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Sato's conjecture for the Weber equation and transformation theory for Schrödinger equations with a merging pair of turning points

By

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Abstract

In [3], together with Aoki and Kawai, I developed the transformation theory for an MTP equation (i.e., a Schrödinger equation with a merging pair of simple turning points) to the Weber equation and combined it with Sato's conjecture to clarify the analytic structure of Borel transformed WKB solutions of an MTP equation. In this paper I present a new proof of Sato's conjecture based on the use of the creation operator for the harmonic oscillator (i.e., the Weber equation) and explain a core part of the transformation theory for an MTP equation developed in [3] with emphasizing the role of Sato's conjecture there.

§ 0. Introduction

In [2] Aoki, Kawai and the author of the present paper developed the exact WKB theoretic transformation theory for a one-dimensional Schrödinger equation near a simple turning point and showed that Voros' connection formula ([12]) for Borel resummed WKB solutions on a Stokes curve emanating from a simple turning point can be obtained from that of the canonical equation, i.e., the Airy equation. In [2] we also constructed a transformation that brings a Schrödinger equation to the Weber equation near two simple turning points. Very recently, in [3] we have succeeded in showing that this transformation near two simple turning points together with what we call Sato's conjecture for the Voros coefficient of the Weber equation (cf. [8], [11]) enables us to analyze the structure of "fixed singularities" (cf. [4], [5], [6]) of Borel transformed WKB

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solutions of a Schrödinger equation with a merging pair of simple turning points ("a merging-turning-points equation" or "an MTP equation" for short). The purpose of this paper is to discuss Sato's conjecture and its analytic implication in details, including its direct proof based on the use of the creation operator for the harmonic oscillator (i.e., the Weber equation), and to explain a core part of the transformation theory for an MTP equation developed in [3] with emphasizing the role of Sato's conjecture there.

The paper is organized as follows: In Section 1 we present a WKB theoretic formulation of Sato's conjecture and give its direct proof. Then, making use of Sato's conjecture, we analyze the structure of fixed singularities of Borel transformed WKB solutions of the Weber equation in Section 2. Finally the transformation theory for an MTP equation is explained in Section 3.

In ending this Introduction, the author expresses his heartiest thanks to Professors T. Kawai, T. Aoki and T. Koike for their kind encouragements and continual stimulating discussions with them. The author sincerely thanks also to Professor H.J. Silverstone for the stimulating discussions with him during his extended stay at RIMS.

§ 1. Sato's conjecture

Sato's conjecture is concerned with WKB solutions

$$\psi_{\pm}(z,\eta)=\exp\left(\pm\int_{z_0}^{z}S^{\pm}dz\right) \tag{1.1}$$

of the Weber equation (i.e., the harmonic oscillator)

$$\left(\frac{d^2}{dz^2}-\eta^2(\frac{z^2}{4}-\lambda)\right)\psi=0. \tag{1.2}$$

Here $\eta>0$ is a large parameter, $\lambda\neq 0$ is a non-zero complex constant, z_0 is an arbitrarily chosen point and

$$S^{\pm}=\pm\eta S_{-1}(z)+S_0(z)\pm\eta^{-1}S_1(z)+\eta^{-2}S_2(z)\pm\cdots \tag{1.3}$$

$$S_{-1}(z)=\sqrt{\frac{z^2}{4}-\lambda},\ \ S_0(z)=-\frac{z}{8(z^2/4-\lambda)},\ \ S_1(z)=-\frac{3z^2/8+\lambda}{16(z^2/4-\lambda)^{5/2}},\ \ldots \tag{1.4}$$

denote WKB solutions of the Riccati equation

$$S^2+\frac{\partial S}{\partial z}=\eta^2\left(\frac{z^2}{4}-\lambda\right) \tag{1.5}$$

associated with (1.2). Note that, if we use the odd part $S_{\rm odd}=(S^+-S^-)/2$ of $S^{\pm}$, WKB solutions (1.1) can be expressed also as

$$\psi_{\pm}(z,\eta)=\frac{1}{\sqrt{S_{\rm odd}}}\exp\left(\pm\int_{z_0}^{z}S_{\rm odd}dz\right) \tag{1.6}$$

since

$$S_{\text{even}} = S^{+} - S_{\text{odd}} = -\frac{1}{2}\frac{d}{dz}\log S_{\text{odd}} \tag{1.7}$$

holds. (In what follows we mainly use the form (1.6) to express WKB solutions.) Sato's conjecture is then explicitly described in the following way:

Theorem 1.1 (Sato's conjecture). *The following relation* (*as formal power series in* η^{-1}) *holds for the Weber equation* (1.2).

$$\int_{2\sqrt{\lambda}}^{\infty}(S_{\text{odd}} - \eta S_{-1})dz = \frac{1}{2}\sum_{n=1}^{\infty}\frac{2^{1-2n}-1}{2n(2n-1)}B_{2n}(\eta\lambda)^{1-2n}, \tag{1.8}$$

where B_{2n} *designates the* $(2n)$*-th Bernoulli number, i.e.,*

$$\frac{w}{e^{w}-1} = 1 - \frac{w}{2} + \sum_{n=1}^{\infty}\frac{B_{2n}}{(2n)!}w^{2n}. \tag{1.9}$$

Remark 1. The original version of Sato's conjecture was described as a relation between the parabolic cylinder function $D_{\eta\lambda-1/2}(\eta^{1/2}z)$, a special solution of (1.2), and a WKB solution (1.6) with z_0 being chosen to be a turning point $z_0 = 2\sqrt{\lambda}$ ([8, p.95]). Recently Shen and Silverstone ([11]) have elucidated its WKB-theoretic meaning and reformulated it in its WKB-theoretic form (1.8). See also [3, Section 3]. Note that after [12] the left-hand side of (1.8) is often called "Voros' coefficient" in exact WKB analysis. Throughout this paper we use Sato's conjecture in the form (1.8).

A clear-cut proof of Theorem 1.1, which is based on the use of some analytic properties of the parabolic cylinder function, is given by Shen and Silverstone ([11]). An equivalent formula was also derived by Voros ([12]) through the asymptotic expansion of the Jost function (i.e., the quantization condition) of (1.2). In what follows we give another proof of (1.8); it is more straightforward in the sense that it directly verifies (1.8) as a relation of formal power series in η^{-1} without resorting to any analytic object corresponding to the left-hand side of (1.8).

Proof of Theorem 1.1.

Let σ denote $\eta\lambda$. A key for the proof of Theorem 1.1 is to consider the following difference equation with respect to σ.

$$F(\sigma+1) - F(\sigma) = 1 + \log(1 + \frac{1}{2\sigma}) - (\sigma+1)\log(1+\frac{1}{\sigma}). \tag{1.10}$$

This equation (1.10) and the Bernoulli number are related in the following manner:

Lemma 1.2. (i) *Let* $F_0(\sigma)$ *denote*

$$F_0(\sigma) = \sum_{n=1}^{\infty} \frac{2^{1-2n}-1}{2n(2n-1)} B_{2n}\sigma^{1-2n}, \tag{1.11}$$

that is, two times the right-hand side of (1.8). *Then* $F_0(\sigma)$ *formally satisfies* (1.10).
(ii) *Conversely, if* $F(\sigma) = \sum_{n\geq 1} c_n\sigma^n$ *is a formal solution of* (1.10), *then* $F(\sigma)$ *must coincide with* $F_0(\sigma)$.

