pmatrix} = -\frac{1050370272\lambda^2(\lambda^2-1)}{\omega^{10}(\lambda^2-8)^7(4\lambda^2-1)} \neq 0.$$

This also holds for all $1 < \lambda < 2\sqrt{2}(0 < \alpha < \infty)$.

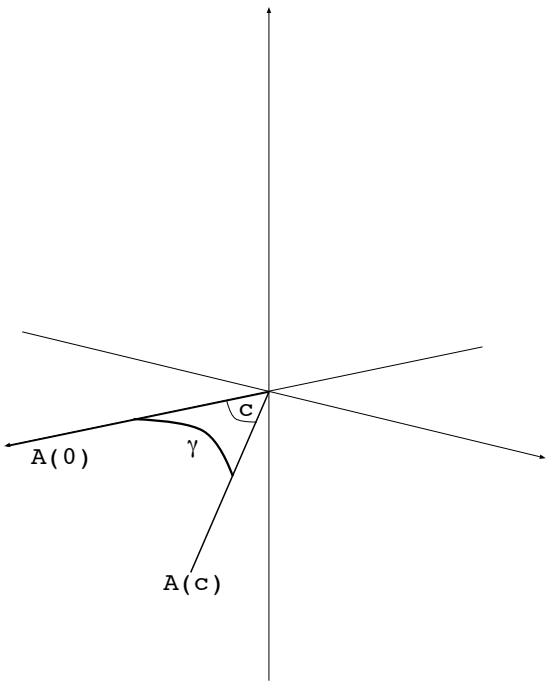
Consequently the circular Euler solution is KAM stable for all $1 < \lambda < 2\sqrt{2}$ $(0 < \alpha < \infty)$ and the elliptic Euler solutions with small eccentricity are KAM stable on the energy surface for all $\lambda \neq 2(\alpha \neq 1/3)$.

§ 3. Existence of symmetric solutions

§ 3.1. Lagrangian formulation

The isosceles three-body problem is equivalent to the variational problem with respect to the action functional
$$\mathcal{A}(\gamma) = \int_0^T L(\gamma, \dot{\gamma})dt.$$

Define
$$A(c) = \{(r\cos c, r\sin c, 0) \mid r \geqq 0\}$$
$$\Omega(c_1, c_2) = \{\gamma \in H^1([0,T], \mathbb{R}^3) \mid \gamma(0) \in A(c_1), \gamma(T) \in A(c_2), \},$$



where $H^1([0,T],\mathbb{R}^3)$ is the Sobolev space with Sobolev norm $\|\cdot\|_{H^1}$:

$$\|\gamma\|_{H^1} = \int_0^T |\gamma(t)|^2 + |\dot{\gamma}(t)|^2 dt$$
$$H^1([0,T],\mathbb{R}^3) = \{\gamma : [0,T] \to \mathbb{R}^3 \mid \gamma \in L^2, \dot{\gamma} \in L^2, \|\gamma\|_{H^1} < \infty\}.$$

We will apply the minimizing method for $\mathcal{A}|_{\Omega(0,c)}$.

§ 3.2. Coercivity of the action functional

In this subsection we prove the coercivity of the action functional $\mathcal{A}|_{\Omega(0,c)}$, that is, if $\|x_n\|_{H^1} \to \infty (x_n \in \Omega(0,c))$, then $\mathcal{A}(x_n) \to \infty$. The coercivity guarantees the existence of a minimizer of $\mathcal{A}|_{\Omega(0,c)}$.

Proposition 3.1. *The action functional $\mathcal{A}|_{\Omega(0,c)}$ is coercive and hence attains its minimum.*

Proof. Our argument is not new and has been used by Chen in his several papers (see for example [1, Section 4]). Consider the function $\delta : \Omega(0,c) \to \mathbb{R}$ defined by

$$\delta(\gamma) := \max_{s_1,s_2 \in [0,T]} |\gamma(s_1) - \gamma(s_2)|.$$

Let

$$\nu = \cos c,$$

which is consistent with $\frac{x(0)\cdot x(T)}{|x(0)||x(T)|}$, greater than -1 and less than 1. We easily obtain

$$|\gamma(0)-\gamma(T)| \geqq C_\nu |\gamma(0)|$$

where $C_\nu = \sqrt{1-\nu^2}$. Note that $C_\nu > 0$. For any $t \in [0,T]$,

$$|\gamma(t)| \leqq |\gamma(0)| + \delta(\gamma) \leqq \frac{1}{C_\nu}|\gamma(0)-\gamma(T)| + \delta(\gamma) \leqq \left(\frac{1}{C_\nu}+1\right)\delta(\gamma),$$

and hence

$$\int_0^T |\gamma|^2 dt \leqq \left(\frac{1}{C_\nu}+1\right)^2 \delta(\gamma)^2 T.$$

