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TOWARD DIRICHLET'S UNIT THEOREM ON ARITHMETIC VARIETIES

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Abstract. In this paper, we would like to propose a fundamental question about a higher dimensional analogue of Dirichlet's unit theorem. We also give a partial answer to the question as an application of the arithmetic Hodge index theorem.

Contents

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Introduction

0.1. **Classical Dirichlet's unit theorem.** Let K be a number field and let O_K be the ring of integers in K. Let $K(\mathbb{C})$ be the set of all embeddings K into $\mathbb{C}$, and let Ξ_K and Ξ_K^0 be real vector spaces given by

$$\Xi_K = \left\{ \xi \in \mathbb{R}^{K(\mathbb{C})} \mid \xi_\sigma = \xi_{\bar{\sigma}} \text{ for all } \sigma \in K(\mathbb{C}) \right\} \quad \text{and} \quad \Xi_K^0 = \left\{ \xi \in \Xi_K \;\middle|\; \sum_{\sigma \in K(\mathbb{C})} \xi_\sigma = 0 \right\}$$

respectively. The classical Dirichlet's unit theorem asserts that the unit group $O_K^\times$ of O_K is a finitely generated abelian group of rank $s = \dim_{\mathbb{R}} \Xi_K^0$. The most essential part of the proof of Dirichlet's unit theorem is to show that Ξ_K^0 is generated by the image of the map $L : O_K^\times \to \Xi_K$ given by $L(u)_\sigma = \log|\sigma(u)|$ ($u \in O_K^\times$) over $\mathbb{R}$, that is, for any $\xi \in \Xi_K^0$, there are $u_1, \ldots, u_r \in O_K^\times$ and $a_1, \ldots, a_r \in \mathbb{R}$ such that

$$\xi_\sigma = a_1 \log|\sigma(u_1)|^2 + \cdots + a_r \log|\sigma(u_r)|^2 \tag{0.1.1}$$

for all $\sigma \in K(\mathbb{C})$.

Let us consider this problem in flavor of Arakelov theory. Let $X = \operatorname{Spec}(O_K)$ and let $\widehat{\operatorname{Div}}(X)_{\mathbb{R}}$ be the real vector space consisting of pairs (D, ξ) of $D \in \operatorname{Div}(X)_{\mathbb{R}} := \operatorname{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ and $\xi \in \Xi_K$. An element of $\widehat{\operatorname{Div}}(X)_{\mathbb{R}}$ is called an *arithmetic $\mathbb{R}$-divisor on X*. For $\overline{D} = (\sum_P a_P P, \xi) \in \widehat{\operatorname{Div}}(X)_{\mathbb{R}}$, the *arithmetic degree $\widehat{\deg}(\overline{D})$ of $\overline{D}$* is given by

$$\widehat{\deg}(\overline{D}) := \sum_P a_P \log \#(O_K/P) + \frac{1}{2} \sum_\sigma \xi_\sigma.$$

The *arithmetic principal divisor* $\widehat{(x)}$ for $x \in K^\times$ is defined to be

$$\widehat{(x)} := \left(\sum_P \operatorname{ord}_P(x) P, \xi(x) \right),$$

where $\xi(x)_\sigma = -\log|\sigma(x)|^2$ for $\sigma \in K(\mathbb{C})$. As the map $\widehat{(\ )} : K^\times \to \widehat{\operatorname{Div}}(X)_{\mathbb{R}}$ given by $x \mapsto \widehat{(x)}$ is a group homomorphism, we have the natural extension

$$\widehat{(\ )}_{\mathbb{R}} : K^\times_{\mathbb{R}} := (K^\times, \times) \otimes_{\mathbb{Z}} \mathbb{R} \to \widehat{\operatorname{Div}}(X)_{\mathbb{R}},$$

that is,

$$(\widehat{x_1^{\otimes a_1} \cdots x_r^{\otimes a_r}}) = a_1 \widehat{(x_1)} + \cdots + a_r \widehat{(x_r)}$$

for $x_1, \ldots, x_r \in K^\times$ and $a_1, \ldots, a_r \in \mathbb{R}$. In particular, $\widehat{\deg}(\widehat{(x)}_{\mathbb{R}}) = 0$ for all $x \in K^\times_{\mathbb{R}}$ by the product formula.

If we set $\overline{D}_\xi = (0, \xi)$ for $\xi \in \Xi_K^0$, then the assertion (0.1.1) is equivalent to show that

$$\overline{D}_\xi + \widehat{(u)}_{\mathbb{R}} = (0, 0)$$

for some $u \in (O_K^\times)_{\mathbb{R}} := (O_K^\times, \times) \otimes_{\mathbb{Z}} \mathbb{R}$. For this purpose, it is actually sufficient to show that

$$\overline{D}_\xi + \widehat{(x)}_{\mathbb{R}} \geq (0, 0)$$

for some $x \in K^\times_{\mathbb{R}}$. Indeed, we choose $x_1, \ldots, x_r \in K^\times$ and $a_1, \ldots, a_r \in \mathbb{R}$ such that $x = x_1^{\otimes a_1} \cdots x_r^{\otimes a_r}$ and $a_1, \ldots, a_r$ are linearly independent over $\mathbb{Q}$. Then, as $\overline{D}_\xi + \widehat{(x)}_{\mathbb{R}} \geq (0, 0)$ and $\widehat{\deg}(\overline{D}_\xi + \widehat{(x)}_{\mathbb{R}}) = 0$, we have $\overline{D}_\xi + \widehat{(x)}_{\mathbb{R}} = (0, 0)$, and hence

$\sum_{i=1}^{r} a_i \operatorname{ord}_P(x_i) = 0$ for all P. Therefore, $\operatorname{ord}_P(x_i) = 0$ for all i and P, which means that $x_i \in O_K^{\times}$ for all i. In this way, the classical Dirichlet's unit theorem can be formulated in the following way:

Theorem 0.1.2 (cf. Proposition 3.4.5)**.** *If $\widehat{\deg}(\overline{D}) \geq 0$ for $\overline{D} \in \widehat{\operatorname{Div}}(X)_{\mathbb{R}}$, then there exists $x \in K_{\mathbb{R}}^{\times}$ such that $\overline{D} + \widehat{(x)}_{\mathbb{R}} \geq (0, 0)$.*

This is an application of the compactness theorem (cf. Corollary 3.3.2) and the arithmetic Riemann-Roch theorem on arithmetic curves, which indicates that the theory of arithmetic $\mathbb{R}$-divisors is not an artificial material, but it actually provides realistic tools for arithmetic problems.

In this paper, we would like to consider a higher dimensional analogue of the above theorem on arithmetic varieties.

0.2. **Arithmetic Cartier divisors.** Let X be an *arithmetic variety*, that is, X is a flat and quasi-projective integral scheme over $\mathbb{Z}$. We say X is *generically smooth* if the generic fiber $X_{\mathbb{Q}}$ of $X \to \operatorname{Spec}(\mathbb{Z})$ is smooth over $\mathbb{Q}$. We assume that X is projective, generically smooth, normal and d-dimensional (i.e. the Krull dimension of X is d, so that $\dim X_{\mathbb{Q}} = d - 1$).

We denote the group of Cartier divisors on X by $\operatorname{Div}(X)$. Let C be a class of real valued continuous functions. As examples of C, we can consider

$$
\begin{aligned}
C^0 &= \text{the class of continuous functions,}\\
C^\infty &= \text{the class of } C^\infty\text{-functions,}\\
C^0 \cap \mathrm{PSH} &= \text{the class of continuous plurisubharmonic functions,}
\end{aligned}
$$

which have good properties as in [20, SubSection 2.3]. Let $\mathbb{K}$ be either $\mathbb{Z}$ or $\mathbb{Q}$ or $\mathbb{R}$. A pair $\overline{D} = (D, g)$ is called an *arithmetic $\mathbb{K}$-Cartier divisor of C-type* if the following conditions are satisfied:

(i) D is a $\mathbb{K}$-Cartier divisor on X, that is, $D = \sum_{i=1}^{r} a_i D_i$ for some $D_1, \ldots, D_r \in \operatorname{Div}(X)$ and $a_1, \ldots, a_r \in \mathbb{K}$.

(ii) $g : X(\mathbb{C}) \to \mathbb{R} \cup \{\pm\infty\}$ is a locally integrable function and $g \circ F_\infty = g$ (*a.e.*), where $F_\infty : X(\mathbb{C}) \to X(\mathbb{C})$ is the complex conjugation map.

(iii) For any point $x \in X(\mathbb{C})$, there are an open neighborhood U_x of x and a function u_x on U_x such that u_x belongs to the class C and

$$
g = u_x + \sum_{i=1}^{r} (-a_i) \log |f_i|^2 \quad (a.e.)
$$

on U_x, where f_i is a local equation of D_i over U_x for each i.

Let $\widehat{\operatorname{Div}}_C(X)_{\mathbb{K}}$ be the set of all arithmetic $\mathbb{K}$-Cartier divisors of C-type. For simplicity, $\widehat{\operatorname{Div}}_C(X)_{\mathbb{Z}}$ is denoted by $\widehat{\operatorname{Div}}_C(X)$. Note that there are natural surjective homomorphisms

$$
\widehat{\operatorname{Div}}_{C^0}(X) \otimes_{\mathbb{Z}} \mathbb{R} \to \widehat{\operatorname{Div}}_{C^0}(X)_{\mathbb{R}} \quad \text{and} \quad \widehat{\operatorname{Div}}_{C^\infty}(X) \otimes_{\mathbb{Z}} \mathbb{R} \to \widehat{\operatorname{Div}}_{C^\infty}(X)_{\mathbb{R}}
$$

and that they are not isomorphisms respectively. For details, see [20].

Let $\mathrm{Rat}(X)$ be the function field of X. The group of *arithmetic principal divisors* on X is denoted by $\widehat{\mathrm{PDiv}}(X)$, that is,

$$\widehat{\mathrm{PDiv}}(X) := \left\{ \widehat{(\phi)} := \left((\phi), -\log|\phi|^2 \right) \in \widehat{\mathrm{Div}}_{C^\infty}(X) \mid \phi \in \mathrm{Rat}(X)^\times \right\}.$$

The homomorphism $\widehat{(\ )} : \mathrm{Rat}(X)^\times \to \widehat{\mathrm{Div}}_{C^\infty}(X)$ given by $\phi \mapsto \widehat{(\phi)}$ has the natural extension

$$\widehat{(\ )}_{\mathbb{K}} : \mathrm{Rat}(X)^\times_{\mathbb{K}} \to \widehat{\mathrm{Div}}_{C^\infty}(X)_{\mathbb{K}},$$

that is,

$$\widehat{(\phi_1^{\otimes a_1} \cdots \phi_l^{\otimes a_l})} = a_1\widehat{(\phi_1)} + \cdots + a_l\widehat{(\phi_l)}$$

for $\phi_1, \ldots, \phi_l \in \mathrm{Rat}(X)^\times$ and $a_1, \ldots, a_l \in \mathbb{K}$. For simplicity, $\widehat{(\ )}_{\mathbb{K}}$ is occasionally denoted by $\widehat{(\ )}$. We define $\widehat{\mathrm{PDiv}}(X)_{\mathbb{K}}$ to be

$$\widehat{\mathrm{PDiv}}(X)_{\mathbb{K}} := \left\{ \widehat{(\varphi)}_{\mathbb{K}} \mid \varphi \in \mathrm{Rat}(X)^\times_{\mathbb{K}} \right\}.$$

Note that

$$\widehat{\mathrm{PDiv}}(X)_{\mathbb{K}} = \left\langle \widehat{\mathrm{PDiv}}(X) \right\rangle_{\mathbb{K}} \subseteq \widehat{\mathrm{Div}}_{C^\infty}(X)_{\mathbb{K}}.$$

An element of $\widehat{\mathrm{PDiv}}(X)_{\mathbb{K}}$ is called an *arithmetic* $\mathbb{K}$*-principal divisor* on X.

Let $\overline{D} = (D, g)$ and $\overline{D}' = (D', g')$ be arithmetic $\mathbb{R}$-Cartier divisors of C^0-type on X. We define $\overline{D} = \overline{D}'$ and $\overline{D} \leq \overline{D}'$ to be

$$\overline{D} = \overline{D}' \quad \Longleftrightarrow \quad D = D' \text{ and } g = g' \text{ (a.e.)}$$

and

$$\overline{D} \leq \overline{D}' \quad \Longleftrightarrow \quad D \leq D' \text{ and } g \leq g' \text{ (a.e.)}.$$

Let C be a reduced and irreducible 1-dimensional closed subschemes of X. The *arithmetic degree* $\widehat{\deg}(\overline{D}\big|_C)$ *of* $\overline{D}$ *along* C is characterized by the following properties (for details, see [20, SubSection 5.3]):

(i) $\widehat{\deg}(\overline{D}\big|_C)$ is linear with respect to $\overline{D}$.

(ii) If $\phi \in \mathrm{Rat}(X)^\times_{\mathbb{R}}$, then $\widehat{\deg}(\widehat{(\phi)}_{\mathbb{R}}\big|_C) = 0$.

(iii) If $C \not\subseteq \mathrm{Supp}(D)$ and C is vertical, then $\widehat{\deg}(\overline{D}\big|_C) = \log(p)\deg(D|_C)$, where C is contained in the fiber over a prime p.

(iv) If $C \not\subseteq \mathrm{Supp}(D)$ and C is horizontal, then $\widehat{\deg}(\overline{D}\big|_C) = \widehat{\deg}\left(D|_{\widetilde{C}}, g|_{\widetilde{C}}\right)$, where $\widetilde{C}$ is the normalization of C and $\widehat{\deg}$ on the right hand side is the arithmetic degree in the sense of SubSection 0.1. (Note that $\widetilde{C} = \mathrm{Spec}(O_K)$ for some number field K.)

The current $dd^c([g]) + \delta_D$ on $X(\mathbb{C})$ is denoted by $c_1(\overline{D})$. Note that $c_1(\overline{D})$ is locally equal to $dd^c([u_x])$ by the Poincaré-Lelong formula. If $\overline{D}$ is of C^∞-type, then $c_1(\overline{D})$ is represented by a C^∞-form. By abuse of notation, we also denote the C^∞-form by $c_1(\overline{D})$.

0.3. **Arithmetic volume function.** Let $\overline{D} = (D, g)$ be an arithmetic $\mathbb{R}$-Cartier divisor of C^0-type on X. We define $H^0(X, D)$ and $\hat{H}^0(X, \overline{D})$ to be

$$H^0(X, D) := \left\{\phi \in \operatorname{Rat}(X)^\times \mid D + (\phi) \geq 0\right\} \cup \{0\}$$

and

$$\hat{H}^0(X, \overline{D}) := \left\{\phi \in \operatorname{Rat}(X)^\times \mid \overline{D} + \widehat{(\phi)} \geq (0, 0)\right\} \cup \{0\}$$

respectively. Note that $H^0(X, D)$ is a finitely generated $\mathbb{Z}$-module and $\hat{H}^0(X, \overline{D})$ is a finite set. It is easy to see that $|\phi| \exp(-g/2)$ is represented by a continuous function $\eta_{\phi,g}$ for $\phi \in H^0(X, D)$ (cf. [20, SubSection 2.5] or Lemma 3.1.1), so that we can define $\|\phi\|_g$ to be

$$\|\phi\|_g := \max\left\{\eta_{\phi,g}(x) \mid x \in X(\mathbb{C})\right\}.$$

Then

$$\hat{H}^0(X, \overline{D}) = \left\{\phi \in H^0(X, D) \mid \|\phi\|_g \leq 1\right\},$$

that is, $\hat{H}^0(X, \overline{D})$ is the set of small sections.

The *arithmetic volume* $\widehat{\operatorname{vol}}(\overline{D})$ *of* $\overline{D}$ is defined to be

$$\widehat{\operatorname{vol}}(\overline{D}) := \limsup_{n\to\infty} \frac{\log \#\hat{H}^0(X, n\overline{D})}{n^d/d!}.$$

As fundamental properties of $\widehat{\operatorname{vol}}$, the following are known (for details, see [20]):

(1) $\widehat{\operatorname{vol}}(\overline{D}) < \infty$ ([17], [18]).

(2) $\widehat{\operatorname{vol}}(\overline{D}) = \lim_{n\to\infty} \dfrac{\log(\#\hat{H}^0(X, n\overline{D}))}{(n^d/d!)}$ ([5], [18]).

(3) $\widehat{\operatorname{vol}}(a\overline{D}) = a^d \widehat{\operatorname{vol}}(\overline{D})$ for $a \in \mathbb{R}_{\geq 0}$ ([17], [18]).

(4) The function $\widehat{\operatorname{Div}}_{C^0}(X)_{\mathbb{R}} \to \mathbb{R}$ given by $\overline{D} \mapsto \widehat{\operatorname{vol}}(\overline{D})$ is continuous in the following sense: Let $\overline{D}_1, \ldots, \overline{D}_r, \overline{A}_1, \ldots, \overline{A}_s$ be arithmetic $\mathbb{R}$-divisors of C^0-type on X. For a compact subset B in $\mathbb{R}^r$ and a positive number ϵ, there are positive numbers δ and δ' such that

$$\left|\widehat{\operatorname{vol}}\left(\sum_{i=1}^r a_i\overline{D}_i + \sum_{j=1}^s \delta_j \overline{A}_j + (0, \phi)\right) - \widehat{\operatorname{vol}}\left(\sum_{i=1}^r a_i \overline{D}_i\right)\right| \leq \epsilon$$

for all $a_1, \ldots, a_r, \delta_1, \ldots, \delta_s \in \mathbb{R}$ and $\phi \in C^0(X)$ with $(a_1, \ldots, a_r) \in B$, $|\delta_1| + \cdots + |\delta_s| \leq \delta$ and $\|\phi\|_{\sup} \leq \delta'$ ([17], [18]).

(5) If $f : Y \to X$ is a birational morphism of generically smooth, normal and projective arithmetic varieties, then $\widehat{\operatorname{vol}}(f^*(\overline{D})) = \widehat{\operatorname{vol}}(\overline{D})$ ([17]).

0.4. **Positivity of arithmetic Cartier divisors.** Let $\overline{D} = (D, g)$ be an arithmetic $\mathbb{R}$-Cartier divisor of C^0-type on X. Here we would like to introduce several kinds of positivity of $\overline{D}$, that is, the effectivity, bigness, pseudo-effectivity, nefness and relative nefness of $\overline{D}$:

- $\overline{D}$ is *effective* $\overset{\text{def}}{\iff}$ $\overline{D} \geq (0, 0)$.

• $\overline{D}$ is *big* $\overset{\text{def}}{\iff}$ $\widehat{\mathrm{vol}}(\overline{D}) > 0$.

• $\overline{D}$ is *pseudo-effective* $\overset{\text{def}}{\iff}$ $\overline{D} + \overline{A}$ is big for any big arithmetic $\mathbb{R}$-divisor $\overline{A}$ of C^0-type.

• $\overline{D}$ is *nef* $\overset{\text{def}}{\iff}$

(1) $\widehat{\deg}(\overline{D}\big|_C) \geq 0$ for all reduced and irreducible 1-dimensional closed subschemes C of X.
(2) $c_1(\overline{D})$ is a positive current.

• $\overline{D}$ is *relatively nef* $\overset{\text{def}}{\iff}$

(1) $\widehat{\deg}(\overline{D}\big|_C) \geq 0$ for all reduced and irreducible 1-dimensional closed vertical subschemes C of X, where "vertical" means "not flat over $\mathbb{Z}$".
(2) $c_1(\overline{D})$ is a positive current.

The set of all nef arithmetic $\mathbb{R}$-Cartier divisors of C^0-type on X is denoted by $\widehat{\mathrm{Nef}}_{C^0}(X)_{\mathbb{R}}$. Note that $\widehat{\mathrm{Nef}}_{C^0}(X)_{\mathbb{R}}$ forms a cone in $\widehat{\mathrm{Div}}_{C^0}(X)_{\mathbb{R}}$.

0.5. **Arithmetic intersection number in terms of the arithmetic volume.** An arithmetic $\mathbb{R}$-Cartier divisor $\overline{D}$ of C^0-type on X is said to be *integrable* if there exist nef arithmetic $\mathbb{R}$-Cartier divisors $\overline{D}_1$ and $\overline{D}_2$ of C^0-type such that $\overline{D} = \overline{D}_1 - \overline{D}_2$. The subspace consisting of integrable arithmetic $\mathbb{R}$-Cartier divisors of C^0-type on X is denoted by $\widehat{\mathrm{Div}}_{C^0}^{\mathrm{Nef}}(X)_{\mathbb{R}}$. Note that $\widehat{\mathrm{Div}}_{C^0}^{\mathrm{Nef}}(X)_{\mathbb{R}}$ is the subspace generated by $\widehat{\mathrm{Nef}}_{C^0}(X)_{\mathbb{R}}$ in $\widehat{\mathrm{Div}}_{C^0}(X)_{\mathbb{R}}$.

By [20, Claim 6.4.2.2], if $\overline{P}$ is a nef arithmetic $\mathbb{R}$-Cartier divisor of C^∞-type, then the arithmetic Hilbert-Samuel formula

$$\widehat{\mathrm{vol}}(\overline{P}) = \widehat{\deg}(\overline{P}^d) \tag{0.5.1}$$

holds. Note that

$$d! X_1 \cdots X_d = \sum_{\emptyset \neq I \subseteq \{1,\ldots,d\}} (-1)^{d-\#(I)} \left(\sum_{i \in I} X_i \right)^d$$

in the polynomial ring $\mathbb{Z}[X_1, \ldots, X_d]$. Thus, for nef arithmetic $\mathbb{R}$-Cartier divisors $\overline{P}_1, \ldots, \overline{P}_d$ of C^∞-type, we have

$$\widehat{\deg}(\overline{P}_1 \cdots \overline{P}_d) = \frac{1}{d!} \sum_{\emptyset \neq I \subseteq \{1,\ldots,d\}} (-1)^{d-\#(I)} \widehat{\mathrm{vol}} \left(\sum_{i \in I} \overline{P}_i \right),$$

so that, for $\overline{D}_1, \ldots, \overline{D}_d \in \widehat{\mathrm{Nef}}_{C^0}(X)_{\mathbb{R}}$, it is very natural to define $\widehat{\deg}(\overline{D}_1 \cdots \overline{D}_d)$ to be

$$\widehat{\deg}(\overline{D}_1 \cdots \overline{D}_d) := \frac{1}{d!} \sum_{\emptyset \neq I \subseteq \{1,\ldots,d\}} (-1)^{d-\#(I)} \widehat{\mathrm{vol}} \left(\sum_{i \in I} \overline{D}_i \right).$$

Using the regularity of quasiplurisubharmonic functions and the continuity of $\widehat{\mathrm{vol}}$, we can see that the above map $\widehat{\deg}(\cdots) : \widehat{\mathrm{Nef}}_{C^0}(X)_{\mathbb{R}} \times \cdots \times \widehat{\mathrm{Nef}}_{C^0}(X)_{\mathbb{R}} \to \mathbb{R}$ is $\mathbb{R}_{\geq 0}$-multilinear, that is,

$$\widehat{\deg}(\overline{D}_1 \cdots (\alpha \overline{D}_i + \alpha' \overline{D}'_i) \cdots \overline{D}_d) = \alpha \widehat{\deg}(\overline{D}_1 \cdots \overline{D}_i \cdots \overline{D}_d) + \alpha' \widehat{\deg}(\overline{D}_1 \cdots \overline{D}'_i \cdots \overline{D}_d)$$

for $\alpha, \alpha' \in \mathbb{R}_{\geq 0}$ (for details, see [20, Claim 6.4.2.4]). Therefore, the map

$$\widehat{\deg}(\cdots) : \widehat{\mathrm{Nef}}_{C^0}(X)_{\mathbb{R}} \times \cdots \times \widehat{\mathrm{Nef}}_{C^0}(X)_{\mathbb{R}} \to \mathbb{R}$$

extends uniquely to an $\mathbb{R}$-multilinear map

$$\widehat{\deg}(\cdots) : \widehat{\mathrm{Div}}_{C^0}^{\mathrm{Nef}}(X)_{\mathbb{R}} \times \cdots \times \widehat{\mathrm{Div}}_{C^0}^{\mathrm{Nef}}(X)_{\mathbb{R}} \to \mathbb{R}.$$

In SubSection 2.1, we will see that the above arithmetic intersection number $\widehat{\deg}(\overline{D}_1 \cdots \overline{D}_d)$ for integrable arithmetic $\mathbb{R}$-Cartier divisors $\overline{D}_1, \ldots, \overline{D}_d$ of C^0-type on X coincides with one due to Zhang ([24, Lemma 6.5], [25, §1]) and Maillot ([13, §5]).

0.6. **Zariski decomposition.** Let $\overline{D} = (D, g)$ be an arithmetic $\mathbb{R}$-Cartier divisor of C^0-type on X. Let us consider the following set:

$$\Upsilon(\overline{D}) := \left\{ \overline{M} \in \widehat{\mathrm{Nef}}_{C^0}(X)_{\mathbb{R}} \mid \overline{M} \leq \overline{D} \right\}.$$

If $\Upsilon(\overline{D}) \neq \emptyset$ and $\Upsilon(\overline{D})$ has the greatest element $\overline{P}$ (that is, $\overline{P} \in \Upsilon(\overline{D})$ and $\overline{M} \leq \overline{P}$ for all $\overline{M} \in \Upsilon(\overline{D})$), then $\overline{D} = \overline{P} + \overline{N}$ is called the *Zariski decomposition of* $\overline{D}$, where $\overline{N} := \overline{D} - \overline{P}$. This decomposition has the following properties:

(1) $\overline{P}$ is nef and $\overline{N}$ is effective.
(2) The natural map $\hat{H}^0(X, n\overline{P}) \to \hat{H}^0(X, n\overline{D})$ is bijective for every $n \geq 0$. In particular, $\widehat{\mathrm{vol}}(\overline{D}) = \widehat{\mathrm{vol}}(\overline{P}) = \widehat{\deg}(\overline{P}^d)$.

In [20, Theorem 9.2.1], we prove that if X is a regular projective arithmetic surface and $\Upsilon(\overline{D}) \neq \emptyset$, then $\Upsilon(\overline{D})$ has the greatest element. Moreover, if we set

$$\begin{cases} X := \mathbb{P}^n_{\mathbb{Z}} = \mathrm{Proj}(\mathbb{Z}[T_0, \ldots, T_n]) \quad (n \geq 2), \\ D := \{T_0 = 0\}, \\ g := \log(1 + |T_1/T_0|^2 + \cdots + |T_n/T_0|^2) - \epsilon \quad (0 < \epsilon < \log(n+1)), \end{cases}$$

then, in [21, Theorem 2.3, Theorem 5.6], we prove that $\overline{D}$ is big and $f^*(\overline{D})$ does not admit the Zariski decomposition for any birational morphism $f : Y \to X$ of generically smooth, normal and projective arithmetic varieties. More generally, a criterion for the existence of the Zariski decomposition on arithmetic toric varieties is known (for details, see [3]).

It is easy to see that if $\Upsilon(\overline{D}) \neq \emptyset$, then $\overline{D}$ is pseudo-effective. The converse is a very interesting question and it is closely related to the fundamental question in the next subsection.

0.7. **Fundamental question.** Let $\overline{D} = (D, g)$ be an arithmetic $\mathbb{R}$-Cartier divisor of C^0-type on X. In this paper, we would like to propose the following fundamental question:

Fundamental question. Are the following conditions (1) and (2) equivalent ?

(1) $\overline{D}$ is pseudo-effective.
(2) $\overline{D} + \widehat{(\varphi)}_{\mathbb{R}}$ is effective for some $\varphi \in \mathrm{Rat}(X)^{\times}_{\mathbb{R}}$.

Obviously (2) implies (1). Moreover, if $\hat{H}^0(X, a\overline{D}) \neq \{0\}$ for some $a \in \mathbb{R}_{>0}$, then (2) holds. Indeed, as we can choose $\phi \in \operatorname{Rat}(X)^{\times}$ with $a\overline{D} + \widehat{(\phi)} \geq 0$, we have $\phi^{1/a} \in \operatorname{Rat}(X)^{\times}_{\mathbb{R}}$ and $\overline{D} + \widehat{(\phi^{1/a})}_{\mathbb{R}} \geq 0$. In the geometric case, (1) does not necessarily imply (2). For example, let ϑ be a divisor on a compact Riemann surface M such that $\deg(\vartheta) = 0$ and the class of ϑ in $\operatorname{Pic}(M)$ is not a torsion element. Then it is easy to see that ϑ is pseudo-effective and there is no element ψ of $\operatorname{Rat}(M)^{\times} \otimes_{\mathbb{Z}} \mathbb{R}$ such that $\vartheta + (\psi)_{\mathbb{R}}$ is effective (cf. Remark 3.1.4). In this sense, the above question is a purely arithmetic problem.

Note that Theorem 0.1.2 yields the answer in the case where $d = 1$ because the pseudo-effectivity of $\overline{D}$ implies $\widehat{\deg}(\overline{D}) \geq 0$. Moreover, as we remarked in SubSection 0.6, if there is $\varphi \in \operatorname{Rat}(X)^{\times}_{\mathbb{R}}$ such that $\overline{D} + \widehat{(\varphi)}_{\mathbb{R}} \geq (0,0)$, then $-\widehat{(\varphi)}_{\mathbb{R}} \in \Upsilon(\overline{D})$.

0.8. **Partial answer to the fundamental question.** One of the main purpose of this paper is to give the following partial answer to the above fundamental question:

Theorem 0.8.1. *If $\overline{D}$ is pseudo-effective and D is numerically trivial on $X_{\mathbb{Q}}$, then there exists $\varphi \in \operatorname{Rat}(X)^{\times}_{\mathbb{R}}$ such that $\overline{D} + \widehat{(\varphi)}_{\mathbb{R}}$ is effective.*

Here we would like to give a sketch of the proof of the above theorem. For simplicity, we restrict ourself to the case where X is regular and $d = 2$, that is, X is a regular projective arithmetic surface. In this case, we can give a simpler proof than the original one by using the recent result on the existence of relative Zariski decomposition. Let $\overline{D} = \overline{Q} + \overline{N}$ be the relative Zariski decomposition of $\overline{D}$ (for details , see [22, Section 1]). In particular, we have the following properties:

(i) $\overline{N}$ is effective and N is vertical.
(ii) $\overline{Q}$ is relatively nef.
(iii) If $\overline{D}$ is pseudo-effective, then $\overline{Q}$ is also pseudo-effective (cf. [22, Proposition A.1]). This part corresponds to Lemma 2.3.5 in the original proof.