Proof. (i) Combining the asymptotic expansion for $\log\Gamma(z)$

$$\log\Gamma(\sigma) - (\sigma - \frac{1}{2})\log\sigma + \sigma - \log\sqrt{2\pi} \sim \sum_{n=1}^{\infty} \frac{B_{2n}}{2n(2n-1)}\sigma^{1-2n} \quad (|\arg\sigma| < \pi) \tag{1.12}$$

([7, Section 1.18, (1)]) with the duplication formula

$$\Gamma(2\sigma) = 2^{2\sigma-1}\pi^{-1/2}\Gamma(\sigma)\Gamma(\sigma+1/2) \tag{1.13}$$

([7, Section 1.2, (15)]), we find

$$\log\frac{\Gamma(\sigma+1/2)}{\sqrt{2\pi}} - \sigma(\log\sigma - 1) \sim F_0(\sigma) \quad (|\arg\sigma| < \pi). \tag{1.14}$$

Replacing σ by $\sigma+1$, we also have

$$\log\frac{\Gamma(\sigma+3/2)}{\sqrt{2\pi}} - (\sigma+1)(\log(\sigma+1) - 1) \sim F_0(\sigma+1) \quad (|\arg\sigma| < \pi). \tag{1.15}$$

Taking the difference of both sides of (1.14) and (1.15), we thus obtain

$$\begin{aligned} &\log\frac{\Gamma(\sigma+3/2)}{\Gamma(\sigma+1/2)} - (\sigma+1)(\log(\sigma+1)-1) + \sigma(\log\sigma - 1) \\ &= 1 + \log(1+\frac{1}{2\sigma}) - (\sigma+1)\log(1+\frac{1}{\sigma}) \\ &\sim F_0(\sigma+1) - F_0(\sigma). \end{aligned} \tag{1.16}$$

This means that $F_0(\sigma)$ formally satisfies (1.10).
(ii) By a simple computation we have

$$(\sigma+1)^{-n} = \sigma^{-n}(1+\frac{1}{\sigma})^{-n} = \sum_{m\geq 0}(-1)^m\frac{(n+m-1)!}{(n-1)!m!}\sigma^{-n-m} \tag{1.17}$$

for $n = 1, 2, \ldots$. Hence

$$\begin{aligned} F(\sigma+1) - F(\sigma) &= \sum_{n\geq 1}\sum_{m\geq 1} c_n(-1)^m\frac{(n+m-1)!}{(n-1)!m!}\sigma^{-n-m} \\ &= \sum_{k\geq 2}\left(\sum_{n=1}^{k-1}\frac{c_n(-1)^{k-n}}{(n-1)!(k-n)!}\right)(k-1)!\sigma^{-k}. \end{aligned} \tag{1.18}$$

Thus, once $F(\sigma+1)-F(\sigma)$ is given (by the right-hand side of (1.10) in our current situation), all the coefficients $\{c_n\}_{n\geq 1}$ of $F(\sigma)$ are uniquely determined in a recursive manner. This completes the proof of Lemma 1.2. □

In view of Lemma 1.2 it now suffices to show that two times the left-hand side of (1.8) satisfies the difference equation (1.10) to prove Theorem 1.1. To confirm this we make use of the creation operator

$$\mathcal{A}=\eta^{-1}\frac{d}{dz}-\frac{z}{2} \tag{1.19}$$

for the harmonic oscillator (1.2). As a matter of fact, using (1.19), we can prove the following

Lemma 1.3. *Let $S^+=S^+(z,\lambda,\eta)$ be the WKB solution* (1.3) (*starting with $S_{-1}=\sqrt{z^2/4-\lambda}$*) *of the Riccati equation* (1.5) *associated with the Weber equation* (1.2)*. Then the following relation holds:*

$$S^+(z,\lambda+\eta^{-1},\eta)-S^+(z,\lambda,\eta)=\frac{d}{dz}\log\left(\eta^{-1}S^+(z,\lambda,\eta)-\frac{z}{2}\right). \tag{1.20}$$

Proof. It follows from the commutation relation

$$\left(\frac{d^2}{dz^2}-\eta^2\frac{z^2}{4}+\eta\right)\mathcal{A}=\mathcal{A}\left(\frac{d^2}{dz^2}-\eta^2\frac{z^2}{4}\right) \tag{1.21}$$

that, if ψ is a solution of (1.2), then $\varphi=\mathcal{A}\psi$ satisfies

$$\left(\frac{d^2}{dz^2}-\eta^2(\frac{z^2}{4}-\lambda-\eta^{-1})\right)\varphi=0. \tag{1.22}$$

In particular, for a WKB solution $\psi_+=\exp(\int^z S^+(z,\lambda,\eta)dz)$ of (1.2),

$$\varphi_+=\mathcal{A}\psi_+=\left(\eta^{-1}S^+(z,\lambda,\eta)-\frac{z}{2}\right)\exp\left(\int^z S^+(z,\lambda,\eta)dz\right) \tag{1.23}$$

becomes a WKB solution of (1.22), that is,

$$\left(\eta^{-1}S^+(z,\lambda,\eta)-\frac{z}{2}\right)\exp\left(\int^z S^+(z,\lambda,\eta)dz\right)=C(\eta)\exp\left(\int^z S^+(z,\lambda+\eta^{-1},\eta)dz\right) \tag{1.24}$$

holds for some constant $C(\eta)$ independent of z. Taking the logarithmic derivative of both sides of (1.24) with respect to z, we then obtain (1.20). □

Using Lemma 1.3, we now finish the proof of Theorem 1.1. Thanks to the square-root character of the coefficients of S_{odd} at $z=2\sqrt{\lambda}$, we can write two times the left-hand side of (1.8) as

$$2\int_{2\sqrt{\lambda}}^{\infty}(S_{\mathrm{odd}}-\eta S_{-1})dz=\int_{\gamma_\infty}(S_{\mathrm{odd}}-\eta S_{-1})dz, \tag{1.25}$$

where γ_∞ is a path that runs from ∞ to $z = 2\sqrt{\lambda} + \varepsilon$ for a sufficiently small number $\varepsilon > 0$, encircles $z = 2\sqrt{\lambda}$ along $\{|z - 2\sqrt{\lambda}| = \varepsilon\}$ in a clockwise manner, and then returns from $z = 2\sqrt{\lambda} + \varepsilon$ to ∞ (cf. Figure 1). Furthermore, since each coefficient of $S_{\mathrm{even}} = S^+ - S_{\mathrm{odd}}$ is single-valued at $z = 2\sqrt{\lambda}$ and

$$\operatorname*{Res}_{z=2\sqrt{\lambda}} S_{\mathrm{even}} = \operatorname*{Res}_{z=2\sqrt{\lambda}} S_0 = -\frac{1}{4} \tag{1.26}$$

holds in view of (1.7) and (1.4), we also find

$$\int_{\gamma_\infty} (S_{\mathrm{odd}} - \eta S_{-1}) dz = \int_{\gamma_\infty} (S^+ - \eta S_{-1} - S_0) dz. \tag{1.27}$$

Thus it suffices to show that the right-hand side of (1.27) satisfies the difference equation (1.10).

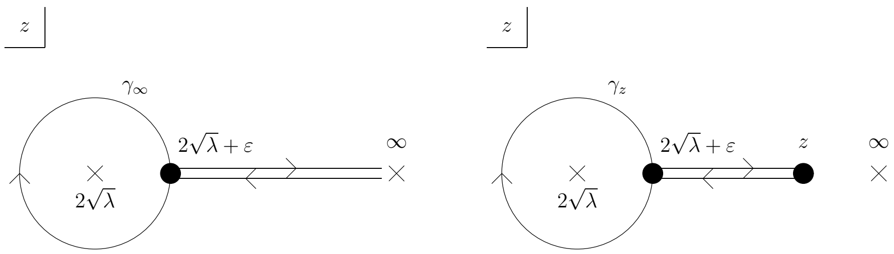

Figure 1. Integration paths γ_∞ and γ_z.

