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SYMMETRIC CRYSTALS AND LLTA TYPE CONJECTURES FOR THE AFFINE HECKE ALGEBRAS OF TYPE B

NAOYA ENOMOTO AND MASAKI KASHIWARA

ABSTRACT. In the previous paper [EK1], we formulated a conjecture on the relations between certain classes of irreducible representations of affine Hecke algebras of type B and symmetric crystals for $\mathfrak{gl}_\infty$. In the first half of this paper (sections 2 and 3), we give a survey of the LLTA type theorem of the affine Hecke algebra of type A. In the latter half (sections 4, 5 and 6), we review the construction of the symmetric crystals and the LLTA type conjectures for the affine Hecke algebra of type B.

1. INTRODUCTION

1.1. The Lascoux-Leclerc-Thibon-Ariki theory connects the representation theory of the affine Hecke algebra of *type A* with representations of the affine quantum enveloping algebra of type A. Recently, we presented the notion of symmetric crystals and conjectured that certain classes of irreducible representations of the affine Hecke algebras of *type B* are described by symmetric crystals for $\mathfrak{gl}_\infty$ or $A^{(1)}_{\ell-1}$ ([EK1]). In this paper, we review the LLTA-theory for the affine Hecke algebra of type A, the symmetric crystals, and then our conjectures for the affine Hecke algebra of type B. *For the sake of simplicity, we restrict ourselves in this note to the case where the parameters of the affine Hecke algebras are not a root of unity.*

This paper is organized as follows. In part I (sections 2 and 3), we review the LLTA-theory for the affine Hecke algebras of type A. In section 2, we recall the representation theory of $U_q(\mathfrak{gl}_\infty)$, especially the PBW basis, the crystal basis and the global basis. In section 3, we recall the representation theory of the affine Hecke algebra of type A and state the LLTA-type theorems. In part II (sections 4, 5 and 6), we explain symmetric crystals for $\mathfrak{gl}_\infty$ and the LLTA type conjectures for the affine Hecke algebras of type B. In section 4, we recall the construction of symmetric crystals based on [EK1] and state the conjecture of existence of the crystal basis and the global basis. In section 5, we explain a combinatorial realization of the symmetric crystals for $\mathfrak{gl}_\infty$ by using the PBW type basis and the θ-restricted multisegments. This section is a new additional part to the announcement [EK1]. The details will appear in [EK2]. In section 6, we explain the representation theory of the affine Hecke algebra of type B and state our LLTA-type conjectures for the affine Hecke algebra of type B. We add proofs of lemmas and propositions in [EK1, section 3.4].

1.2. Let us recall the LLTA-theory for the affine Hecke algebra of type A.

The representation theory of quantum enveloping algebras and the representation theory of affine Hecke algebras have developed independently. G. Lusztig [L] constructed the PBW type basis and canonical basis of $U_q^-(\mathfrak{g})$ for the A, D, E cases. The second author [Kas]

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defined the crystal basis $B(\infty)$ and the (lower and upper) global bases $\{G^{\mathrm{low}}(b)\}_{b\in B(\infty)}$, $\{G^{\mathrm{up}}(b)\}_{b\in B(\infty)}$ of $U_q^-(\mathfrak{g})$. The lower global basis coincides with Lusztig's canonical basis. On the other hand, A. V. Zelevinsky [Z] gave a parametrization of the irreducible representations of the affine Hecke algebra of type A by using multisegments. Chriss-Ginzburg [CG] and Kazhdan-Lusztig [KL] constructed all the irreducible representations of the affine Hecke algebras in geometric methods.