Therefore, we may assume that $\overline{D}$ is relatively nef. By the Hodge index theorem (cf. Theorem 2.2.3), we have $\widehat{\deg}(\overline{D}^2) \leq 0$. Here we assume that $\widehat{\deg}(\overline{D}^2) < 0$. Let $\overline{A}$ be an ample arithmetic $\mathbb{R}$-divisor of C^{∞}-type on X. Then $\widehat{\deg}(\overline{D} + \epsilon\overline{A} \cdot \overline{D}) < 0$ for a sufficiently small positive number ϵ. As $D + \epsilon A$ is ample, we can find a positive number c such that $\overline{D} + \epsilon\overline{A} + (0, c)$ is nef. In particular,

$$\widehat{\deg}(\overline{D} + \epsilon\overline{A} + (0,c) \cdot \overline{D}) \geq 0$$

because $\overline{D}$ is pseudo-effective. On the other hand, as $\deg(D_{\mathbb{Q}}) = 0$,

$$\widehat{\deg}(\overline{D} + \epsilon\overline{A} + (0,c) \cdot \overline{D}) = \widehat{\deg}(\overline{D} + \epsilon\overline{A} \cdot \overline{D}) + \frac{c}{2}\deg(D_{\mathbb{Q}}) = \widehat{\deg}(\overline{D} + \epsilon\overline{A} \cdot \overline{D}) < 0.$$

This is a contradiction, so that $\widehat{\deg}(\overline{D}^2) = 0$, and hence, by the equality condition of the Hodge index theorem (cf. Remark 2.2.4), there are $\phi \in \operatorname{Rat}(X)^{\times}_{\mathbb{R}}$ and a locally constant function λ on $X(\mathbb{C})$ such that $\overline{D} = \widehat{(\phi)}_{\mathbb{R}} + (0, \lambda)$. Let $X \to \operatorname{Spec}(O_K)$ be the Stein factorization of $X \to \operatorname{Spec}(\mathbb{Z})$, where K is a number field and O_K is the ring of integers in K. Let X_{σ} be the connected component of $X(\mathbb{C})$ corresponding to $\sigma \in K(\mathbb{C})$ (cf. Conventions and terminology 3). We set $\lambda_\sigma = \lambda|_{X_\sigma}$. As $\overline{D}$ is pseudo-effective,

$$0 \leq \widehat{\deg}(\overline{A} \cdot \overline{D}) = \frac{\deg(A_{\mathbb{Q}})}{2[K:\mathbb{Q}]} \sum_{\sigma} \lambda_\sigma,$$

so that $\sum_\sigma \lambda_\sigma \geq 0$. If we set

$$\lambda'_\sigma = \lambda_\sigma - \frac{1}{[K:\mathbb{Q}]} \sum_\sigma \lambda_\sigma$$

for each σ and we consider a locally constant function $\lambda' : X(\mathbb{C}) \to \mathbb{R}$ given by $\lambda'|_{X_\sigma} = \lambda'_\sigma$, then $\lambda' \leq \lambda$ on $X(\mathbb{C})$ and $\sum_\sigma \lambda'_\sigma = 0$. Thus, by the classical Dirichlet's unit theorem, there exists $u \in (O_K^\times)_{\mathbb{R}}$ such that $(0, \lambda') = \widehat{(u)}_{\mathbb{R}}$. Thus

$$\overline{D} = \widehat{(\phi)}_{\mathbb{R}} + (0, \lambda) \geq \widehat{(\phi)}_{\mathbb{R}} + (0, \lambda') = \widehat{(\phi)}_{\mathbb{R}} + \widehat{(u)}_{\mathbb{R}} = \widehat{(\phi \cdot u)}_{\mathbb{R}},$$

as required.

0.9. **Further discussions.** Theorem 0.8.1 treats only the case where D is scanty. For example, if D is ample, the problem seems to be difficult to get a solution. For this purpose, we would like introduce a notion of multiplicative generators of approximately smallest sections.

Here we define $\Gamma^\times_{\mathbb{R}}(X, D)$ to be

$$\Gamma^\times_{\mathbb{R}}(X, D) := \left\{ \varphi \in \mathrm{Rat}(X)^\times_{\mathbb{R}} \mid D + (\varphi)_{\mathbb{R}} \geq 0 \right\}.$$

Let $\ell : \mathrm{Rat}(X)^\times \to L^1_{loc}(X(\mathbb{C}))$ be a homomorphism given by $\varphi \mapsto \log|\varphi|$. It extends to a linear map $\ell_{\mathbb{R}} : \mathrm{Rat}(X)^\times_{\mathbb{R}} \to L^1_{loc}(X(\mathbb{C}))$. For $\varphi \in \mathrm{Rat}(X)^\times_{\mathbb{R}}$, we denote $\exp(\ell_{\mathbb{R}}(\varphi))$ by $|\varphi|$. If $\varphi \in \Gamma^\times_{\mathbb{R}}(X, D)$, then $|\varphi| \exp(-g/2)$ is represented by a continuous function $\eta_{\varphi,g}$ (cf. Lemma 3.1.1), so that we define $\|\varphi\|_{g,\sup}$ to be

$$\|\varphi\|_{g,\sup} := \max \left\{ \eta_{\varphi,g}(x) \mid x \in X(\mathbb{C}) \right\}.$$

Let $\varphi_1, \ldots, \varphi_l$ be elements of $\mathrm{Rat}(X)^\times_{\mathbb{R}}$. We say $\varphi_1, \ldots, \varphi_l$ are *multiplicative generators of approximately smallest sections for* $\overline{D}$ if, for a given $\epsilon > 0$, there is $n_0 \in \mathbb{Z}_{>0}$ such that, for any integer n with $n \geq n_0$ and $H^0(X, nD) \neq \{0\}$, we can find $a_1, \ldots, a_l \in \mathbb{R}$ satisfying $\varphi_1^{\otimes a_1} \cdots \varphi_l^{\otimes a_l} \in \Gamma^\times_{\mathbb{R}}(X, nD)$ and

$$\|\varphi_1^{\otimes a_1} \cdots \varphi_l^{\otimes a_l}\|_{ng,\sup} \leq e^{\epsilon n} \min \left\{ \|\phi\|_{ng,\sup} \mid \phi \in H^0(X, nD) \setminus \{0\} \right\}.$$

The advantage of the existence of multiplicative generators of approximately smallest sections is the following theorem.

Theorem 0.9.1 (cf. Theorem 3.6.3)**.** *If we admit the existence of multiplicative generators of approximately smallest sections, then we can find* $\varphi \in \Gamma^\times_{\mathbb{R}}(X, D)$ *such that*

$$\|\varphi\|_{g,\sup} = \inf \left\{ \|\psi\|_{g,\sup} \mid \psi \in \Gamma^\times_{\mathbb{R}}(X, D) \right\}.$$

For the proof, we need the following compactness theorem.

Theorem 0.9.2. *Let* $\overline{H}$ *be an ample arithmetic* $\mathbb{R}$*-Cartier divisor on* X*. Let* Λ *be a finite set and let* $\left\{ \overline{D}_\lambda \right\}_{\lambda \in \Lambda}$ *be a family of arithmetic* $\mathbb{R}$*-Cartier divisors of* C^∞*-type with the following properties:*

(i) $\widehat{\deg}(\overline{H}^{d-1} \cdot \overline{D}_\lambda) = 0$ *for all* $\lambda \in \Lambda$.
(ii) *For each* $\lambda \in \Lambda$, *there is an* F_∞*-invariant locally constant function* ρ_λ *on* $X(\mathbb{C})$ *such that*

$$c_1(\overline{D}_\lambda) \wedge c_1(\overline{H})^{\wedge d-2} = \rho_\lambda c_1(\overline{H})^{\wedge d-1}.$$

(iii) $\left\{\overline{D}_\lambda\right\}_{\lambda\in\Lambda}$ *is linearly independent in* $\widehat{\mathrm{Div}}_{C^\infty}(X)_{\mathbb{R}}$.

Then the set

$$\left\{ \boldsymbol{a} \in \mathbb{R}^\Lambda \;\middle|\; \overline{D} + \sum_{\lambda\in\Lambda} \boldsymbol{a}_\lambda \overline{D}_\lambda \geq 0 \right\}$$

is convex and compact for $\overline{D} \in \widehat{\mathrm{Div}}_{C^0}(X)_{\mathbb{R}}$.

As a consequence, we have the following partial answer to the fundamental question.

Theorem 0.9.3. *If* $\overline{D}$ *is pseudo-effective,* D *is big on the generic fiber of* $X \to \mathrm{Spec}(\mathbb{Z})$ *and* $\overline{D}$ *possesses multiplicative generators of approximately smallest sections, then there exists* $\varphi \in \mathrm{Rat}(X)^\times_{\mathbb{R}}$ *such that* $\overline{D} + \widehat{(\varphi)}_{\mathbb{R}} \geq 0$.

Here we would like to give the following question:

Question 0.9.4. If D is big on the generic fiber of $X \to \mathrm{Spec}(\mathbb{Z})$, then does $\overline{D}$ have multiplicative generators of approximately smallest sections ?

For example, if $d = 0$, then $\overline{D}$ has multiplicative generators of approximately smallest sections (cf. Corollary 3.4.6). Moreover, if

$$\begin{cases} X := \mathbb{P}^n_{\mathbb{Z}} = \mathrm{Proj}(\mathbb{Z}[T_0, \ldots, T_n]) \quad (n \geq 1), \\ D := \{T_0 = 0\}, \\ g := \log(a_0 + a_1|T_1/T_0|^2 + \cdots + a_n|T_n/T_0|^2) \quad (a_0, a_1, \ldots, a_n \in \mathbb{R}_{>0}), \end{cases}$$

then $\overline{D}$ has also multiplicative generators of approximately smallest sections (cf. Example 3.6.8). More generally, a toric arithmetic $\mathbb{R}$-Cartier divisor on an arithmetic toric variety has multiplicative generators of approximately smallest sections (for details, see [3]).

Finally I would like to express thanks to the referee for giving me several comments and remarks.

0.10. **Conventions and terminology.** We basically use the same notation as in [20]. Here we fix several conventions and the terminology of this paper. Let $\mathbb{K}$ be either $\mathbb{Q}$ or $\mathbb{R}$. Moreover, in the following 3 and 4, X is a d-dimensional, generically smooth, normal and projective arithmetic variety.

1. Let M be a k-equidimensional complex manifold. The space of real valued continuous functions (reps C^∞-functions) on M is denoted by $C^0(M)$ (resp $C^\infty(M)$). Moreover, the space of currents of bidegree (p, q) is denoted by $D^{p,q}(M)$. Let $N^{p,q}(M)$ be the space of currents T of bidegree (p, q) such that $T(\eta) = 0$ for all d-closed C^∞ $(k - p, k - q)$-forms with compact support.

2. Let S be a normal and integral noetherian scheme. We denote the group of Cartier divisors (resp. Weil divisors) on S by $\mathrm{Div}(S)$ (resp. $\mathrm{WDiv}(S)$). We set

$$\mathrm{Div}(S)_{\mathbb{K}} := \mathrm{Div}(S) \otimes_{\mathbb{Z}} \mathbb{K} \quad \text{and} \quad \mathrm{WDiv}(S)_{\mathbb{K}} := \mathrm{WDiv}(S) \otimes_{\mathbb{Z}} \mathbb{K}.$$

An element of $\mathrm{Div}(S)_{\mathbb{K}}$ (resp. $\mathrm{WDiv}(S)_{\mathbb{K}}$) is called a $\mathbb{K}$-*Cartier divisor* (resp. $\mathbb{K}$-*Weil divisor*) on S. We denote the group of principal divisors on S by $\mathrm{PDiv}(S)$. Let $\mathrm{Rat}(S)^{\times}_{\mathbb{K}} := \mathrm{Rat}(S)^{\times} \otimes_{\mathbb{Z}} \mathbb{K}$, that is,

$$\mathrm{Rat}(S)^{\times}_{\mathbb{K}} = \left\{ \phi_1^{\otimes a_1} \cdots \phi_l^{\otimes a_l} \mid \phi_1, \ldots, \phi_l \in \mathrm{Rat}(S)^{\times} \text{ and } a_1, \ldots, a_l \in \mathbb{K} \right\}.$$

The homomorphism $\mathrm{Rat}(S)^{\times} \to \mathrm{Div}(S)$ given by $\phi \mapsto (\phi)$ naturally extends to a homomorphism

$$(\ )_{\mathbb{K}} : \mathrm{Rat}(S)^{\times}_{\mathbb{K}} \to \mathrm{Div}(S)_{\mathbb{K}},$$

i.e. $(\phi_1^{\otimes a_1} \cdots \phi_l^{\otimes a_l}) = a_1(\phi_1) + \cdots + a_l(\phi_l)$. By abuse of notation, we sometimes denote $(\ )_{\mathbb{K}}$ by $(\ )$. We define $\mathrm{PDiv}(S)_{\mathbb{K}}$ to be

$$\mathrm{PDiv}(S)_{\mathbb{K}} := \left\{ (\varphi)_{\mathbb{K}} \mid \varphi \in \mathrm{Rat}(S)^{\times}_{\mathbb{K}} \right\}.$$

Note that

$$\mathrm{PDiv}(S)_{\mathbb{K}} := \langle \mathrm{PDiv}(S) \rangle_{\mathbb{K}} \subseteq \mathrm{Div}(S)_{\mathbb{K}}.$$

An element of $\mathrm{PDiv}(S)_{\mathbb{K}}$ is called a $\mathbb{K}$-*principal divisor* on S.

3. Let $X \xrightarrow{\pi} \mathrm{Spec}(O_K) \to \mathrm{Spec}(\mathbb{Z})$ be the Stein factorization of $X \to \mathrm{Spec}(\mathbb{Z})$, where K is a number field and O_K is the ring of integers in K. We denote by $K(\mathbb{C})$ the set of all embedding of K into $\mathbb{C}$. For $\sigma \in K(\mathbb{C})$, we set $X_\sigma := X \times^{\sigma}_{\mathrm{Spec}(O_K)} \mathrm{Spec}(\mathbb{C})$, where $\times^{\sigma}_{\mathrm{Spec}(O_K)}$ means the fiber product over $\mathrm{Spec}(O_K)$ with respect to σ. Then $\{X_\sigma\}_{\sigma \in K(\mathbb{C})}$ gives rise to the set of all connected components of $X(\mathbb{C})$. For a locally constant function λ on $X(\mathbb{C})$ and $\sigma \in K(\mathbb{C})$, the value of λ on the connected component X_σ is denoted by λ_σ. Clearly the set of all locally constant real valued functions on $X(\mathbb{C})$ can be identified with $\mathbb{R}^{K(\mathbb{C})}$. The complex conjugation map $X(\mathbb{C}) \to X(\mathbb{C})$ is denoted by F_∞. Note that $F_\infty(X_\sigma) = X_{\bar{\sigma}}$.

4. An *arithmetic* $\mathbb{K}$-*Weil divisor of* C^0-*type* (*resp.* C^∞-*type*) *on* X is a pair $\overline{D} = (D, g)$ consisting of a $\mathbb{K}$-Weil divisor D on X and a D-Green function g of C^0-type (resp. C^∞-type). We denote the group of arithmetic $\mathbb{K}$-Weil divisors of C^0-type (resp. of C^∞-type) on X by $\widehat{\mathrm{WDiv}}_{C^0}(X)_{\mathbb{K}}$ (resp. $\widehat{\mathrm{WDiv}}_{C^\infty}(X)_{\mathbb{K}}$). It is easy to see that there is a unique multi-linear form

$$\alpha : \left(\widehat{\mathrm{Div}}_{C^\infty}(X)_{\mathbb{K}}\right)^{d-1} \times \mathrm{WDiv}(X)_{\mathbb{K}} \to \mathbb{R}$$

such that $\alpha(\overline{D}_1, \ldots, \overline{D}_{d-1}, \Gamma) = \widehat{\deg}(\overline{D}_1\big|_{\widetilde{\Gamma}} \cdots \overline{D}_{d-1}\big|_{\widetilde{\Gamma}})$ for $\overline{D}_1, \ldots, \overline{D}_{d-1} \in \widehat{\mathrm{Div}}_{C^\infty}(X)$ and a prime divisor Γ with $\Gamma \not\subseteq \mathrm{Supp}(D_1) \cup \cdots \cup \mathrm{Supp}(D_{d-1})$, where $\widetilde{\Gamma}$ is the normalization of Γ. We denote $\alpha(\overline{D}_1, \ldots, \overline{D}_{d-1}, D)$ by $\widehat{\deg}(\overline{D}_1 \cdots \overline{D}_{d-1} \cdot (D, 0))$. Further, for $\overline{D}_1, \ldots, \overline{D}_{d-1} \in \widehat{\mathrm{Div}}_{C^\infty}(X)_{\mathbb{K}}$ and $\overline{D} = (D, g) \in \widehat{\mathrm{WDiv}}_{C^0}(X)_{\mathbb{K}}$, we define $\widehat{\deg}(\overline{D}_1 \cdots \overline{D}_{d-1} \cdot \overline{D})$ to be

$$\widehat{\deg}(\overline{D}_1 \cdots \overline{D}_{d-1} \cdot \overline{D}) := \widehat{\deg}(\overline{D}_1 \cdots \overline{D}_{d-1} \cdot (D, 0)) + \frac{1}{2} \int_{X(\mathbb{C})} g c_1(\overline{D}_1) \wedge \cdots \wedge c_1(\overline{D}_{d-1}).$$

5. For a set Λ, let $\mathbb{R}^{\Lambda}$ be the set of all maps from Λ to $\mathbb{R}$. The vector space generated by Λ over $\mathbb{R}$ is denoted by $\mathbb{R}(\Lambda)$, that is,

$$\mathbb{R}(\Lambda) = \{\boldsymbol{a} \in \mathbb{R}^{\Lambda} \mid \boldsymbol{a}(\lambda) = 0 \text{ except finitely many } \lambda \in \Lambda\}.$$

For $\boldsymbol{a} \in \mathbb{R}^{\Lambda}$ and $\lambda \in \Lambda$, we often denote $\boldsymbol{a}(\lambda)$ by $\boldsymbol{a}_{\lambda}$.

6. Let V be a vector space over $\mathbb{R}$ and let $\langle\ ,\ \rangle$ be an inner product on V. For a finite subset $\{x_1, \ldots, x_r\}$ of V, we define $\mathrm{vol}(\{x_1, \ldots, x_r\})$ to be the square root of the Gramian of $x_1, \ldots, x_r$ with respect to $\langle\ ,\ \rangle$, that is,

$$\mathrm{vol}(\{x_1, \ldots, x_r\}) = \sqrt{\det \begin{pmatrix} \langle x_1, x_1\rangle & \langle x_1, x_2\rangle & \cdots & \langle x_1, x_r\rangle \\ \langle x_2, x_1\rangle & \langle x_2, x_2\rangle & \cdots & \langle x_2, x_r\rangle \\ \cdots & \cdots & \cdots & \cdots \\ \langle x_r, x_1\rangle & \langle x_r, x_2\rangle & \cdots & \langle x_r, x_r\rangle \end{pmatrix}}.$$

For convenience, we set $\mathrm{vol}(\emptyset) = 1$. Note that if $V = \mathbb{R}^n$ and $\langle\ ,\ \rangle$ is the standard inner product, then $\mathrm{vol}(\{x_1, \ldots, x_r\})$ is the volume of the parallelotope given by $\{a_1x_1 + \cdots + a_rx_r \mid 0 \leq a_1 \leq 1, \ldots, 0 \leq a_r \leq 1\}$.

1. Preliminaries

In this section, we prepare several materials for later sections. In SubSection 1.1, we consider elementary results on linear algebra. In Subsection 1.2, we introduce the notion of proper currents and investigate several properties, which will be used to see that the arithmetic intersection number treated in [20, SubSection 6.4] coincides with the classical one due to Zhang and Maillot (cf. [24], [25], [13]). They will be also used to establish the equality condition of the arithmetic Hodge index theorem in a general context. SubSection 1.3 is devoted to the proof of a variant of Gromov's inequality for $\mathbb{R}$-Cartier divisors.

1.1. **Lemmas of linear algebra.**

Here we would like to provide the following four lemmas of linear algebra.

Lemma 1.1.1. *Let M be a $\mathbb{Z}$-module. Then we have the following:*

(1) *For $x \in M \otimes_{\mathbb{Z}} \mathbb{R}$, there are $x_1, \ldots, x_l \in M$ and $a_1, \ldots, a_l \in \mathbb{R}$ such that $a_1, \ldots, a_l$ are linearly independent over $\mathbb{Q}$ and $x = x_1 \otimes a_1 + \cdots + x_l \otimes a_l$.*

(2) *Let $x_1, \ldots, x_l \in M$ and $a_1, \ldots, a_l \in \mathbb{R}$ such that $a_1, \ldots, a_l$ are linearly independent over $\mathbb{Q}$. If $x_1 \otimes a_1 + \cdots + x_l \otimes a_l = 0$ in $M \otimes_{\mathbb{Z}} \mathbb{R}$, then $x_1, \ldots, x_l$ are torsion elements in M.*

(3) *If N is a submodule of M, then $(M \otimes_{\mathbb{Z}} \mathbb{Q}) \cap (N \otimes_{\mathbb{Z}} \mathbb{R}) = N \otimes_{\mathbb{Z}} \mathbb{Q}$.*

Proof. (1) As $x \in M \otimes_{\mathbb{Z}} \mathbb{R}$, there are $a'_1, \ldots, a'_r \in \mathbb{R}$ and $x'_1, \ldots, x'_r \in M$ such that $x = x'_1 \otimes a'_1 + \cdots + x'_r \otimes a'_r$. Let $a_1, \ldots, a_l$ be a basis of $\langle a'_1, \ldots, a'_r\rangle_{\mathbb{Q}}$ over $\mathbb{Q}$. Then there are $c_{ij} \in \mathbb{Q}$ such that $a'_i = \sum_{j=1}^{l} c_{ij}a_j$. Replacing a_j by a_j/n ($n \in \mathbb{Z}_{>0}$) if necessarily, we may assume that $c_{ij} \in \mathbb{Z}$. If we set $x_j = \sum_{i=1}^{r} c_{ij}x'_i$, then $x_1, \ldots, x_l \in M$, $x = x_1 \otimes a_1 + \cdots + x_s \otimes a_s$ and $a_1, \ldots, a_s$ are linearly independent over $\mathbb{Q}$.

(2) We set $M' = \mathbb{Z}x_1 + \cdots + \mathbb{Z}x_l$. Then, since $\mathbb{R}$ is flat over $\mathbb{Z}$, the natural homomorphism $M' \otimes \mathbb{R} \to M \otimes \mathbb{R}$ is injective, and hence we may assume that M is finitely generated. Let M_{tor} be the set of all torsion elements in M. Considering

M/M_{tor}, we may further assume that M is free. Note that the natural homomorphism $\mathbb{Z}a_1 \oplus \cdots \oplus \mathbb{Z}a_l \to \mathbb{R}$ is injective. Thus $M \otimes_{\mathbb{Z}} (\mathbb{Z}a_1 \oplus \cdots \oplus \mathbb{Z}a_l) \to M \otimes_{\mathbb{Z}} \mathbb{R}$ is also injective because M is flat over $\mathbb{Z}$. Namely,

$$(M \otimes_{\mathbb{Z}} \mathbb{Z}a_1) \oplus \cdots \oplus (M \otimes_{\mathbb{Z}} \mathbb{Z}a_l) \to M \otimes_{\mathbb{Z}} \mathbb{R}$$

is injective. Therefore, $x_1 \otimes a_1 = \cdots = x_l \otimes a_l = 0$. Thus $x_1 = \cdots = x_l = 0$ because the homomorphism $M \to M \otimes \mathbb{R}$ given by $x \mapsto x \otimes a_i$ is also injective for each i.

(3) It actually follows from [19, Lemma 1.1.3]. For reader's convenience, we continue its proof in an elementary way. Let us consider the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N \otimes_{\mathbb{Z}} \mathbb{Q} & \xrightarrow{\iota_{\mathbb{Q}}} & M \otimes_{\mathbb{Z}} \mathbb{Q} & \xrightarrow{\varrho_{\mathbb{Q}}} & (M/N) \otimes_{\mathbb{Z}} \mathbb{Q} & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle \tau_N} & & \big\downarrow{\scriptstyle \tau_M} & & \big\downarrow{\scriptstyle \tau_{M/N}} & & \\ 0 & \longrightarrow & N \otimes_{\mathbb{Z}} \mathbb{R} & \xrightarrow{\iota_{\mathbb{R}}} & M \otimes_{\mathbb{Z}} \mathbb{R} & \xrightarrow{\varrho_{\mathbb{R}}} & (M/N) \otimes_{\mathbb{Z}} \mathbb{R} & \longrightarrow & 0 \end{array}$$

Note that horizontal sequences are exact and vertical homomorphisms are injective. Therefore, we have

$$(M \otimes_{\mathbb{Z}} \mathbb{Q}) \cap (N \otimes_{\mathbb{Z}} \mathbb{R}) = \operatorname{Ker}(\varrho_{\mathbb{R}} \circ \tau_M) = \operatorname{Ker}(\tau_{M/N} \circ \varrho_{\mathbb{Q}}) = \operatorname{Ker}(\varrho_{\mathbb{Q}}) = N \otimes_{\mathbb{Z}} \mathbb{Q}.$$

□

Lemma 1.1.2. *Let V be a finite dimensional vector space over $\mathbb{R}$ and let $\langle\ ,\ \rangle$ be an inner product on V. Let Σ be a non-empty finite subset of V and $x \in \Sigma$. Let h be the distance between x and $\langle \Sigma \setminus \{x\} \rangle_{\mathbb{R}}$ (note that $\langle \emptyset \rangle_{\mathbb{R}} = \{0\}$). Then we have the following (for the definition of* $\operatorname{vol}(\Sigma)$*, see Conventions and terminology* 6*):*

1. $\operatorname{vol}(\Sigma) = \operatorname{vol}(\Sigma \setminus \{x\})h$.
2. $\operatorname{vol}(\Sigma) \leq \operatorname{vol}(\Sigma \setminus \{x\}) \sqrt{\langle x, x \rangle}$. *In the case where $\Sigma \setminus \{x\}$ consists of linearly independent vectors, the equality holds if and only if x is orthogonal to $\langle \Sigma \setminus \{x\} \rangle_{\mathbb{R}}$.*
3. *We assume that $\Sigma \setminus \{x\}$ consists of linearly independent vectors and $x \neq 0$. If θ is the angle between x and $\langle \Sigma \setminus \{x\} \rangle_{\mathbb{R}}$, then*

$$\frac{\operatorname{vol}(\Sigma)}{\sqrt{\langle x, x \rangle} \operatorname{vol}(\Sigma \setminus \{x\})} = \sin(\theta).$$

Proof. (1) If $\#(\Sigma) = 1$, then the assertion is obvious, so that we may set $\Sigma = \{x_1, \ldots, x_n\}$, where $x_1 = x$ and $n = \#(\Sigma) \geq 2$. If $x_2, \ldots, x_n$ are linearly dependent, then $\operatorname{vol}(\Sigma) = \operatorname{vol}(\Sigma \setminus \{x_1\}) = 0$. Thus the assertion is also obvious for this case. Moreover, if $x_1 \in \langle x_2, \ldots, x_r \rangle_{\mathbb{R}}$, then $h = \operatorname{vol}(\Sigma) = 0$. Thus we may assume that $x_1, x_2, \ldots, x_n$ are linearly independent. Let $\{e_1, e_2, \ldots, e_r\}$ be an orthonormal basis of $\langle x_1, x_2, \ldots, x_r \rangle_{\mathbb{R}}$ such that $\{e_2, \ldots, e_r\}$ yields an orthonormal basis of $\langle x_2, \ldots, x_r \rangle_{\mathbb{R}}$. We set $x_i = \sum_{j=1}^{r} a_{ij} e_j$. Then $h = |a_{11}|$ and $a_{i1} = 0$ for $i = 2, \ldots, r$. Further, if we set $A = (a_{ij})_{1 \leq i,j \leq r}$ and $A' = (a_{ij})_{2 \leq i,j \leq r}$, then $\operatorname{vol}(\Sigma) = |\det(A)|$ and $\operatorname{vol}(\Sigma \setminus \{x_1\}) = |\det(A')|$. Thus the assertion follows.

(2) and (3) follow from (1). □

Lemma 1.1.3. *Let V be a vector space over $\mathbb{R}$ and let $\langle\ ,\ \rangle : V \times V \to \mathbb{R}$ be a negative semi-definite symmetric bi-linear form, that is, $\langle v, v \rangle \leq 0$ for all $v \in V$. For $x \in V$, the following are equivalent:*

1. $\langle x, x \rangle = 0$.

(2) $\langle x, y\rangle = 0$ *for all* $y \in V$.

Proof. Clearly (2) implies (1). We assume $\langle x, x\rangle = 0$ and $\langle x, y\rangle \neq 0$ for some $y \in V$. First of all,

$$0 \geq \langle y + tx, y + tx\rangle = \langle y, y\rangle + 2t\langle x, y\rangle$$

for all $t \in \mathbb{R}$. Thus, if we set $t = -\langle y, y\rangle/\langle x, y\rangle$, then the above implies $\langle y, y\rangle \geq 0$, and hence $\langle y, y\rangle = 0$. Therefore, if we set $t = \langle x, y\rangle/2$, then we have $\langle x, y\rangle^2 \leq 0$, which is a contradiction because $\langle x, y\rangle \neq 0$. □

Lemma 1.1.4 (Zariski's lemma for vector spaces)**.** *Let* $\mathbb{K}$ *be either* $\mathbb{Q}$ *or* $\mathbb{R}$*. Let* V *be a finite dimensional vector space over* $\mathbb{K}$*, and let* $Q : V \times V \to \mathbb{R}$ *be a symmetric bi-linear form. We assume that there are* $e \in V$ *and generators* $e_1, \ldots, e_n$ *of* V *with the following properties:*

(i) $e = a_1e_1 + \cdots + a_ne_n$ *for some* $a_1, \ldots, a_n \in \mathbb{K}_{>0}$.
(ii) $Q(e, e_i) \leq 0$ *for all* i.
(iii) $Q(e_i, e_j) \geq 0$ *for all* $i \neq j$.
(iv) *If we set* $S = \{(i, j) \mid i \neq j \text{ and } Q(e_i, e_j) > 0\}$*, then, for any* $i \neq j$*, there is a sequence* $i_1, \ldots, i_l$ *such that* $i_1 = i$*,* $i_l = j$*, and* $(i_t, i_{t+1}) \in S$ *for all* $1 \leq t < l$.

Then we have the following:

(1) *If* $Q(e, e_i) < 0$ *for some* i*, then* Q *is negative definite, that is,* $Q(x, x) \leq 0$ *for all* $x \in V$*, and* $Q(x, x) = 0$ *if and only if* $x = 0$.
(2) *If* $Q(e, e_i) = 0$ *for all* i*, then* Q *is negative semi-definite and its kernel is* $\mathbb{K}e$*, that is,* $Q(x, x) \leq 0$ *for all* $x \in V$*, and* $Q(x, x) = 0$ *if and only if* $x \in \mathbb{K}e$.

Proof. Replacing e_i by a_ie_i, we may assume that $a_1 = \cdots = a_n = 1$. If we set $x = x_1e_1 + \cdots + x_ne_n$ for some $x_1, \ldots, x_n \in \mathbb{K}$, then we can show

$$Q(x, x) = \sum_i x_i^2 Q(e_i, e) - \sum_{i<j} (x_i - x_j)^2 Q(e_i, e_j).$$

Thus our assertions follow from easy observations. □

1.2. **Proper currents and admissible continuous functions.** Throughout this subsection, we fix a k-equidimensional complex manifold M. A current of bidegree (l, l) on M is said to be *proper* if, for any $x \in M$, there are an open neighborhood U_x of x and d-closed positive currents T_1, T_2 of bidegree (l, l) on U_x such that $T = T_1 - T_2$ over U_x. We denote the space of proper currents of bidegree (l, l) by $D^{l,l}_{\mathrm{pr}}(M)$. As a proper current is of order 0, for $f \in C^0(M)$ and $T \in D^{l,l}_{\mathrm{pr}}(M)$, we define the wedge product $dd^c([f]) \wedge T$ of $dd^c([f])$ and T to be

$$dd^c([f]) \wedge T := dd^c(fT),$$

that is, $(dd^c([f]) \wedge T)(\eta) = T(f dd^c(\eta))$ for a C^∞-form η of bidegree $(k-l-1, k-l-1)$. It is easy to see that the map

$$C^0(M) \times D^{l,l}_{\mathrm{pr}}(M) \to D^{l+1,l+1}(M)$$

given by $(f, T) \mapsto dd^c([f]) \wedge T$ is multi-linear.

A continuous function $f : M \to \mathbb{R}$ is said to be *admissible* if, for any point $x \in M$, there are an open neighborhood U_x of x and continuous plurisubharmonic functions ϕ_1, ϕ_2 on U_x such that $f = \phi_1 - \phi_2$ over U_x. Note that $dd^c([f])$ is a proper current of bidegree $(1, 1)$. The space of admissible continuous functions on M is denoted by $C^0_{\mathrm{ad}}(M)$. It is easy to see that $C^\infty(M) \subseteq C^0_{\mathrm{ad}}(M)$ (cf. the proof of (3) in Lemma 1.2.1). Moreover, let $B^{1,1}_{\mathrm{ad}}(M)$ be the space of currents T of bidegree $(1,1)$ such that $T = dd^c([\varphi])$ locally for some admissible continuous function φ on each local open neighborhood. As a d-closed positive C^∞-form of bidegree $(1,1)$ can be locally written as $dd^c(C^\infty\text{-function})$ (cf. [7, Chapter 3, (1.18)]), any d-closed real C^∞-form of bidegree $(1,1)$ on M belongs to $B^{1,1}_{\mathrm{ad}}(M)$.