Let γ_z be a path that runs from z to $2\sqrt{\lambda} + \varepsilon$, encircles $2\sqrt{\lambda}$ in a clockwise manner and returns from $2\sqrt{\lambda} + \varepsilon$ to z (cf. Figure 1), and let $I(z, \sigma)$ and $I_j(z, \sigma)$ denote

$$I(z, \sigma) = \int_{\gamma_z} S^+ dz \Big|_{\lambda = \eta^{-1}(\sigma+1)} - \int_{\gamma_z} S^+ dz \Big|_{\lambda = \eta^{-1}\sigma}, \tag{1.28}$$

$$I_j(z, \sigma) = \int_{\gamma_z} S_j dz \Big|_{\lambda = \eta^{-1}(\sigma+1)} - \int_{\gamma_z} S_j dz \Big|_{\lambda = \eta^{-1}\sigma}, \tag{1.29}$$

respectively. It then follows from Lemma 1.3 that

$$I(z, \sigma) = \log\left(\eta^{-1} S^+(z, \eta^{-1}\sigma, \eta) - \frac{z}{2}\right) - \log\left(\eta^{-1} S^+(\hat{z}, \eta^{-1}\sigma, \eta) - \frac{z}{2}\right). \tag{1.30}$$

(Note that the branch of $S^+(z, \lambda, \eta)$ at the starting point of γ_z is different from the branch at its end point. To distinguish these two different branches, we use the notation $\hat{z}$ in (1.30) to specify the branch of S^+ at the starting point of γ_z.) Using (1.4) and

$$S_j = O(\frac{1}{z^3}) \quad \text{as } z \to \infty \text{ for } j \geq 1, \tag{1.31}$$

we thus find that

$$I(z, \sigma) = \log \frac{\eta^{-1}\sigma}{z^2} + \log(1 + \frac{1}{2\sigma}) + O(\frac{1}{z^2}) \quad \text{as } z \to \infty. \tag{1.32}$$

On the other hand, since

$$(1.33) \quad \int^z S_{-1}dz = \frac{z}{4}\sqrt{z^2-4\lambda} - \lambda\log(z+\sqrt{z^2-4\lambda}), \quad \int^z S_0 dz = -\frac{1}{4}\log(z^2-4\lambda),$$

we can confirm

$$(1.34) \qquad \eta I_{-1}(z,\sigma) = -1 + \log\frac{\eta^{-1}\sigma}{z^2} + (\sigma+1)\log(1+\frac{1}{\sigma}) + O(\frac{1}{z^2}) \quad (\text{as } z\to\infty),$$

$$(1.35) \qquad I_0(z,\sigma) = 0$$

by straightforward computations. Hence we obtain

$$(1.36) \quad \lim_{z\to\infty}(I(z,\sigma) - \eta I_{-1}(z,\sigma) - I_0(z,\sigma)) = 1 + \log(1+\frac{1}{2\sigma}) - (\sigma+1)\log(1+\frac{1}{\sigma}).$$

Relation (1.36) means that the right-hand side of (1.27) satisfies the difference equation (1.10). This completes the proof of Theorem 1.1. □

§ 2. Fixed singularities of Borel transformed WKB solutions of the Weber equation

In this section we discuss analytic implications of Sato's conjecture.

In what follows, rotating the variables as

$$(2.1) \qquad z = \exp(\pi i/4)x, \quad \lambda = \exp(\pi i/2)E_0$$

(where we adopt to use E_0 instead of E to denote the new parameter $\exp(-\pi i/2)\lambda$ in order that it may be consistent with the notation in the subsequent section), we deal with the Schrödinger equation with the inverted-parabola potential

$$(2.2) \qquad \left(\frac{d^2}{dx^2} - \eta^2 Q(x)\right)\psi = 0 \quad \text{with} \quad Q(x) = E_0 - \frac{x^2}{4},$$

which is equivalent to (1.2) via (2.1), and its WKB solutions normalized at a simple turning point $x = 2\sqrt{E_0}$

(2.3)

$$\begin{aligned}\psi_\pm(x,\eta) &= \frac{1}{\sqrt{S_{\text{odd}}}}\exp\left(\pm\int_{2\sqrt{E_0}}^x S_{\text{odd}}dx\right)\\ &= \exp(\eta y_\pm(x))\sum_{n=0}^\infty \psi_{\pm,n}(x)\eta^{-(n+1/2)} \quad \text{where} \quad y_\pm(x) = \pm\int_{2\sqrt{E_0}}^x \sqrt{Q(x)}dx.\end{aligned}$$

Here and in what follows the branch of $S_{-1}(x) = \sqrt{Q(x)} = \sqrt{E_0 - x^2/4}$ is chosen so that $\exp(-\pi i/2)\sqrt{E_0 - x^2/4} > 0$ holds for $E_0 > 0$, $x > 2\sqrt{E_0}$. In exact WKB analysis we endow $\psi_\pm(x,\eta)$ with an analytic meaning through Borel resummation method (with respect to the large parameter η), that is, we first define the Borel transform of $\psi_\pm(x,\eta)$, denoted by $\psi_{\pm,B}(x,y)$, by

$$\psi_{\pm,B}(x,y)=\sum_{n=0}^{\infty}\frac{\psi_{\pm,n}(x)}{\Gamma(n+1/2)}(y+y_\pm(x))^{n-1/2}, \tag{2.4}$$

and then consider its Borel sum

$$\int_{-y_\pm(x)}^{\infty}\exp(-y\eta)\psi_{\pm,B}(x,y)dy \tag{2.5}$$

as an analytic substitute of $\psi_\pm$. Here the path of integration for (2.5) is conventionally taken to be parallel to the positive real axis.

The Borel sum of a WKB solution is well-defined in a Stokes region, i.e., a region surrounded by Stokes curves

$$\operatorname{Im}\int_a^x S_{-1}(x)dx=\operatorname{Im}\int_a^x\sqrt{Q(x)}dx=0 \tag{2.6}$$

(where a is a turning point $\pm 2\sqrt{E_0}$ of (2.2)), provided that there is no Stokes curve connecting two turning points. Note that the relations between WKB solutions in adjacent two Stokes regions are described by Voros' connection formula and that Voros' connection formula takes the simplest form when we choose $\psi_\pm(x,\eta)$ normalized as (2.3) as a basis of WKB solutions (cf. [12]). In the case of (2.2) two turning points $\pm 2\sqrt{E_0}$ are connected by a Stokes curve when and only when $E_0\in\mathbb{R}$ (cf. Figure 2). Such a

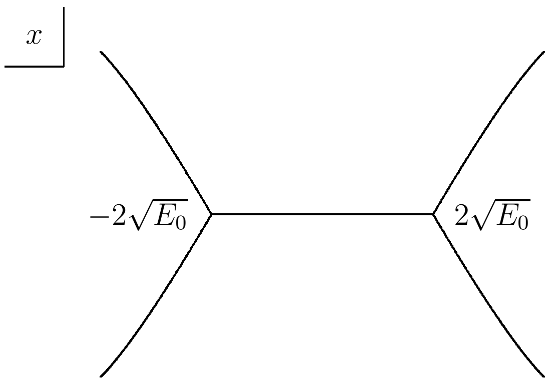

Figure 2. Stokes curves of (2.2) for $E_0>0$.