Lascoux-Leclerc-Thibon conjectured in [LLT] that certain composition multiplicities (called the decomposition numbers) of the Hecke algebra of type A can be written by the transition matrices (specialized at $q = 1$) between the upper global basis and a standard basis of the level 1 fundamental representation of $U_q(\widehat{\mathfrak{sl}_\ell})$. In [A], S. Ariki generalized and solved the conjecture for the cyclotomic Hecke algebra and the affine Hecke algebra of type A by using the geometric representation theory of the affine Hecke algebra of type A. In [GV], I. Grojnowski and M. Vazirani proved the multiplicity-one results for the socle of certain restriction functors and the cosocle of certain induction functors on the category of the finite-dimensional representations of the affine Hecke algebras $\mathcal{H}^A$ of type A. By using these functors, Grojnowski ([G]) gave the crystal structure on the set of irreducible modules over the affine Hecke algebras $\mathcal{H}^A$ of type A. In [V], Vazirani combinatorially constructed the crystal operators on the set of multisegments and proved the compatibility between her actions and Grojnowski's actions.

For $p \in \mathbb{C}^*$, let $\mathcal{H}_n^A(p)$ be the affine Hecke algebra of type A of degree n generated by T_i $(1 \leqslant i \leqslant n-1)$ and $X_j^{\pm 1}$ $(1 \leqslant j \leqslant n)$. For a subset J of $\mathbb{C}^*$, we say that a finite-dimensional $\mathcal{H}_n^A$-module is of type J if all the eigenvalues of X_j $(1 \leqslant j \leqslant n)$ belong to J. We can prove that in order to study the irreducible modules over the affine Hecke algebras of type A, it is enough to treat those of type J for an orbit J with respect to the $\mathbb{Z}$-action on $\mathbb{C}^*$ generated by $a \mapsto ap^2$ (see Lemma 3.3). For a $\mathbb{Z}$-orbit J, let $K_J(\mathcal{H}_n^A)$ be the Grothendieck group of the abelian category of finite-dimensional $\mathcal{H}_n^A$-modules of type J, and $K_J^A = \oplus_{n\geqslant 0} K_J(\mathcal{H}_n^A)$. The LLTA-theory gives the following correspondence between the notions in the representation theory of a quantum enveloping algebra $U_q(\mathfrak{gl}_\infty)$ and the ones in the representation theory of affine Hecke algebras of type A.

the quantum enveloping algebra $U_q(\mathfrak{gl}_\infty)$	the affine Hecke algebra of type A $\mathcal{H}_n^A(p)$ $(n \geqslant 0)$
$U_q^-(\mathfrak{gl}_\infty)$	$K_J^A = \oplus_{n\geqslant 0} K_J(\mathcal{H}_n^A(p))$
e'_a, f_a	certain restrictions e_a and inductions f_a
the crystal basis $B(\infty)$	$\mathcal{M} = \{\text{the multisegments}\}$
the upper global basis $\{G^{\mathrm{up}}(b)\}_{b\in B(\infty)}$	the irreducible modules $\{L_b\}_{b\in B(\infty)}$
the modified root operators $\widetilde{e}_a, \widetilde{f}_a$	$\widetilde{e}_a = \mathrm{soc}(e_a), \widetilde{f}_a = \mathrm{cosoc}(f_a)$ $\widetilde{e}_a L_b = L_{\widetilde{e}_a b}, \widetilde{f}_a L_b = L_{\widetilde{f}_a b}$
the PBW basis $\{P(b)\}_{b\in B(\infty)}$	the standard modules $\{M(b)\}_{b\in B(\infty)}$

FIGURE 1. Lascoux-Leclerc-Thibon-Ariki correspondence in type A

The additive group K_J^A has a structure of Hopf algebra by the restriction and the induction. The set J may be regarded as a Dynkin diagram with J as the set of vertices and with edges between $a \in J$ and ap^2. Let $\mathfrak{g}_J$ be the associated Lie algebra, and $\mathfrak{g}_J^-$ the unipotent Lie subalgebra. Hence $\mathfrak{g}_J$ is isomorphic to $\mathfrak{gl}_\infty$ if p has an infinite order. Let U_J be the group associated to $\mathfrak{g}_J^-$. Then $\mathbb{C} \otimes \mathrm{K}_J^A$ is isomorphic to the algebra $\mathscr{O}(U_J)$ of regular functions on U_J. Let $U_q(\mathfrak{g}_J)$ be the associated quantized enveloping algebra. Then $U_q^-(\mathfrak{g}_J)$ has a crystal basis $B(\infty)$ and an upper global basis $\{G^{\mathrm{up}}(b)\}_{b\in B(\infty)}$. By specializing $\bigoplus \mathbb{C}[q,q^{-1}]G^{\mathrm{up}}(b)$ at $q=1$, we obtain $\mathscr{O}(U_J)$. Then the LLTA-theory says that the elements associated to the irreducible $\mathcal{H}^A$-modules correspond to the image of the upper global basis. Namely, each $b \in B(\infty)$, an irreducible $\mathcal{H}^A$-module L_b is associated and we have