An upper semicontinuous function $f : M \to \mathbb{R} \cup \{-\infty\}$ is called a *quasiplurisubharmonic function* on M if f is locally a sum of a plurisubharmonic function and a C^∞-function. We denote the space of all continuous quasiplurisubharmonic functions on M by $(C^0 \cap \mathrm{QPSH})(M)$. Clearly $(C^0 \cap \mathrm{QPSH})(M) \subseteq C^0_{\mathrm{ad}}(M)$. The subspace generated by $(C^0 \cap \mathrm{QPSH})(M)$ in $C^0_{\mathrm{ad}}(M)$ is denoted by $\langle (C^0 \cap \mathrm{QPSH})(M)\rangle_{\mathbb{R}}$. For a real continuous form α of bidegree $(1,1)$, we define $C^0_{\mathrm{ad}}(M;\alpha)$ to be

$$C^0_{\mathrm{ad}}(M;\alpha) := \left\{ f \in C^0_{\mathrm{ad}}(M) \mid dd^c([f]) + \alpha \geq 0 \right\}.$$

Note that $C^0_{\mathrm{ad}}(M;\alpha) \subseteq (C^0 \cap \mathrm{QPSH})(M)$ (cf. the proof of (3) in Lemma 1.2.1). Let us begin with the following lemma.

Lemma 1.2.1. (1) *If $A \in B^{1,1}_{\mathrm{ad}}(X)$ and $T \in D^{l,l}_{\mathrm{pr}}(X)$, then $A \wedge T \in D^{l+1,l+1}_{\mathrm{pr}}(X)$. Moreover, if A and T are positive, then $A \wedge T$ is also positive.*

(2) *For $A_1, \ldots, A_r \in B^{1,1}_{\mathrm{ad}}(M)$ and $T \in D^{l,l}_{\mathrm{pr}}(M)$, the wedge product*

$$A_1 \wedge \cdots \wedge A_r \wedge T$$

of currents $A_1, \ldots, A_r$ and T is defined inductively as an element of $D^{r+l,r+l}_{\mathrm{pr}}(M)$ by using (1)*, that is,*

$$A_1 \wedge \cdots \wedge A_r \wedge T = A_1 \wedge (A_2 \wedge \cdots \wedge A_r \wedge T).$$

Then the map $B^{1,1}_{\mathrm{ad}}(M)^r \to D^{r+l,r+l}_{\mathrm{pr}}(M)$ given by

$$(A_1, \ldots, A_r) \mapsto A_1 \wedge \cdots \wedge A_r \wedge T$$

is multi-linear and symmetric.

(3) *Let α be a real continuous form of bidegree $(1,1)$. Let $\{f_{1,n}\}_{n=1}^\infty, \ldots, \{f_{r,n}\}_{n=1}^\infty$ be sequences in $C^0_{\mathrm{ad}}(M;\alpha)$ such that $\{f_{i,n}\}_{n=1}^\infty$ converges locally uniformly to $f_i \in C^0_{\mathrm{ad}}(M;\alpha)$ for each i. Then, for $T \in D^{l,l}_{\mathrm{pr}}(M)$, a sequence*

$$\{f_{1,n} dd^c([f_{2,n}]) \wedge \cdots \wedge dd^c([f_{r,n}]) \wedge T\}_{n=1}^\infty$$

converges weakly to

$$f_1 dd^c([f_2]) \wedge \cdots \wedge dd^c([f_r]) \wedge T.$$

Proof. (1) This is a local question, so that we may assume that there are continuous plurisubharmonic functions ϕ_1, ϕ_2 and d-closed positive currents T_1, T_2 such that $A = dd^c([\phi_1]) - dd^c([\phi_2])$ and $T = T_1 - T_2$. Therefore,

$$A \wedge T = \left(dd^c([\phi_1]) \wedge T_1 + dd^c([\phi_2]) \wedge T_2\right) - \left(dd^c([\phi_1]) \wedge T_2 + dd^c([\phi_2]) \wedge T_1\right),$$

as required. The second assertion is obvious.

(2) The multi-linearity of $B^{1,1}_{\mathrm{ad}}(M)^r \to D^{r+l,r+l}_{\mathrm{pr}}(M)$ is obvious. For symmetry, it is sufficient to see that following claim:

Claim 1.2.1.1. *Let f and g be continuous plurisubharmonic functions on M and let T be a proper current on M. Then $dd^c([f]) \wedge dd^c([g]) \wedge T = dd^c([g]) \wedge dd^c([f]) \wedge T$.*

Proof. If f is C^∞, then, for a C^∞-form η,

$$\begin{aligned}(dd^c(f) \wedge dd^c([g]) \wedge T)(\eta) &= (dd^c([g]) \wedge T)(dd^c(f) \wedge \eta) = T(gdd^c(dd^c(f) \wedge \eta)) \\ &= T(gdd^c(f) \wedge dd^c(\eta)) = (dd^c(f) \wedge T)(gdd^c(\eta)) = (dd^c([g]) \wedge dd^c(f) \wedge T)(\eta).\end{aligned}$$

Otherwise, as the question is a local problem, we can find a sequence of C^∞ plurisubharmonic functions $\{f_n\}$ such that $\{f_n\}$ converges locally uniformly to f. Then $\{dd^c(f_n) \wedge dd^c([g]) \wedge T\}$ and $\{dd^c([g]) \wedge dd^c(f_n) \wedge T\}$ converge weakly to $dd^c([f]) \wedge dd^c([g]) \wedge T$ and $dd^c([g]) \wedge dd^c([f]) \wedge T$ respectively (cf. [7, Corollary 3.6 in Chapter 3]), and hence the assertion follows. □

(3) This is also a local question. For $x \in M$, let us consider a local coordinate $(z_1, \ldots, z_k)$ over an open neighborhood U_x of x. As $dd^c(\log(1 + |z_1|^2 + \cdots + |z_k|^2))$ is a positive form, shrinking U_x if necessarily, we can find $\lambda > 0$ such that

$$\lambda dd^c(\log(1 + |z_1|^2 + \cdots + |z_k|^2)) \geq \alpha$$

over U_x. Thus, if we set $\psi = \lambda \log(1 + |z_1|^2 + \cdots + |z_k|^2)$, then $f_i + \psi$, $g_i + \psi$, $f_{i,n} + \psi$ and $g_{i,n} + \psi$ are continuous and plurisubharmonic over U_x for all i and n. Therefore, (3) is a consequence of the convergence theorem for plurisubharmonic functions (cf. [7, Corollary 3.6 in Chapter 3]). □

Next we consider the following lemma.

Lemma 1.2.2. *We assume that M is compact.*

(1) *Let α be a positive continuous form of bidegree* $(1, 1)$. *If $f \in (C^0 \cap \mathrm{QPSH})(M)$, then there is a positive number t_0 such that $f \in C^0_{\mathrm{ad}}(M; t\alpha)$ for all $t \geq t_0$.*
(2) *For $f, g \in \langle (C^0 \cap \mathrm{QPSH})(M) \rangle_{\mathbb{R}}$ and $T \in D^{l,l}_{\mathrm{pr}}(M)$,*

$$f dd^c([g]) \wedge T \equiv g dd^c([f]) \wedge T \mod N^{l+1,l+1}(M)$$

(*for the definition of $N^{l+1,l+1}(M)$, see Conventions and terminology* 1).
(3) *Let T be a d-closed positive current of bidegree* $(k - 1, k - 1)$. *Then*

$$\int_M f dd^c([f]) \wedge T \leq 0$$

for $f \in \langle (C^0 \cap \mathrm{QPSH})(M) \rangle_{\mathbb{R}}$.

Proof. (1) For each point $x \in M$, there are an open neighborhood U_x of x, a plurisubharmonic function p_x on U_x and a C^∞-function q_x on U_x such that $f = p_x + q_x$ over U_x. If we consider a smaller U_x, then we can write α and $dd^c(q_x)$ as follows:

$$\alpha = \sqrt{-1} \sum_{ij} \alpha_{ij} dz_i \wedge d\bar{z}_j \quad \text{and} \quad dd^c(q_x) = \sqrt{-1} \sum_{ij} \beta_{ij} dz_i \wedge d\bar{z}_j,$$

where $(z_1, \ldots, z_k)$ is a local coordinate on U_x. As $(\alpha_{ij}(x))$ is a positive definite hermitian matrix, we can find a positive number s_x such that $s_x(\beta_{ij}(x)) + (\alpha_{ij}(x))$ is positive. Note that $s_x(\beta_{ij}) + (\alpha_{ij})$ is continuous on U_x. Thus, shrinking U_x if necessarily, $s_x(\beta_{ij}) + (\alpha_{ij})$ is positive on U_x, and hence, for $t \geq t_x := 1/s_x$,

$$dd^c(q_x) + t\alpha = (t - t_x)\alpha + t_x(s_x dd^c(q_x) + \alpha) \geq 0$$

on U_x. Because of the compactness of X, there are finitely many $x_1, \ldots, x_r \in X$ with $X = U_{x_1} \cup \cdots \cup U_{x_r}$. If we set $t_0 = \max\{t_{x_1}, \ldots, x_{x_r}\}$, then, for $t \geq t_0$,

$$dd^c([f]) + t\alpha = dd^c([p_{x_i}]) + (dd^c(q_{x_i}) + t\alpha)$$

is positive over U_{x_i}, as required.

(2) By our assumption, there are $f_1, f_2, g_1, g_2 \in (C^0 \cap \mathrm{QPSH})(M)$ such that $f = f_1 - f_2$ and $g = g_1 - g_2$. Therefore, we may assume that $f, g \in (C^0 \cap \mathrm{QPSH})(M)$. If f is C^∞, then, for a d-closed C^∞-form η of bidegree $(k-l-1, k-l-1)$,

$$(f dd^c([g]) \wedge T)(\eta) = T(g dd^c(f\eta)) = T(g dd^c(f) \wedge \eta) = (g dd^c(f) \wedge T)(\eta).$$

Otherwise, by (1), we can take a positive C^∞-form α of bidegree $(1,1)$ with $f \in C^0_{\mathrm{ad}}(X;\alpha)$. Thus, by [1] or [20, Lemma 4.2], we can find a sequence of C^∞-functions $\{f_n\}$ in $C^0_{\mathrm{ad}}(M;\alpha)$ such that $\{f_n\}$ converges uniformly to f. Therefore, by (3) in Lemma 1.2.1,

$$f_n dd^c([g]) \wedge T \quad \text{and} \quad g dd^c(f_n) \wedge T$$

converges weakly to $f dd^c([g]) \wedge T$ and $g dd^c([f]) \wedge T$ respectively. Thus (2) follows from the case where f is C^∞.

(3) First we assume f is C^∞. Then, as

$$\partial\left(\frac{\sqrt{-1}}{2\pi} f \bar{\partial}(f)\right) = \frac{\sqrt{-1}}{2\pi}\partial(f) \wedge \bar{\partial}(f) + f dd^c(f)$$

and T is ∂-closed, we have

$$0 = -(\partial T)\left(\frac{\sqrt{-1}}{2\pi} f \bar{\partial}(f)\right) = T\left(\partial\left(\frac{\sqrt{-1}}{2\pi} f \bar{\partial}(f)\right)\right) = T\left(\frac{\sqrt{-1}}{2\pi}\partial(f) \wedge \bar{\partial}(f)\right) + T(f dd^c(f)).$$

Note that

$$T\left(\frac{\sqrt{-1}}{2\pi}\partial(f) \wedge \bar{\partial}(f)\right) \geq 0.$$

Thus we have the assertion in the case where f is C^∞.

In general, by using (1), we can find continuous functions g, h on M and a positive C^∞-form α such that $g, h \in C^0_{\mathrm{ad}}(M;\alpha)$ and $f = g - h$. Thus, by [1] or [20, Lemma 4.2], there are sequences $\{g_n\}_{n=1}^\infty$ and $\{h_n\}_{n=1}^\infty$ of C^∞-functions on M such that $g_n, h_n \in C^0_{\mathrm{ad}}(M;\alpha)$ for all $n \geq 1$ and

$$\lim_{n\to\infty} \|g_n - g\|_{\sup} = \lim_{n\to\infty} \|h_n - h\|_{\sup} = 0.$$

Then, by (3) in Lemma 1.2.1, a sequence $\{(g_n - h_n) dd^c(g_n - h_n) \wedge T\}$ of currents converge weakly to $(g-h) dd^c([g-h]) \wedge T = f dd^c([f]) \wedge T$. Thus, (3) follows from the previous case. □

From now on, we assume that M is compact and Kähler. Let T be a d-closed positive current of bidegree $(k-1, k-1)$. For $f, g \in C^0_{\mathrm{ad}}(M)$, we define $I_T(f,g)$ to be

$$I_T(f,g) := \int_M f dd^c([g]) \wedge T,$$

which will be used to see the equality condition of the Hodge index theorem (cf. Theorem 2.2.3 and Theorem 2.2.5). Then we have the following proposition.

Proposition 1.2.3. *I_T is a symmetric and negative semidefinite bi-linear form on*

$$\langle (C^0 \cap \mathrm{QPSH})(M)\rangle_{\mathbb{R}},$$

that is, the following properties are satisfied:

(1) $I_T(af + bf', g) = aI_T(f, g) + bI_T(f', g)$ *and* $I_T(f, ag + bg') = aI_T(f, g) + bI_T(f, g')$ *hold for all* $f, f', g, g' \in C^0_{\mathrm{ad}}(M)$ *and* $a, b \in \mathbb{R}$.
(2) $I_T(f, g) = I_T(g, f)$ *for all* $f, g \in \langle (C^0 \cap \mathrm{QPSH})(M)\rangle_{\mathbb{R}}$.
(3) $I_T(f, f) \leq 0$ *for all* $f \in \langle (C^0 \cap \mathrm{QPSH})(M)\rangle_{\mathbb{R}}$.

Moreover, let $A_1, \ldots, A_{k-1} \in B^{1,1}_{\mathrm{ad}}(M)$ *and let* ω *be a Kähler form of* M. *We assume that, for each* $i = 1, \ldots, k-1$, *there is* $\epsilon_i \in \mathbb{R}_{>0}$ *with* $A_i \geq \epsilon_i\omega$. *If* $T = A_1 \wedge \cdots \wedge A_{k-1}$, *then*

$$I_T(f, f) = 0 \quad \Longleftrightarrow \quad f \text{ is a constant.}$$

Proof. (1) is obvious. (2) follows from (2) in Lemma 1.2.2. (3) is a consequence of (3) in Lemma 1.2.2. Finally we consider the last assertion. Clearly if f is a constant, then $I_T(f, f) = 0$. We set

$$T' = (\epsilon_1^{-1}A_1) \wedge \cdots \wedge (\epsilon_{k-1}^{-1}A_{k-1}) = (\epsilon_1 \cdots \epsilon_{k-1})^{-1}T.$$

Then, as $\epsilon_i^{-1}A_i - \omega$ is positive, by (1) in Lemma 1.2.1, there is a d-closed positive current T'' of bidegree $(k-1, k-1)$ such that $T' = \omega^{k-1} + T''$. In particular, by (3),

$$I_{T'}(f, f) \leq I_{\omega^{k-1}}(f, f) \leq 0$$

for $f \in \langle (C^0 \cap \mathrm{QPSH})(M)\rangle_{\mathbb{R}}$. Note that we can define a Laplacian $\square_\omega$ by the equation:

$$-dd^c(f) \wedge \omega^{k-1} = \square_\omega(f)\omega^k \quad (f \in C^\infty(M)).$$

Let us see that $\square_\omega$ is elliptic. This is a local question. Let $\theta_1, \ldots, \theta_k$ be a local orthonormal frame of the holomorphic cotangent bundle Ω^1_M with respect to the metric arising from the Kähler form ω so that $\omega = \sqrt{-1}\sum_i \theta_1 \wedge \bar{\theta}_j$. If we set $dd^c(f) = \sqrt{-1}\sum_{i,j} a_{ij}\theta_i \wedge \bar{\theta}_j$, then

$$\square_\omega(f) = -\frac{1}{k}\sum_i a_{ii}.$$

On the other hand, we set $dz_s = \sum_i c_{si}\theta_i$ for $s = 1, \ldots, k$, where $(z_1, \ldots, z_k)$ is a local coordinate. Then

$$dd^c(f) = \frac{\sqrt{-1}}{2\pi}\sum_{s,t}\frac{\partial^2(f)}{\partial z_s \partial \bar{z}_t}dz_s \wedge d\bar{z}_t = \frac{\sqrt{-1}}{2\pi}\sum_{s,t,i,j}\frac{\partial^2(f)}{\partial z_s \partial \bar{z}_t}c_{si}\bar{c}_{tj}\theta_i \wedge \bar{\theta}_j,$$

so that

$$\square_\omega(f) = -\frac{1}{2k\pi}\sum_{s,t}\left(\sum_i c_{si}\bar{c}_{ti}\right)\frac{\partial^2(f)}{\partial z_s \partial \bar{z}_t}.$$

Thus it is sufficient to show that a matrix $D = (\sum_i c_{si}\bar{c}_{ti})_{1 \leq s,t \leq k}$ is positive-definite. This is obvious because $D = C \cdot$ (the transpose of $\bar{C}$) and $\det(C) \neq 0$, where $C = (c_{si})_{1 \leq s,i \leq k}$.

Therefore,

$$
\begin{aligned}
I_T(f,f) = 0 \quad &\Longrightarrow \quad I_{T'}(f,f) = 0 \quad \Longrightarrow \quad I_{\omega^{k-1}}(f,f) = 0 \\
&\Longrightarrow \quad I_{\omega^{k-1}}(g,f) = 0 \text{ for all } g \in C^\infty(M) \ (\because \text{ Lemma 1.1.3}) \\
&\Longrightarrow \quad dd^c([f]) \wedge \omega^{k-1} = 0 \text{ as a current} \\
&\Longrightarrow \quad \square_\omega([f]) = 0 \\
&\Longrightarrow \quad f \text{ is harmonic } (\because \text{ the regularity of elliptic operators}) \\
&\Longrightarrow \quad f \text{ is a constant,}
\end{aligned}
$$

as required. □

1.3. **A variant of Gromov's inequality for $\mathbb{R}$-Cartier divisors.** In this subsection, we would like to consider a generalization of [17, Lemma 1.1.4] to $\mathbb{R}$-Cartier divisors.

Lemma 1.3.1. *Let X be a d-dimensional compact Kähler manifold and let ω be a Kähler form on X. Let $D_1, \ldots, D_l$ be $\mathbb{R}$-Cartier divisors on X. For each $i = 1, \ldots, l$, let g_i be a D_i-Green function of C^∞-type. Let U be an open set of X such that U is not empty on each connected component of X. Then there are constants $C_1, \ldots, C_l \geq 1$ such that C_i depends only on g_i and U, and that*

$$
\sup_{x \in X}\{|s|_{m_1 g_1 + \cdots + m_l g_l}(x)\} \leq C_1^{m_1} \cdots C_l^{m_l} \sup_{x \in U}\{|s|_{m_1 g_1 + \cdots + m_l g_l}(x)\}.
$$

for all $m_1, \ldots, m_l \in \mathbb{R}_{\geq 0}$ and all $s \in H^0(X, m_1 D_1 + \cdots + m_l D_l)$. Moreover, if $D_i = 0$ and g_i is a constant function, then $C_i = 1$.

Proof. Clearly we may assume that X is connected. Shrinking U if necessarily, we may identify U with $\{x \in \mathbb{C}^d \mid |x| < 1\}$. We set $W = \{x \in \mathbb{C}^d \mid |x| < 1/2\}$. In this proof, we define a Laplacian $\square_\omega$ by the formula:

$$
-\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}(g) \wedge \omega^{\wedge(d-1)} = \square_\omega(g)\omega^{\wedge d}.
$$

Let ω_i be a C^∞-form of $(1,1)$-type given by $dd^c([g_i]) + \delta_{D_i} = [\omega_i]$. Let a_i be a C^∞-function given by $\omega_i \wedge \omega^{\wedge(d-1)} = a_i \omega^{\wedge d}$. We choose a C^∞-function ϕ_i on X such that

$$
\int_X a_i \omega^{\wedge d} = \int_X \phi_i \omega^{\wedge d}
$$

and that ϕ_i is identically zero on $X \setminus W$. Thus we can find a C^∞-function F_i with $\square_\omega(F_i) = a_i - \phi_i$. Note that $\square_\omega(F_i) = a_i$ on $X \setminus W$.

Let $s \in H^0(X, m_1 D_1 + \cdots + m_l D_l)$. We set

$$
f = |s|^2_{m_1 g_1 + \cdots + m_l g_l} \exp(-(m_1 F_1 + \cdots + m_l F_l)).
$$

Note that f is continuous over X and $\log(f)$ is C^∞ over $X \setminus Z_s$, where

$$
Z_s = \operatorname{Supp}((s) + m_1 D_1 + \cdots + m_l D_l).
$$

Claim 1.3.1.1. $\max_{x \in X \setminus W}\{f(x)\} = \max_{x \in \partial(W)}\{f(x)\}$.

If f is a constant over $X \setminus W$, then our assertion is obvious, so that we assume that f is not a constant over $X \setminus W$. In particular, $s \neq 0$. Since

$$-\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}(\log(|s|^2_{m_1 g_1+\cdots+m_l g_l})) = m_1\omega_1 + \cdots + m_l\omega_l \quad \text{over } X \setminus Z_s,$$

we have $\square_{\omega}(\log(f)) = 0$ on $X \setminus (W \cup Z_s)$. Let us choose $x_0 \in X \setminus W$ such that the continuous function f over $X \setminus W$ takes the maximum value at x_0. Note that

$$x_0 \in X \setminus (W \cup Z_s).$$

For, if $Z_s = \emptyset$, then our assertion is obvious. Otherwise, f is zero at any point of Z_s. Since $\log(f)$ is harmonic over $X \setminus (W \cup Z_s)$, $\log(f)$ takes the maximum value at x_0 and $\log(f)$ is not a constant, we have $x_0 \in \partial(W)$ by virtue of the maximum principle of harmonic functions. Thus the claim follows. □

We set

$$b_i = \min_{x \in X \setminus W}\{\exp(-F_i)\}, \quad B_i = \max_{x \in \partial(W)}\{\exp(-F_i)\} \quad \text{and} \quad C_i = B_i/b_i.$$

Then

$$b_1^{m_1}\cdots b_l^{m_l}|s|^2_{m_1 g_1+\cdots+m_l g_l} \leq f$$

over $X \setminus W$ and

$$f \leq B_1^{m_1}\cdots B_l^{m_l}|s|^2_{m_1 g_1+\cdots+m_l g_l}$$

over $\partial(W)$. Hence

$$\max_{x \in X\setminus W}\{|s|^2_{m_1 g_1+\cdots+m_l g_l}\} \leq C_1^{m_1}\cdots C_l^{m_l}\max_{x\in\partial(W)}\{|s|^2_{m_1 g_1+\cdots+m_l g_l}\} \leq C_1^{m_1}\cdots C_l^{m_l}\max_{x\in\overline{W}}\{|s|^2_{m_1 g_1+\cdots+m_l g_l}\}.$$

which implies that

$$\max_{x\in X}\{|s|^2_{m_1 g_1+\cdots+m_l g_l}\} \leq C_1^{m_1}\cdots C_l^{m_l}\max_{x\in\overline{W}}\{|s|^2_{m_1 g_1+\cdots+m_l g_l}\},$$

as required. The last assertion is obvious by our construction because $F_i = 0$ in this case. □

2. Hodge index theorem for arithmetic $\mathbb{R}$-Cartier divisors

In this section, we would like to observe the Hodge index theorem for arithmetic $\mathbb{R}$-Cartier divisors and apply it to the pseudo-effectivity of arithmetic divisors. A negative definite quadric form over $\mathbb{Q}$ does not necessarily extend to a negative definite quadric form over $\mathbb{R}$. For example, the quadric form $q(x,y) = -(x+\sqrt{2}y)^2$ on $\mathbb{Q}^2$ is negative definite, but it is not negative definite on $\mathbb{R}^2$. In this sense, the equality condition of the Hodge index theorem for arithmetic $\mathbb{R}$-Cartier divisors is not an obvious generalization. In addition, the equality condition is crucial to consider the pseudo-effectivity of $\mathbb{R}$-Cartier divisors.

In SubSection 2.1, we compare the arithmetic intersection number in [20, SubSection 6.4] with the classical one due to Zhang and Maillot (cf. [24], [25], [13]). SubSection 2.2 is devoted to the Hodge index theorem for arithmetic $\mathbb{R}$-Cartier divisors. Especially its equality condition is treated carefully. In SubSection 2.3, we consider a necessary condition for the pseudo-effectivity of arithmetic $\mathbb{R}$-Cartier divisors as an application of the equality condition of the arithmetic Hodge index theorem.

Throughout this section, X will be a d-dimensional, generically smooth, normal projective arithmetic variety. Moreover, let

$$X \xrightarrow{\pi} \operatorname{Spec}(O_K) \to \operatorname{Spec}(\mathbb{Z})$$

be the Stein factorization of $X \to \operatorname{Spec}(\mathbb{Z})$, where K is a number field and O_K is the ring of integers in K.

2.1. **Generalized intersection pairing on arithmetic varieties.** Let $\widehat{\operatorname{Div}}_{C^0}^{\operatorname{Nef}}(X)_{\mathbb{R}}$ be the subspace of $\widehat{\operatorname{Div}}_{C^0}(X)_{\mathbb{R}}$ consisting of integrable arithmetic $\mathbb{R}$-Cartier divisors of C^0-type on X, that is, $\widehat{\operatorname{Div}}_{C^0}^{\operatorname{Nef}}(X)_{\mathbb{R}}$ is the subspace generated by $\widehat{\operatorname{Nef}}_{C^0}(X)_{\mathbb{R}}$. For $\overline{D}_1, \ldots, \overline{D}_d \in \widehat{\operatorname{Div}}_{C^0}^{\operatorname{Nef}}(X)_{\mathbb{R}}$, we can define the intersection number $\widehat{\operatorname{deg}}(\overline{D}_1 \cdots \overline{D}_d)$ as follows: If $\overline{D}_1, \ldots, \overline{D}_d \in \widehat{\operatorname{Nef}}_{C^0}(X)_{\mathbb{R}}$, then it is given by

$$\widehat{\operatorname{deg}}(\overline{D}_1 \cdots \overline{D}_d) = \frac{1}{d!} \sum_{\emptyset \neq I \subseteq \{1,\ldots,d\}} (-1)^{d-\#(I)} \widehat{\operatorname{vol}}\left(\sum_{i\in I} \overline{D}_i\right).$$

In general, we extend the above by multi-linearity (for details, see [20, SubSection 6.4]). Note that if $\overline{D}_1, \ldots, \overline{D}_d \in \widehat{\operatorname{Div}}_{C^\infty}(X)_{\mathbb{R}}$, then $\widehat{\operatorname{deg}}(\overline{D}_1 \cdots \overline{D}_d)$ coincides with the usual arithmetic intersection number because the self intersection number of a nef arithmetic $\mathbb{R}$-Cartier divisor of C^∞-type in the usual sense is equal to its arithmetic volume (cf. [20, Claim 6.4.2.2]). The following proposition is the main result of this subsection. Especially, (3) means that the above intersection number coincides with other definitions [24, Lemma 6.5], [25, §1] and [13, §5]. In this sense, this subsection provides a quick introduction to the generalized intersection pairing on arithmetic varieties.

Here we need to fix a notation. Let $u_1, \ldots, u_p \in \langle (C^0 \cap \operatorname{QPSH})(X(\mathbb{C}))\rangle_{\mathbb{R}}$ and $B_1, \ldots, B_p \in B^{1,1}_{\mathrm{ad}}(X(\mathbb{C}))$. Let I be a non-empty subset of $\{1, \ldots, p\}$ and $J = \{1, \ldots, p\} \setminus I$. If we set $I = \{i_1, \ldots, i_k\}$ and $J = \{j_1, \ldots, j_l\}$, then, by Lemma 1.2.1, the class of

$$u_{i_1} dd^c([u_{i_2}]) \wedge \cdots \wedge dd^c([u_{i_k}]) \wedge B_{j_1} \wedge \cdots \wedge B_{j_l}$$

in $D^{p-1,p-1}(X(\mathbb{C}))/N^{p-1,p-1}(X(\mathbb{C}))$ does not depend on the choice of $i_1, \ldots, i_k$ and $j_1, \ldots, j_l$, so that it is denoted by $udd^c(u_I) \wedge B_J$.

Proposition 2.1.1. (1) *If $\overline{D} = \overline{D}' + (0, \eta)$ for $\overline{D}, \overline{D}' \in \widehat{\operatorname{Div}}_{C^0}^{\operatorname{Nef}}(X)_{\mathbb{R}}$ and $\eta \in C^0(X)$, then $\eta \in \langle (C^0 \cap \operatorname{QPSH})(X(\mathbb{C}))\rangle_{\mathbb{R}}$.*

(2) *Let $\overline{D}_1, \ldots, \overline{D}_d \in \widehat{\operatorname{Div}}_{C^0}^{\operatorname{Nef}}(X)_{\mathbb{R}}$, $\overline{A}_1, \ldots, \overline{A}_d \in \widehat{\operatorname{Div}}_{C^\infty}(X)_{\mathbb{R}}$ and $u_1, \ldots, u_d \in C^0(X)$ such that $\overline{D}_i = \overline{A}_i + (0, u_i)$ for $i = 1, \ldots, d$. Then the quantity*

$$\widehat{\operatorname{deg}}(\overline{A}_1 \cdots \overline{A}_d) + \frac{1}{2} \sum_{\emptyset \neq I \subseteq \{1,\ldots,d\}} \int_{X(\mathbb{C})} udd^c(u_I) \wedge c_1(\overline{A}_J)$$

does not depend on the choice of $\overline{A}_1, \ldots, \overline{A}_d$ and $u_1, \ldots, u_d$. If we denote the above number by $\widehat{\operatorname{deg}}'(\overline{D}_1 \cdots \overline{D}_d)$, then the map

$$\left(\widehat{\operatorname{Div}}_{C^0_{\mathrm{ad}}}(X)_{\mathbb{R}}\right)^d \to \mathbb{R}$$

given by $(\overline{D}_1, \ldots, \overline{D}_d) \mapsto \widehat{\operatorname{deg}}'(\overline{D}_1 \cdots \overline{D}_d)$ is symmetric and multi-linear.

(3) $\widehat{\deg}(\overline{D}_1 \cdots \overline{D}_d) = \widehat{\deg}'(\overline{D}_1 \cdots \overline{D}_d)$ *for* $\overline{D}_1, \ldots, \overline{D}_d \in \widehat{\mathrm{Div}}_{C^0}^{\mathrm{Nef}}(X)_{\mathbb{R}}$.

(4) *Let* $\overline{D}_1, \cdots, \overline{D}_d, \overline{D}'_1, \cdots, \overline{D}'_d \in \widehat{\mathrm{Div}}_{C^0}^{\mathrm{Nef}}(X)_{\mathbb{R}}$ *and* $\eta_1, \ldots, \eta_d \in C^0(X)$ *such that* $\overline{D}_i = \overline{D}'_i + (0, \eta_i)$ *for* $i = 1, \ldots, d$. *Then*

$$\widehat{\deg}(\overline{D}_1 \cdots \overline{D}_d) = \widehat{\deg}(\overline{D}'_1 \cdots \overline{D}'_d) + \frac{1}{2} \sum_{\emptyset \neq I \subseteq \{1,\ldots,d\}} \int_{X(\mathbb{C})} \eta dd^c(\eta_I) \wedge c_1(\overline{D}'_J).$$

Proof. (1) We can find $\overline{E}, \overline{F}, \overline{E}', \overline{F}' \in \widehat{\mathrm{Nef}}_{C^0}(X)_{\mathbb{R}}$ such that $\overline{D} = \overline{E} - \overline{F}$ and $\overline{D}' = \overline{E}' - \overline{F}'$. Then, as $\overline{E} + \overline{F}' = \overline{E}' + \overline{F} + (0, \eta)$, the assertion of (1) is obvious if we compare two local equations of the Green functions in $\overline{E} + \overline{F}'$ and $\overline{E}' + \overline{F}$.