degenerate configuration of Stokes curves then causes a kind of Stokes phenomenon to occur with the Borel resummed WKB solutions $\psi_\pm$ normalized as (2.3) and it can be explicitly analyzed by using Sato's conjecture in the following manner: The degenerate

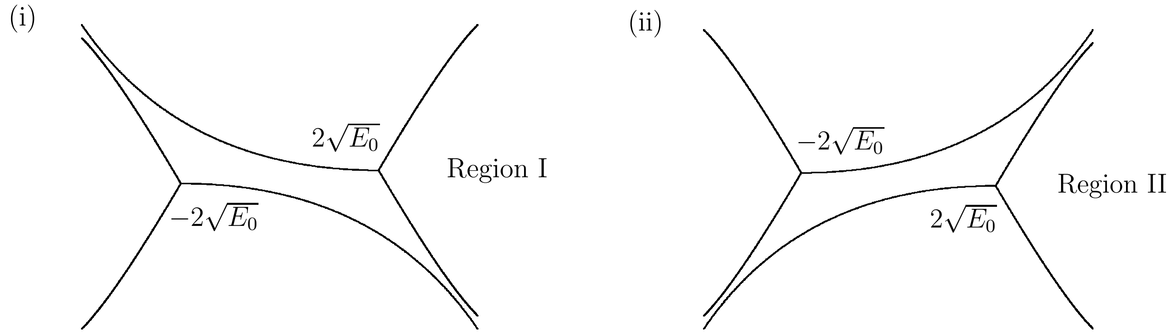

Figure 3. Stokes curves of (2.2) when (i) Im $E_0 > 0$, and (ii) Im $E_0 < 0$.

configuration observed for, say, $E_0 > 0$ is resolved in two different ways by adding a small imaginary part to E_0, as is indicated in Figure 3. Let ψ_+^{I} denote the Borel sum of the WKB solution ψ_+ defined by (2.3) in, say, Region I of Figure 3, (i) (i.e., when Re $E_0 > 0$ and Im $E_0 > 0$) and $\psi_\pm^{\mathrm{II}}$ the Borel sum of the same WKB solution in the corresponding Region II of Figure 3, (ii) (i.e., when Re $E_0 > 0$ and Im $E_0 < 0$). Then these two Borel sums define different analytic functions. As a matter of fact, Sato's conjecture (Theorem 1.1) analytically implies the following

Theorem 2.1. *Between the Borel sums ψ_+^{I} and ψ_+^{II} of the WKB solution ψ_+ defined by* (2.3) *the following relation holds:*

$$\psi_+^{\mathrm{I}} = (1 + \exp(-2\pi E_0 \eta))^{1/2}\, \psi_+^{\mathrm{II}}. \tag{2.7}$$

Formula (2.7) describes the Stokes phenomenon for the WKB solution ψ_+ of (2.2) when the parameter E_0 crosses the positive real axis. Although an equivalent formula is already discussed in [11] (cf. [11, Formula (50)]; note that $\hbar \pm i0$ in [11] correspond to Im $E_0 \to \mp 0$ in this article, respectively) and the essential part of its proof was given by [8, Proposition 2.2], we present the proof of Theorem 2.1 here for the reader's convenience.

Proof. We factorize $\psi_+(x, \eta)$ as

$$\psi_+(x, \eta) = \exp\left(\int_{2\sqrt{E_0}}^{\infty} (S_{\mathrm{odd}} - \eta S_{-1}) dx \right) \psi_+^{(\infty)}(x, \eta). \tag{2.8}$$

Here, thanks to Sato's conjecture (Theorem 1.1), the first factor of the right-hand side can be written as $\exp \phi(E_0, \eta)$ with

$$\phi(E_0, \eta) = \frac{1}{2} \sum_{n=1}^{\infty} \frac{2^{1-2n} - 1}{2n(2n-1)} B_{2n} (iE_0\eta)^{1-2n}, \tag{2.9}$$

i.e., the right-hand side of (1.8) (after the rotation of variables (2.1) is substituted), and

$$\psi_{\pm}^{(\infty)}(x,\eta)=\frac{1}{\sqrt{S_{\rm odd}}}\exp\left(\pm\left[\eta\int_{2\sqrt{E_0}}^{x}S_{-1}dx+\int_{\infty}^{x}(S_{\rm odd}-\eta S_{-1})dx\right]\right) \tag{2.10}$$

is a WKB solution of (2.2) that is normalized at infinity in the sense of [5] and [6]. Let us define two Borel sums $\psi_{+}^{(\infty),\mathrm{I}}$ and $\psi_{+}^{(\infty),\mathrm{II}}$ of $\psi_{+}^{(\infty)}$ similarly to ψ_{+}^{I} and ψ_{+}^{II}. It is then known that $\psi_{+}^{(\infty),\mathrm{I}}$ and $\psi_{+}^{(\infty),\mathrm{II}}$ coincide since $\psi_{+,B}^{(\infty)}(x,y)$, the Borel transform of $\psi_{+}^{(\infty)}(x,\eta)$, is free from singularities on the half line

$$\{y\in\mathbb{C}\,;\,y=-\int_{2\sqrt{E_0}}^{x}\sqrt{E_0-\frac{x^2}{4}}dx+\rho,\,\rho>0\} \tag{2.11}$$

([5], [6, Theorem 1.2.2 (c)]). Hence, to verify (2.7), it suffices to compare the Borel sums ϕ^{I} (i.e., the Borel sum of ϕ for Im $E_0>0$) and ϕ^{II} (i.e., that for Im $E_0<0$).

Let us now compute the Borel transform $\phi_B(E_0,y)$ of ϕ. It follows from the definition of the Borel transformation and (2.9) that

$$\begin{aligned}\phi_B(E_0,y)&=\frac{1}{2}\sum_{n=1}^{\infty}\frac{2^{1-2n}-1}{2n(2n-1)}B_{2n}(iE_0)^{1-2n}\frac{y^{2n-2}}{(2n-2)!}\\&=\frac{iE_0}{y^2}\sum_{n=1}^{\infty}\frac{B_{2n}}{(2n)!}(2iE_0)^{-2n}y^{2n}-\frac{iE_0}{2y^2}\sum_{n=1}^{\infty}\frac{B_{2n}}{(2n)!}(iE_0)^{-2n}y^{2n}\\&=\frac{iE_0}{y^2}\left(\frac{y/(2iE_0)}{\exp(y/(2iE_0))-1}-1+\frac{y}{4iE_0}\right)\\&\quad-\frac{iE_0}{2y^2}\left(\frac{y/(iE_0)}{\exp(y/(iE_0))-1}-1+\frac{y}{2iE_0}\right)\\&=\frac{1}{4y}\left(\frac{1}{\exp(y/(2iE_0))-1}+\frac{1}{\exp(y/(2iE_0))+1}-\frac{2iE_0}{y}\right).\end{aligned} \tag{2.12}$$

To simplify the computation we introduce the following auxiliary infinite series

$$\tilde{\phi}=\phi+\frac{1}{4}-\frac{iE_0\eta}{2}\log(1+\frac{1}{2iE_0\eta})=\phi-\frac{1}{4}\sum_{n=0}^{\infty}\frac{1}{n+2}(-2iE_0\eta)^{-(n+1)}. \tag{2.13}$$