On the other hand, by the Cauchy-Schwarz inequality

$$\delta(\gamma)^2 \leqq \left(\int_0^T |\dot\gamma| dt\right)^2 \leqq T\int_0^T |\dot\gamma|^2 dt.$$

Therefore the H^1 norm of γ is controlled by its action:

$$\begin{aligned}
\|\gamma\|_{H^1}^2 &= \int_0^T |\gamma|^2 + |\dot\gamma|^2 dt \\
&\leqq \left(\left(\frac{1}{C_\nu}+1\right)^2 T^2 + 1\right)\int_0^T |\dot\gamma|^2 dt \\
&< \left(\left(\frac{1}{C_\nu}+1\right)^2 T^2 + 1\right)\frac{\alpha+2}{\alpha}\mathcal{A}(\gamma).
\end{aligned}$$

This implies that $\mathcal{A}|_\Omega(0,c)$ is coercive. □

The minimizer γ satisfies the first variational formula:

$$\delta\mathcal{A}(\gamma) = \left[\frac{\partial L}{\partial \dot{\mathbf{q}}}\cdot\delta\gamma\right]_{t=0}^T - \int_0^T \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} - \frac{\partial L}{\partial \mathbf{q}}\right)\cdot\delta\gamma dt = 0 \tag{3.1}$$

for any $\delta\gamma \in \Omega(0,c)$. Hence by considering any variations $\delta\gamma$ with $\delta\gamma(0) = \delta\gamma(T) = 0$, we see that the minimizer satisfies the Euler-Lagrange equation:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\mathbf{q}}} - \frac{\partial L}{\partial \mathbf{q}} = 0$$

which is equivalent to the equation of the isosceles three-body problem. We again consider (3.1), and then we obtain

$$\frac{\partial L}{\partial \dot{\mathbf{q}}}(\gamma(T),\dot\gamma(T))\cdot\delta\gamma(T) - \frac{\partial L}{\partial \dot{\mathbf{q}}}(\gamma(0),\dot\gamma(0))\cdot\delta\gamma(0) = 0.$$

Note

$$\frac{\partial L}{\partial \dot{\mathbf{q}}} = \left(2\dot{x}, 2\dot{y}, \frac{2\alpha}{\alpha+2}\dot{z}\right).$$

Since $\delta\gamma(0)$ and $\delta\gamma(T)$ can have any point in $A(0)$ and $A(c)$ respectively, it follows that

$$\dot{\gamma}(0) \perp A(0), \dot{\gamma}(T) \perp A(c). \tag{3.2}$$

§ 3.3. Collision-free minimizer

In this subsection we show that the minimizer has no collision. First we discuss the isosceles symmetry in order to explain "the rotating circle property" introduced by Ferrario and Terracini [2]. They dealt with more general case (the planar and spatial N-body problem and more general group actions) but here we just consider the isosceles symmetry in the spatial three-body problem. Let $\mathcal{X}$ be the configuration space of the spatial three-body problem:

$$\mathcal{X} = \{(q_1, q_2, q_3) \in (\mathbb{R}^3)^3 \mid m_1 q_1 + m_2 q_2 + m_3 q_3 = 0\}.$$

We consider a finite group G and representations

$$\rho : G \to O(3)$$
$$\sigma : G \to \mathfrak{S}_3,$$

such that for $g \in G, q = (q_1, q_2, q_3) \in \mathcal{X}$

$$g \cdot (q_1, q_2, q_3) = (\rho(g) q_{\sigma(g^{-1})(1)}, \rho(g) q_{\sigma(g^{-1})(2)}, \rho(g) q_{\sigma(g^{-1})(3)}).$$

The symmetric configuration space is defined by $\mathcal{X}^G = \{q \in \mathcal{X} \mid g \cdot q = q \ (\forall g \in G)\}$. In our situation $G = \{\pm 1\}$ and the isosceles symmetry is determined by

$$\rho(-1) = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \sigma(-1) = (1\ 2). \tag{3.3}$$

For $i = 1, 2, 3$, let G^i be the isotropy subgroup of G at i under the σ-action, namely,

$$G^i = \{g \in G | \sigma(g) i = i\}.$$

Definition 3.2. We say a finite group G acts with the rotating circle property, if for at least 2 indices i there exists a circle in $\mathbb{R}^3$ such that G acts on the circle by rotation and that the circle is contained in $(\mathbb{R}^3)^{G^i} = \{w \in \mathbb{R}^3 \mid \rho(g) \cdot w = w (\forall g \in G^i)\}$.