$$[e_aL_b : L_{b'}] = e'_{a,b,b'}|_{q=1}, \quad [f_aL_b : L_{b'}] = f_{a,b,b'}|_{q=1}.$$

Here $[e_aL_b : L_{b'}]$ and $[f_aL_b : L_{b'}]$ are the composition multiplicities of $L_{b'}$ of e_aL_b and f_aL_b in K_J^A. (For the definition of the functors e_a and f_a for $a \in J$, see Definition 3.4.) The Laurent polynomials $e'_{a,b,b'}$ and $f_{a,b,b'}$ are defined by

$$e'_aG^{\mathrm{up}}(b) = \sum_{b'\in B(\infty)} e'_{a,b,b'}G^{\mathrm{up}}(b'), \quad f_aG^{\mathrm{up}}(b) = \sum_{b'\in B(\infty)} f_{a,b,b'}G^{\mathrm{up}}(b').$$

1.3. Let us explain our analogous conjectures for the affine Hecke algebras of type B.

For $p_0, p_1 \in \mathbb{C}^*$, let $\mathcal{H}_n^B(p_0,p_1)$ be the affine Hecke algebra of type B generated by T_i $(0 \leqslant i \leqslant n-1)$ and X_j $(1 \leqslant j \leqslant n)$. The representation theory of $\mathcal{H}_n^B(p_0,p_1)$ of type B are studied by V. Miemietz and Syu Kato. In [M], V. Miemietz defined certain restriction functors E_a and the induction functors F_a on the category of the finite-dimensional representations of the affine Hecke algebras of type B, which are analogous to Grojnowski-Vazirani's construction, and proved the multiplicity-one results (see sections 6.3 and 6.4). On the other hand, S. Kato obtained in [Kat] a geometric parametrization of the irreducible representations of the affine Hecke algebra $\mathcal{H}_n^B(p_0,p_1)$, which is an analogue to geometric methods of Kazhdan-Lusztig and Chriss-Ginzburg.

We say that a finite-dimensional $\mathcal{H}_n^B$-module is of type $J \subset \mathbb{C}^*$ if all the eigenvalues of X_j $(1 \leqslant j \leqslant n)$ belong to J. Let us consider the $\mathbb{Z} \rtimes \mathbb{Z}_2$-action on $\mathbb{C}^*$ generated by $a \mapsto ap_1^2$ and $a \mapsto a^{-1}$. We can prove that in order to study $\mathcal{H}^B$-modules, it is enough to study irreducible modules of type J for a $\mathbb{Z} \rtimes \mathbb{Z}_2$-orbit J in $\mathbb{C}^*$ such that J is a $\mathbb{Z}$-orbit or J contains one of $\pm 1, \pm p_0$ (see Proposition 6.4). Let $I = \mathbb{Z}_{\mathrm{odd}}$ be the set of odd integers. In this paper, we consider the case $J = \{p_1^k \mid k \in I\}$ such that $\pm 1, \pm p_0 \notin J$. Let $K_J(\mathcal{H}_n^B)$ be the Grothendieck group of the abelian category of finite-dimensional representations of $\mathcal{H}_n^B(p_0,p_1)$ of type J.