(2) In order to proceed with arguments, we need several notations. Let $\widehat{Z}^p(X)_{\mathbb{R}}$ be the set of all pairs (Z, T) such that Z is a codimension p $\mathbb{R}$-cycle on X (i.e. $Z = a_1 Z_1 + \cdots + a_r Z_r$ for some $a_1, \ldots, a_r \in \mathbb{R}$ and codimension p integral subschemes $Z_1, \ldots, Z_r$ of X) and T is a real current of bidegree $(p-1, p-1)$ on $X(\mathbb{C})$. Let $\widehat{R}^p(X)'_{\mathbb{R}}$ be the vector subspace generated by the following elements:

(a) $((f), -[\log |f|^2])$, where f is a rational function on some integral closed subscheme Y of codimension $p-1$ and $[\log |f|^2]$ is the current defined by

$$[\log |f|^2](\gamma) = \int_{Y(\mathbb{C})} (\log |f|^2) \gamma.$$

(b) $(0, T)$, where T is a real current in $N^{p-1,p-1}(X(\mathbb{C}))$. (for the definition of $N^{p-1,p-1}(X(\mathbb{C}))$, see Conventions and terminology 1).

We set

$$\widehat{\mathrm{CH}}^p(X)'_{\mathbb{R}} := \widehat{Z}^p(X)_{\mathbb{R}} / \widehat{R}^p(X)'_{\mathbb{R}}.$$

Let $\overline{A}$ be an arithmetic $\mathbb{R}$-Cartier divisor of C^∞-type. Then we can define a homomorphism

$$\widehat{c}_1(\overline{A}) \cdot : \widehat{\mathrm{CH}}^p(X)'_{\mathbb{R}} \to \widehat{\mathrm{CH}}^{p+1}(X)'_{\mathbb{R}}$$

given by $\widehat{c}_1(\overline{A}) \cdot (Z, T) = \widehat{c}_1(\overline{A}) \cdot (Z, 0) + (0, c_1(\overline{A}) \wedge T)$. Note that

$$\widehat{c}_1(\overline{A}) \cdot \widehat{c}_1(\overline{B}) \cdot = \widehat{c}_1(\overline{B}) \cdot \widehat{c}_1(\overline{A}) \cdot$$

for arithmetic $\mathbb{R}$-Cartier divisors $\overline{A}$ and $\overline{B}$ of C^∞-type.

Claim 2.1.1.1. *The class of*

$$Z(\overline{A}_1, \ldots . \overline{A}_p, u_1, \ldots, u_p) := \widehat{c}_1(\overline{A}_1) \cdots \widehat{c}_1(\overline{A}_p) + \sum_{\emptyset \neq I \subseteq \{1,\ldots,p\}} (0, u dd^c(u_I) \wedge c_1(\overline{A}_J))$$

in $\widehat{\mathrm{CH}}^p(X)'_{\mathbb{R}}$ *does not depend on the choice of* $\overline{A}_1, \ldots, \overline{A}_p$ *and* $u_1, \ldots, u_p$ *for* $p = 1, \ldots, d$.

Proof. Let $\overline{B}_1, \ldots, \overline{B}_p$ be arithmetic $\mathbb{R}$-Cartier divisors of C^∞-type and $v_1, \ldots, v_p \in C^0_{\mathrm{ad}}(X)$ such that $\overline{D}_i = \overline{B}_i + (0, v_i)$ for $i = 1, \ldots, p$. Then we can find C^∞-function $\phi_1, \ldots, \phi_p$ such that $u_i = v_i + \phi_i$ and $\overline{B}_i = \overline{A}_i + (0, \phi_i)$ for $i = 1, \ldots, p$. We need to see that

$$Z(\overline{A}_1, \ldots, \overline{A}_p, u_1, \ldots, u_p) = Z(\overline{B}_1, \ldots, \overline{B}_p, v_1, \ldots, v_p)$$

in $\widehat{\mathrm{CH}}^{p}(X)'_{\mathbb{R}}$. We prove it by induction on p. If $p = 1$, then the assertion is obvious, so that we assume $p > 1$. By the hypothesis of induction, we have

$$Z(\overline{A}_2, \ldots, \overline{A}_p, u_2, \ldots, u_p) = Z(\overline{B}_2, \ldots, \overline{B}_p, v_2, \ldots, v_p)$$

in $\widehat{\mathrm{CH}}^{p-1}(X)'_{\mathbb{R}}$, which implies

$$\widehat{c}_1(\overline{A}_1) \cdot Z(\overline{A}_2, \ldots, \overline{A}_p, u_2, \ldots, u_p) = (\widehat{c}_1(\overline{B}_1) - \widehat{c}_1(0, \phi_1)) \cdot Z(\overline{B}_2, \ldots, \overline{B}_p, v_2, \ldots, v_p)$$

in $\widehat{\mathrm{CH}}^{p-1}(X)'_{\mathbb{R}}$. The left hand side is equal to

$$\begin{aligned} Z(\overline{A}_1, \ldots, \overline{A}_p, u_1, \ldots, u_p) &- \sum_{1 \in I \subseteq \{1, \ldots, p\}} (0, udd^c(u_I) \wedge c_1(\overline{A}_J)) \\ &= Z(\overline{A}_1, \ldots, \overline{A}_p, u_1, \ldots, u_p) - \sum_{I' \subseteq \{2, \ldots, p\}} (0, u_1 dd^c(u_{I'}) \wedge c_1(\overline{A}_{J'})), \end{aligned}$$

where $J' = \{2, \ldots, p\} \setminus I'$. Moreover, the right hand side is equal to

$$\begin{aligned} Z(\overline{B}_1, \ldots, \overline{B}_p, v_1, \ldots, v_p) &- \sum_{I' \subseteq \{2, \ldots, p\}} (0, v_1 dd^c(v_{I'}) \wedge c_1(\overline{B}_{J'})) \\ &- \widehat{c}_1(\overline{B}_2) \cdots \widehat{c}_1(\overline{B}_p) \cdot \widehat{c}_1(0, \phi_1) - \sum_{\emptyset \neq I' \subseteq \{2, \ldots, p\}} \widehat{c}_1(0, \phi_1) \cdot (0, vdd^c(v_{I'}) \wedge c_1(\overline{B}_{J'})) \\ = Z(\overline{B}_1, \ldots, \overline{B}_p, v_1, \ldots, v_p) &- \sum_{I' \subseteq \{2, \ldots, p\}} (0, v_1 dd^c(v_{I'}) \wedge c_1(\overline{B}_{J'})) - \sum_{I' \subseteq \{2, \ldots, p\}} (0, \phi_1 dd^c(v_{I'}) \wedge c_1(\overline{B}_{J'})) \\ &= Z(\overline{B}_1, \ldots, \overline{B}_p, v_1, \ldots, v_p) - \sum_{I' \subseteq \{2, \ldots, p\}} (0, u_1 dd^c(v_{I'}) \wedge c_1(\overline{B}_{J'})). \end{aligned}$$

in $\widehat{\mathrm{CH}}^{p-1}(X)'_{\mathbb{R}}$. Therefore, we can see that

$$Z(\overline{A}_1, \ldots, \overline{A}_p, u_1, \ldots, u_p) - Z(\overline{B}_1, \ldots, \overline{B}_p, v_1, \ldots, v_p)$$

is equal to

$$\left(0, u_1 \sum_{I' \subseteq \{2, \ldots, p\}} \left(dd^c(u_{I'}) \wedge c_1(\overline{A}_{J'}) - dd^c(v_{I'}) \wedge c_1(\overline{B}_{J'})\right)\right),$$

which is zero by the following Lemma 2.1.2. □

Applying the above claim to the case where $p = d$, the first assertion follows. The second assertion can be easily checked by using its definition.

(3) For this purpose, it is sufficient to show that $\widehat{\deg}'(\overline{D}^d) = \widehat{\mathrm{vol}}(\overline{D})$ for $\overline{D} = (D, g) \in \widehat{\mathrm{Nef}}_{C^0}(X)_{\mathbb{R}}$. Let $\overline{A}$ be an ample arithmetic Cartier divisor of C^∞-type. We assume

$$\widehat{\deg}'((\overline{D} + (1/n)\overline{A})^{d+1}) = \widehat{\mathrm{vol}}(\overline{D} + (1/n)\overline{A})$$

for all $n > 0$. Then, using the continuity of $\widehat{\mathrm{vol}}$, we can see $\widehat{\deg}'(\overline{D}^d) = \widehat{\mathrm{vol}}(\overline{D})$. Thus we may assume D is ample, so that there is a D-Green function h such that $\alpha := c_1(D, h)$ is positive. We set $\overline{D}' = (D, h)$ and $\phi = g - h$. Then ϕ is continuous and

$dd^c([\phi]) + \alpha \geq 0$. Therefore, by [1] or [20, Lemma 4.2], we can take a sequence of C^∞-functions $\{\phi_n\}$ such that $\lim_{n\to\infty} \|\phi_n - \phi\|_{\sup} = 0$, and that $\phi \leq \phi_n$ and $\phi_n \in C^0_{\mathrm{ad}}(X;\alpha)$ for all n. We set $\overline{D}_n = \overline{D}' + (0, \phi_n)$. Then $\overline{D}_n$ is a nef arithmetic $\mathbb{R}$-Cartier divisor of C^∞-type, and hence $\widehat{\deg}'(\overline{D}_n^d) = \widehat{\mathrm{vol}}(\overline{D}_n)$ for all n by [20, Claim 6.4.2.2]. As $\lim_{n\to\infty} \widehat{\mathrm{vol}}(\overline{D}_n) = \widehat{\mathrm{vol}}(\overline{D})$ by the continuity of $\widehat{\mathrm{vol}}$, it is sufficient to see that

$$\lim_{n\to\infty} \widehat{\deg}'(\overline{D}_n^d) = \widehat{\deg}'(\overline{D}^d).$$

Note that

$$\widehat{\deg}'(\overline{D}_n^d) = \widehat{\deg}'((\overline{D}' + (0,\phi_n))^d) = \widehat{\deg}'(\overline{D}'^d) + \sum_{i=1}^d \binom{d}{i} \int_{X(\mathbb{C})} \phi_n dd^c(\phi_n)^{i-1} \wedge \alpha^{d-i}.$$

In addition, by (3) in Lemma 1.2.1, $\{\phi_n dd^c(\phi_n)^{i-1} \wedge \alpha^{d-i}\}$ converges weakly to

$$\phi dd^c([\phi])^{i-1} \wedge \alpha^{d-i}$$

for each i. Thus we have the assertion.

(3) By using the symmetry and multi-linearity of $\widehat{\deg}(\overline{D}_1 \cdots \overline{D}_d)$, it is sufficient to see that

$$\widehat{\deg}((0,\eta_1)\cdot \overline{D}_2 \cdots \overline{D}_d) = \frac{1}{2} \sum_{I \subseteq \{2,\ldots,d\}} \int_{X(\mathbb{C})} \eta_1 dd^c(u_I) \wedge c_1(\overline{D}_J),$$

which is a straightforward calculation by using the definition in (2). □

Lemma 2.1.2. *Let V and W be vector spaces over $\mathbb{R}$ and let $f : V^s \to W$ be a symmetric multi-linear map. Let $a_1, \ldots, a_s, b_1, \ldots, b_s$ be elements of V. For a subset I of $\{1, \ldots, s\}$, we set $I = \{i_1, \ldots, i_k\}$ and $J = \{j_1, \ldots, j_l\}$, where $J = \{1, \ldots, s\} \setminus I$ and $k + l = s$. Then*

$$f(a_{i_1}, \ldots, a_{i_k}, b_{j_1}, \ldots, b_{j_l})$$

does not depend on the choice of $i_1, \ldots, i_k$ and $j_1, \ldots, j_l$, so that it is denoted by $f(a_I, b_J)$. Let $a_1, \ldots, a_s, b_1, \ldots, b_s, c_1, \ldots, c_s, d_1, \ldots, d_s$ be elements of V. We assume that there are $u_1, \ldots, u_s \in V$ such that $a_i = c_i + u_i$ and $b_i = d_i - u_i$ for all $i = 1, \ldots, s$. Then

$$\sum_{I \subseteq \{1,\ldots,s\}} f(a_I, b_J) = \sum_{I \subseteq \{1,\ldots,s\}} f(c_I, d_J).$$

Proof. We prove the lemma by induction on s. If $s = 1$, then

$$\sum_{I \subseteq \{1,\ldots,s\}} f(a_I, b_J) = f(a_1) + f(b_1) = f(c_1 + u_1) + f(d_1 - u_1) = f(c_1) + f(d_1) = \sum_{I \subseteq \{1,\ldots,s\}} f(c_I, d_J).$$

Thus we assume $s > 1$. By the hypothesis of induction, we have

$$\sum_{I' \subseteq \{2,\ldots,s\}} f(a_1, a_{I'}, b_{J'}) = \sum_{I' \subseteq \{2,\ldots,s\}} f(a_1, c_{I'}, d_{J'})$$

and

$$\sum_{I' \subseteq \{2,\ldots,s\}} f(b_1, a_{I'}, b_{J'}) = \sum_{I' \subseteq \{2,\ldots,s\}} f(b_1, c_{I'}, d_{J'}),$$

where $J' = \{2, \ldots, s\} \setminus I'$. The first equation and the second equation imply that

$$\sum_{1 \in I \subseteq \{1,\ldots,s\}} f(a_I, b_J) = \sum_{1 \in I \subseteq \{1,\ldots,s\}} f(c_I, d_J) + \sum_{I' \subseteq \{2,\ldots,s\}} f(u_1, c_{I'}, d_{J'})$$

and

$$\sum_{1 \notin I \subseteq \{1,\ldots,s\}} f(a_I, b_J) = \sum_{1 \notin I \subseteq \{1,\ldots,s\}} f(c_I, d_J) - \sum_{I' \subseteq \{2,\ldots,s\}} f(u_1, c_{I'}, d_{J'})$$

respectively. Thus the lemma follows. □

2.2. **Hodge index theorem for arithmetic $\mathbb{R}$-Cartier divisors.** First of all, let us fix notation. Let $\mathbb{K}$ be either $\mathbb{Q}$ or $\mathbb{R}$. Let H be an ample $\mathbb{K}$-Cartier divisor on X. Let D be a $\mathbb{K}$-Cartier divisor on X and let E be a vertical $\mathbb{K}$-Weil divisor on X. We set $E = \sum_{i=1}^{l} a_i \Gamma_i$, where $a_1, \ldots, a_l \in \mathbb{K}$ and $\Gamma_1, \ldots, \Gamma_l$ are vertical prime divisors. Then a quantity

$$\sum_{i=1}^{l} a_i \deg\left(\left(H|_{\Gamma_i}\right)^{d-2} \cdot \left(D|_{\Gamma_i}\right)\right)$$

is denoted by $\deg_H(D \cdot E)$. Note that if X is regular and D and E are vertical, then $\deg_H(D \cdot E) = \deg_H(E \cdot D)$. We say D is *divisorially π-nef with respect to H* if $\deg_H(D \cdot \Gamma) \geq 0$ for all vertical prime divisors Γ on X. Moreover, D is said to be *divisorially π-numerically trivial with respect to H* if D and $-D$ is divisorially π-nef with respect to H, that is, $\deg_H(D \cdot \Gamma) = 0$ for all vertical prime divisors Γ on X.

Lemma 2.2.1. *We assume that X is regular. Let $P \in \operatorname{Spec}(O_K)$ and let $\pi^{-1}(P) = a_1\Gamma_1 + \cdots + a_n\Gamma_n$ be the irreducible decomposition as a cycle, that is, $a_1, \ldots, a_n \in \mathbb{Z}_{>0}$ and $\Gamma_1, \ldots, \Gamma_n$ are prime divisors. Let us consider a linear map $T_P : \mathbb{K}^n \to \mathbb{K}^n$ given by*

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} \deg_H(\Gamma_1 \cdot \Gamma_1) & \cdots & \deg_H(\Gamma_1 \cdot \Gamma_n) \\ \vdots & \ddots & \vdots \\ \deg_H(\Gamma_n \cdot \Gamma_1) & \cdots & \deg_H(\Gamma_n \cdot \Gamma_n) \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Then $\operatorname{Ker}(T_P) = \langle (a_1, \ldots, a_n) \rangle_{\mathbb{K}}$ *and* $T_P(\mathbb{K}^n) = \{(y_1, \ldots, y_n) \in \mathbb{K}^n \mid a_1 y_1 + \cdots + a_n y_n = 0\}$.

Proof. This is a consequence of Zariski's lemma (cf. Lemma 1.1.4). □

Lemma 2.2.2. *We assume that X is regular. Let D be a $\mathbb{K}$-Cartier divisor on X with* $\deg(H_{\mathbb{Q}}^{d-2} \cdot D_{\mathbb{Q}}) = 0$. *Then there is a vertical effective $\mathbb{K}$-Cartier divisor E such that $D + E$ is divisorially π-numerically trivial with respect to H.*

Proof. We can choose $P_1, \ldots, P_n \in \operatorname{Spec}(O_K)$ such that $\deg_H(D \cdot \Gamma) = 0$ for all vertical prime divisors Γ with $\pi(\Gamma) \notin \{P_1, \ldots, P_n\}$. We set $\pi^{-1}(P_k) = \sum_{i=1}^{n_k} a_{ki}\Gamma_{ki}$ for each $k = 1, \ldots, n$, where $a_{ki} \in \mathbb{Z}_{>0}$ and Γ_{ki} is a vertical prime divisor over P_k. Since

$$\sum_{j=1}^{n_k} a_{kj} \deg_H\left(D \cdot \Gamma_{kj}\right) = \deg_H\left(D \cdot \pi^{-1}(P_k)\right) = 0,$$

by virtue of Lemma 2.2.1, we can find $x_{ki} \in \mathbb{K}$

$$\sum_{i=1}^{n_k} x_{ki} \deg_H\left(\Gamma_{ki} \cdot \Gamma_{kj}\right) = -\deg_H(D \cdot \Gamma_{kj})$$

for all k. Moreover, replacing x_{ki} by $x_{ki} + na_{ki}$ ($n \gg 1$), we may assume that $x_{ki} > 0$. Here we set

$$E = \sum_{k=1}^{n} \sum_{i=1}^{n_k} x_{ki}\Gamma_{ki}.$$

Then $D + E$ is divisorially π-numerically trivial. □

First let us consider the Hodge index theorem for $\mathbb{R}$-Cartier divisors on an arithmetic surface. It was actually treated in [2, Theorem 5.5]. Here we would like to present a slightly different version.

Theorem 2.2.3. *We assume $d = 2$. Let $\operatorname{Div}_0(X_{\mathbb{Q}})_{\mathbb{R}}$ be a vector subspace of $\operatorname{Div}(X_{\mathbb{Q}})_{\mathbb{R}}$ given by*

$$\operatorname{Div}_0(X_{\mathbb{Q}})_{\mathbb{R}} := \{\vartheta \in \operatorname{Div}(X_{\mathbb{Q}})_{\mathbb{R}} \mid \deg(\vartheta) = 0\}.$$

Let $\overline{D} = (D, g)$ be an arithmetic $\mathbb{R}$-Cartier divisor in $\widehat{\operatorname{Div}}^{\mathrm{Nef}}_{C^0}(X)_{\mathbb{R}}$ with $D_{\mathbb{Q}} \in \operatorname{Div}_0(X_{\mathbb{Q}})_{\mathbb{R}}$. Then

$$\widehat{\deg}(\overline{D}^2) \leq -2[K : \mathbb{Q}]\langle D_{\mathbb{Q}}, D_{\mathbb{Q}}\rangle_{NT},$$

where $\langle\ ,\ \rangle_{NT}$ is the Néron-Tate pairing on $\operatorname{Div}_0(X_{\mathbb{Q}})_{\mathbb{R}}$ (cf. Remark 2.2.4). Moreover, the equality holds if and only if the following conditions (a), (b) and (c) hold:

(a) *D is divisorially π-numerically trivial.*
(b) *g is of C^∞-type.*
(c) *$c_1(\overline{D}) = 0$.*

Proof. Let $\mu : X' \to X$ be a resolution of singularities of X (cf. [12]). Then, since the arithmetic volume function is invariant under birational morphisms (cf. [17, Theorem 4.3]), we can see $\widehat{\deg}(\overline{D}^2) = \widehat{\deg}(\mu^*(\overline{D})^2)$. Thus we may assume that X is regular.

Let g' be an F_∞-invariant D-Green function of C^∞-type with $c_1(D, g') = 0$. Let η be an F_∞-invariant continuous function on $X(\mathbb{C})$ with $g = g' + \eta$. Then, by (1) in Proposition 2.1.1, $\eta \in \langle (C^0 \cap \mathrm{QPSH})(X(\mathbb{C}))\rangle_{\mathbb{R}}$

By Lemma 2.2.2, we can find an effective and vertical $\mathbb{R}$-Cartier divisor E such that $D + E$ is divisorially π-numerically trivial. If we set $\overline{D}' = (D + E, g')$, then $\overline{D}'$ satisfies the above conditions (a), (b) and (c). Moreover, as $\overline{D} = \overline{D}' - (E, 0) + (0, \eta)$,

$$\widehat{\deg}(\overline{D}^2) = \widehat{\deg}(\overline{D}'^2) + \widehat{\deg}((E, 0)^2) + \frac{1}{2}\int_{X(\mathbb{C})} \eta dd^c(\eta).$$

Thus, by Proposition 1.2.3 and Zariski's lemma (cf. Lemma 1.1.4), in order to prove the assertions of the theorem, it is sufficient to see

$$\widehat{\deg}(\overline{D}^2) = -2[K : \mathbb{Q}]\langle D_K, D_K\rangle_{NT}.$$

under the assumptions (a), (b) and (c).

By (1) in Lemma 1.1.1, we can choose $D_1, \ldots, D_l \in \operatorname{Div}(X)$ and $a_1, \ldots, a_l \in \mathbb{R}$ such that $D = a_1 D_1 + \cdots + a_l D_l$ and $a_1, \ldots, a_l$ are linearly independent over $\mathbb{Q}$. Let C be a 1-dimensional vertical closed integral subscheme. Since

$$0 = \deg(D|_C) = a_1 \deg(D_1|_C) + \cdots + a_n \deg(D_n|_C),$$

we have $\deg(D_i|_C) = 0$ for all i, and hence D_i is divisorially π-numerically trivial for every i, so that we can also choose a D_i-Green function h_i of C^∞-type such that

$\overline{D} = a_1\overline{D}_1 + \cdots + a_l\overline{D}_l$ and $c_1(\overline{D}_i) = 0$ for all i, where $\overline{D}_i = (D_i, h_i)$ for $i = 1, \ldots, l$. We need to show

$$\widehat{\deg}\left((a_1\overline{D}_1 + \cdots + a_l\overline{D}_l)^2\right) = -2[K : \mathbb{Q}]\langle a_1D_1 + \cdots + a_lD_l, a_1D_1 + \cdots + a_lD_l\rangle_{NT}.$$

Note that it holds for $a_1, \ldots, a_l \in \mathbb{Q}$ by Faltings-Hriljac ([8], [10]). Moreover, each hand side is continuous with respect to $a_1, \ldots, a_l$. Thus the equality follows in general. □

Remark 2.2.4. (1) Let $\mathrm{Div}_0(X_{\mathbb{Q}})$ be the group of divisors ϑ on $X_{\mathbb{Q}}$ with $\deg(\vartheta) = 0$. By using (1) in Lemma 1.1.1, we can see $\mathrm{Div}_0(X_{\mathbb{Q}}) \otimes_{\mathbb{Z}} \mathbb{R} = \mathrm{Div}_0(X_{\mathbb{Q}})_{\mathbb{R}}$. Let

$$\langle\ ,\ \rangle_{NT} : \mathrm{Div}_0(X_{\mathbb{Q}}) \times \mathrm{Div}_0(X_{\mathbb{Q}}) \to \mathbb{R}$$

be the Néron-Tate height pairing on $\mathrm{Div}_0(X_{\mathbb{Q}})$, which extends to

$$\mathrm{Div}_0(X_{\mathbb{Q}})_{\mathbb{R}} \times \mathrm{Div}_0(X_{\mathbb{Q}})_{\mathbb{R}} \to \mathbb{R}$$

in the natural way. By abuse of notation, the above bi-linear map is also denoted by $\langle\ ,\ \rangle_{NT}$. By virtue of [9, Proposition B.5.3], we can see that

$$\mathrm{PDiv}(X_{\mathbb{Q}})_{\mathbb{R}} = \{\vartheta \in \mathrm{Div}_0(X_{\mathbb{Q}})_{\mathbb{R}} \mid \langle\vartheta, \vartheta\rangle_{NT} = 0\}.$$

(2) Let $\overline{D} = (D, g)$ be an integrable arithmetic $\mathbb{R}$-Cartier divisor of C^0-type on X. If $D_{\mathbb{Q}} \in \mathrm{Div}_0(X_{\mathbb{Q}})_{\mathbb{R}}$ and $\widehat{\deg}(\overline{D}^2) = 0$, there are $\varphi \in \mathrm{Rat}(X)^{\times}_{\mathbb{R}}$ and an F_∞-invariant locally constant function η on $X(\mathbb{C})$ such that $\overline{D} = \widehat{(\varphi)}_{\mathbb{R}} + (0, \eta)$. Indeed, by Theorem 2.2.3 and the above (1), D is divisorially π-numerically trivial, g is of C^∞-type, $c_1(\overline{D}) = 0$ and $D_{\mathbb{Q}} \in \mathrm{PDiv}(X_{\mathbb{Q}})_{\mathbb{R}}$. Therefore, there exist $\varphi \in \mathrm{Rat}(X)^{\times}_{\mathbb{R}}$, a vertical $\mathbb{R}$-Cartier divisor E and an F_∞-invariant continuous function η on $X(\mathbb{C})$ such that $\overline{D} = \widehat{(\varphi)}_{\mathbb{R}} + (E, \eta)$. As D and $(\varphi)_{\mathbb{R}}$ is divisorially π-numerically trivial, by using Zariski's lemma, we can find $\vartheta \in \widehat{\mathrm{Div}}(\mathrm{Spec}(O_K))_{\mathbb{R}}$ such that $E = \pi^*(\vartheta)$. Note that the class group of O_K is finite, so that $\vartheta \in \mathrm{PDiv}(\mathrm{Spec}(O_K))_{\mathbb{R}}$, and hence $E \in \mathrm{PDiv}(X)_{\mathbb{R}}$. Therefore, we may assume that $E = 0$. Thus

$$0 = \widehat{\deg}(\overline{D}^2) = \frac{1}{2}\int_{X(\mathbb{C})} \eta dd^c(\eta),$$

which implies that η is locally constant by Proposition 1.2.3.

Finally let us consider the Hodge index theorem on a higher dimensional arithmetic variety. The proof is almost same as [16], but we need a careful treatment at the final step.

Theorem 2.2.5. *Let $\overline{D} = (D, g)$ be an arithmetic $\mathbb{R}$-Cartier divisor in $\widehat{\mathrm{Div}}^{\mathrm{Nef}}_{C^0}(X)_{\mathbb{R}}$ and let $\overline{H} = (H, h)$ be an ample arithmetic $\mathbb{Q}$-Cartier divisor on X. If $\deg(D_{\mathbb{Q}} \cdot H_{\mathbb{Q}}^{d-2}) = 0$, then*

$$\widehat{\deg}(\overline{D}^2 \cdot \overline{H}^{d-2}) \leq 0.$$

Moreover, if the equality holds, then $D_{\mathbb{Q}} \in \mathrm{PDiv}(X_{\mathbb{Q}})_{\mathbb{R}}$.

Proof. By (1) in Lemma 1.1.1, we can choose $D_1, \ldots, D_l \in \mathrm{Div}(X)$ and $a_1, \ldots, a_l \in \mathbb{R}$ such that $a_1, \ldots, a_l$ are linearly independent over $\mathbb{Q}$ and $D = a_1 D_1 + \cdots + a_l D_l$. Since

$$0 = \deg(D_{\mathbb{Q}} \cdot H_{\mathbb{Q}}^{d-2}) = \sum_{i=1}^{l} a_i \deg(D_{i\mathbb{Q}} \cdot H_{\mathbb{Q}}^{d-2})$$

and $\deg(D_{i\mathbb{Q}} \cdot H_{\mathbb{Q}}^{d-2}) \in \mathbb{Q}$ for all i, we have $\deg(D_{i\mathbb{Q}} \cdot H_{\mathbb{Q}}^{d-2}) = 0$ for all i. Let us also choose an F_∞-invariant D_i-Green function g_i of C^∞-type such that $c_1(D_i, g_i) \wedge c_1(\overline{H})^{d-2} = 0$. If we set $g' = a_1 g_1 + \cdots + a_l g_l$, then, by (1) in Proposition 2.1.1, there is $\eta \in \langle (C^0 \cap \mathrm{QPSH})(X(\mathbb{C})) \rangle_{\mathbb{R}}$ such that $g = g' + \eta$. By (4) in Proposition 2.1.1,

$$\widehat{\deg}(\overline{D}^2 \cdot \overline{H}^{d-2}) = \widehat{\deg}((D, g')^2 \cdot \overline{H}^{d-2}) + \frac{1}{2} \int_{X(\mathbb{C})} \eta dd^c(\eta) c_1(\overline{H})^{d-2}$$

because $c_1(D, g') \wedge c_1(\overline{H})^{d-2} = 0$. Therefore, by Proposition 1.2.3,

$$\widehat{\deg}(\overline{D}^2 \cdot \overline{H}^{d-2}) \leq \widehat{\deg}((D, g')^2 \cdot \overline{H}^{d-2})$$

and the equality holds if and only if η is a constant. Thus we may assume that η is a constant, that is, $g = g'$ by replacing g_l by $g_l + \eta / a_l$.

By virtue of [16, Theorem 1.1],

$$\widehat{\deg}\left((\alpha_1 (D_1, g_1) + \cdots + \alpha_l (D_l, g_l))^2 \cdot \overline{H}^{d-2} \right) \leq 0$$

for all $\alpha_1, \ldots, \alpha_l \in \mathbb{Q}$, and hence $\widehat{\deg}(\overline{D}^2 \cdot \overline{H}^{d-2}) \leq 0$.

We need to check the equality condition. We prove it by induction on d. If $d = 2$, then the assertion follows from Theorem 2.2.3 and Remark 2.2.4. We assume that $d > 2$ and $\widehat{\deg}(\overline{D}^2 \cdot \overline{H}^{d-2}) = 0$. By using arithmetic Bertini's theorem (cf. [15]), we can find $m \in \mathbb{Z}_{>0}$ and $f \in \mathrm{Rat}(X)^\times$ with the following properties:

(i) If we set $\overline{H}' = (H', h') = m\overline{H} + \widehat{(f)}$, then $(H', h') \in \widehat{\mathrm{Div}}_{C^\infty}(X)$, H' is effective, $h' > 0$ and H' is smooth over $\mathbb{Q}$.

(ii) If $H' = Y' + c_1 F_1 + \cdots + c_r F_r$ is the irreducible decomposition such that Y' is horizontal and F_i's are vertical, then F_i's are connected components of smooth fibers over $\mathbb{Z}$.

(iii) D and H' have no common irreducible component.