Then (2.12) implies

$$\begin{aligned}\tilde{\phi}_B=&\phi_B+\frac{1}{8iE_0}\sum_{n=0}^{\infty}\frac{1}{(n+2)n!}\left(-\frac{y}{2iE_0}\right)^n\\=&\phi_B+\frac{1}{8iE_0}\left(-\frac{2iE_0}{y}\right)^2\left[\left(-\frac{y}{2iE_0}-1\right)\exp\left(-\frac{y}{2iE_0}\right)+1\right]\\=&\frac{1}{4y}\left[\frac{1}{\exp(y/(2iE_0))-1}+\frac{1}{\exp(y/(2iE_0))+1}-\frac{2iE_0}{y}\right]\end{aligned} \tag{2.14}$$

$$
\begin{aligned}
&-\frac{1}{4y}\left[\left(1+\frac{2iE_0}{y}\right)\exp\left(-\frac{y}{2iE_0}\right)-\frac{2iE_0}{y}\right]\\
=&\frac{1}{2y}\exp\left(-\frac{y}{2iE_0}\right)\left[\frac{1}{\exp(y/(iE_0))-1}+\frac{1}{2}-\frac{iE_0}{y}\right].
\end{aligned}
$$

Hence, making a change of variable $y/(iE_0)=t$ of integration and using an integral representation of the logarithm of the Γ-function

$$\int_0^\infty\left(\frac{1}{e^t-1}+\frac{1}{2}-\frac{1}{t}\right)e^{-t\theta}\frac{dt}{t}=\log\frac{\Gamma(\theta)}{\sqrt{2\pi}}-(\theta-\frac{1}{2})\log\theta+\theta\quad(\text{for Re }\theta>0)\tag{2.15}$$

([7, Section 1.9, (5)]), we find that the Borel sum of $\tilde{\phi}$ is given by

$$\frac{1}{2}\left(\log\frac{\Gamma(iE_0\eta+1/2)}{\sqrt{2\pi}}-iE_0\eta\log(iE_0\eta+1/2)+iE_0\eta+\frac{1}{2}\right)\tag{2.16}$$

when Im $E_0<0$. We thus obtain

$$\phi^{\mathrm{II}}=\frac{1}{2}\log\frac{\Gamma(iE_0\eta+1/2)}{\sqrt{2\pi}}-\frac{iE_0\eta}{2}(\log(iE_0\eta)-1).\tag{2.17}$$

On the other hand, when Im $E_0>0$, making use of the relation

$$\phi(E_0,\eta)=-\phi(-E_0,\eta)\tag{2.18}$$

and employing the above reasoning for $\phi(-E_0,\eta)$, we find

$$
\begin{aligned}
\phi^{\mathrm{I}}&=-\left[\frac{1}{2}\log\frac{\Gamma(-iE_0\eta+1/2)}{\sqrt{2\pi}}+\frac{iE_0\eta}{2}(\log(-iE_0\eta)-1)\right]\\
&=-\frac{1}{2}\log\frac{\Gamma(-iE_0\eta+1/2)}{\sqrt{2\pi}}-\frac{iE_0\eta}{2}(\log(iE_0\eta)-1)-\frac{\pi E_0\eta}{2}.
\end{aligned}\tag{2.19}
$$

Comparison of (2.17) and (2.19) entails

$$
\begin{aligned}
\phi^{\mathrm{II}}-\phi^{\mathrm{I}}&=\frac{1}{2}\log\frac{\Gamma(iE_0\eta+1/2)\Gamma(-iE_0\eta+1/2)}{2\pi}+\frac{\pi E_0\eta}{2}\\
&=-\frac{1}{2}\log\left(2\cos(i\pi E_0\eta)\right)+\frac{\pi E_0\eta}{2}\\
&=-\frac{1}{2}\log\left(1+\exp(-2\pi E_0\eta)\right).
\end{aligned}\tag{2.20}
$$

Hence we conclude

$$
\begin{aligned}
\psi_+^{\mathrm{I}}&=(\exp\phi^{\mathrm{I}})\psi_+^{(\infty),\mathrm{I}}\\
&=\left(1+\exp(-2\pi E_0\eta)\right)^{1/2}(\exp\phi^{\mathrm{II}})\psi_+^{(\infty),\mathrm{II}}\\
&=\left(1+\exp(-2\pi E_0\eta)\right)^{1/2}\psi_+^{\mathrm{II}}.
\end{aligned}\tag{2.21}
$$

This completes the proof of Theorem 2.1. □

In view of (2.12) we find that $\phi_B(E_0, y)$ is holomorphic at $y = 0$ whereas it has simple poles at $y = 2m\pi E_0$ for every non-zero integer $m = \pm 1, \pm 2, \ldots$. This implies that the Borel transform $\psi_{+,B}(x, y)$ of the WKB solution (2.3) has singularities at

$$y = -y_{\pm}(x) + 2m\pi E_0 \quad \text{where} \quad y_{\pm}(x) = \pm \int_{2\sqrt{E_0}}^{x} \sqrt{E_0 - \frac{x^2}{4}}\, dx, \quad m \in \mathbb{Z} \tag{2.22}$$

(cf. Figures 4 and 5). Among them $y = -y_+(x) + 2m\pi E_0$ ($m \in \mathbb{Z}$) are called "fixed

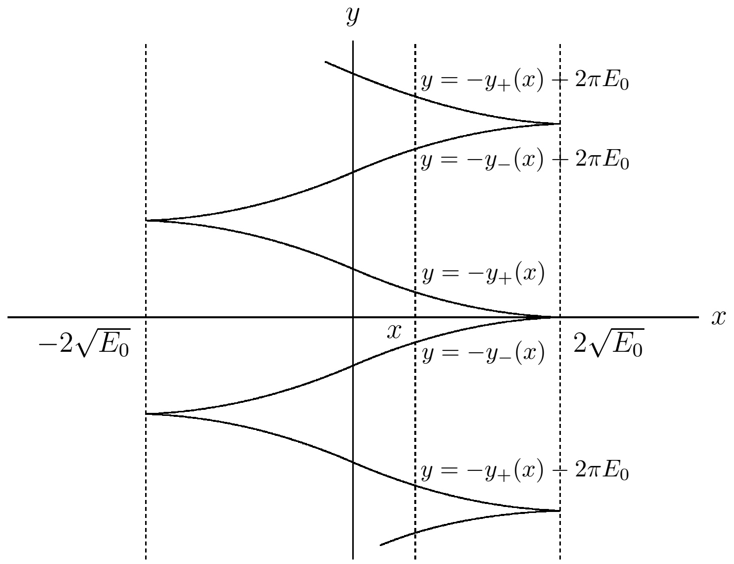

Figure 4. Singularity locus of $\psi_{+,B}(x, y)$ in $\mathbb{C}_x \times \mathbb{C}_y$.

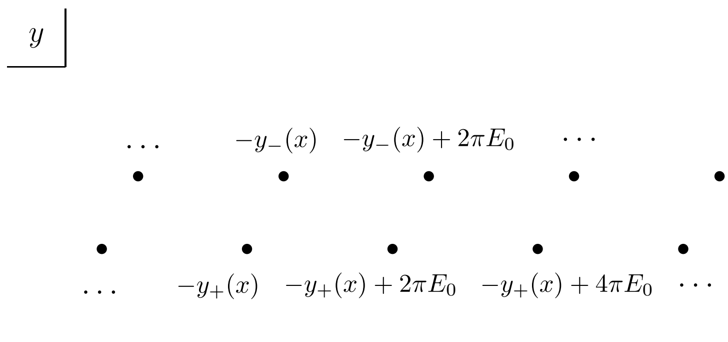

Figure 5. Singular points of $\psi_{+,B}(x, y)$ in y-plane for fixed $x \neq \pm 2\sqrt{E_0}$.

singularities" of the WKB solution $\psi_+(x, \eta)$ (or $\psi_{+,B}(x, y)$), since their relative location with respect to the reference point $y = -y_+(x)$ is not changed as x varies. These fixed singularities lie on the path of integration for the Borel sum of $\psi_+(x,\eta)$ when $E_0 \in \mathbb{R}$; this is the origin of the Stokes phenomenon (2.7) for $E_0 \in \mathbb{R}$.