In our case (3.3), $G^1 = G^2 = \{1\}$. If we take a circle $\mathbb{S}$ on xy-plane whose center is the origin, then $g \in G$ acts on $\mathbb{S}$ by rotation and $\mathbb{S}$ is contained in $(\mathbb{R}^3)^{G^i} = \mathbb{R}^3$ for $i = 1, 2$. Therefore the group action satisfies the rotating circle property.

Proposition 3.3 ([2] Theorem 10.10). *Consider a finite group G acting with the rotating circle property. Then a minimizer of the fixed-ends problem on $\mathcal{X}^G$ is free of collisions.*

The proposition can be applied to our situation. Let γ be a minimizer of $\mathcal{A}|_{\Omega(0,c)}$. Of course γ is also a minimizer of $\mathcal{A}$ on

$$\{\rho : [0,T] \to \mathbb{R}^3 \mid \rho(0) = \gamma(0), \rho(T) = \gamma(T)\}$$

that is, a minimizer of fixed-end problem. From Proposition 3.3, γ has no collision in $(0,T)$. So it is enough to show γ does not have a collision at $t = 0, T$.

We will estimate the value of the action functional with respect to curves with collisions at $t = 0$ (the case of $t = T$ is similar) and find a test path with less value of the action functional than any collision path. It follows that the minimizer has no collision.

We use the circular solution as a test path:

$$\gamma_{\mathrm{cir}}(t) = \sqrt[3]{\frac{T^2(4\alpha+1)}{4c^2\alpha}}\left(\cos\frac{c}{T}t, \sin\frac{c}{T}t, 0\right). \tag{3.4}$$

By an easy calculation, the value of the action functional with respect to the orbit is

$$\begin{aligned} A_{\mathrm{cir}}(c,\alpha,T) := \mathcal{A}(\gamma_{\mathrm{cir}}) &= \int_0^T \left(\frac{c(4\alpha+1)}{4T\alpha}\right)^{2/3} + \left(\frac{1}{2\alpha}+2\right)\left(\frac{4c^2\alpha}{T^2(4\alpha+1)}\right)^{1/3} dt \\ &= 2^{-4/3}\cdot 3c^{2/3}T^{1/3}\alpha^{-2/3}(4\alpha+1)^{2/3}. \end{aligned}$$

We assume that a curve $\gamma_{\mathrm{col}}(t) = (x(t), y(t), z(t)) \in \Omega(0,c)$ has a collision at $t = 0$. Since γ_{col} belongs to $A(0)$ at $t = 0$, $(x(0), y(0), z(0)) = (0,0,0)$. Let $(x,y) = r(\cos\theta, \sin\theta)$. Then

$$\begin{aligned} L &= \dot{r}^2 + r^2\dot{\theta}^2 + \frac{\alpha}{\alpha+2}\dot{z}^2 + \frac{1}{2\alpha}r^{-1} + 2(r^2+z^2)^{-1/2} \\ &\geqq \dot{r}^2 + \frac{\alpha}{\alpha+2}\dot{z}^2 + \frac{1}{2\alpha}r^{-1} + 2(r^2+z^2)^{-1/2} \\ &= \left(\frac{2}{\alpha+2}\dot{r}^2 + \frac{1}{2\alpha}r^{-1}\right) + \left(\frac{\alpha}{\alpha+2}(\dot{r}^2+\dot{z}^2) + 2(r^2+z^2)^{-1/2}\right). \end{aligned}$$