Let Y be the normalization of Y'. Then

$$\begin{aligned} 0 = m^{d-2} \widehat{\deg}(\overline{D}^2 \cdot \overline{H}^{d-2}) = \widehat{\deg}(\overline{D}^2 \cdot \overline{H}'^{d-2}) = \widehat{\deg}(\overline{D}\big|_Y^2 \cdot \overline{H}'\big|_Y^{d-3}) \\ + \sum c_i \deg(D|_{F_i}^2 \cdot H'|_{F_i}^{d-3}) + \frac{1}{2} \int_{X(\mathbb{C})} h' c_1(\overline{D})^2 c_1(\overline{H}')^{d-3}. \end{aligned}$$

Therefore, by using [16, Lemma 1.1.2], we can see that $\widehat{\deg}(\overline{D}\big|_Y^2 \cdot \overline{H}'\big|_Y^{d-3}) = 0$ and $c_1(\overline{D}) = 0$. In particular, by hypothesis of induction, $D_{\mathbb{Q}}\big|_Y \in \widehat{\mathrm{PDiv}}(Y_{\mathbb{Q}})_{\mathbb{R}}$. Let C be a closed and integral curve of $X_{\mathbb{Q}}$. Then, since

$$0 = \int_{C(\mathbb{C})} c_1(\overline{D}) = \deg(D_{\mathbb{Q}} \cdot C) = \sum a_i \deg(D_{i\mathbb{Q}} \cdot C)$$

and $a_1, \ldots, a_l$ are linearly independent over $\mathbb{Q}$, we have $\deg(D_{i\mathbb{Q}} \cdot C) = 0$ for all i. Therefore, if we set $L_i = \mathcal{O}_{X_{\mathbb{Q}}}(D_i)$, then L_i is numerically trivial, and hence $(L_i)_{\mathbb{C}}$ is also numerically trivial on $X(\mathbb{C})$. This means that $(L_i)_{\mathbb{C}}$ comes from a representation $\rho_i : \pi_1(X(\mathbb{C})) \to \mathbb{C}^{\times}$. Let ι be the natural homomorphism $\iota : \pi_1(Y(\mathbb{C})) \to \pi_1(X(\mathbb{C}))$ and let

$$\rho'_i = \rho_i \circ \iota : \pi_1(Y(\mathbb{C})) \xrightarrow{\iota} \pi_1(X(\mathbb{C})) \xrightarrow{\rho_i} \mathbb{C}^{\times}.$$

Then ρ'_i yields $(L_i)_{\mathbb{C}}|_{Y(\mathbb{C})}$. Let

$$\rho : \pi_1(X(\mathbb{C})) \to \mathbb{C}^{\times} \otimes_{\mathbb{Z}} \mathbb{R} \quad \text{and} \quad \rho' : \pi_1(Y(\mathbb{C})) \to \mathbb{C}^{\times} \otimes_{\mathbb{Z}} \mathbb{R}$$

be homomorphisms given by $\rho = \rho_1^{\otimes a_1} \cdots \rho_l^{\otimes a_l}$ and $\rho' = \rho_1'^{\otimes a_1} \cdots \rho_l'^{\otimes a_l}$. Since

$$\left((L_1)_{\mathbb{C}}|_{Y(\mathbb{C})}\right)^{\otimes a_1} \otimes \cdots \otimes \left((L_l)_{\mathbb{C}}|_{Y(\mathbb{C})}\right)^{\otimes a_l} = 1$$

in $\operatorname{Pic}(Y_{\mathbb{Q}}) \otimes \mathbb{R}$, we have $\rho' = 1$. Note that ι is surjective (cf. [14, Theorem 7.4] and the homotopy exact sequence). Thus $\rho = 1$ because $\rho' = \rho \circ \iota$. Therefore, by (2) in Lemma 1.1.1, the image of ρ_i is finite for all i. This means that there is a positive integer n such that $(L_i)_{\mathbb{C}}^{\otimes n} \simeq \mathcal{O}_{X(\mathbb{C})}$ for all i. If we fix $\sigma \in K(\mathbb{C})$, then

$$\dim_K H^0(X_{\mathbb{Q}}, L_i^{\otimes n}) = \dim_{\mathbb{C}} H^0(X_{\mathbb{Q}} \times^{\sigma}_{\operatorname{Spec}(K)} \operatorname{Spec}(\mathbb{C}), L_i^{\otimes n} \otimes^{\sigma}_K \mathbb{C}) = 1,$$

and hence $L_i^{\otimes n} \simeq \mathcal{O}_{X_{\mathbb{Q}}}$ because $\deg(L_i \cdot H_{\mathbb{Q}}^{d-2}) = 0$. Therefore,

$$L_1^{\otimes a_1} \otimes \cdots \otimes L_l^{\otimes a_l} = (L_1^{\otimes n})^{\otimes a_1/n} \otimes \cdots \otimes (L_l^{\otimes n})^{\otimes a_l/n} = 1$$

in $\operatorname{Pic}(X_{\mathbb{Q}})_{\mathbb{R}}$. Thus $D_{\mathbb{Q}} \in \operatorname{PDiv}(X_{\mathbb{Q}})_{\mathbb{R}}$. □

Remark 2.2.6. There is a typo in [16, Lemma 1.1.2]. The form ω should be real, that is, $\bar{\omega} = \omega$.

2.3. **Hodge index theorem and pseudo-effectivity.** In this subsection, let us observe the pseudo-effectivity of arithmetic $\mathbb{R}$-Cartier divisors as an application of Hodge index theorem. Let us begin with the following lemma:

Lemma 2.3.1. *We assume that X is regular. Let $\overline{D} = (D, g)$ be an arithmetic $\mathbb{R}$-Cartier divisor of C^0-type. If D is semi-ample on $X_{\mathbb{Q}}$ (that is, there are semi-ample divisors $A_1, \ldots, A_r$ on $X_{\mathbb{Q}}$ and $a_1, \ldots, a_r \in \mathbb{R}_{>0}$ such that $D_{\mathbb{Q}} = a_1 A_1 + \cdots + a_r A_r$), then there are $\varphi_1, \ldots, \varphi_l \in \operatorname{Rat}(X)^{\times}_{\mathbb{R}}$ and $c \in \mathbb{R}$ such that $\overline{D} + \widehat{(\varphi_i)}_{\mathbb{R}} + (0, c) \geq 0$ for all i and*

$$\bigcap_{i=1}^{l} \operatorname{Supp}(D + (\varphi_i)_{\mathbb{R}}) = \emptyset$$

on $X_{\mathbb{Q}}$ (for the definition of $\operatorname{Rat}(X)^{\times}_{\mathbb{R}}$ and arithmetic $\mathbb{R}$-principal divisors, see SubSection 0.2 in Introduction and Conventions and terminology 2).

Proof. Let us consider the assertion of the lemma for $\overline{D} = (D, g)$:

(∗) There exist $\varphi_1, \ldots, \varphi_l \in \operatorname{Rat}(X)^{\times}_{\mathbb{R}}$ and $c \in \mathbb{R}$ such that $\overline{D} + \widehat{(\varphi_i)}_{\mathbb{R}} + (0, c) \geq 0$ for all i and $\bigcap_{i=1}^{l} \operatorname{Supp}(D + (\varphi_i)_{\mathbb{R}}) = \emptyset$ on $X_{\mathbb{Q}}$.

Claim 2.3.1.1. (1) *If D is a $\mathbb{Q}$-Cartier divisor and D is semi-ample on $X_{\mathbb{Q}}$ (i.e. nD is base-point free on $X_{\mathbb{Q}}$ for some $n > 0$), then (∗) holds for $\overline{D}$.*

(2) *If D is vertical, then* $(*)$ *holds for* $\overline{D}$.
(3) *If $a \in \mathbb{R}_{>0}$ and* $(*)$ *holds for $\overline{D}$, then so does for $a\overline{D}$.*
(4) *If* $(*)$ *holds for $\overline{D}$ and $\overline{D}'$, so does for $\overline{D} + \overline{D}'$.*

Proof. (1) Since D is a semi-ample $\mathbb{Q}$-Cartier divisor on $X_{\mathbb{Q}}$, there are a positive integer n and $\phi_1, \ldots, \phi_l \in H^0(X, nD) \setminus \{0\}$ such that $\bigcap_{i=1}^l \operatorname{Supp}(nD + (\phi_i)) = \emptyset$ on $X_{\mathbb{Q}}$. Since $D + (\phi_i^{1/n})_{\mathbb{R}}$ is effective, we can find $c \in \mathbb{R}$ such that $\overline{D} + \widehat{(\phi_i^{1/n})}_{\mathbb{R}} + (0, c) \geq 0$ for all i.

(2) We choose $x \in O_K \setminus \{0\}$ such that $D + (x) \geq 0$, and hence there is $c \in \mathbb{R}$ such that $\overline{D} + \widehat{(x)} + (0, c) \geq 0$.

(3) Let $\varphi_1, \ldots, \varphi_l \in \operatorname{Rat}(X)^{\times}_{\mathbb{R}}$ and $c \in \mathbb{R}$ such that $\overline{D} + \widehat{(\varphi_i)}_{\mathbb{R}} + (0, c) \geq 0$ for all i and $\bigcap_{i=1}^l \operatorname{Supp}(D + (\varphi_i)_{\mathbb{R}}) = \emptyset$ on $X_{\mathbb{Q}}$. Then $a\overline{D} + \widehat{(\varphi_i^a)}_{\mathbb{R}} + (0, ac) \geq 0$ for all i and $\bigcap_{i=1}^l \operatorname{Supp}(aD + (\varphi_i^a)_{\mathbb{R}}) = \emptyset$ on $X_{\mathbb{Q}}$.

(4) By our assumption, there exist $\varphi_1, \ldots, \varphi_l, \varphi'_1, \ldots, \varphi'_{l'} \in \operatorname{Rat}(X)^{\times}_{\mathbb{R}}$ and $c, c' \in \mathbb{R}$ such that

$$\begin{cases} \overline{D} + \widehat{(\varphi_i)}_{\mathbb{R}} + (0, c) \geq 0 \text{ for all } i. \\ \bigcap_{i=1}^l \operatorname{Supp}(D + (\varphi_i)_{\mathbb{R}}) = \emptyset \text{ on } X_{\mathbb{Q}}, \\ \overline{D}' + \widehat{(\varphi'_j)}_{\mathbb{R}} + (0, c') \geq 0 \text{ for all } j, \\ \bigcap_{j=1}^{l'} \operatorname{Supp}(D' + (\varphi'_j)_{\mathbb{R}}) = \emptyset \text{ on } X_{\mathbb{Q}}. \end{cases}$$

Then $\overline{D} + \overline{D}' + \widehat{(\varphi_i \varphi'_j)}_{\mathbb{R}} + (0, c + c') \geq 0$ for all i, j and

$$\bigcap_{i,j} \operatorname{Supp}(D + D' + (\varphi_i \varphi'_j)_{\mathbb{R}}) = \emptyset$$

on $X_{\mathbb{Q}}$ because

$$\bigcap_{i,j} \operatorname{Supp}(D + D' + (\varphi_i \varphi'_j)_{\mathbb{R}}) \subseteq \bigcap_{i,j} \left(\operatorname{Supp}(D + (\varphi_i)_{\mathbb{R}}) \cup \operatorname{Supp}(D' + (\varphi'_j)_{\mathbb{R}}) \right).$$

□

Let us go back to the proof of the lemma. Since X is regular and D is semi-ample on $X_{\mathbb{Q}}$, there are arithmetic $\mathbb{Q}$-Cartier divisors $\overline{D}_1, \ldots, \overline{D}_r$ of C^0-type, $a_1, \ldots, a_r \in \mathbb{R}_{>0}$, a vertical $\mathbb{R}$-Cartier divisor E and an F_∞-invariant continuous function η on $X(\mathbb{C})$ such that D_i's are semi-ample on $X_{\mathbb{Q}}$ and $\overline{D} = a_1\overline{D}_1 + \cdots + a_r\overline{D}_r + (E, \eta)$. Thus the assertion follows from the above claim. □

Let us fix an ample arithmetic $\mathbb{Q}$-Cartier divisor $\overline{H}$ on X. For arithmetic $\mathbb{R}$-Cartier divisors $\overline{D}_1$ and $\overline{D}_2$ of C^∞-type on X, we denote $\widehat{\deg}(\overline{H}^{d-2} \cdot \overline{D}_1 \cdot \overline{D}_2)$ by $\widehat{\deg}_{\overline{H}}(\overline{D}_1 \cdot \overline{D}_2)$. Let us consider the following lemma, which is a useful criterion of pseudo-effectivity.

Lemma 2.3.2. *We assume that X is regular. Let $\overline{D} = (D, g)$ be an arithmetic $\mathbb{R}$-Cartier divisor of C^∞-type on X with the following properties:*
(1) *D is nef on $X_{\mathbb{Q}}$ and* $\deg(D_{\mathbb{Q}} \cdot H_{\mathbb{Q}}^{d-2}) = 0$.
(2) *$c_1(\overline{D})$ is semipositive.*
(3) *D is divisorially π-nef with respect to H.*

(4) $\widehat{\deg}_{\overline{H}}(\overline{D}^2) < 0$.

Then $\overline{D}$ is not pseudo-effective.

Proof. First we claim the following:

Claim 2.3.2.1. *There is an arithmetic $\mathbb{R}$-Cartier divisor $\overline{L} = (L, h)$ of C^∞-type with the following properties:*

(a) *L is ample on $X_{\mathbb{Q}}$.*
(b) *$c_1(\overline{L})$ is positive.*
(c) *L is divisorially π-nef with respect to H.*
(d) $\widehat{\deg}_{\overline{H}}(\overline{L} \cdot \overline{D}) < 0$.

Proof. Since $\widehat{\deg}_{\overline{H}}(\overline{D}^2) < 0$, we have

$$\widehat{\deg}_{\overline{H}}(\overline{D} + \epsilon \overline{H} \cdot \overline{D}) < 0$$

for a sufficiently small positive number ϵ. Thus, if we set $\overline{L} = \overline{D} + \epsilon \overline{H}$, then $\overline{L}$ satisfies all properties (a) – (d). □

Let us go back to the proof of the lemma. Since L is ample on $X_{\mathbb{Q}}$, by Lemma 2.3.1, there are $\varphi_1, \ldots, \varphi_l \in \operatorname{Rat}(X)^{\times}_{\mathbb{R}}$ and $c \in \mathbb{R}$ such that $\overline{L} + \widehat{(\varphi_i)}_{\mathbb{R}} + (0, c) \geq 0$ for all i and $\bigcap_{i=1}^{l} \operatorname{Supp}(L + (\varphi_i)_{\mathbb{R}}) = \emptyset$ on $X_{\mathbb{Q}}$. Let Γ be a horizontal prime divisor. Then we can find i such that $\Gamma \not\subseteq \operatorname{Supp}(L + (\varphi_i)_{\mathbb{R}})$. Thus

$$\begin{aligned}\widehat{\deg}_{\overline{H}}((\overline{L} + (0,c)) \cdot (\Gamma, 0)) &= \widehat{\deg}_{\overline{H}}((\overline{L} + \widehat{(\varphi_i)}_{\mathbb{R}} + (0,c)) \cdot (\Gamma, 0)) \\ &= \widehat{\deg}\left(\overline{H}\Big|_{\Gamma}^{d-2} \cdot (\overline{L} + \widehat{(\varphi_i)}_{\mathbb{R}} + (0,c))\Big|_{\Gamma} \right) \geq 0.\end{aligned}$$

Furthermore, the above inequality also holds for a vertical prime divisor Γ because L is divisorially π-nef with respect to H. Therefore, if $\overline{G} = (G, k)$ is an effective arithmetic $\mathbb{R}$-Cartier divisor of C^0-type, then

$$\widehat{\deg}_{\overline{H}}((\overline{L} + (0,c)) \cdot \overline{G}) = \widehat{\deg}_{\overline{H}}((\overline{L} + (0,c)) \cdot (G, 0)) + \frac{1}{2} \int_{X(\mathbb{C})} k c_1(\overline{H})^{d-2} c_1(\overline{L}) \geq 0.$$

In particular, if $\overline{D}$ is pseudo-effective, then

$$\widehat{\deg}_{\overline{H}}((\overline{L} + (0,c)) \cdot \overline{D}) \geq 0.$$

On the other hand, as $\deg(D_{\mathbb{Q}} \cdot H_{\mathbb{Q}}^{d-2}) = 0$,

$$\begin{aligned}\widehat{\deg}_{\overline{H}}((\overline{L} + (0,c)) \cdot \overline{D}) &= \widehat{\deg}_{\overline{H}}(\overline{L} \cdot \overline{D}) + \frac{c}{2} \deg(D_{\mathbb{Q}} \cdot H_{\mathbb{Q}}^{d-2}) \\ &= \widehat{\deg}_{\overline{H}}(\overline{L} \cdot \overline{D}) < 0.\end{aligned}$$

This is a contradiction. □

As consequence of Hodge index theorem and the above lemma, we have the following theorem on pseudo-effectivity:

Theorem 2.3.3. *We assume that X is regular and $d \geq 2$. Let $\overline{D} = (D, g)$ be an arithmetic $\mathbb{R}$-Cartier divisor of C^0-type. If $\overline{D}$ is pseudo-effective and D is numerically trivial on $X_{\mathbb{Q}}$, then $D_{\mathbb{Q}} \in \operatorname{PDiv}(X_{\mathbb{Q}})_{\mathbb{R}}$.*

Proof. We assume that $D_{\mathbb{Q}} \notin \mathrm{PDiv}(X_{\mathbb{Q}})_{\mathbb{R}}$. Since D is numerically trivial on $X_{\mathbb{Q}}$, by Lemma 2.2.2, we can find an effective vertical $\mathbb{R}$-Cartier divisor E such that $D + E$ is divisorially π-numerically trivial with respect to H. Moreover, we can find an F_∞-invariant D-Green function g_0 of C^∞-type with $c_1(D, g_0) = 0$. Then there is an F_∞-invariant continuous function η on $X(\mathbb{C})$ such that $g + \eta = g_0$. Replacing g_0 by $g_0 + c$ ($c \in \mathbb{R}$), we may assume that $\eta \geq 0$. By the Hodge index theorem,

$$\widehat{\deg}_{\overline{H}}((D + E, g_0)^2) < 0.$$

Thus $(D + E, g_0)$ is not pseudo-effective by Lemma 2.3.2, and hence

$$\overline{D} = (D + E, g_0) - (E, \eta)$$

is also not pseudo-effective. This is a contradiction. □

Finally let us consider the following lemmas on pseudo-effectivity.

Lemma 2.3.4. *For $\overline{D} \in \widehat{\mathrm{Div}}_{C^0}(X)_{\mathbb{R}}$ and $z \in \widehat{\mathrm{PDiv}}(X)_{\mathbb{R}}$, if $\overline{D}$ is pseudo-effective, then $\overline{D} + z$ is also pseudo-effective.*

Proof. Let $\overline{A}$ be an ample arithmetic $\mathbb{R}$-Cartier divisor on X. Since $\overline{D}$ is pseudo-effective, $\overline{D} + (1/2)\overline{A}$ is big. Moreover, $z + (1/2)\overline{A}$ is ample because z is nef. Therefore,

$$(\overline{D} + z) + \overline{A} = (\overline{D} + (1/2)\overline{A}) + (z + (1/2)\overline{A})$$

is big, as required. □

Lemma 2.3.5. *Let D be a vertical $\mathbb{R}$-Cartier divisor on X and let η be an F_∞-invariant continuous function on $X(\mathbb{C})$. Let λ be an element of $\mathbb{R}^{K(\mathbb{C})}$ given by $\lambda_\sigma = \inf_{x \in X_\sigma} \eta(x)$ for all $\sigma \in K(\mathbb{C})$. We can view λ as a locally constant function on $X(\mathbb{C})$, that is, $\lambda|_{X_\sigma} = \lambda_\sigma$. If (D, η) is pseudo-effective, then (D, λ) is also pseudo-effective.*

Proof. Let us begin with the following claim:

Claim 2.3.5.1. *We may assume that λ is a constant function.*

Proof. We set $\lambda' = (1/[K : \mathbb{Q}]) \sum_{\sigma \in K(\mathbb{C})} \lambda_\sigma$ and $\xi_\sigma = \lambda' - \lambda_\sigma$ for each $\sigma \in K(\mathbb{C})$. Then $\sum_{\sigma \in K(\mathbb{C})} \xi_\sigma = 0$ and $\xi_\sigma = \xi_{\bar{\sigma}}$ for all $\sigma \in K(\mathbb{C})$. Thus, by Dirichlet's unit theorem (cf. Corollary 3.4.7), there are $a_1, \ldots, a_s \in \mathbb{R}$ and $u_1, \ldots, u_s \in O_K^\times$ such that

$$\xi_\sigma = a_1 \log|\sigma(u_1)| + \cdots + a_s \log|\sigma(u_s)|$$

for all $\sigma \in K(\mathbb{C})$. If we set

$$(D, \eta') = (D, \eta) - \pi^*\left((a_1/2)\widehat{(u_1)} + \cdots + (a_s/2)\widehat{(u_s)}\right),$$

then $\inf_{x \in X_\sigma} \eta'(x) = \lambda'$ for all $\sigma \in K(\mathbb{C})$. Moreover, by Lemma 2.3.4, (D, η') is pseudo-effective. If the lemma holds for η', then (D, λ') is pseudo-effective, and hence

$$(D, \lambda) = (D, \lambda') + \pi^*\left((a_1/2)\widehat{(u_1)} + \cdots + (a_s/2)\widehat{(u_s)}\right)$$

is also pseudo-effective by Lemma 2.3.4. □

For a given positive number ϵ, we set

$$U_\sigma = \{x \in X_\sigma \mid \eta(x) < \lambda_\sigma + (\epsilon/2)\}$$

and $U = \coprod_{\sigma \in K(\mathbb{C})} U_\sigma$. Let $\overline{A} = (A, h)$ be an ample arithmetic Cartier divisor on X. Then, by Lemma 1.3.1, there is a constant $C \geq 1$ depending only on ϵ and h such that

$$\sup_{x \in X(\mathbb{C})} \left\{ |s|^2_{t+bh}(x) \right\} \leq C^b \sup_{x \in U} \left\{ |s|^2_{t+bh}(x) \right\} \tag{2.3.5.2}$$

for all $s \in H^0(X(\mathbb{C}), bA)$, $b \in \mathbb{R}_{\geq 0}$ and all constant functions t on $X(\mathbb{C})$. Let n be an arbitrary positive integer with $n \geq (2\log(C))/\epsilon$. Since $(D, \eta) + (1/n)\overline{A}$ is big, there are a positive integer m and $s \in H^0(X, mD + (m/n)A) \setminus \{0\}$ such that $|s|_{m\eta+(m/n)h} \leq 1$, which implies that

$$|s|^2_{(m/n)h} \leq \exp(m\eta).$$

Therefore, $|s|^2_{(m/n)h} \leq \exp(m(\lambda + (\epsilon/2)))$ over U, that is,

$$\sup_{x \in U} \left\{ |s|^2_{m(\lambda+(\epsilon/2))+(m/n)h} \right\} \leq 1.$$

Thus, by the estimation (2.3.5.2), we have

$$C^{-(m/n)} \sup_{x \in X(\mathbb{C})} \left\{ |s|^2_{m(\lambda+(\epsilon/2))+(m/n)h} \right\} \leq 1.$$

Since $\log(C)/n \leq \epsilon/2$,

$$\begin{aligned} \sup_{x \in X(\mathbb{C})} \left\{ |s|^2_{m(\lambda+\epsilon)+(m/n)h} \right\} &\leq \sup_{x \in X(\mathbb{C})} \left\{ |s|^2_{(m/n)\log(C)+m(\lambda+(\epsilon/2))+(m/n)h} \right\} \\ &= C^{-(m/n)} \sup_{x \in X(\mathbb{C})} \left\{ |s|^2_{m(\lambda+(\epsilon/2))+(m/n)h} \right\} \leq 1 \end{aligned}$$

which yields $\hat{H}^0(X, m((1/n)\overline{A} + (D, \lambda + \epsilon))) \neq \{0\}$. Thus $(D, \lambda + \epsilon) + (1/n)\overline{A}$ is big if $n \gg 1$. As a consequence, $(D, \lambda + \epsilon)$ is pseudo-effective for any positive number ϵ, and hence (D, λ) is also pseudo-effective. □

3. Dirichlet's unit theorem on arithmetic varieties

In this section, we propose the fundamental question of this paper, which is a higher dimensional analogue of Dirichlet's unit theorem on arithmetic varieties. In SubSection 3.4, we give the proof of the fundamental question on arithmetic curves by using the arithmetic Riemann-Roch theorem and the compactness theorem in SubSection 3.3. By the observations in this subsection, we can realize why the fundamental question is related to the classical Dirichlet's unit theorem. We can also recognize that the theory of arithmetic $\mathbb{R}$-divisors is not an artificial material. In SubSection 3.5, we consider a partial answer to the fundamental question, that is, Dirichlet's unit theorem under the assumption of the numerical triviality of divisors on the generic fiber. Many results in the previous sections will be used for the partial answer. Especially the equality condition of the Hodge index theorem is crucial for our proof. In SubSection 3.6, we introduce the notion of multiplicative generators of approximately smallest sections for further discussions of the fundamental question. It gives rise to many examples in which

Dirichlet's unit theorem holds. SubSection 3.2 is devoted to the technical results on the continuity of norms.

Let us fix notation throughout this section. Let X be a d-dimensional, generically smooth, normal and projective arithmetic variety. Let

$$X \xrightarrow{\pi} \mathrm{Spec}(O_K) \to \mathrm{Spec}(\mathbb{Z})$$

be the Stein factorization of $X \to \mathrm{Spec}(\mathbb{Z})$, where K is a number field and O_K is the ring of integers in K.

3.1. **Fundamental question.** Let $\mathbb{K}$ be either $\mathbb{Q}$ or $\mathbb{R}$. As in Conventions and terminology 2, we set

$$\mathrm{Rat}(X)^{\times}_{\mathbb{K}} := \mathrm{Rat}(X)^{\times} \otimes_{\mathbb{Z}} \mathbb{K},$$

whose element is called a $\mathbb{K}$-*rational function* on X. Note that the zero function is not a $\mathbb{K}$-rational function. Let

$$(\ )_{\mathbb{K}} : \mathrm{Rat}(X)^{\times}_{\mathbb{K}} \to \mathrm{Div}(X)_{\mathbb{K}} \quad \text{and} \quad \widehat{(\ )}_{\mathbb{K}} : \mathrm{Rat}(X)^{\times}_{\mathbb{K}} \to \widehat{\mathrm{Div}}_{C^{\infty}}(X)_{\mathbb{K}}$$

be the natural extensions of the homomorphisms

$$\mathrm{Rat}(X)^{\times} \to \mathrm{Div}(X) \quad \text{and} \quad \mathrm{Rat}(X)^{\times} \to \widehat{\mathrm{Div}}_{C^{\infty}}(X)$$

given by $\phi \mapsto (\phi)$ and $\phi \mapsto \widehat{(\phi)}$ respectively. Note that

$$\mathrm{PDiv}(X)_{\mathbb{K}} = \left\{ (\varphi)_{\mathbb{K}} \mid \varphi \in \mathrm{Rat}(X)^{\times}_{\mathbb{K}} \right\} \quad \text{and} \quad \widehat{\mathrm{PDiv}}(X)_{\mathbb{K}} = \left\{ \widehat{(\varphi)}_{\mathbb{K}} \mid \varphi \in \mathrm{Rat}(X)^{\times}_{\mathbb{K}} \right\}$$

(cf. SubSection 0.2 in Introduction and Conventions and terminology 2). Let $\overline{D} = (D, g)$ be an arithmetic $\mathbb{R}$-Cartier divisor of C^0-type. We define $\Gamma^{\times}(X, D)$, $\widehat{\Gamma}^{\times}(X, \overline{D})$, $\Gamma^{\times}_{\mathbb{K}}(X, D)$ and $\widehat{\Gamma}^{\times}_{\mathbb{K}}(X, \overline{D})$ to be

$$\begin{cases} \Gamma^{\times}(X, D) := \left\{ \phi \in \mathrm{Rat}(X)^{\times} \mid D + (\phi) \geq 0 \right\} = H^0(X, D) \setminus \{0\}, \\ \widehat{\Gamma}^{\times}(X, \overline{D}) := \left\{ \phi \in \mathrm{Rat}(X)^{\times} \mid \overline{D} + \widehat{(\phi)} \geq 0 \right\} = \hat{H}^0(X, \overline{D}) \setminus \{0\}, \\ \Gamma^{\times}_{\mathbb{K}}(X, D) := \left\{ \varphi \in \mathrm{Rat}(X)^{\times}_{\mathbb{K}} \mid D + (\varphi)_{\mathbb{K}} \geq 0 \right\}, \\ \widehat{\Gamma}^{\times}_{\mathbb{K}}(X, \overline{D}) := \left\{ \varphi \in \mathrm{Rat}(X)^{\times}_{\mathbb{K}} \mid \overline{D} + \widehat{(\varphi)}_{\mathbb{K}} \geq 0 \right\}. \end{cases}$$

Let us consider a homomorphism

$$\ell : \mathrm{Rat}(X)^{\times} \to L^1_{loc}(X(\mathbb{C}))$$

given by $\phi \mapsto \log|\phi|$. It extends to a linear map

$$\ell_{\mathbb{K}} : \mathrm{Rat}(X)^{\times}_{\mathbb{K}} \to L^1_{loc}(X(\mathbb{C})).$$

For $\varphi \in \mathrm{Rat}(X)^{\times}_{\mathbb{K}}$, we denote $\exp(\ell_{\mathbb{K}}(\varphi))$ by $|\varphi|$. First let us consider the following lemma.