The comparison, such as (2.7), between two Borel sums of WKB solutions of Schrödinger equations with polynomial potentials is investigated in [5] and [6] under the name of "Stokes automorphism" or "connection automorphism" from the viewpoint of the theory of Ecalle's resurgent functions. In discussing formulas like (2.7) in the framework of resurgent functions theory, the fixed singularities of Borel transformed WKB solutions and the alien derivatives there play a crucially important role. The principal aim of [3] and this paper is to verify (2.7) for WKB solutions of an MTP equation (a merging-turning-points equation) by using Theorem 2.1 above and the transformation theory to the Weber equation, as will be explained in Section 3 below. In what follows, as preliminaries for Section 3, we reformulate Theorem 2.1 in terms of the Borel transform $\psi_{+,B}(x,y)$, that is, in the language of the fixed singularities of $\psi_{+,B}(x,y)$ and the alien derivatives there.

Here we briefly review the definition of the alien derivative. Recall that the fixed singularities $y = -y_+(x) + 2m\pi E_0$ ($m \in \mathbb{Z}$) of $\psi_{+,B}(x,y)$ lie on $\{y \in \mathbb{C}\,;\, y = -y_+(x) + \rho,\ \rho \in \mathbb{R}\}$ when $E_0 > 0$. Under this situation the alien derivative $\Delta\psi_+$ of $\psi_+(x,\eta)$ is defined by

$$\begin{aligned} \Delta\psi_+ &= \mathcal{B}^{-1}\log(\mathcal{L}_-^{-1}\mathcal{L}_+)\mathcal{B}\psi_+ \\ &= \mathcal{B}^{-1}\log(1 + (\mathcal{L}_-^{-1}\mathcal{L}_+ - 1))\mathcal{B}\psi_+ \\ &= \mathcal{B}^{-1}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}(\mathcal{L}_-^{-1}\mathcal{L}_+ - 1)^n\mathcal{B}\psi_+, \end{aligned} \tag{2.23}$$

where $\mathcal{B}$ denotes the Borel transformation (2.4) and $\mathcal{L}_+$ (resp., $\mathcal{L}_-$) denotes the Laplace transformation (2.5) along a path which avoids the singular points from the above (resp., from the below). It is also known (cf., e.g., [6]) that the alien derivative (2.23) is decomposed as

$$\Delta\psi_+ = \sum_{m=1}^{\infty}\Delta_{y=-y_+(x)+2m\pi E_0}\,\psi_+ \tag{2.24}$$

with

$$\Delta_{y=-y_+(x)+2m\pi E_0}\,\psi_+ = \mathcal{B}^{-1}\left[(\gamma_+^{(m)} - \gamma_-^{(m)})\sum_{\varepsilon_k=\pm}\frac{p_+!p_-!}{m!}\gamma_{\varepsilon_{m-1}}^{(m-1)}\cdots\gamma_{\varepsilon_1}^{(1)}\right]\mathcal{B}\psi_+, \tag{2.25}$$

where $\gamma_+^{(k)}$ (resp., $\gamma_-^{(k)}$) designates analytic continuation along a path avoiding the k-th singular point $y = -y_+(x) + 2k\pi E_0$ from the above (resp., from the below) and p_+ (resp., p_-) denotes the number of indices k satisfying $1 \le k \le m-1$ and $\varepsilon_k = +$ (resp.,

$\varepsilon_k = -$). Note that, in terms of $\mathcal{L}_\pm$ and $\mathcal{B}$, formula (2.7) is expressed as

$$\mathcal{B}^{-1}\mathcal{L}_+^{-1}\mathcal{L}_-\mathcal{B}\psi_+ = (1+\exp(-2\pi E_0\eta))^{1/2}\psi_+. \tag{2.26}$$

Thus the verification of formula (2.7) can be achieved also through the computation of the alien derivative $\Delta\psi_+$.

The computation of $\Delta\psi_+$ is done in [3, Section 3] in the following manner: Since the Borel transform $\phi_B(E_0, y)$ of $\phi(E_0,\eta)$ is a single-valued analytic function with simple poles at $y = 2m\pi E_0$ ($m \neq 0$) and its residue there is equal to $(-1)^{m-1}/(4\pi i m)$ in view of (2.12), we find

$$\Delta_{y=2m\pi E_0}\,\phi = \frac{(-1)^m}{2m}. \tag{2.27}$$

(Cf. [4], [6], [10].) Then the alien calculus leads to

$$\Delta_{y=2m\pi E_0}(\exp\phi) = \frac{(-1)^m}{2m}\exp\phi. \tag{2.28}$$

On the other hand, as we have observed above, the Borel transform of $\psi_+^{(\infty)}(x,\eta)$ is free from singularities on (2.11). This implies that

$$\Delta\left(\exp(-y_+(x)\eta)\psi_+^{(\infty)}(x,\eta)\right) = 0 \tag{2.29}$$

holds when x is in Region I and Region II, i.e., the Stokes regions in question. Hence, combining (2.28) and (2.29), we obtain

$$\begin{aligned}
&\Delta_{y=2m\pi E_0}\Big(\exp(-y_+(x)\eta)\psi_+(x,\eta)\Big)\\
&\quad= \Delta_{y=2m\pi E_0}\left(\exp(-y_+(x)\eta)\exp(\phi(E_0,\eta))\psi_+^{(\infty)}(x,\eta)\right)\\
&\quad= \frac{(-1)^m}{2m}\left(\exp(-y_+(x)\eta)\exp(\phi(E_0,\eta))\psi_+^{(\infty)}(x,\eta)\right)\\
&\quad= \frac{(-1)^m}{2m}\Big(\exp(-y_+(x)\eta)\psi_+(x,\eta)\Big).
\end{aligned} \tag{2.30}$$

We have thus verified the following Theorem 2.2, which is equivalent to Theorem 2.1, on the singularity structure of $\psi_{+,B}(x,y)$ expressed in terms of its alien derivatives.

Theorem 2.2. *Let $\psi_+(x,\eta)$ denote the WKB solution of the Weber equation* (2.2) *that is normalized as in* (2.3)*. Then its Borel transform $\psi_{+,B}(x,y)$ is singular at*

$$y = -y_+(x) + 2m\pi E_0 \quad (m = 0, \pm1, \pm2, \ldots), \tag{2.31}$$

where $y_+(x)$ is given by (2.22)*, and its alien derivative $\Delta_{y=-y_+(x)+2m\pi E_0}\psi_+$ there satisfies the following relation* (2.32) *for x in Region I and Region II:*

$$(\Delta_{y=-y_+(x)+2m\pi E_0}\,\psi_+)_B(x,y) = \frac{(-1)^m}{2m}\psi_{+,B}(x, y-2m\pi E_0). \tag{2.32}$$

§ 3. Transformation theory for an MTP equation

The transformation theory to the Weber equation developed in [3] enables us to extend Theorems 2.1 and 2.2 to a wider class of Schrödinger equations, that is, MTP equations (Schrödinger equations with a merging pair of simple turning points). In this section we explain the core part of [3] and discuss an extension of Theorems 2.1 and 2.2 to an MTP equation.