It is known [3] that the minimizer of Lagrangian of collinear or planar Kepler problem

$$L = a\dot{\xi}^2 + b\xi^{-1} \text{ or } L = a(\dot{\xi}^2+\dot{\eta}^2) + b(\xi^2+\eta^2)^{-1/2}$$

with $\xi(0) = \eta(0) = 0$ is attained by the collision-ejection orbit and that the value of the action functional is

$$g_{ab} = 2^{-2/3}\cdot 3\pi^{2/3}a^{1/3}b^{2/3}T^{1/3}.$$

Therefore since $r(0) = s(0) = 0$,

$$\begin{aligned}
\mathcal{A}(\gamma_{\mathrm{col}}) &\geqq \int_0^T \left(\frac{2}{\alpha+2}\dot{r}^2 + \frac{1}{2\alpha}r^{-1}\right) dt + \int_0^T \left(\frac{\alpha}{\alpha+2}(\dot{r}^2+\dot{s}^2) + 2(r^2+s^2)^{-1/2}\right) dt \\
&\geqq g_{\frac{2}{\alpha+2},\frac{1}{2\alpha}} + g_{\frac{\alpha}{\alpha+2},2} \\
&= 2^{-2/3}\cdot 3\pi^{2/3}T^{1/3}\left(\left(\frac{2}{\alpha+2}\right)^{1/3}\left(\frac{1}{2\alpha}\right)^{2/3} + \left(\frac{\alpha}{\alpha+2}\right)^{1/3}\cdot 2^{2/3}\right) \\
&= 2^{-1}\cdot 3\pi^{2/3}T^{1/3}\alpha^{-2/3}\left(\alpha+2\right)^{-1/3}\left(2\alpha+1\right) =: A_{\mathrm{col}}(\alpha,T).
\end{aligned}$$

If

$$A_{\mathrm{cir}}(c,\alpha,T) < A_{\mathrm{col}}(\alpha,T)$$

(that is, $c < g(\alpha)$), then the minimizer does not have collision at $t = 0$.

Remark. We can obtain better estimate by finding another test path. But the calculation is very complicated. We show that in the appendix.

§ 3.4. Non-constancy of the vertical component

In this subsection we show that z-component of the minimizer is not identically zero and hence that the obtained minimizer is a non-trivial (non-Keplerian) orbit.

Proposition 3.4. *Let $\gamma = (\gamma_x, \gamma_y, \gamma_z)$ be the minimizer. If $f(\alpha) < c$, $\gamma_z \not\equiv 0$.*

Proof. If $\gamma_z \equiv 0$, the motion is Keplerian. From (3.2), this is the circular motion. We can solve the motion:

$$\gamma(t) = \sqrt[3]{\frac{T^2(4\alpha+1)}{4c^2\alpha}}\left(\cos\frac{c}{T}t, \sin\frac{c}{T}t, 0\right). \tag{3.5}$$

Define a modified curve $\gamma^{(\varepsilon)}$ by

$$\gamma^{(\varepsilon)}(t) = \gamma(t) + \left(0, 0, \varepsilon\sin\frac{\pi t}{T}\right).$$

The difference of the values of the action functional is

$$\begin{aligned}
&\mathcal{A}(\gamma) - \mathcal{A}(\gamma^{(\varepsilon)}) \\
&= \int_0^T -\frac{\alpha}{\alpha+2}\left(\dot{\gamma}_z^{(\varepsilon)}\right)^2 + \frac{2}{\sqrt{\gamma_x^2+\gamma_y^2}} - \frac{2}{\sqrt{\gamma_x^2+\gamma_y^2+\left(\gamma_z^{(\varepsilon)}\right)^2}} dt \\
&= \int_0^T -\frac{\pi^2\alpha\varepsilon^2}{(\alpha+2)T^2}\cos^2\frac{\pi t}{T} + \frac{4c^2\alpha\varepsilon^2}{(4\alpha+1)T^2}\sin^2\frac{\pi t}{T} + O(\varepsilon^4) dt \\
&= \frac{2\alpha\varepsilon^2}{(4\alpha+1)T}\left(c^2 - \frac{\pi^2(4\alpha+1)}{4\left(\alpha+2\right)}\right) + O(\varepsilon^4).
\end{aligned}$$

Therefore if $f(\alpha) < c$, $A(\gamma) > A(\gamma^{(\varepsilon)})$ for small $\varepsilon > 0$ and hence the minimizer is not an Keplerian orbit. $\square$

This proposition means that if $f(\alpha) < c < g(\alpha)$, the obtained solutions are new solutions.

Remark. The circular orbit (3.5) exists for any $c > 0$. The proposition means that if $f(\alpha) < c < g(\alpha)$, the circular orbit is a critical point of $\mathcal{A}|_{\Omega_{0,c}}$ but not a minimizer.