Lemma 3.1.1. (1) *If $\varphi \in \Gamma^{\times}_{\mathbb{K}}(X, D)$, then $|\varphi| \exp(-g/2)$ is represented by a continuous function $\eta_{\varphi,g}$ on $X(\mathbb{C})$, so that we define $\|\varphi\|_{g,\sup}$ to be*

$$\|\varphi\|_{g,\sup} := \max\left\{ \eta_{\varphi,g}(x) \mid x \in X(\mathbb{C}) \right\}.$$

(2) $\widehat{\Gamma}^{\times}_{\mathbb{K}}(X, \overline{D}) = \left\{ \varphi \in \Gamma^{\times}_{\mathbb{K}}(X, D) \mid \|\varphi\|_{g,\sup} \leq 1 \right\}.$

(3) *We have the following formulae in* $\mathrm{Rat}(X)^{\times}_{\mathbb{Q}}$ *or* $\mathrm{Rat}(X)^{\times}_{\mathbb{R}}$*:*

$$\begin{cases} \Gamma^{\times}_{\mathbb{Q}}(X,D) = \bigcup_{n>0} \Gamma^{\times}(X,nD)^{1/n}, & \widehat{\Gamma}^{\times}_{\mathbb{Q}}(X,\overline{D}) = \bigcup_{n>0} \widehat{\Gamma}^{\times}(X,n\overline{D})^{1/n}, \\ \Gamma^{\times}_{\mathbb{Q}}(X,\alpha D) = \Gamma^{\times}_{\mathbb{Q}}(X,D)^{\alpha}, & \widehat{\Gamma}^{\times}_{\mathbb{Q}}(X,\alpha D) = \widehat{\Gamma}^{\times}_{\mathbb{Q}}(X,D)^{\alpha} \qquad (\alpha \in \mathbb{Q}_{>0}), \\ \Gamma^{\times}_{\mathbb{R}}(X,aD) = \Gamma^{\times}_{\mathbb{R}}(X,D)^{a}, & \widehat{\Gamma}^{\times}_{\mathbb{R}}(X,aD) = \widehat{\Gamma}^{\times}_{\mathbb{R}}(X,D)^{a} \qquad (a \in \mathbb{R}_{>0}). \end{cases}$$

Proof. (1) We set $D = a_1D_1 + \cdots + a_nD_n$ and $\varphi = \varphi_1^{x_1}\cdots\varphi_l^{x_l}$, where $D_1,\ldots,D_n$ are prime divisors, $\varphi_1,\ldots,\varphi_l \in \mathrm{Rat}(X)^{\times}$ and $a_1,\ldots,a_n,x_1,\ldots,x_l \in \mathbb{K}$. Let $f_1,\ldots,f_n$ be local equations of $D_1,\ldots,D_n$ around $P \in X(\mathbb{C})$. Then there is a local continuous function h such that $g = -\sum_{i=1}^n a_i \log|f_i|^2 + h$ (a.e.) around P. Here let us see that $|\varphi_1|^{x_1}\cdots|\varphi_l|^{x_l}|f_1|^{a_1}\cdots|f_n|^{a_n}$ is continuous around P. We set $f_i = u_i t_1^{\alpha_{i1}}\cdots t_r^{\alpha_{ir}}$ and $\varphi_j = v_j t_1^{\beta_{j1}}\cdots t_r^{\beta_{jr}}$, where $\alpha_{ik},\beta_{jk} \in \mathbb{Z}$, $u_1,\ldots,u_n,v_1,\ldots,v_l$ are units of $\mathscr{O}_{X(\mathbb{C}),P}$ and $t_1,\ldots,t_r$ are prime elements of $\mathscr{O}_{X(\mathbb{C}),P}$. Then

$$\begin{aligned} &|\varphi_1|^{x_1}\cdots|\varphi_l|^{x_l}|f_1|^{a_1}\cdots|f_n|^{a_n} \\ &\qquad\qquad = |u_1|^{a_1}\cdots|u_n|^{a_n}|v_1|^{x_1}\cdots|v_l|^{x_l}|t_1|^{\sum_i a_i\alpha_{i1}+\sum_j x_j\beta_{j1}}\cdots|t_r|^{\sum_i a_i\alpha_{ir}+\sum_j x_j\beta_{jr}}. \end{aligned}$$

On the other hand, as

$$D + (\varphi)_{\mathbb{K}} = \left(\sum_i a_i\alpha_{i1} + \sum_j x_j\beta_{j1}\right)(t_1) + \cdots + \left(\sum_i a_i\alpha_{ir} + \sum_j x_j\beta_{jr}\right)(t_r) \geq 0$$

around P, we have

(3.1.1.1) $$\sum_i a_i\alpha_{i1} + \sum_j x_j\beta_{j1} \geq 0,\ \ldots\ ,\sum_i a_i\alpha_{ir} + \sum_j x_j\beta_{jr} \geq 0.$$

Thus the assertion follows. Therefore, $|\varphi_1|^{x_1}\cdots|\varphi_l|^{x_l}|f_1|^{a_1}\cdots|f_n|^{a_n}\exp(-h/2)$ is also continuous around P, and hence we obtain (1) because

$$|\varphi|\exp(-g/2) = |\varphi_1|^{x_1}\cdots|\varphi_l|^{x_l}|f_1|^{a_1}\cdots|f_n|^{a_n}\exp(-h/2) \text{ (a.e.)}.$$

(2) We use the same notation as in (1). Note that

$$\overline{D} + \widehat{(\varphi)}_{\mathbb{K}} = \left(D + (\varphi)_{\mathbb{K}}, g + \sum_{i=1}^n x_i(-\log|\varphi_i|^2)\right).$$

Moreover,

$$g + \sum_{i=1}^n x_i(-\log|\varphi_i|^2) = -\log(|\varphi_1|^{2x_1}\cdots|\varphi_l|^{2x_l}|f_1|^{2a_1}\cdots|f_n|^{2a_n}\exp(-h)) \text{ (a.e.)}$$

locally. Thus $\|\varphi\|_{g,\sup} \leq 1$ if and only if $g + \sum_{i=1}^n x_i(-\log|\varphi_i|^2) \geq 0$ (a.e.), and hence (2) follows.

(3) For $\varphi \in \mathrm{Rat}(X)^{\times}_{\mathbb{R}}$ and $a \in \mathbb{R}_{>0}$, $D + (\varphi)_{\mathbb{R}} \geq 0$ (resp. $\overline{D} + \widehat{(\varphi)}_{\mathbb{R}} \geq 0$) if and only if $aD + (\varphi^a)_{\mathbb{R}} \geq 0$ (resp. $a\overline{D} + \widehat{(\varphi^a)}_{\mathbb{R}} \geq 0$). Thus the assertions in (3) are obvious. □

Remark 3.1.2. We assume $d = 1$, that is, $X = \mathrm{Spec}(O_K)$. For $P \in \mathrm{Spec}(O_K) \setminus \{0\}$ and $\sigma \in K(\mathbb{C})$, the homomorphisms $\mathrm{ord}_P : K^{\times} \to \mathbb{Z}$ and $|\cdot|_{\sigma} : K^{\times} \to \mathbb{R}^{\times}$ given by $\phi \mapsto \mathrm{ord}_P(\phi)$ and $\phi \mapsto |\sigma(\phi)|$ naturally extend to homomorphisms $K^{\times} \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{R}$ and $K^{\times} \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{R}^{\times}$ respectively. By abuse of notation, we denote them by ord_P

and $|\cdot|_\sigma$ respectively. Clearly, for $\varphi \in K^\times \otimes_{\mathbb{Z}} \mathbb{R}$, $|\varphi|_\sigma$ is the value of $|\varphi|$ at σ. Moreover, by using the product formula on $K^\times$, we can see

$$\prod_{\sigma \in K(\mathbb{C})} |\varphi|_\sigma = \prod_{P \in \mathrm{Spec}(O_K) \setminus \{0\}} \#(O_K/P)^{\mathrm{ord}_P(\varphi)} \tag{3.1.2.1}$$

for $\varphi \in K^\times \otimes_{\mathbb{Z}} \mathbb{R}$

Finally we would like to propose the fundamental question as in SubSection 0.7 of Introduction.

Fundamental question. Let $\overline{D}$ be an arithmetic $\mathbb{R}$-Cartier divisor of C^0-type. Are the following equivalent ?

(1) $\overline{D}$ is pseudo-effective.
(2) $\widehat{\Gamma}^\times_{\mathbb{R}}(X, \overline{D}) \neq \emptyset$.

Clearly (2) implies (1). Indeed, let φ be an element of $\widehat{\Gamma}^\times_{\mathbb{R}}(X, \overline{D})$. Let $\overline{A}$ be an ample $\mathbb{R}$-Cartier divisor on X. Since $-\widehat{(\varphi)}_{\mathbb{R}}$ is a nef $\mathbb{R}$-Cartier divisor of C^∞-type, $\overline{A} - \widehat{(\varphi)}_{\mathbb{R}}$ is ample, and hence $\overline{D} + \overline{A}$ is big because $\overline{D} + \overline{A} \geq \overline{A} - \widehat{(\varphi)}_{\mathbb{R}}$. The observations in Subsection 3.4 show that the fundamental question is nothing more than a generalization of Dirichlet's unit theorem. Moreover, the above question does not hold in the geometric case as indicated in the following remark.

Remark 3.1.4. Let C be a smooth algebraic curve over an algebraically closed field. For $\vartheta \in \mathrm{Div}(C)_{\mathbb{Q}}$ with $\deg(\vartheta) = 0$, the following are equivalent:

(1) $\vartheta \in \mathrm{PDiv}(C)_{\mathbb{Q}}$.
(2) There is $\varphi \in \mathrm{Rat}(C)^\times_{\mathbb{R}}$ such that $\vartheta + (\varphi)_{\mathbb{R}} \geq 0$.

Indeed, "(1) $\Longrightarrow$ (2)" is obvious. Conversely we assume (2). Then if we set $\theta = \vartheta + (\varphi)_{\mathbb{R}}$, then θ is effective and $\deg(\theta) = 0$, and hence $\theta = 0$. Thus $\vartheta = (\varphi^{-1})_{\mathbb{R}}$. Therefore, by (3) in Lemma 1.1.1, $\vartheta \in \mathrm{PDiv}(C)_{\mathbb{Q}}$.

The above observation shows that if ϑ is a divisor on C such that $\deg(\vartheta) = 0$ and ϑ is not a torsion element in $\mathrm{Pic}(C)$, then there is no $\varphi \in \mathrm{Rat}(C)^\times_{\mathbb{R}}$ with $\vartheta + (\varphi)_{\mathbb{R}} \geq 0$.

3.2. **Continuity of norms.** Let us fix $p \in \mathbb{R}_{\geq 1}$ and an F_∞-invariant continuous volume form Ω on X with $\int_{X(\mathbb{C})} \Omega = 1$. For $\varphi \in \Gamma^\times_{\mathbb{R}}(X, D)$, we define the *$L^p$-norm of φ with respect to g* to be

$$\|\varphi\|_{g,L^p} := \left(\int_{X(\mathbb{C})} (|\varphi| \exp(-g/2))^p \, \Omega \right)^{1/p}.$$

In this subsection, we consider the following proposition.

Proposition 3.2.1. *Let $\varphi_1, \ldots, \varphi_l \in \mathrm{Rat}(X)^\times_{\mathbb{R}}$. If we set*

$$\Phi = \{(x_1, \ldots, x_l) \in \mathbb{R}^l \mid \varphi_1^{x_1} \cdots \varphi_l^{x_l} \in \Gamma^\times_{\mathbb{R}}(X, D)\},$$

then the map $v_p : \Phi \to \mathbb{R}$ given by $(x_1, \ldots, x_l) \mapsto \|\varphi_1^{x_1} \cdots \varphi_l^{x_l}\|_{g,L^p}$ is uniformly continuous on $K \cap \Phi$ for any compact set K of $\mathbb{R}^l$. Moreover, the map $v_{\sup} : \Phi \to \mathbb{R}$ given by $(x_1, \ldots, x_l) \mapsto \|\varphi_1^{x_1} \cdots \varphi_l^{x_l}\|_{g,\sup}$ is also uniformly continuous on $K \cap \Phi$ for any compact set K of $\mathbb{R}^l$.

Proof. In order to obtain the first assertion, we may clearly assume that $\varphi_1, \ldots, \varphi_l \in \mathrm{Rat}(X)^{\times}$. Let us begin with the following claim:

Claim 3.2.1.1. *There is a constant M such that*

$$|\varphi_1|^{x_1} \cdots |\varphi_l|^{x_l} \exp(-g/2) \leq M \text{ (a.e.)}$$

on $X(\mathbb{C})$ for all $(x_1, \ldots, x_l) \in K \cap \Phi$.

Proof. Since $X(\mathbb{C})$ is compact, it is sufficient to see that the above assertion holds locally. We set $D = a_1D_1 + \cdots + a_nD_n$, where $a_1, \ldots, a_n \in \mathbb{R}$ and $D_1, \ldots, D_n$ are prime divisors. Let us fix $P \in X(\mathbb{C})$ and let $f_1, \ldots, f_n$ be local equations of $D_1, \ldots, D_n$ around P respectively. Let $g = \sum_i (-a_i) \log |f_i|^2 + h$ (a.e.) be the local expression of g with respect to $f_1, \ldots, f_r$, where h is a continuous function around P. We set $f_i = u_i t_1^{\alpha_{i1}} \cdots t_r^{\alpha_{ir}}$ and $\phi_j = v_j t_1^{\beta_{j1}} \cdots t_r^{\beta_{jr}}$, where $\alpha_{ik}, \beta_{jk} \in \mathbb{Z}$, $u_1, \ldots, u_n, v_1, \ldots, v_l$ are units of $\mathscr{O}_{X(\mathbb{C}),P}$ and $t_1, \ldots, t_r$ are prime elements of $\mathscr{O}_{X(\mathbb{C}),P}$. Then

$$\begin{aligned}
&|\phi_1|^{x_1} \cdots |\phi_l|^{x_l} \exp(-g/2) \\
&\quad = |u_1|^{a_1} \cdots |u_n|^{a_n} |v_1|^{x_1} \cdots |v_l|^{x_l} |t_1|^{\sum_i a_i\alpha_{i1} + \sum_j x_j\beta_{j1}} \cdots |t_r|^{\sum_i a_i\alpha_{ir} + \sum_j x_j\beta_{jr}} \exp(-h/2) \text{ (a.e.)}.
\end{aligned}$$

Note that $\sum_i a_i\alpha_{ik} + \sum_j x_j\beta_{jk}$ ($k = 1, \ldots, r$) are bounded non-negative numbers (cf. (3.1.1.1) in the proof of Lemma 3.1.1). Thus the claim follows. $\square$

By the above claim, we obtain

$$\begin{aligned}
&\left| \|\varphi_1^{x_1} \cdots \varphi_l^{x_l}\|_{g,L^p}^p - \|\varphi_1^{y_1} \cdots \varphi_l^{y_l}\|_{g,L^p}^p \right| \\
&\qquad \leq \int_{X(\mathbb{C})} \left| 1 - |\varphi_1|^{p(y_1 - x_1)} \cdots |\varphi_l|^{p(y_l - x_l)} \right| \left(|\varphi_1|^{x_1} \cdots |\varphi_l|^{x_l} \exp(-g/2) \right)^p \Omega \\
&\qquad\qquad \leq \int_{X(\mathbb{C})} \left| 1 - |\varphi_1|^{p(y_1 - x_1)} \cdots |\varphi_l|^{p(y_l - x_l)} \right| M^p \Omega
\end{aligned}$$

for $(x_1, \ldots, x_l), (y_1, \ldots, y_l) \in \Phi$. Thus the first assertion follows from the following Lemma 3.2.2.

For the second assertion, note that $\lim_{p \to \infty} \|\varphi_1^{x_1} \cdots \varphi_l^{x_l}\|_{g,L^p} = \|\varphi_1^{x_1} \cdots \varphi_l^{x_l}\|_{g,\sup}$ for $(x_1, \ldots, x_l) \in \Phi$ (cf. [11, the proof of Corollary 19.9]). Thus it follows from the first assertion. $\square$

Lemma 3.2.2. *Let M be a d-equidimensional complex manifold and let ω be a continuous (d, d)-form on M such that $\omega = \nu\Omega$, where Ω is a volume form on M and ν is a non-negative real valued continuous function on M. Let $\varphi_1, \ldots, \varphi_d$ be meromorphic functions such that φ_i's are non-zero on each connected component of M. Then*

$$\lim_{(x_1, \ldots, x_l) \to (0, \ldots, 0)} \int_M \left| 1 - |\varphi_1|^{x_1} \cdots |\varphi_l|^{x_l} \right| \omega = 0.$$

Proof. Clearly we may assume that M is connected. Let $\mu : M' \to M$ be a proper bimeromorphic morphism of compact complex manifolds such that the principal divisors $(\mu^*(\varphi_1)), \ldots, (\mu^*(\varphi_l))$ are normal crossing. Note that there are a volume form Ω' on M' and a non-negative real valued continuous function ν' on M' such that $\mu^*(\omega) = \nu'\Omega'$. Moreover,

$$\int_{M'} \left| 1 - |\mu^*(\varphi_1)|^{x_1} \cdots |\mu^*(\varphi_l)|^{x_l} \right| \mu^*(\omega) = \int_M \left| 1 - |\varphi_1|^{x_1} \cdots |\varphi_l|^{x_l} \right| \omega.$$

Thus we may assume that the principal divisors $(\varphi_1), \dots, (\varphi_l)$ are normal crossing. Here let us consider the following claim:

Claim 3.2.2.1. *Let $\varphi_1, \dots, \varphi_l$ be meromorphic functions on*

$$\Delta^d = \{(z_1, \dots, z_d) \in \mathbb{C}^d \mid |z_1| < 1, \dots, |z_d| < 1\}$$

such that $\varphi_i = z_1^{c_{1i}} \cdots z_d^{c_{di}} \cdot u_i$ $(i = 1, \dots, l)$, where $c_{ji} \in \mathbb{Z}$ and u_i's are nowhere vanishing holomorphic functions on $\{(z_1, \dots, z_d) \in \mathbb{C}^d \mid |z_1| < 1 + \delta, \dots, |z_d| < 1 + \delta\}$ for some $\delta \in \mathbb{R}_{>0}$. Then

$$\lim_{(x_1,\dots,x_l)\to(0,\dots,0)} \int_{\Delta^d} \left|1 - |\varphi_1|^{x_1} \cdots |\varphi_l|^{x_l}\right| \left(\frac{\sqrt{-1}}{2}\right)^d dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_d \wedge d\bar{z}_d = 0.$$

Proof. If we set $y_j = \sum_{i=1}^l c_{ji}x_i$, then

$$|\varphi_1|^{x_1} \cdots |\varphi_l|^{x_l} = |z_1|^{y_1} \cdots |z_d|^{y_d} |u_1|^{x_1} \cdots |u_l|^{x_l}.$$

Thus, if we put $z_i = r_i \exp(\sqrt{-1}\theta_i)$, then

$$\begin{aligned} &\int_{\Delta^d} \left|1 - |\varphi_1|^{x_1} \cdots |\varphi_l|^{x_l}\right| \left(\frac{\sqrt{-1}}{2}\right)^d dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_d \wedge d\bar{z}_d \\ &\quad = \int_{([0,1]\times[0,2\pi])^d} \left|r_1 \cdots r_d - r_1^{1+y_1} \cdots r_d^{1+y_d} |u_1|^{x_1} \cdots |u_l|^{x_l}\right| dr_1 \wedge d\theta_1 \wedge \cdots \wedge dr_d \wedge d\theta_d. \end{aligned}$$

Note that $r_1^{1+y_1} \cdots r_d^{1+y_d} |u_1|^{x_1} \cdots |u_l|^{x_l} \to r_1 \cdots r_d$ uniformly, as $(x_1, \dots, x_l) \to (0, \dots, 0)$, on $([0,1] \times [0, 2\pi])^d$. Thus the claim follows. □

Let us choose a covering $\{U_j\}_{j=1}^N$ of M with the following properties:

(a) For each j, there is a local parameter $(w_1, \dots, w_d)$ of U_j such that U_j can be identified with Δ^d in terms of $(w_1, \dots, w_d)$.
(b) $\mathrm{Supp}((\phi_i)) \cap U_j \subseteq \{w_1 \cdots w_d = 0\}$ for all i and j.

Let $\{\rho_j\}_{j=1}^N$ be a partition of unity subordinate to the covering $\{U_j\}_{j=1}^N$. Then

$$\int_M \left|1 - |\varphi_1|^{x_1} \cdots |\varphi_l|^{x_l}\right| \omega = \sum_{j=1}^N \int_M \left|1 - |\varphi_1|^{x_1} \cdots |\varphi_l|^{x_l}\right| \rho_j \omega.$$

Note that there is a positive constant C_j such that

$$\rho_j \omega \leq C_j \left(\frac{\sqrt{-1}}{2}\right)^d dw_1 \wedge d\bar{w}_1 \wedge \cdots \wedge dw_d \wedge d\bar{w}_d.$$

Thus the lemma follows from the above claim. □

3.3. **Compactness theorem.** Let $\overline{H}$ be an ample arithmetic $\mathbb{R}$-Cartier divisor on X. Let Γ be a prime divisor on X and let g_Γ be an F_∞-invariant Γ-Green function of C^0-type such that

$$\int_{X_\sigma} g_\Gamma c_1(\overline{H})^{d-1} = -\frac{2\widehat{\deg}(\overline{H}^{d-1} \cdot (\Gamma, 0))}{[K : \mathbb{Q}]}$$

for each $\sigma \in K(\mathbb{C})$. We set $\overline{\Gamma} = (\Gamma, g_{\Gamma})$. Note that

$$\overline{\Gamma} \in \widehat{\mathrm{WDiv}}_{C^0}(X)_{\mathbb{R}} \quad \text{and} \quad \widehat{\deg}(\overline{H}^{d-1} \cdot \overline{\Gamma}) = 0$$

(see, Conventions and terminology 4). Moreover, let $C^0_0(X)$ be the space of F_∞-invariant real valued continuous functions η on $X(\mathbb{C})$ with $\int_{X(\mathbb{C})} \eta c_1(\overline{H})^{d-1} = 0$.

The following theorem will provide a useful tool to find an element of $\widehat{\Gamma}^{\times}_{\mathbb{R}}(X, \overline{D})$.

Theorem 3.3.1. *Let $X^{(1)}$ be the set of all prime divisors on X. For an arithmetic $\mathbb{R}$-Weil divisor $\overline{D}$ of C^0-type* (*cf. Conventions and terminology* 4), *we set*

$$\Upsilon(\overline{D}) = \left\{ (\boldsymbol{a}, \eta) \in \mathbb{R}(X^{(1)}) \oplus C^0_0(X) \;\middle|\; \overline{D} + \sum_{\Gamma} \boldsymbol{a}_{\Gamma} \overline{\Gamma} + (0, \eta) \geq 0 \right\},$$

where $\mathbb{R}(X^{(1)})$ is the vector space generated by $X^{(1)}$ over $\mathbb{R}$ (*cf. Conventions and terminology* 5). *Then $\Upsilon(\overline{D})$ has the following boundedness:*

(1) *For each $\Gamma \in X^{(1)}$, $\{\boldsymbol{a}_{\Gamma}\}_{(\boldsymbol{a}, \eta) \in \Upsilon(\overline{D})}$ is bounded.*
(2) *For each $\sigma \in K(\mathbb{C})$,*

$$\left\{ \int_{X_\sigma} \eta c_1(\overline{H})^{d-1} \right\}_{(\boldsymbol{a}, \eta) \in \Upsilon(\overline{D})}$$

is bounded.

Proof. We set $\overline{D} = (\sum_{\Gamma} d_{\Gamma} \Gamma, g)$. Here we claim the following:

Claim 3.3.1.1. (1) *For all $(\boldsymbol{a}, \eta) \in \Upsilon(\overline{D})$ and $\Gamma \in X^{(1)}$,*

$$-d_{\Gamma} \leq \boldsymbol{a}_{\Gamma} \leq \frac{\dfrac{1}{2}\displaystyle\int_{X(\mathbb{C})} g c_1(\overline{H})^{\wedge d-1} + \sum_{\Gamma' \in X^{(1)} \setminus \{\Gamma\}} d_{\Gamma'} \widehat{\deg}(\overline{H}^{d-1} \cdot (\Gamma', 0))}{\widehat{\deg}(\overline{H}^{d-1} \cdot (\Gamma, 0))}.$$

(2) *For all $(\boldsymbol{a}, \eta) \in \Upsilon(\overline{D})$ and $\sigma \in K(\mathbb{C})$,*

$$-\frac{2\widehat{\deg}(\overline{H}^{d-1} \cdot (D, 0))}{[K : \mathbb{Q}]} - \int_{X_\sigma} g c_1(\overline{H})^{d-1} \leq \int_{X_\sigma} \eta c_1(\overline{H})^{d-1}.$$

Proof. (1) The first inequality is obvious because $-d_{\Gamma} \leq \boldsymbol{a}_{\Gamma}$ for $(\boldsymbol{a}, \eta) \in \Upsilon(\overline{D})$ and $\Gamma \in X^{(1)}$. Moreover, for $\Gamma' \in X^{(1)}$,

$$0 = \widehat{\deg}(\overline{H}^{d-1} \cdot \overline{\Gamma}') = \widehat{\deg}(\overline{H}^{d-1} \cdot (\Gamma', 0)) + \frac{1}{2} \int_{X(\mathbb{C})} g_{\Gamma'} c_1(\overline{H})^{\wedge d-1}.$$

Thus, as $\sum_{\Gamma'} \boldsymbol{a}_{\Gamma'} g_{\Gamma'} + \eta + g \geq 0$, we have

$$
\begin{aligned}
\sum_{\Gamma'} \boldsymbol{a}_{\Gamma'} \widehat{\deg}(\overline{H}^{d-1} \cdot (\Gamma', 0)) &\leq \sum_{\Gamma'} \boldsymbol{a}_{\Gamma'} \widehat{\deg}(\overline{H}^{d-1} \cdot (\Gamma', 0)) + \frac{1}{2} \int_{X(\mathbb{C})} \left(\sum_{\Gamma'} \boldsymbol{a}_{\Gamma'} g_{\Gamma'} + \eta + g \right) c_1(\overline{H})^{\wedge d-1} \\
&= \sum_{\Gamma'} \boldsymbol{a}_{\Gamma'} \left(\widehat{\deg}(\overline{H}^{d-1} \cdot (\Gamma', 0)) + \frac{1}{2} \int_{X(\mathbb{C})} g_{\Gamma'} c_1(\overline{H})^{\wedge d-1} \right) \\
&\qquad + \frac{1}{2} \int_{X(\mathbb{C})} \eta c_1(\overline{H})^{\wedge d-1} + \frac{1}{2} \int_{X(\mathbb{C})} g c_1(\overline{H})^{\wedge d-1} \\
&= \frac{1}{2} \int_{X(\mathbb{C})} g c_1(\overline{H})^{\wedge d-1},
\end{aligned}
$$

and hence

$$
\begin{aligned}
\boldsymbol{a}_{\Gamma} \widehat{\deg}(\overline{H}^{d-1} \cdot (\Gamma, 0)) &= \sum_{\Gamma' \in X^{(1)}} \boldsymbol{a}_{\Gamma'} \widehat{\deg}(\overline{H}^{d-1} \cdot (\Gamma', 0)) + \sum_{\Gamma' \in X^{(1)} \setminus \{\Gamma\}} (-\boldsymbol{a}_{\Gamma'}) \widehat{\deg}(\overline{H}^{d-1} \cdot (\Gamma', 0)) \\
&\leq \frac{1}{2} \int_{X(\mathbb{C})} g c_1(\overline{H})^{\wedge d-1} + \sum_{\Gamma' \neq \Gamma} d_{\Gamma'} \widehat{\deg}(\overline{H}^{d-1} \cdot (\Gamma', 0))
\end{aligned}
$$

for all Γ, which shows the second inequality.

(2) Since $\sum_{\Gamma} \boldsymbol{a}_{\Gamma} \Gamma + D \geq 0$, we obtain

$$
0 \leq \widehat{\deg} \left(\overline{H}^{d-1} \cdot \left(\left(\sum_{\Gamma} \boldsymbol{a}_{\Gamma} \Gamma + D, 0 \right) \right) \right) = \sum_{\Gamma} \boldsymbol{a}_{\Gamma} \widehat{\deg}(\overline{H}^{d-1} \cdot (\Gamma, 0)) + \widehat{\deg}(\overline{H}^{d-1} \cdot (D, 0)).
$$

Therefore, as

$$
\int_{X_\sigma} g_{\Gamma} c_1(\overline{H})^{d-1} = \frac{-2\widehat{\deg}(\overline{H}^{d-1}(\Gamma, 0))}{[K : \mathbb{Q}]},
$$

$$
\begin{aligned}
0 &\leq \int_{X_\sigma} \left(\sum_{\Gamma} \boldsymbol{a}_{\Gamma} g_{\Gamma} + \eta + g \right) c_1(\overline{H})^{d-1} \\
&= \frac{-2 \sum_{\Gamma} \boldsymbol{a}_{\Gamma} \widehat{\deg}(\overline{H}^{d-1}(\Gamma, 0))}{[K : \mathbb{Q}]} + \int_{X_\sigma} \eta c_1(\overline{H})^{d-1} + \int_{X_\sigma} g c_1(\overline{H})^{d-1} \\
&\leq \frac{2\widehat{\deg}(\overline{H}^{d-1} \cdot (D, 0))}{[K : \mathbb{Q}]} + \int_{X_\sigma} \eta c_1(\overline{H})^{d-1} + \int_{X_\sigma} g c_1(\overline{H})^{d-1},
\end{aligned}
$$

as required. □

By (1) in the above claim, $\{\boldsymbol{a}_{\Gamma}\}_{(\boldsymbol{a}, \eta) \in \Upsilon(\overline{D})}$ is bounded for each Γ. Further, by (2), there is a constant M such that

$$
\int_{X_\sigma} \eta c_1(\overline{H})^{d-1} \geq M
$$

for all $(\boldsymbol{a}, \eta) \in \Upsilon(\overline{D})$ and $\sigma \in K(\mathbb{C})$, and hence

$$M \leq \int_{X_\sigma} \eta c_1(\overline{H})^{d-1} = \sum_{\sigma' \in K(\mathbb{C}) \setminus \{\sigma\}} - \int_{X_{\sigma'}} \eta c_1(\overline{H})^{d-1} \leq (\#(K(\mathbb{C})) - 1)(-M),$$

as desired. □

Corollary 3.3.2. *Let Λ be a finite set and let $\{\overline{D}_\lambda\}_{\lambda \in \Lambda}$ be a family of arithmetic $\mathbb{R}$-Weil divisors of C^∞-type with the following properties:*

(a) $\widehat{\deg}(\overline{H}^{d-1} \cdot \overline{D}_\lambda) = 0$ *for $\lambda \in \Lambda$.*
(b) *For each $\lambda \in \Lambda$, there is an F_∞-invariant locally constant function ρ_λ such that*

$$c_1(\overline{D}_\lambda) \wedge c_1(\overline{H})^{\wedge d-2} = \rho_\lambda c_1(\overline{H})^{\wedge d-1}.$$

(c) *$\{\overline{D}_\lambda\}_{\lambda \in \Lambda}$ is linearly independent in $\widehat{\mathrm{WDiv}}_{C^\infty}(X)_{\mathbb{R}}$.*

Then, for $\overline{D} \in \widehat{\mathrm{WDiv}}_{C^0}(X)_{\mathbb{R}}$, the set

$$\left\{ \boldsymbol{a} \in \mathbb{R}(\Lambda) \;\middle|\; \overline{D} + \sum_{\lambda \in \Lambda} \boldsymbol{a}_\lambda \overline{D}_\lambda \geq 0 \right\}$$

is convex and compact.