Let us begin with recalling the definition of an MTP equation.

Definition 3.1. A Schrödinger equation of the form

$$\left(\frac{d^2}{dx^2} - \eta^2 Q(x,t)\right)\psi = 0 \quad (\eta > 0:\ \text{a large parameter}) \tag{3.1}$$

is called an MTP equation if the potential $Q(x,t)$ is holomorphic and has the following form on a sufficiently small neighborhood of the origin $(x,t) = (0,0)$:

$$Q(x,t) = Q^{(0)}(x) + tQ^{(1)}(x) + t^2 Q^{(2)}(x) + \cdots \tag{3.2}$$

with

$$Q^{(0)}(x) = cx^2 + O(x^3) \quad (c:\ \text{a non-zero constant}), \tag{3.3}$$

$$Q^{(1)}(0) \neq 0. \tag{3.4}$$

Under the conditions (3.3) and (3.4) we can confirm that the equation $Q(x,t) = 0$ in x has two distinct simple zeros $s_\pm(t)$ in a neighborhood of $x = 0$ for each t $(\neq 0)$, whereas the other zeros of the equation stay uniformly away from 0 for sufficiently small t. Furthermore, these two simple zeros (i.e., simple turning points) merge together at $t = 0$ with the merging speed

$$s_\pm(t) = O(\sqrt{t}), \quad |s_+(t) - s_-(t)| \geq \sigma_0\sqrt{t} \quad \text{for some positive constant } \sigma_0. \tag{3.5}$$

Thus it is reasonable to call equation (3.1) satisfying the conditions (3.3) and (3.4) "an equation with a merging pair of simple turning points".

Remark 2. In [3] we defined an MTP equation as an equation that has a merging pair $\{s_\pm(t)\}$ of simple turning points satisfying (3.5). As was discussed in [3, Proposition 2.1], the definition adopted in [3] is equivalent to the above Definition 3.1.

One of the main results of [3] is the following transformation theorem of an MTP equation to the "∞-Weber equation", i.e., the Weber equation containing an infinite series as its parameter:

Theorem 3.2 ([3, Theorem 2.2]). *Let $Q(x,t)$ be the potential of an MTP equation. Then we can find a positive constant δ, holomorphic functions $X_k(x,t)$ $(k \geq 0)$ of (x,t) on $\{(x,t); |x|, |t| < \delta\}$, holomorphic functions $E_k(t)$ $(k \geq 0)$ of t on $\{t; |t| < \delta\}$ such that the formal series*

$$X(x,t,\eta) = \sum_{k \geq 0} X_k(x,t)\eta^{-k}, \tag{3.6}$$

$$E(t,\eta) = \sum_{k \geq 0} E_k(t)\eta^{-k} \tag{3.7}$$

satisfy the following relations (3.8) $\sim$ (3.12)*:*

$$Q(x,t) = \left(\frac{\partial X(x,t,\eta)}{\partial x}\right)^2 \left(E(t,\eta) - \frac{X(x,t,\eta)^2}{4}\right) - \frac{\eta^{-2}}{2}\{X(x,t,\eta); x\}, \tag{3.8}$$

$$X_0(0,0) = 0, \quad \frac{\partial X_0}{\partial x}(0,0) \neq 0, \tag{3.9}$$

$$E_0(0) = 0, \quad \frac{\partial E_0}{\partial t}(0) \neq 0, \tag{3.10}$$

$$X_0(s_+(t),t) = 2\sqrt{E_0(t)}, \tag{3.11}$$

$$X_{2p+1}(x,t) = 0, \quad E_{2p+1}(t) = 0 \quad \textit{for } p = 0,1,2,\ldots, \tag{3.12}$$

where $\{X(x,t,\eta); x\}$ designates the Schwarzian derivative

$$\left(\frac{d^3X}{dx^3}\Big/\frac{dX}{dx}\right) - \frac{3}{2}\left(\frac{d^2X}{dx^2}\Big/\frac{dX}{dx}\right)^2. \tag{3.13}$$

Otherwise stated, an MTP equation (3.1) *can be transformed into the ∞-Weber equation*

$$\left(\frac{d^2}{dX^2} - \eta^2(E(t,\eta) - \frac{X^2}{4})\right)\Psi = 0 \tag{3.14}$$

by the formal transformation

$$X = X(x,t,\eta) \quad \textit{and} \quad \psi(x,t,\eta) = \left(\frac{\partial X(x,t,\eta)}{\partial x}\right)^{-1/2} \Psi(X(x,t,\eta),\eta; E(t,\eta)) \tag{3.15}$$

on a neighborhood of the origin $(x,t) = (0,0)$.

Remark 3. By taking a smaller δ if necessary, we can also verify the following estimates for $X_k(x,t)$ and $E_k(t)$: There exist positive constants M and C_0 so that

$$\sup_{|x|,|t| \leq \delta} |X_k(x,t)| \leq MC_0^k k! \tag{3.16}$$

$$\sup_{|t| \leq \delta} |E_k(t)| \leq MC_0^k k! \tag{3.17}$$

hold for $k = 0,1,2,\ldots$.

Remark 4. Let $S_{\rm odd}(x,t,\eta)$ denote the odd part of WKB solutions of the Riccati equation associated with (3.1). Then the infinite series $E(t,\eta)$ in Theorem 3.2 for $t \neq 0$ is given by the following contour integral of $S_{\rm odd}(x,t,\eta)$:

$$E(t,\eta) = \frac{1}{2\pi i}\oint_{\gamma(t)} S_{\rm odd}(x,t,\eta)dx, \tag{3.18}$$

where $\gamma(t)$ designates the closed curve in the cut plane shown in Figure 6.

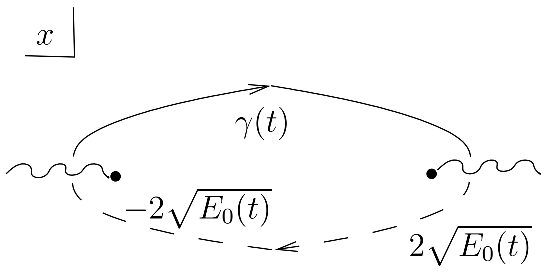

Figure 6. Closed curve $\gamma(t)$.

See Section 2 and Appendix B of [3] for the proof of Theorem 3.2, Remark 3 and Remark 4.

As is discussed in [3], the formal transformation (3.15) can be endowed with an analytic meaning by considering its action on the Borel transform of WKB solutions. Let $\psi_\pm(x,t,\eta)$ be a WKB solution of an MTP equation (3.1) for $t \neq 0$ that is normalized as follows:

$$\psi_\pm(x,t,\eta) = \frac{1}{\sqrt{S_{\rm odd}}}\exp\left(\pm\int_{s_+(t)}^{x} S_{\rm odd}dx\right). \tag{3.19}$$

Then, if $T_{\rm odd} = T_{\rm odd}(X,\eta)$ (or, if we use more specific notation, $T_{\rm odd}(X,\eta;E(t,\eta))$) denotes the odd part of WKB solutions of the Riccati equation associated with the ∞-Weber equation (3.14), $\psi_\pm(x,t,\eta)$ corresponds to a WKB solution

$$\Psi_\pm(X,\eta) = \Psi_\pm(X,\eta;E(t,\eta)) = \frac{1}{\sqrt{T_{\rm odd}}}\exp\left(\pm\int_{2\sqrt{E_0(t)}}^{X} T_{\rm odd}dX\right) \tag{3.20}$$

of (3.14) normalized at a simple turning point $X = 2\sqrt{E_0(t)}$ through the formal transformation (3.15), that is, $\psi_\pm$ and $\Psi_\pm$ are related by

$$\psi_\pm(x,t,\eta) = \left(\frac{\partial X(x,t,\eta)}{\partial x}\right)^{-1/2}\Psi_\pm(X(x,t,\eta),\eta). \tag{3.21}$$