Define

$$R(\theta) = \begin{pmatrix} \cos 2\theta & \sin 2\theta & 0 \\ \sin 2\theta & -\cos 2\theta & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Let $\gamma \in \Omega(0, c)$ be a minimizer of $\mathcal{A}|_{\Omega(0,c)}$. For $n \in \mathbb{Z}$, $\gamma_n := R(nc)\gamma$ belongs to $\Omega(nc, (n+1)c)$ and is a minimizer of $\mathcal{A}|_{\Omega(nc,(n+1)c)}$. From (3.2), the curves $\gamma_n (n \in \mathbb{Z})$ can be smoothly connected and the connected curve is a solution defined for all time. This is the solution we desired.

§ 4. Future direction

Further research is in progress on the existence and bifurcations of families of periodic solutions. It will be shown ([5, 6]) that the Euler orbits and the obtained minimizing orbits are complicatedly connected by families of periodic solutions and complicated bifurcations occur and in particular many bifurcated orbits connect to heteroclinic connections of the triple collision.

Appendix. Better estimate than g

We use the curve

$$\gamma(t) = \left(d \cos \frac{c}{T} t, d \sin \frac{c}{T} t, \zeta(t)\right)$$

as a test path instead of the circular motion, where c and d are constans and

$$\zeta = \varepsilon \sin \frac{\pi t}{T}.$$

From the inequality

$$\frac{1}{\sqrt{1+x^2}} \leqq 1 - \frac{x^2}{2} + \frac{3x^4}{8},$$

$$
\begin{aligned}
\mathcal{A}(\gamma) &= \int_0^T d^2\left(\frac{c}{T}\right)^2 + \frac{\alpha}{\alpha+2}\dot{\zeta}^2 + \frac{1}{2d\alpha} + \frac{2}{\sqrt{d^2+\zeta^2}}dt \\
&\leqq \int_0^T d^2\left(\frac{c}{T}\right)^2 + \frac{\alpha}{\alpha+2}\dot{\zeta}^2 + \frac{1}{2d\alpha} + \frac{2}{d} - \frac{\zeta^2}{d^3} + \frac{3\zeta^4}{4d^5}dt \\
&= \int_0^T d^2\left(\frac{c}{T}\right)^2 + \frac{1}{2d\alpha} + \frac{2}{d} + \frac{\varepsilon^2\pi^2\alpha}{(\alpha+2)\,T^2}\cos^2\frac{\pi t}{T} - \frac{\varepsilon^2}{d^3}\sin^2\frac{\pi t}{T} + \frac{3\varepsilon^4}{4d^5}\sin^4\frac{\pi t}{T}dt \\
&= T\left(d^2\left(\frac{c}{T}\right)^2 + \frac{1}{2d\alpha} + \frac{2}{d}\right) + \varepsilon^2\left(\frac{\pi^2\alpha}{2\,(\alpha+2)\,T} - \frac{T}{2d^3}\right) + \frac{9\varepsilon^4 T}{32d^5} \\
&=: B(c, T, \alpha^{-1}, \varepsilon, d).
\end{aligned}
$$

The minimum attains at

$$
\begin{aligned}
\varepsilon = \varepsilon_0 &:= \frac{2d_0}{3T}\sqrt{2T^2 - \frac{2\pi^2 d_0^3\alpha}{\alpha+2}} \\
d = d_0 &:= \left\{\left(\frac{(\alpha+2)T^2}{10\pi^4\alpha^2}\right) \times \big(18c^2 + 4\pi^2\alpha + 9c^2\alpha \right. \\
&\left. -3\sqrt{36c^4\alpha + 16\pi^2c^2\alpha^2 + 36c^4\alpha 2 - 16\pi^4 + 8\pi^2c^2\alpha^3 + 9c^4\alpha^3 - 5\pi^4\alpha^2}\big)\right\}^{1/3}.
\end{aligned}
$$

The minimum value is

$$A_{\mathrm{test}}(c, \alpha, T) = B(c, T, \alpha, \varepsilon_0, d_0).$$

and the explicit representation is very complicated. Define an implicit function $h(\alpha)$ by

$$A_{\mathrm{test}}(h(\alpha), \alpha, T) = A_{\mathrm{col}}(\alpha, T).$$

We show the graph obtained numerically(figure 2). If $f(\alpha) < c < h(\alpha)$, there is a new (quasi-)periodic solution.

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