Proof. The convexity of the above set is obvious, so that we need to show compactness. We pose more conditions to the Γ-Green function g_Γ, that is, we further assume that g_Γ is of C^∞-type and $c_1(\overline{\Gamma}) \wedge c_1(\overline{H})^{\wedge d-2} = \nu_\Gamma c_1(\overline{H})^{\wedge d-1}$ for some locally constant function ν_Γ on $X(\mathbb{C})$. Note that this is actually possible. We set

$$\Xi_X := \left\{ \xi : X(\mathbb{C}) \to \mathbb{R} \;\middle|\; \xi \text{ is locally constant, } F_\infty\text{-invariant and } \sum_{\sigma \in K(\mathbb{C})} \xi_\sigma = 0 \right\}.$$

Then there are $\alpha_{\lambda\Gamma} \in \mathbb{R}$ and $\xi_\lambda \in \Xi_X$ such that

$$\overline{D}_\lambda = \sum_\Gamma \alpha_{\lambda\Gamma} \overline{\Gamma} + (0, \xi_\lambda)$$

for each λ. Therefore,

$$\sum_\lambda \boldsymbol{a}_\lambda \overline{D}_\lambda = \sum_\Gamma \left(\sum_\lambda \boldsymbol{a}_\lambda \alpha_{\lambda\Gamma} \right) \overline{\Gamma} + \sum_\lambda \boldsymbol{a}_\lambda \xi_\lambda.$$

Let us consider a linear map

$$T : \mathbb{R}(\Lambda) \to \mathbb{R}(X^{(1)}) \oplus \Xi_X$$

given by $T(\boldsymbol{a}) = (T_1(\boldsymbol{a}), T_2(\boldsymbol{a}))$, where

$$T_1(\boldsymbol{a})_\Gamma = \sum_\lambda \boldsymbol{a}_\lambda \alpha_{\lambda\Gamma} \; (\Gamma \in X^{(1)}) \quad \text{and} \quad T_2(\boldsymbol{a}) = \sum_\lambda \boldsymbol{a}_\lambda \xi_\lambda.$$

Then T is injective. Indeed, if $T(\boldsymbol{a}) = 0$, then

$$\sum_\lambda \boldsymbol{a}_\lambda \alpha_{\lambda\Gamma} = 0 \; (\forall \Gamma) \quad \text{and} \quad \sum_\lambda \boldsymbol{a}_\lambda \xi_\lambda = 0.$$

Thus $\sum_{\lambda} \boldsymbol{a}_{\lambda}\overline{D}_{\lambda} = 0$, and hence $\boldsymbol{a} = 0$. Since Λ is finite, we can find a finite subset Λ' of $X^{(1)}$ such that the image of T is contained in $\mathbb{R}(\Lambda') \oplus \Xi_X$. Moreover, by the previous theorem, $\Upsilon(\overline{D}) \cap (\mathbb{R}(\Lambda') \oplus \Xi_X)$ is compact. Thus

$$\left\{\boldsymbol{a} \in \mathbb{R}(\Lambda) \;\middle|\; \overline{D} + \sum_{\lambda\in\Lambda} \boldsymbol{a}_{\lambda}\overline{D}_{\lambda} \geq 0\right\} = T^{-1}\left(\Upsilon(\overline{D}) \cap (\mathbb{R}(\Lambda') \oplus \Xi_X)\right)$$

is also compact. □

Corollary 3.3.3. *Let $\varphi_1, \ldots, \varphi_l$ be $\mathbb{R}$-rational functions on X (i.e. $\varphi_1, \ldots, \varphi_l \in \mathrm{Rat}(X)^{\times}_{\mathbb{R}}$) and let $\overline{D} = (D, g)$ be an arithmetic $\mathbb{R}$-Cartier divisor of C^0-type on X. If*

$$\Phi = \left\{(a_1, \ldots, a_l) \in \mathbb{R}^l \mid \varphi_1^{a_1} \cdots \varphi_l^{a_l} \in \Gamma^{\times}_{\mathbb{R}}(X, D)\right\} \neq \emptyset,$$

then there exists $(b_1, \ldots, b_l) \in \Phi$ such that

$$\|\varphi_1^{b_1} \cdots \varphi_l^{b_l}\|_{g,\sup} = \inf_{(a_1,\ldots,a_l)\in\Phi} \left\{\|\varphi_1^{a_1} \cdots \varphi_l^{a_l}\|_{g,\sup}\right\}.$$

Proof. Clearly we may assume that $\varphi_1, \ldots, \varphi_l$ are linearly independent in $\mathrm{Rat}(X)^{\times}_{\mathbb{R}}$. Replacing g by $g + \lambda$ ($\lambda \in \mathbb{R}$) if necessarily, we may further assume that

$$\left\{(a_1, \ldots, a_l) \in \mathbb{R}^l \mid \varphi_1^{a_1} \cdots \varphi_l^{a_l} \in \widehat{\Gamma}^{\times}_{\mathbb{R}}(X, \overline{D})\right\} \neq \emptyset.$$

We denote the above set by $\widehat{\Phi}$. As

$$\widehat{\Phi} = \left\{(a_1, \ldots, a_l) \in \Phi \mid \|\varphi_1^{a_1} \cdots \varphi_l^{a_l}\|_{g,\sup} \leq 1\right\},$$

we have

$$\inf_{(a_1,\ldots,a_l)\in\Phi} \left\{\|\varphi_1^{a_1} \cdots \varphi_l^{a_l}\|_{g,\sup}\right\} = \inf_{(a_1,\ldots,a_l)\in\widehat{\Phi}} \left\{\|\varphi_1^{a_1} \cdots \varphi_l^{a_l}\|_{g,\sup}\right\}.$$

On the other hand, $\widehat{\Phi}$ is compact by Corollary 3.3.2. Thus the assertion of the corollary follows from Proposition 3.2.1. □

3.4. **Dirichlet's unit theorem on arithmetic curves.** We assume $d = 1$, that is, $X = \mathrm{Spec}(O_K)$. In this subsection, we would like to give a proof of Dirichlet's unit theorem in flavor of Arakelov theory (cf. [23]). Of course, the contents of this subsection are nothing new, but it provides the background of this paper and a usage of the compactness theorem (cf. Corollary 3.3.2). The referee points out that Chambert-Loir give a similar proof based on a certain kind of compactness in [4, §1.4, D]. Let us begin with the following weak version of Dirichlet's unit theorem, which is much easier than Dirichlet's unit theorem.

Lemma 3.4.1. *$O_K^{\times}$ is a finitely generated abelian group.*

Proof. This is a standard fact. Indeed, let us consider a homomorphism $L : O_K^{\times} \to \mathbb{R}^{K(\mathbb{C})}$ given by $L(x)_{\sigma} = \log|\sigma(x)|$ for $\sigma \in K(\mathbb{C})$. It is easy to see that, for any bounded set B in $\mathbb{R}^{K(\mathbb{C})}$, the set $\{x \in O_K^{\times} \mid L(x) \in B\}$ is a finite set. Thus the assertion of the lemma is obvious. □

We denote the set of all maximal ideals of O_K by M_K. For an $\mathbb{R}$-Cartier divisor $E = \sum_{P\in M_K} e_P P$ on X, we define $\deg(E)$ and $\mathrm{Supp}(E)$ to be

$$\deg(E) = \sum_{P\in M_K} e_P \log(\#(O_K/P)) \quad \text{and} \quad \mathrm{Supp}(E) := \{P \in M_K \mid e_P \neq 0\}.$$

Lemma 3.4.2. *For a constant C, the set $\{E \in \operatorname{Div}(X) \mid E \geq 0$ and $\deg(E) \leq C\}$ is finite.*

Proof. This is obvious. □

Lemma 3.4.3. *If we set $K^{\times}_{\Sigma} = \{x \in K^{\times} \mid \operatorname{Supp}((x)) \subseteq \Sigma\}$ for a finite subset Σ of M_K, then $K^{\times}_{\Sigma}$ is a finitely generated subgroup of $K^{\times}$.*

Proof. Let us consider a homomorphism $\alpha : K^{\times}_{\Sigma} \to \mathbb{Z}^{\Sigma}$ given by $\alpha(x)_P = \operatorname{ord}_P(x)$ for $P \in \Sigma$. Then $\operatorname{Ker}(\alpha) = O_K^{\times}$ and the image of α is a finitely generated. Thus the lemma follows from the above weak version of Dirichlet's unit theorem. □

Lemma 3.4.4. *We set $C_K = \log\left((2/\pi)^{r_2}\sqrt{|d_{K/\mathbb{Q}}|}\right)$, where r_2 is the number of complex embeddings of K into $\mathbb{C}$ and $d_{K/\mathbb{Q}}$ is the discriminant of K over $\mathbb{Q}$. If $\widehat{\deg}(\overline{D}) \geq C_K$ for $\overline{D} \in \widehat{\operatorname{Div}}(X)$, then there is $x \in K^{\times}$ such that $\overline{D} + \widehat{(x)} \geq 0$.*

Proof. This is a consequence of Minkowski's theorem and the arithmetic Riemann-Roch theorem on arithmetic curves. □

The following proposition is a core part of Dirichlet's unit theorem in terms of Arakelov theory, and can be proved by using arithmetic Riemann-Roch theorem and the compactness theorem (cf. Corollary 3.3.2 and Corollary 3.3.3). As a corollary, it actually implies Dirichlet's unit theorem itself (cf. Corollary 3.4.7).

Proposition 3.4.5. *Let $\overline{D} = (D, g)$ be an arithmetic $\mathbb{R}$-Cartier divisor on X. Then the following are equivalent:*

(i) $\widehat{\deg}(\overline{D}) = 0$.
(ii) $\overline{D} \in \widehat{\operatorname{PDiv}}(X)_{\mathbb{R}}$.
(iii) $\widehat{\deg}(\overline{D}) = 0$ *and* $\widehat{\Gamma}^{\times}_{\mathbb{R}}(X, \overline{D}) \neq \emptyset$.

Proof. "(iii) $\Longrightarrow$ (ii)" : By our assumption, $\overline{D} + z \geq 0$ for some $z \in \widehat{\operatorname{PDiv}}(X)_{\mathbb{R}}$. If we set $\overline{E} = \overline{D} + z$, then $\overline{E}$ is effective and $\widehat{\deg}(\overline{E}) = \widehat{\deg}(\overline{D}) + \widehat{\deg}(z) = 0$. Thus $\overline{E} = 0$, and hence $\overline{D} = -z \in \widehat{\operatorname{PDiv}}(X)_{\mathbb{R}}$.

"(ii) $\Longrightarrow$ (i)" is obvious.

"(i) $\Longrightarrow$ (iii)" : First of all, we can find $\alpha_1, \ldots, \alpha_l \in \mathbb{R}_{>0}$ and $\overline{D}_1, \ldots, \overline{D}_l \in \widehat{\operatorname{Div}}(X)$ such that $\overline{D} = \alpha_1\overline{D}_1 + \cdots + \alpha_l\overline{D}_l$ and $\deg(\overline{D}_i) = 0$ for all i. If we can choose $\psi_i \in \widehat{\Gamma}^{\times}_{\mathbb{R}}(X, \overline{D}_i)$ for all i, then $\psi_1^{\alpha_1} \cdots \psi_l^{\alpha_l} \in \widehat{\Gamma}^{\times}_{\mathbb{R}}(X, \overline{D})$. Thus we may assume that $\overline{D} \in \widehat{\operatorname{Div}}(X)$ in order to show "(i) $\Longrightarrow$ (iii)". For a positive integer n, we set

$$\overline{D}_n = \overline{D} + \left(0, \frac{2C_K}{n[K:\mathbb{Q}]}\right).$$

Since $\widehat{\deg}(n\overline{D}_n) = C_K$, by Lemma 3.4.4, there is $x_n \in K^{\times}$ such that $n\overline{D}_n + \widehat{(x_n)} \geq 0$. In particular, $nD + (x_n) \geq 0$ and

$$\deg(nD + (x_n)) \leq \widehat{\deg}(n\overline{D}_n + \widehat{(x_n)}) = C_K.$$

Thus, by Lemma 3.4.2, there is a finite subset Σ' of M_K such that

$$\operatorname{Supp}(nD + (x_n)) \subseteq \Sigma'$$

for all $n \geq 1$. Note that $\mathrm{Supp}((x_n)) \subseteq \mathrm{Supp}((x_n) + nD) \cup \mathrm{Supp}(D)$. Therefore, we can find a finite subset Σ of M_K such that $x_n \in K_\Sigma^\times$ for all $n \geq 1$. By Lemma 3.4.3, we can take a basis $\varphi_1, \ldots, \varphi_s$ of $K_\Sigma^\times \otimes_{\mathbb{Z}} \mathbb{R}$ over $\mathbb{R}$. Then, by Corollary 3.3.3, if we set

$$\Phi = \{(a_1, \ldots, a_s) \in \mathbb{R}^s \mid \varphi_1^{a_1} \cdots \varphi_s^{a_s} \in \Gamma_{\mathbb{R}}^\times(X, D)\},$$

then there exists $(c_1, \ldots, c_s) \in \Phi$ such that

$$\|\varphi_1^{c_1} \cdots \varphi_s^{c_s}\|_{g,\sup} = \inf_{(a_1,\ldots,a_s)\in\Phi} \{\|\varphi_1^{a_1} \cdots \varphi_s^{a_s}\|_{g,\sup}\},$$

that is, if we set $\psi = \varphi_1^{c_1} \cdots \varphi_s^{c_s}$, then $\|\psi\|_{g,\sup} = \inf_{\varphi\in\Gamma_{\mathbb{R}}^\times(X,D)\cap(K_\Sigma^\times\otimes_{\mathbb{Z}}\mathbb{R})}\{\|\varphi\|_{g,\sup}\}$. On the other hand, as $\overline{D}_n + \widehat{(x_n^{1/n})}_{\mathbb{R}} \geq 0$, we have $x_n^{1/n} \in \Gamma_{\mathbb{R}}^\times(X, D) \cap (K_\Sigma^\times \otimes_{\mathbb{Z}} \mathbb{R})$ and $\|x_n^{1/n}\|_{g,\sup} \leq \exp(C_K/n[K : \mathbb{Q}])$, so that $\|\psi\|_{g,\sup} \leq \exp(C_K/n[K : \mathbb{Q}])$ for all $n > 0$, and hence $\|\psi\|_{g,\sup} \leq 1$, as required. $\square$

As corollaries, we have the following. The second one is nothing more than of Dirichlet's unit theorem.

Corollary 3.4.6. *Let $\overline{D} = (D, g)$ be an arithmetic $\mathbb{R}$-Cartier divisor on X. Then there exists $\psi \in \Gamma_{\mathbb{R}}^\times(X, D)$ such that*

$$\|\psi\|_{g,\sup} = \inf\left\{\|\phi\|_{g,\sup} \mid \phi \in \Gamma_{\mathbb{R}}^\times(X, D)\right\}.$$

Proof. Clearly if the assertion holds for $\overline{D}$, then so does for $\overline{D} + (0, c)$ for all $c \in \mathbb{R}$. Thus we may assume that $\widehat{\deg}(\overline{D}) = 0$. We set $D = \sum_{P\in M_K} d_P P$. Then, for $\phi \in \Gamma_{\mathbb{R}}^\times(X, D)$, by using the product formula (3.1.2.1) in Remark 3.1.2,

$$\prod_{\sigma\in K(\mathbb{C})} |\phi|_\sigma \exp(-g_\sigma/2) = \prod_{P\in X^{(1)}} \#(O_K/P)^{\mathrm{ord}_P(\phi)+d_P} \geq 1,$$

and hence $\|\phi\|_{g,\sup} \geq 1$. On the other hand, by Proposition 3.4.5, there is $\psi \in \Gamma_{\mathbb{R}}^\times(X, D)$ with $\|\psi\|_{g,\sup} \leq 1$, as required. $\square$

Corollary 3.4.7 (Dirichlet's unit theorem). *Let ξ be an element of $\mathbb{R}^{K(\mathbb{C})}$ such that*

$$\sum_{\sigma\in K(\mathbb{C})} \xi_\sigma = 0 \quad \textit{and} \quad \xi_\sigma = \xi_{\bar\sigma}\ (\forall \sigma \in K(\mathbb{C})).$$

Then there are $u_1, \ldots, u_s \in O_K^\times$ and $a_1, \ldots, a_s \in \mathbb{R}$ such that

$$\xi_\sigma = a_1 \log|u_1|_\sigma + \cdots + a_s \log|u_s|_\sigma$$

for all $\sigma \in K(\mathbb{C})$, that is, $(0, \xi) + (a_1/2)\widehat{(u_1)} + \cdots + (a_s/2)\widehat{(u_s)} = 0$.

Proof. Since $\widehat{\deg}((0, \xi)) = 0$, by virtue of Proposition 3.4.5 and (1) in Lemma 1.1.1, there are $a'_1, \ldots, a'_s \in \mathbb{R}$ and $u_1, \ldots, u_s \in K^\times$ such that $a'_1, \ldots, a'_s$ are linearly independent over $\mathbb{Q}$ and $(0, \xi) = a'_1\widehat{(u_1)} + \cdots + a'_s\widehat{(u_s)}$. We set $(u_j) = \sum_{k=1}^l \alpha_{jk} P_k$ for each j, where $\alpha_{jk} \in \mathbb{Z}$ and $P_1, \ldots, P_l$ are distinct maximal ideals of O_K. Then

$$0 = a'_1(u_1) + \cdots + a'_s(u_s) = \left(\sum_{j=1}^s a'_j\alpha_{j1}\right) P_1 + \cdots + \left(\sum_{j=1}^s a'_j\alpha_{jl}\right) P_l.$$

Thus $\sum_{j=1}^s a'_j\alpha_{jk} = 0$ for all k, and hence $\alpha_{jk} = 0$ for all j, k, which means that $u_1, \ldots, u_s \in O_K^\times$. Therefore, if we set $a_j = -2a'_j$, then the corollary follows. $\square$

Remark 3.4.8. Similarly, the finiteness of $\operatorname{Div}(X)/\operatorname{PDiv}(X)$ is also a consequence of Lemma 3.4.2 and Lemma 3.4.4 (cf. [23]). Indeed, if we set

$$\Theta = \{E \in \operatorname{Div}(X) \mid E \geq 0 \text{ and } \deg(E) \leq C_K\},$$

then Θ is a finite set by Lemma 3.4.2. Thus it is sufficient to show that, for $D \in \operatorname{Div}(X)$, there is $x \in K^\times$ such that $D + (x) \in \Theta$. Since

$$\widehat{\deg}\left(D, \frac{2(C_K - \deg(D))}{[K : \mathbb{Q}]}\right) = C_K,$$

by Lemma 3.4.4, there is $x \in K^\times$ such that $\left(D, \frac{2(C_K-\deg(D))}{[K:\mathbb{Q}]}\right) + \widehat{(x)} \geq 0$, that is, $D + (x) \geq 0$ and $\log|x|_\sigma \leq \frac{C_K-\deg(D)}{[K:\mathbb{Q}]}$ for all $\sigma \in K(\mathbb{C})$. By using the product formula,

$$\deg(D + (x)) = \deg(D) + \sum_\sigma \log|x|_\sigma \leq \deg(D) + \sum_\sigma \frac{C_K - \deg(D)}{[K : \mathbb{Q}]} = C_K.$$

Therefore, $D + (x) \in \Theta$, as required.

3.5. **Dirichlet's unit theorem on higher dimensional arithmetic varieties.** In this subsection, we will give a partial answer to the fundamental question as an application of Hodge index theorem. First we consider the case where $d = 1$.

Proposition 3.5.1. *We assume $d = 1$, that is, $X = \operatorname{Spec}(O_K)$. For an arithmetic $\mathbb{R}$-Cartier divisor $\overline{D}$ on X, the following are equivalent:*

(i) *$\overline{D}$ is pseudo-effective.*
(ii) $\widehat{\deg}(\overline{D}) \geq 0$.
(iii) $\widehat{\Gamma}^\times_{\mathbb{R}}(X, \overline{D}) \neq \emptyset$.

Proof. "(i) $\Longrightarrow$ (ii)" : Let $\overline{A}$ is an ample arithmetic Cartier divisor on X. Then $\overline{D} + \epsilon\overline{A}$ is big for any $\epsilon > 0$, that is, $\widehat{\deg}(\overline{D} + \epsilon\overline{A}) > 0$. Therefore, $\widehat{\deg}(\overline{D}) \geq 0$.

"(ii) $\Longrightarrow$ (iii)" : If $\widehat{\deg}(\overline{D}) > 0$, then the assertion is obvious because $\hat{H}^0(X, n\overline{D}) \neq \{0\}$ for $n \gg 1$, so that we assume $\widehat{\deg}(\overline{D}) = 0$. Then $\overline{D} \in \widehat{\operatorname{PDiv}}(X)_{\mathbb{R}}$ by Proposition 3.4.5.

"(iii) $\Longrightarrow$ (i)" is obvious. □

To proceed with further arguments, we need the following lemma.

Lemma 3.5.2. *We assume that X is regular. Let us fix an ample $\mathbb{Q}$-Cartier divisor H on X. Let $P_1, \ldots, P_l \in \operatorname{Spec}(O_K)$ and let $F_{P_1}, \ldots, F_{P_l}$ be prime divisors on X such that $F_{P_i} \subseteq \pi^{-1}(P_i)$ for all i. If A is an ample $\mathbb{Q}$-Cartier divisor on X, then there is an effective $\mathbb{Q}$-Cartier divisor M on X with the following properties:*

(a) $\operatorname{Supp}(M) \subseteq \pi^{-1}(P_1) \cup \cdots \cup \pi^{-1}(P_l)$.
(b) *$A - M$ is divisorially π-nef with respect to H, that is, $\deg_H(A - M \cdot \Gamma) \geq 0$ for all vertical prime divisors Γ on X (cf. Subsection 2.2).*
(c) *$\deg_H(A - M \cdot F) = 0$ for all closed integral integral curve F on X with $F \subseteq \pi^{-1}(P_1) \cup \cdots \cup \pi^{-1}(P_l)$ and $F \neq F_{P_i}$ ($\forall i$).*

Proof. Let us begin with the following claim:

Claim 3.5.2.1. *Let $\pi^{-1}(P_k) = a_1F_1 + \cdots + a_nF_n$ be the irreducible decomposition as a cycle, where $a_i \in \mathbb{Z}_{>0}$. Renumbering $F_1, \ldots, F_n$, we may assume $F_{P_k} = F_1$. Then there are $x_1, \ldots, x_n \in \mathbb{Q}_{>0}$ such that if we set $M_k = x_1F_1 + \cdots + x_nF_n$, then $\deg_H(A - M_k \cdot F_1) > 0$ and $\deg_H(A - M_k \cdot F_i) = 0$ for $i = 2, \ldots, n$.*

Proof. By Lemma 2.2.1, there are $x_1, \ldots, x_n \in \mathbb{Q}$ such that

$$\begin{pmatrix} \deg_H(F_2 \cdot F_1) & \deg_H(F_2 \cdot F_2) & \cdots & \deg_H(F_2 \cdot F_n) \\ \vdots & \vdots & \ddots & \vdots \\ \deg_H(F_n \cdot F_1) & \deg_H(F_n \cdot F_2) & \cdots & \deg_H(F_n \cdot F_n) \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \deg_H(A \cdot F_2) \\ \vdots \\ \deg_H(A \cdot F_n) \end{pmatrix}$$

Replacing x_i by $x_i + ta_i$, we may assume that $x_i > 0$ for all i. We set $M_k = x_1F_1 + \cdots + x_nF_n$. Then $\deg_H(A - M_k \cdot F_i) = 0$ for all $i = 2, \ldots, n$. Here we assume that $\deg_H(A - M_k \cdot F_1) \leq 0$. Then

$$0 < \deg_H(A \cdot F_1) \leq \deg_H(M_k \cdot F_1),$$

and hence

$$\begin{aligned} \deg_H(M_k \cdot M_k) &= \sum_{i=1}^{n} x_i \deg_H(M_k \cdot F_i) \\ &= x_1 \deg_H(M_k \cdot F_1) + \sum_{i=2}^{n} x_i \deg_H(A \cdot F_i) > 0. \end{aligned}$$

This contradicts to Zariski's lemma (cf. Lemma 1.1.4). □

Let $M_1, \ldots, M_n$ be effective $\mathbb{Q}$-Cartier divisors as the above claim. If we set

$$M = M_1 + \cdots + M_l,$$

then M is our desired $\mathbb{Q}$-Cartier divisor. □

The following theorem is a partial answer to the fundamental question.

Theorem 3.5.3. *Let $\overline{D}$ be a pseudo-effective arithmetic $\mathbb{R}$-Cartier divisor of C^0-type. If $d \geq 2$ and D is numerically trivial on $X_{\mathbb{Q}}$, then $\widehat{\Gamma}^{\times}_{\mathbb{R}}(X, \overline{D}) \neq \emptyset$.*

Proof. Let us begin with the following claim:

Claim 3.5.3.1. *We may assume that X is regular.*

Proof. By [6, Theorem 8.2], there is a generically finite morphism $\mu : Y \to X$ of projective arithmetic varieties such that Y is regular. Clearly we have the following:

$$\begin{cases} \overline{D} \text{ is pseudo-effective} \implies \mu^*(\overline{D}) \text{ is pseudo-effective}, \\ D \text{ is numerically trivial on } X_{\mathbb{Q}} \implies \mu^*(D) \text{ is numerically trivial on } Y_{\mathbb{Q}} \end{cases}$$

because $\widehat{\mathrm{vol}}(\mu^*(\overline{L})) \geq \widehat{\mathrm{vol}}(\overline{L})$ for any arithmetic $\mathbb{R}$-Cartier divisor $\overline{L}$ of C^0-type on X. Let $\widehat{\mathrm{Div}}_{\mathrm{Cur}}(X)_{\mathbb{R}}$ be the vector space over $\mathbb{R}$ consisting of pairs (D, T), where D is an $\mathbb{R}$-Cartier divisor D and T is an F_∞-invariant $(0,0)$-current of real type. We can assign an ordering $\geq$ to $\widehat{\mathrm{Div}}_{\mathrm{Cur}}(X)_{\mathbb{R}}$ in following way:

$$(D_1, T_1) \geq (D_2, T_2) \quad \iff \quad D_1 \geq D_1 \text{ and } T_1 \geq T_2.$$

In the same way, we can define $\widehat{\mathrm{Div}}_{\mathrm{Cur}}(Y)_{\mathbb{R}}$ and the ordering on $\widehat{\mathrm{Div}}_{\mathrm{Cur}}(Y)_{\mathbb{R}}$. Let

$$\mu_* : \widehat{\mathrm{Div}}_{\mathrm{Cur}}(Y)_{\mathbb{R}} \to \widehat{\mathrm{Div}}_{\mathrm{Cur}}(X)_{\mathbb{R}}$$

be a homomorphism given by $\mu_*(D, T) = (\mu_*(D), \mu_*(T))$. Let

$$N : \mathrm{Rat}(Y)^{\times} \to \mathrm{Rat}(X)^{\times}$$

be the norm map. Then it is easy to see the following:

$$\begin{cases} \mu_*(\widehat{\psi}) = (\widehat{N(\psi)}) \text{ for } \psi \in \mathrm{Rat}(Y)^{\times}, \\ \mu_*(\mu^*(\overline{D})) = \deg(Y \to X)\overline{D} \text{ for } \overline{D} \in \widehat{\mathrm{Div}}_{C^0}(X)_{\mathbb{R}}, \\ (D_1, T_1) \geq (D_2, T_2) \implies \mu_*(D_1, T_1) \geq \mu_*(D_2, T_2). \end{cases}$$

The first equation yields a homomorphism

$$\mu_* : \widehat{\mathrm{PDiv}}(Y)_{\mathbb{R}} \to \widehat{\mathrm{PDiv}}(X)_{\mathbb{R}}.$$

Thus the claim follows from the above formulae. □

First of all, by Theorem 2.3.3, $D_{\mathbb{Q}} \in \mathrm{PDiv}(X_{\mathbb{Q}})_{\mathbb{R}}$. Thus there are $z \in \widehat{\mathrm{PDiv}}(X)_{\mathbb{R}}$, a vertical $\mathbb{R}$-Cartier divisor E and an F_∞-invariant continuous function η on $X(\mathbb{C})$ such that $\overline{D} = z + (E, \eta)$.

Claim 3.5.3.2. *We may assume the following:*

(a) *E is effective.*
(b) *There are $P_1, \ldots, P_l \in \mathrm{Spec}(O_K)$ such that $\mathrm{Supp}(E) \subseteq \pi^{-1}(P_1) \cup \cdots \cup \pi^{-1}(P_l)$.*
(c) *For each $i = 1, \ldots, l$, there is a closed integral curve F_{P_i} on X such that $F_{P_i} \subseteq \pi^{-1}(P_i)$ and $F_{P_i} \not\subseteq \mathrm{Supp}(E)$.*

Proof. Clearly we can choose $P_1, \ldots, P_l \in \mathrm{Spec}(O_K)$ and $\beta_1, \ldots, \beta_l \in \mathbb{R}$ such that if we set $E' = E + \beta_1\pi^{-1}(P_1) + \cdots + \beta_l\pi^{-1}(P_l)$, then E' satisfy the above (a), (b) and (c). Moreover, since the class group of O_K is finite (cf. Remark 3.4.8), there are $n_i \in \mathbb{Z}_{>0}$ and $f_i \in O_K$ such that $n_i P_i = f_i O_K$. Thus $\beta_1\pi^{-1}(P_1) + \cdots + \beta_l\pi^{-1}(P_l) \in \mathrm{PDiv}(X)_{\mathbb{R}}$, and hence the claim follows. □

Note that (E, η) is pseudo-effective by Lemma 2.3.4. By Lemma 2.3.5, there is a locally constant function λ on $X(\mathbb{C})$ such that $(E, \eta) \geq (E, \lambda)$ and (E, λ) is pseudo-effective. Let us fix an ample arithmetic Cartier divisor $\overline{H} = (H, h)$ on X. Then, by Lemma 3.5.2, there is an effective vertical $\mathbb{Q}$-Cartier divisor M such that

$$\deg_H(H - M \cdot E) = 0 \quad \text{and} \quad \deg_H(H - M \cdot \Gamma) \geq 0$$

for all vertical prime divisors Γ.

Claim 3.5.3.3. *There is a constant c such that if we set $h' = h + c$, then*

$$\widehat{\deg}\left((H - M, h') \cdot \overline{H}^{d-2} \cdot (\Gamma, 0)\right) \geq 0$$

for all horizontal prime divisors Γ on X.

Proof. Note that $\widehat{\deg}((H, h) \cdot \overline{H}^{d-2} \cdot (\Gamma, 0)) \geq 0$. Thus it is sufficient to find a constant c such that

$$\widehat{\deg}((M, -c) \cdot \overline{H}^{d-2} \cdot (\Gamma, 0)) \leq 0$$

for all horizontal prime divisors Γ on X. We choose $Q_1, \ldots, Q_m \in \operatorname{Spec}(O_K)$ and $\alpha_1, \ldots, \alpha_m \in \mathbb{R}_{>0}$ such that $M \leq \sum_{i=1}^m \alpha_i \pi^{-1}(Q_i)$. We also choose a constant c such that

$$c[K:\mathbb{Q}] \geq \sum_{i=1}^m \alpha_i \log \#(O_K/Q_i).$$

Then

$$\begin{aligned}
&\widehat{\deg}((M, -c) \cdot \overline{H}^{d-2} \cdot (\Gamma, 0)) \\
&\qquad \leq \widehat{\deg}\left(\left(\sum_{i=1}^m \alpha_i \pi^{-1}(Q_i), -c\right) \cdot \overline{H}^{d-2} \cdot (\Gamma, 0)\right) \\
&\qquad \leq \sum_{i=1}^m \alpha_i \frac{\deg(H_{\mathbb{Q}}^{d-2} \cdot \Gamma_{\mathbb{Q}})}{[K:\mathbb{Q}]} \log \#(O_K/Q_i) - c \deg(H_{\mathbb{Q}}^{d-2} \cdot \Gamma_{\mathbb{Q}}) \leq 0.
\end{aligned}$$

□

Let $\overline{L} = (L, k)$ be an effective $\mathbb{R}$-Cartier divisor of C^0-type. Then, since

$$\widehat{\deg}\left((H - M, h') \cdot \overline{H}^{d-2} \cdot (L, 0)\right) \geq 0$$

by the above claim, we have

$$\begin{aligned}
&\widehat{\deg}\left((H - M, h') \cdot \overline{H}^{d-2} \cdot (L, k)\right) \\
&\qquad \geq \widehat{\deg}\left((H - M, h') \cdot \overline{H}^{d-2} \cdot (0, k)\right) = \frac{1}{2} \int_{X(\mathbb{C})} k c_1(\overline{H})^{d-1} \geq 0.
\end{aligned}$$

In particular,

$$\widehat{\deg}((H - M, h') \cdot \overline{H}^{d-2} \cdot (E, \lambda)) \geq 0$$

because (E, λ) is pseudo-effective. Note that

$$\widehat{\deg}((H - M, h') \cdot \overline{H}^{d-2} \cdot (E, \lambda)) = \frac{1}{2}\left(\sum_{\sigma \in K(\mathbb{C})} \lambda_\sigma\right) \int_{X(\mathbb{C})} c_1(\overline{H})^{d-1}.$$

Therefore, $\sum_{\sigma \in K(\mathbb{C})} \lambda_\sigma \geq 0$, and hence, by Proposition 3.5.1, there are $u_1, \ldots, u_s \in K^\times$ and $\gamma_1, \ldots, \gamma_s \in \mathbb{R}$ such that $\gamma_1 \widehat{(u_1)} + \cdots + \gamma_s \widehat{(u_s)} \leq (0, \lambda)$. Thus

$$\overline{D} = z + (E, \eta) \geq z + (0, \lambda) \geq z + \gamma_1 \widehat{(u_1)} + \cdots + \gamma_s \widehat{(u_s)}.$$

□

Corollary 3.5.4. *If* $d = 2$*,* $\overline{D}$ *is pseudo-effective and* $\deg(D_{\mathbb{Q}}) = 0$*, then the Zariski decomposition of* $\overline{D}$ *exists.*

3.6. **Multiplicative generators of approximately smallest sections.** In this subsection, we define a notion of multiplicative generators of approximately smallest sections and observe its properties. It is a sufficient condition to guarantee the fundamental question (cf. Corollary 3.6.4). Let $\overline{D}$ be an arithmetic $\mathbb{R}$-Cartier divisor of C^0-type on X. Let us begin with its definition.