Note that each factor of the right-hand side of (3.21) can be formally expressed as

$$\left(\frac{\partial X}{\partial x}\right)^{-1/2} = \left(\frac{\partial g}{\partial X}\right)^{1/2}\left(1+\frac{\partial r}{\partial X}\right)^{-1/2}\Bigg|_{X=X_0(x,t)} \tag{3.22}$$

and

$$\Psi_\pm(X(x,t,\eta),\eta) = \sum_{n=0}^{\infty}\frac{(r_1\eta^{-1}+r_2\eta^{-2}+\cdots)^n}{n!}\left(\frac{\partial}{\partial X}\right)^n \Psi_\pm\Bigg|_{X=X_0(x,t)}, \tag{3.23}$$

where $g(X,t)$ is the inverse function of $X = X_0(x,t)$, i.e., $X_0(g(X,t),t) = X$ near $(X,t) = (0,0)$ and $r = r(X,t,\eta)$ is defined by

$$r(X,t,\eta) = \sum_{k=1}^{\infty} r_k(X,t)\eta^{-k} \quad \text{with} \quad r_k(X,t) = X_k(g(X,t),t). \tag{3.24}$$

Hence the Borel transformation of (3.21) with respect to the large parameter η provides us with the following microdifferential relation

$$\psi_{\pm,B}(x,t,y)\Big|_{x=g(X,t)} = \mathcal{X}\left(X,t,\frac{\partial}{\partial X},\frac{\partial}{\partial y}\right)\Psi_{\pm,B}(X,y) \tag{3.25}$$

with $\mathcal{X} = \mathcal{X}(X,t,\partial/\partial X,\partial/\partial y)$ being a microdifferential operator in the sense of [9] defined by

$$\mathcal{X} = :\left(\frac{\partial g}{\partial X}(X,t)\right)^{1/2}\left(1+\frac{\partial r}{\partial X}\right)^{-1/2}\exp(r(X,t,\eta)\Xi) : \tag{3.26}$$

where the ideograph : : designates the normal ordered product (cf. [1]) and Ξ denotes the symbol of $\partial/\partial X$.

Remark 5. In this case the action of $\mathcal{X}$ upon the multi-valued analytic function $\Psi_{\pm,B}(X,t,y)$ can be represented as an integro-differential operator of the following form

$$\mathcal{X}\Psi_{\pm,B} = \int_{y_0}^{y} K\left(X,t,y-y',\frac{d}{dX}\right)\Psi_{\pm,B}(X,t,y')dy', \tag{3.27}$$

where $K(X,t,y,d/dX)$ is a differential operator of infinite order in X and y_0 is a constant that is chosen arbitrarily to fix the action of $(\partial/\partial y)^{-1}$ as an integral operator. For details see [3, Theorem 2.7 and Appendix C].

Furthermore, if we use X and Φ instead of x and ψ to express the independent variable and the unknown function, respectively, of the ordinary Weber equation (2.2) in this section, we also find that the Borel transform of the normalized WKB solution (3.20) of the ∞-Weber equation (3.14) and that of the WKB solution $\Phi_\pm(X,\eta)$ of the ordinary Weber equation (2.2) normalized as in (2.3) are related by another microdifferential operator $\mathcal{E}(E_0, \partial/\partial y, \partial/\partial E_0)$ (in the variables (y, E_0)) as

$$\Psi_{\pm,B}(X,y) = \mathcal{E}\left(E_0, \frac{\partial}{\partial y}, \frac{\partial}{\partial E_0}\right)\Phi_{\pm,B}(X,y), \tag{3.28}$$

where $\mathcal{E}$ is explicitly given by

$$\mathcal{E}\left(E_0, \frac{\partial}{\partial y}, \frac{\partial}{\partial E_0}\right) = \,:\sum_{n=0}^{\infty} \frac{(E_1\eta^{-1} + E_2\eta^{-2} + \cdots)^n \theta^n}{n!}: \tag{3.29}$$

$$= \sum_{n=0}^{\infty} \frac{(E_1(\partial/\partial y)^{-1} + E_2(\partial/\partial y)^{-2} + \cdots)^n}{n!}\left(\frac{\partial}{\partial E_0}\right)^n \tag{3.30}$$

(with θ denoting the symbol of $\partial/\partial E_0$; cf. [3, Section 4]). Thanks to the microdifferential relations (3.25) and (3.28) we find that the singularity structure of $\Phi_{+,B}$ is inherited to that of $\psi_{+,B}$. In particular, Theorem 2.2 entails the following Theorem 3.3, an extension of Theorem 2.2 to an MTP equation (3.1).

Theorem 3.3 ([3, Theorem 5.1]). *Let $\psi_{+,B}$ be the Borel transform of the WKB solution ψ_+ of an MTP equation* (3.1) *that is normalized as in* (3.19). *Then $\psi_{+,B}$ are singular at*

$$y = -y_+(x,t) + 2m\pi E_0(t) \quad (m = 0, \pm 1, \pm 2, \ldots) \tag{3.31}$$

in a sufficiently small neighborhood of the origin $(x,y,t) = (0,0,0)$, where

$$y_+(x,t) = \int_{s_+(t)}^{x} \sqrt{Q(x,t)}dx. \tag{3.32}$$

Furthermore, its alien derivative there satisfies the following relation (3.33) *for sufficiently small $t(\neq 0)$.*

(3.33)

$$\begin{aligned}&(\Delta_{y=-y_+(x,t)+2m\pi E_0(t)}\psi_+)_B(x,t,y)\\ &= \frac{(-1)^m}{2m} : \exp(-2m\pi(E_2(t)\eta^{-1} + E_4(t)\eta^{-3} + \cdots)) : \psi_{+,B}(x,t,y - 2m\pi E_0(t)),\end{aligned}$$

where

$$E_j = \frac{1}{2\pi i}\oint_{\gamma(t)} S_j(x,t)dx \tag{3.34}$$

with $\gamma(t)$ being the closed path given in Figure 6 and with S_j denoting the coefficient of η^{-j} in S_{odd}, the odd part of WKB solutions $S^\pm$ of the Riccati equation associated with (3.1).

Remark 6. The operator : $\exp(-2m\pi(E_2(t)\eta^{-1} + E_4(t)\eta^{-3} + \cdots))$: in (3.33) originates from the comparison (3.28) between the Borel transform of the WKB solution of the ∞-Weber equation and that of the ordinary Weber equation. See [3, Section 4] for details.

Finally, as a corollary of Theorem 3.3, we also obtain the following formula (3.35), an extension of (2.7) to an MTP equation:

$$\psi_+^{\mathrm{I}} = \left(1 + \exp(-2\pi E(t,\eta)\eta)\right)^{1/2} \psi_+^{\mathrm{II}}, \tag{3.35}$$

where ψ_+^{I} (resp. ψ_+^{II}) denotes the Borel sum of the WKB solution of an MTP equation (3.1) that is normalized as in (3.19) in the region corresponding to Region I (resp. Region II) through the coordinate change $X = X_0(x,t)$ for t ($\neq 0$) satisfying Re $E_0(t) > 0$ and Im $E_0(t) > 0$ (resp. Re $E_0(t) > 0$ and Im $E_0(t) < 0$).

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