Definition 3.6.1. We assume that $\Gamma^{\times}_{\mathbb{Q}}(X, D) \neq \emptyset$. Let $\varphi_1, \ldots, \varphi_l$ be $\mathbb{R}$-rational functions on X (i.e. $\varphi_1, \ldots, \varphi_l \in \operatorname{Rat}(X)^{\times}_{\mathbb{R}}$). We say $\varphi_1, \ldots, \varphi_l$ are *multiplicative generators of approximately smallest sections for* $\overline{D}$ if, for a given $\epsilon > 0$, there is $n_0 \in \mathbb{Z}_{>0}$ such that, for any integer n with $n \geq n_0$ and $\Gamma^{\times}(X, nD) \neq \emptyset$, we can find $a_1, \ldots, a_l \in \mathbb{R}$ satisfying $\varphi_1^{a_1} \cdots \varphi_l^{a_l} \in \Gamma^{\times}_{\mathbb{R}}(X, nD)$ and

$$\|\varphi_1^{a_1} \cdots \varphi_l^{a_l}\|_{ng,\sup} \leq e^{\epsilon n} \min\left\{\|\phi\|_{ng,\sup} \mid \phi \in \Gamma^{\times}(X, nD)\right\}.$$

First let us see the following proposition.

Proposition 3.6.2. *We assume that $\Gamma^{\times}_{\mathbb{Q}}(X, D) \neq \emptyset$. Let $\varphi_1, \ldots, \varphi_l$ be $\mathbb{R}$-rational functions on X. Then the following are equivalent:*

(1) *$\varphi_1, \ldots, \varphi_l$ are multiplicative generators of approximately smallest sections for $\overline{D}$.*
(2) *There are $x_1, \ldots, x_l \in \mathbb{R}$ such that $\varphi_1^{x_1} \cdots \varphi_l^{x_l} \in \Gamma^{\times}_{\mathbb{R}}(X, D)$ and*

$$\|\varphi_1^{x_1} \cdots \varphi_l^{x_l}\|_{g,\sup} \leq \inf\left\{\|f\|_{g,\sup} \mid f \in \Gamma^{\times}_{\mathbb{Q}}(X, D)\right\}.$$

Note that if we set $\psi = \varphi_1^{x_1} \cdots \varphi_l^{x_l}$ in (2)*, then ψ forms a multiplicative generator of approximately smallest sections for $\overline{D}$.*

Proof. It is obvious that (2) implies (1), so that we assume (1). Let m be a positive integer with $\Gamma^{\times}(X, mD) \neq \emptyset$. Here, let us check the following claim:

Claim 3.6.2.1. $\lim_{n\to\infty}\left(\min\{\|h\|_{nmg,\sup} \mid h \in \Gamma^{\times}(X, nmD)\}\right)^{1/nm}$ *exists and*

$$\lim_{n\to\infty}\left(\min\{\|h\|_{nmg,\sup} \mid h \in \Gamma^{\times}(X, nmD)\}\right)^{1/nm} = \inf\left\{\|f\|_{g,\sup} \mid f \in \Gamma^{\times}_{\mathbb{Q}}(X, D)\right\}.$$

Proof. If we set

$$a_n = \min\{\|h\|_{nmg,\sup} \mid h \in \Gamma^{\times}(X, nmD)\},$$

then $a_{n+n'} \leq a_n a_{n'}$ for all $n, n' > 0$. Thus it is easy to see that $\lim_{n\to\infty} a_n^{1/n} = \inf_{n>0}\{a_n^{1/n}\}$, which means

$$\lim_{n\to\infty}\left(\min\{\|h\|_{nmg,\sup} \mid h \in \Gamma^{\times}(X, nmD)\}\right)^{1/nm} = \inf_{n>0}\left\{\min\{\|h^{1/nm}\|_{g,\sup} \mid h \in \Gamma^{\times}(X, nmD)\}\right\}.$$

On the other hand, by (3) in Lemma 3.1.1,

$$\Gamma^{\times}_{\mathbb{Q}}(X, D) = \Gamma^{\times}_{\mathbb{Q}}(X, mD)^{1/m} = \bigcup_{n>0} \Gamma^{\times}(X, nmD)^{1/nm},$$

and hence the claim follows. □

By Corollary 3.3.3, there exist $x_1, \dots, x_l \in \mathbb{R}$ such that if we set

$$\Phi = \left\{(a_1, \dots, a_l) \in \mathbb{R}^l \mid \varphi_1^{a_1} \cdots \varphi_l^{a_l} \in \Gamma^{\times}_{\mathbb{R}}(X, D)\right\},$$

then $(x_1, \dots, x_l) \in \Phi$ and

$$\|\varphi_1^{x_1} \cdots \varphi_l^{x_l}\|_{g,\sup} = \inf_{(a_1,\dots,a_l)\in\Phi} \left\{\|\varphi_1^{a_1} \cdots \varphi_l^{a_l}\|_{g,\sup}\right\}.$$

On the other hand, by definition, for a given $\epsilon > 0$, there is $n_0 \in \mathbb{Z}_{>0}$ such that, for any integer $n \geq n_0$, we can find $c_1, \dots, c_l \in \mathbb{R}$ satisfying $\varphi_1^{c_1} \cdots \varphi_l^{c_l} \in \Gamma^{\times}_{\mathbb{R}}(X, nmD)$ and

$$\|\varphi_1^{c_1} \cdots \varphi_l^{c_l}\|_{nmg,\sup} \leq e^{\epsilon nm} \min\{\|h\|_{nmg,\sup} \mid h \in \Gamma^{\times}(X, nmD)\}.$$

Thus, as $(c_1/nm, \dots, c_l/nm) \in \Phi$,

$$\begin{aligned}\|\varphi_1^{x_1} \cdots \varphi_l^{x_l}\|_{g,\sup} &\leq \|\varphi_1^{c_1/nm} \cdots \varphi_l^{c_l/nm}\|_{g,\sup} \\ &\leq e^{\epsilon} \left(\min\{\|h\|_{nmg,\sup} \mid h \in \Gamma^{\times}(X, nmD)\}\right)^{1/nm}\end{aligned}$$

for $n \geq n_0$. Therefore, by Claim 3.6.2.1,

$$\begin{aligned}\|\varphi_1^{x_1} \cdots \varphi_l^{x_l}\|_{g,\sup} &\leq e^{\epsilon} \lim_{n\to\infty} \left(\min\{\|h\|_{nmg,\sup} \mid h \in \Gamma^{\times}(X, nmD)\}\right)^{1/nm} \\ &= e^{\epsilon} \inf\left\{\|f\|_{g,\sup} \mid f \in \Gamma^{\times}_{\mathbb{Q}}(X, D)\right\}.\end{aligned}$$

Thus (2) follows because ϵ is arbitrary. □

By Corollary 3.4.6, if $d = 1$, then we can find $\psi \in \Gamma^{\times}_{\mathbb{R}}(X, D)$ such that

$$\|\psi\|_{g,\sup} = \inf\left\{\|\phi\|_{g,\sup} \mid \phi \in \Gamma^{\times}_{\mathbb{R}}(X, D)\right\}.$$

Note that the above ψ yields a multiplicative generator of approximately smallest sections. The same assertion holds if we assume the existence of multiplicative generators of approximately smallest sections.

Theorem 3.6.3. *We assume that $\Gamma^{\times}_{\mathbb{Q}}(X, D) \neq \emptyset$. If $\overline{D}$ has multiplicative generators of approximately smallest sections, then there exists $\psi \in \Gamma^{\times}_{\mathbb{R}}(X, D)$ such that*

$$\|\psi\|_{g,\sup} = \inf\left\{\|\phi\|_{g,\sup} \mid \phi \in \Gamma^{\times}_{\mathbb{R}}(X, D)\right\}.$$

Proof. By Proposition 3.6.2, it is sufficient to see the following inequality:

$$\inf\left\{\|f\|_{g,\sup} \mid f \in \Gamma^{\times}_{\mathbb{Q}}(X, D)\right\} \leq \inf\left\{\|\phi\|_{g,\sup} \mid \phi \in \Gamma^{\times}_{\mathbb{R}}(X, D)\right\}. \tag{3.6.3.1}$$

Let $\eta \in \Gamma^{\times}_{\mathbb{Q}}(X, D)$, $D' = D + (\eta)$ and $g' = g - \log|\eta|^2$. Then

$$\begin{cases}\Gamma^{\times}_{\mathbb{Q}}(X, D') = \left\{f/\eta \mid f \in \Gamma^{\times}_{\mathbb{Q}}(X, D)\right\}, \\ \Gamma^{\times}_{\mathbb{R}}(X, D') = \left\{\phi/\eta \mid \phi \in \Gamma^{\times}_{\mathbb{R}}(X, D)\right\}, \\ \|\phi/\eta\|_{g',\sup} = \|\phi\|_{g,\sup} \text{ for } \phi \in \Gamma^{\times}_{\mathbb{R}}(X, D),\end{cases}$$

and hence

$$\begin{cases}\inf\left\{\|f'\|_{g',\sup} \mid f' \in \Gamma^{\times}_{\mathbb{Q}}(X, D')\right\} = \inf\left\{\|f\|_{g,\sup} \mid f \in \Gamma^{\times}_{\mathbb{Q}}(X, D)\right\}, \\ \inf\left\{\|\phi'\|_{g',\sup} \mid \phi' \in \Gamma^{\times}_{\mathbb{R}}(X, D')\right\} = \inf\left\{\|\phi\|_{g,\sup} \mid \phi \in \Gamma^{\times}_{\mathbb{R}}(X, D)\right\}.\end{cases}$$

Therefore, in order to see (3.6.3.1), we may assume that D is effective, that is, if we set $D = \sum d_\Gamma \Gamma$, then $d_\Gamma \geq 0$ for all Γ.

Let ϕ be an arbitrary element of $\Gamma^{\times}_{\mathbb{R}}(X, D)$. Then we can find $f_1, \ldots, f_r \in \operatorname{Rat}(X)^{\times}_{\mathbb{Q}}$ and $a_1, \ldots, a_r \in \mathbb{R}$ such that $\phi = f_1^{a_1} \cdots f_r^{a_r}$ and $a_1, \ldots, a_r$ are linearly independent over $\mathbb{Q}$. Let S be the set of codimension one points of $\bigcup_{i=1}^{r} \operatorname{Supp}((f_i))$.

Claim 3.6.3.2. *If ϵ is a positive number, then $\operatorname{ord}_{\Gamma}(\phi^{1/(1+\epsilon)}) + d_{\Gamma} > 0$ for all $\Gamma \in S$.*

Proof. It is sufficient to show that $\operatorname{ord}_{\Gamma}(\phi) + (1+\epsilon)d_{\Gamma} > 0$ for all $\Gamma \in S$. First of all, note that $\operatorname{ord}_{\Gamma}(\phi) + d_{\Gamma} \geq 0$. If either $\operatorname{ord}_{\Gamma}(\phi) > 0$ or $d_{\Gamma} > 0$, then the assertion is obvious, so that we assume $\operatorname{ord}_{\Gamma}(\phi) \leq 0$ and $d_{\Gamma} = 0$. Then

$$\operatorname{ord}_{\Gamma}(\phi) = a_1 \operatorname{ord}_{\Gamma}(f_1) + \cdots + a_r \operatorname{ord}_{\Gamma}(f_r) = 0,$$

which yields $\operatorname{ord}_{\Gamma}(f_1) = \cdots = \operatorname{ord}_{\Gamma}(f_r) = 0$. This is a contradiction because $\Gamma \in S$. □

As $\phi^{1/(1+\epsilon)} = f_1^{a_1/(1+\epsilon)} \cdots f_1^{a_r/(1+\epsilon)}$, by Claim 3.6.3.2, we can find $\delta > 0$ such that $f_1^{x_1} \cdots f_r^{x_r} \in \Gamma^{\times}_{\mathbb{R}}(X, D)$ for all $(x_1, \ldots, x_r) \in \mathbb{R}^r$ with

$$|x_1 - a_1/(1+\epsilon)| + \cdots + |x_r - a_r/(1+\epsilon)| \leq \delta.$$

We choose a sequence $\{\boldsymbol{t}_n = (t_{n1}, \ldots, t_{nr})\}_{n=1}^{\infty}$ of $\mathbb{Q}^r$ such that

$$|t_{n1} - a_1/(1+\epsilon)| + \cdots + |t_{nr} - a_r/(1+\epsilon)| \leq \delta$$

and $\lim_{n\to\infty} \boldsymbol{t}_n = (a_1/(1+\epsilon), \ldots, a_r/(1+\epsilon))$. Then

$$\inf\left\{\|f\|_{g,\sup} \mid f \in \Gamma^{\times}_{\mathbb{Q}}(X, D)\right\} \leq \|f_1^{t_{n1}} \cdots f_r^{t_{nr}}\|_{g,\sup}$$

because $f_1^{t_{n1}} \cdots f_r^{t_{nr}} \in \Gamma^{\times}_{\mathbb{Q}}(X, D)$. Thus, by using Proposition 3.2.1, we obtain

$$\inf\left\{\|f\|_{g,\sup} \mid f \in \Gamma^{\times}_{\mathbb{Q}}(X, D)\right\} \leq \|\phi^{1/(1+\epsilon)}\|_{g,\sup},$$

which implies $\inf\left\{\|f\|_{g,\sup} \mid f \in \Gamma^{\times}_{\mathbb{Q}}(X, D)\right\} \leq \|\phi\|_{g,\sup}$ by Proposition 3.2.1 again. Therefore, we have (3.6.3.1). □

As a corollary, we have the following:

Corollary 3.6.4. *We assume the following:*

(1) $\widehat{\Gamma}^{\times}_{\mathbb{Q}}(X, \overline{D} + (0, \epsilon)) \neq \emptyset$ *for any* $\epsilon > 0$.

(2) $\overline{D}$ *has multiplicative generators of approximately smallest sections.*

Then $\widehat{\Gamma}^{\times}_{\mathbb{R}}(X, \overline{D}) \neq \emptyset$.

Proof. By the above theorem, there exists $\psi \in \Gamma^{\times}_{\mathbb{R}}(X, D)$ such that

$$\|\psi\|_{g,\sup} = \inf\left\{\|\phi\|_{g,\sup} \mid \phi \in \Gamma^{\times}_{\mathbb{R}}(X, D)\right\}.$$

Since $\widehat{\Gamma}^{\times}_{\mathbb{Q}}(X, \overline{D} + (0, \epsilon)) \neq \emptyset$, we can find $\phi \in \Gamma^{\times}_{\mathbb{Q}}(X, D)$ with $\|\phi\|_{g,\sup} \leq e^{\epsilon/2}$, and hence $\|\psi\|_{g,\sup} \leq e^{\epsilon/2}$. Therefore, $\|\psi\|_{g,\sup} \leq 1$, as required. □

Remark 3.6.5. (1) We assume that $D \in \operatorname{Div}(X)_{\mathbb{Q}}$. Then $\Gamma^{\times}_{\mathbb{Q}}(X, D)$ is dense in $\Gamma^{\times}_{\mathbb{R}}(X, D)$, that is, for $f_1^{a_1} \cdots f_r^{a_r} \in \Gamma^{\times}_{\mathbb{R}}(X, D)$ with $a_1, \ldots, a_r \in \mathbb{R}$ and $f_1, \ldots, f_r \in \operatorname{Rat}(X)^{\times}_{\mathbb{Q}}$, there is a sequence $\{(a_{1n}, \ldots, a_{rn})\}_{n=1}^{\infty}$ in $\mathbb{Q}^r$ such that $f_1^{a_{1n}} \cdots f_r^{a_{rn}} \in \Gamma^{\times}_{\mathbb{Q}}(X, D)$ and $\lim_{n\to\infty}(a_{1n}, \ldots, a_{rn}) = (a_1, \ldots, a_r)$. In particular, $\Gamma^{\times}_{\mathbb{Q}}(X, D) \neq \emptyset$ if and only if $\Gamma^{\times}_{\mathbb{R}}(X, D) \neq \emptyset$. This fact can be checked as follows. Clearly we may assume that $a_1, \ldots, a_r$ are linearly independent over $\mathbb{Q}$. Let S be the set of codimension one points of $\bigcup_i \operatorname{Supp}((f_i))$ and $D = \sum_\Gamma d_\Gamma \Gamma$ ($d_\Gamma \in \mathbb{Q}$). If $(\mathbb{Q}a_1 + \cdots + \mathbb{Q}a_r) \cap \mathbb{Q} = \{0\}$, then it is easy to see that $\operatorname{ord}_\Gamma(f_1^{a_1} \cdots f_r^{a_r}) + d_\Gamma > 0$ for all $\Gamma \in S$. Thus the assertion follows. If $(\mathbb{Q}a_1 + \cdots + \mathbb{Q}a_r) \cap \mathbb{Q} = \mathbb{Q}$, then we may assume that $a_1 \in \mathbb{Q}$ and $(\mathbb{Q}a_2 + \cdots + \mathbb{Q}a_r) \cap \mathbb{Q} = \{0\}$. Thus, as before, we can find a sequence $\{(a_{2n}, \ldots, a_{rn})\}_{n=1}^{\infty}$ in $\mathbb{Q}^{r-1}$ such that $f_2^{a_{2n}} \cdots f_r^{a_{rn}} \in \Gamma^\times_{\mathbb{Q}}(X, (f_1^{a_1}) + D)$ and $\lim_{n\to\infty}(a_{2n}, \ldots, a_{rn}) = (a_2, \ldots, a_r)$, as required.

(2) The assertion of (1) does not hold in general. For example, let $\mathbb{P}^1_{\mathbb{Z}} = \operatorname{Proj}(\mathbb{Z}[T_0, T_1])$ and $a \in \mathbb{R}_{>0} \setminus \mathbb{Q}$. Then $\Gamma^\times_{\mathbb{R}}(X, a(T_1/T_0)) \neq \emptyset$ and $\Gamma^\times_{\mathbb{Q}}(X, a(T_1/T_0)) = \emptyset$. Indeed, $z^a \in \Gamma^\times_{\mathbb{R}}(X, a(T_1/T_0))$, where $z = T_0/T_1$. Moreover, if $\Gamma^\times_{\mathbb{Q}}(X, a(T_1/T_0)) \neq \emptyset$, then there are $n \in \mathbb{Z}_{>0}$ and $f \in \mathbb{Q}(z)$ such that $(f) \geq na(z)$. In particular, $f \in \mathbb{Q}[z]$, so that we can set $f(z) = \sum_{i=s}^{t} c_i z^i$, where $0 \leq s \leq t$, $c_s \neq 0$ and $c_t \neq 0$. Note that $\operatorname{ord}_0(f) = s$ and $\operatorname{ord}_\infty(f) = -t$. Thus $na \leq s \leq t \leq na$, and hence $na = s = t$. This is a contraction because $a \in \mathbb{R}_{>0} \setminus \mathbb{Q}$.

Finally let us consider a sufficient condition for multiplicative generators of approximately smallest sections. Let us fix an F_∞-invariant continuous volume form Ω on X with $\int_{X(\mathbb{C})} \Omega = 1$. We assume that $\Gamma^\times_{\mathbb{Q}}(X, D) \neq \emptyset$. The natural inner product $\langle\ ,\ \rangle_{n\overline{D}}$ on $H^0(X, nD) \otimes \mathbb{R}$ is given by

$$\langle \varphi, \psi \rangle_{n\overline{D}} := \int_{X(\mathbb{C})} \varphi\bar{\psi} \exp(-ng)\Omega \qquad (\varphi, \psi \in H^0(X, nD)).$$

For $\varphi_1, \ldots, \varphi_l \in H^0(X, D)$ and $A = (a_1, \ldots, a_l) \in \mathbb{Z}^l_{\geq 0}$, $\varphi_1^{a_1} \cdots \varphi_l^{a_l}$ is denoted by $\boldsymbol{\varphi}^A$ for simplicity. Note that $\boldsymbol{\varphi}^A \in H^0(X, |A|D)$, where $|A| = a_1 + \cdots + a_l$.

Definition 3.6.6. We say $\varphi_1, \ldots, \varphi_l \in H^0(X, D) \setminus \{0\}$ is a *well-posed generators for* $\overline{D}$ if, for $n \gg 1$, there is a subset Σ_n of $\{A = (a_1, \ldots, a_l) \in \mathbb{Z}^l_{\geq 0} \mid a_1 + \cdots + a_l = n\}$ with the following properties:

(1) $\{\boldsymbol{\varphi}^A \mid A \in \Sigma_n\}$ forms a basis of $H^0(X, nD) \otimes \mathbb{Q}$ over $\mathbb{Q}$.

(2) Let $\langle \boldsymbol{\varphi}^A \mid A \in \Sigma_n \rangle_{\mathbb{Z}}$ be the $\mathbb{Z}$-submodule generated by $\{\boldsymbol{\varphi}^A \mid A \in \Sigma_n\}$ in $H^0(X, nD)$, that is, $\langle \boldsymbol{\varphi}^A \mid A \in \Sigma_n \rangle_{\mathbb{Z}} = \sum_{A \in \Sigma_n} \mathbb{Z}\, \boldsymbol{\varphi}^A$. Then

$$\limsup_{n\to\infty} \left(\#\left(H^0(X, nD)/\langle \boldsymbol{\varphi}^A \mid A \in \Sigma_n \rangle_{\mathbb{Z}}\right)\right)^{1/n} = 1.$$

(3) For a finite subset $\{\psi_1, \ldots, \psi_r\}$ of $H^0(X, nD)_{\mathbb{R}}$, the square root of the Gramian of $\psi_1, \ldots, \psi_r$ with respect to $\langle\ ,\ \rangle_{n\overline{D}}$ is denoted by $\operatorname{vol}(\{\psi_1, \ldots, \psi_r\})$ (for details, see Conventions and terminology 6). Then

$$\liminf_{n\to\infty} \min \left\{ \left(\frac{\operatorname{vol}(\{\boldsymbol{\varphi}^B \mid B \in \Sigma_n\})}{\sqrt{\langle \boldsymbol{\varphi}^A, \boldsymbol{\varphi}^A \rangle_{n\overline{D}}}\, \operatorname{vol}(\{\boldsymbol{\varphi}^B \mid B \in \Sigma_n \setminus \{A\}\})} \right)^{1/n} \,\middle|\, A \in \Sigma_n \right\} = 1.$$

Proposition 3.6.7. *We assume that $\overline{D}$ is of C^∞-type. If $\varphi_1, \ldots, \varphi_l \in H^0(X, D) \setminus \{0\}$ are well-posed generators for $\overline{D}$, then $\varphi_1, \ldots, \varphi_l$ are multiplicative generators of approximately smallest sections for $\overline{D}$*

Proof. For a given $\epsilon > 0$, we set $\epsilon' = \epsilon/6$. First of all, there is a positive integer n_0 such that

$$r_n = \#\left(H^0(X, nD)/\langle \boldsymbol{\varphi}^A \mid A \in \Sigma_n \rangle_{\mathbb{Z}}\right) \leq e^{n\epsilon'}$$

and

$$\frac{\operatorname{vol}(\{\boldsymbol{\varphi}^B \mid B \in \Sigma_n\})}{\sqrt{\langle \boldsymbol{\varphi}^A, \boldsymbol{\varphi}^A \rangle} \operatorname{vol}(\{\boldsymbol{\varphi}^B \mid B \in \Sigma_n \setminus \{A\}\})} \geq e^{-n\epsilon'}$$

for all $n \geq n_0$ and $A \in \Sigma_n$. Let W_A be the subspace generated by $\{\boldsymbol{\varphi}^B\}_{B \in \Sigma_n \setminus \{A\}}$ over $\mathbb{R}$. If θ_A is the angle between $\boldsymbol{\varphi}^A$ and W_A, then, by Lemma 1.1.2

$$\sin(\theta_A) = \frac{\operatorname{vol}(\{\boldsymbol{\varphi}^B \mid B \in \Sigma_n\})}{\sqrt{\langle \boldsymbol{\varphi}^A, \boldsymbol{\varphi}^A \rangle} \operatorname{vol}(\{\boldsymbol{\varphi}^B \mid B \in \Sigma_n \setminus \{A\}\})},$$

and hence

$$\begin{aligned} \cos(\theta_A) &= \sqrt{1 - \sin^2(\theta_A)} \\ &\leq \sqrt{1 - e^{-2n\epsilon'}} \leq 1 - (1/2)e^{-2n\epsilon'} \end{aligned}$$

for all $A \in \Sigma_n$ because $\sqrt{1-x} \leq 1 - (1/2)x$ for $x \in [0,1]$. Let $y \in W_A$ and let θ be the angle between $\boldsymbol{\varphi}^A$ and y. Then, as $\theta_A \leq \min\{\theta, \pi - \theta\}$,

$$\begin{aligned} |\langle \boldsymbol{\varphi}^A, y \rangle| &\leq \cos(\theta_A) \sqrt{\langle \boldsymbol{\varphi}^A, \boldsymbol{\varphi}^A \rangle} \sqrt{\langle y, y \rangle} \\ &\leq (1 - (1/2)e^{-2n\epsilon'}) \sqrt{\langle \boldsymbol{\varphi}^A, \boldsymbol{\varphi}^A \rangle} \sqrt{\langle y, y \rangle}. \end{aligned}$$

Let $\phi \in \Gamma^\times(X, n\overline{D})$. Then we can find $a_A \in \mathbb{Q}$ ($A \in \Sigma_n$) such that $\phi = \sum_{A \in \Sigma_n} a_A \boldsymbol{\varphi}^A$. Note that $r_n a_A \in \mathbb{Z}$ for all $A \in \Sigma_n$. Let us choose $A_0 \in \Sigma_n$ such that $a_{A_0} \neq 0$. We set $y = \sum_{A \in \Sigma_n \setminus \{A_0\}} a_A \boldsymbol{\varphi}^A$. Then $\phi = a_{A_0} \boldsymbol{\varphi}^{A_0} + y$. Since $e^{n\epsilon'}|a_{A_0}| \geq |r_n a_{A_0}| \geq 1$,

$$\begin{aligned} \langle \phi, \phi \rangle &= a_{A_0}^2 \langle \boldsymbol{\varphi}^{A_0}, \boldsymbol{\varphi}^{A_0} \rangle + 2a_{A_0} \langle \boldsymbol{\varphi}^{A_0}, y \rangle + \langle y, y \rangle \\ &\geq a_{A_0}^2 \langle \boldsymbol{\varphi}^{A_0}, \boldsymbol{\varphi}^{A_0} \rangle + \langle y, y \rangle - 2|a_{A_0}| \cdot |\langle \boldsymbol{\varphi}^{A_0}, y \rangle| \\ &\geq a_{A_0}^2 \langle \boldsymbol{\varphi}^{A_0}, \boldsymbol{\varphi}^{A_0} \rangle + \langle y, y \rangle - 2|a_{A_0}| \sqrt{\langle \boldsymbol{\varphi}^{A_0}, \boldsymbol{\varphi}^{A_0} \rangle} \sqrt{\langle y, y \rangle}(1 - (1/2)e^{-2n\epsilon'}) \\ &= (1 - (1/2)e^{-2n\epsilon'}) \left(|a_{A_0}| \sqrt{\langle \boldsymbol{\varphi}^{A_0}, \boldsymbol{\varphi}^{A_0} \rangle} - \sqrt{\langle y, y \rangle} \right)^2 \\ &\qquad\qquad + (1/2)e^{-2n\epsilon'} \left(a_{A_0}^2 \langle \boldsymbol{\varphi}^{A_0}, \boldsymbol{\varphi}^{A_0} \rangle + \langle y, y \rangle \right) \\ &\geq (1/2)e^{-2n\epsilon'} a_{A_0}^2 \langle \boldsymbol{\varphi}^{A_0}, \boldsymbol{\varphi}^{A_0} \rangle = (1/2)e^{-4n\epsilon'} (e^{n\epsilon'} a_{A_0})^2 \langle \boldsymbol{\varphi}^{A_0}, \boldsymbol{\varphi}^{A_0} \rangle \\ &\geq (1/2)e^{-4n\epsilon'} \langle \boldsymbol{\varphi}^{A_0}, \boldsymbol{\varphi}^{A_0} \rangle. \end{aligned}$$

On the other hand, by Gromov's inequality (cf. [20, Proposition 3.1.1]), choosing a larger n_0 if necessarily, $\|\psi\|_{\sup}^2 \leq e^{n\epsilon'} \langle \psi, \psi \rangle$ for all $n \geq n_0$ and $\psi \in H^0(X, nD)$. Moreover, we may also assume that $2 \leq e^{n\epsilon'}$ for all $n \geq n_0$. Thus, as $\|\phi\|_{\sup}^2 \geq \langle \phi, \phi \rangle$,

$$\begin{aligned} e^{n\epsilon} \|\phi\|_{\sup}^2 &= e^{6n\epsilon'} \|\phi\|_{\sup}^2 \geq 2e^{5n\epsilon'} \|\phi\|_{\sup}^2 \geq 2e^{5n\epsilon'} \langle \phi, \phi \rangle \\ &\geq 2e^{5n\epsilon'} \left((1/2)e^{-4n\epsilon'} \langle \boldsymbol{\varphi}^{A_0}, \boldsymbol{\varphi}^{A_0} \rangle \right) = e^{n\epsilon'} \langle \boldsymbol{\varphi}^{A_0}, \boldsymbol{\varphi}^{A_0} \rangle \geq \|\boldsymbol{\varphi}^{A_0}\|_{\sup}^2, \end{aligned}$$

as required. □

Example 3.6.8. Let $\mathbb{P}^d_{\mathbb{Z}} = \operatorname{Proj}(\mathbb{Z}[T_0, T_1, \ldots, T_d])$, $H_i = \{T_i = 0\}$ and $z_i = T_i/T_0$ for $i = 0, 1, \ldots, d$. Let $\overline{D} = (H_0, g)$ be an arithmetic Cartier divisor of C^∞-type on $\mathbb{P}^d_{\mathbb{Z}}$. Moreover, let Ω be an F_∞-invariant continuous volume form on $\mathbb{P}^d(\mathbb{C})$. We assume that there are continuous functions a and b on $\mathbb{R}^d_{\geq 0}$ such that $g(z_1, \dots, z_d) = a(|z_1|, \dots, |z_d|)$ and

$$\Omega = \left(\frac{\sqrt{-1}}{2\pi}\right)^d b(|z_1|, \dots, |z_d|) dz_1 \wedge d\bar{z}_1 \wedge \cdots dz_d \wedge d\bar{z}_d.$$

Arithmetic Cartier divisors considered in [21] satisfy the above condition.

Here let us see that $1, z_1, \dots, z_d$ are well-posed generator for $\overline{D}$. We set

$$\Sigma_n = \{(a_1, \dots, a_d) \in \mathbb{Z}^d_{\geq 0} \mid a_1 + \cdots + a_d \leq n\}.$$

Then $\{\boldsymbol{z}^A\}_{A \in \Sigma_n}$ forms a free basis of $H^0(\mathbb{P}^d_{\mathbb{Z}}, nH_0)$. Moreover, if we set

$$z_i = r_i \exp(2\pi \sqrt{-1}\theta_i) \quad (i = 1, \dots, d),$$

then

$$\langle \boldsymbol{z}^A, \boldsymbol{z}^{A'} \rangle_{ng} = \int_{\mathbb{R}^d_{\geq 0} \times [0,1]^d} \left(\prod_{i=1}^d 2r_i^{A_i + A'_i + 1} \exp(2\pi\sqrt{-1}(A_i - A'_i)) \right) \\ \times \exp(-na(r_1, \dots, r_d)) b(r_1, \dots, r_d) dr_1 \cdots dr_d d\theta_1 \cdots d\theta_d,$$

which implies $\langle \boldsymbol{z}^A, \boldsymbol{z}^{A'} \rangle_{ng} = 0$ for $A, A' \in \Sigma_n$ with $A \neq A'$, and hence

$$\mathrm{vol}(\{\boldsymbol{z}^B \mid B \in \Sigma_n\}) = \sqrt{\langle \boldsymbol{z}^A, \boldsymbol{z}^A \rangle}\, \mathrm{vol}(\{\boldsymbol{z}^B \mid B \in \Sigma_n \setminus \{A\}\})$$

for all $A \in \Sigma_n$.

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