Let α_a $(a \in J)$ be the simple roots with

$$(\alpha_a, \alpha_b) = \begin{cases} 2 & \text{if } a = b, \\ -1 & \text{if } b = ap_1^{\pm 2}, \\ 0 & \text{otherwise.} \end{cases}$$

Then the corresponding Lie algebra is $\mathfrak{gl}_\infty$. Let θ be the involution of J given by $\theta(a) = a^{-1}$. In sections 4 and 5, we introduce the ring $\mathcal{B}_\theta(\mathfrak{gl}_\infty)$ and the $\mathcal{B}_\theta(\mathfrak{gl}_\infty)$-module $V_\theta(0)$. They are analogues of the reduced q-analogue $\mathcal{B}_q(\mathfrak{gl}_\infty)$ generated by e'_a and f_a, and the $\mathcal{B}_q(\mathfrak{gl}_\infty)$-module $U_q^-(\mathfrak{gl}_\infty)$. We can prove that $V_\theta(0)$ has the PBW type basis $\{P_\theta(b)\}_{b\in B_\theta(0)}$, the crystal basis $(L_\theta(0), B_\theta(0))$, the lower global basis $\{G_\theta^{\mathrm{low}}(b)\}_{b\in \mathrm{B}_\theta(0)}$ and the upper global basis $\{G_\theta^{\mathrm{up}}(b)\}_{b\in B_\theta(0)}$. Moreover we can combinatorially describe the crystal structure by using the θ-restricted multisegments.

We conjecture that the irreducible $\mathcal{H}^B$-modules of type J are parametrized by $B_\theta(0)$ and if L_b is an irreducible $\mathcal{H}^B$-module associated to $b \in B_\theta(0)$, then we have $\widetilde{E}_aL_b = L_{\widetilde{E}_ab}$,

$\widetilde{F}_a L_b = L_{\widetilde{F}_a b}$ and $[E_a L_b : L_{b'}] = E_{a,b,b'}|_{q=1}$, $[F_a L_b : L_{b'}] = F_{a,b,b'}|_{q=1}$. (For the definition of the functors E_a, F_a, $\widetilde{E}_a$ and $\widetilde{F}_a$ for $a \in J$, see Definition 6.5.) Here the Laurent polynomials $E_{a,b,b'}$ and $F_{a,b,b'}$ are defined by

$$E_a G^{\mathrm{up}}_\theta(b) = \sum_{b' \in B_\theta(0)} E_{a,b,b'} G^{\mathrm{up}}_\theta(b'), \quad F_a G^{\mathrm{up}}_\theta(b) = \sum_{b' \in B_\theta(0)} F_{a,b,b'} G^{\mathrm{up}}_\theta(b').$$

the quantum enveloping algebra $U_q(\mathfrak{gl}_\infty)$ with θ	the affine Hecke algebra of type B $\mathcal{H}^B_n(p_0, p_1)$ $(n \geqslant 0)$
$V_\theta(0) = U_q^-(\mathfrak{gl}_\infty)/\sum_i U_q^-(\mathfrak{gl}_\infty)(f_i - f_{\theta(i)})$	$K^B_J = \oplus_{n \geqslant 0} K_J(\mathcal{H}^B_n(p_0, p_1))$
E_a, F_a	certain inductions E_a and restrictions F_a
the crystal basis $B_\theta(0)$	$\mathcal{M}_\theta = \{$the θ-restricted multisegments$\}$
the upper global basis $\{G^{\mathrm{up}}_\theta(b)\}_{b \in B_\theta(0)}$	the irreducible modules $\{L_b\}_{b \in B_\theta(0)}$
the modified root operators $\widetilde{E}_a, \widetilde{F}_a$	$\widetilde{E}_a = \mathrm{soc}(E_a), \widetilde{F}_a = \mathrm{cosoc}(F_a)$ $\widetilde{E}_a L_b = L_{\widetilde{E}_a b}, \widetilde{F}_a L_b = L_{\widetilde{F}_a b}$
the PBW basis $\{P_\theta(b)\}_{b \in B_\theta(0)}$	the standard modules

FIGURE 2. Conjectural correspondence in type B

Part I. Review on Lascoux-Leclerc-Thibon-Ariki Theory

2. Representation Theory of $U_q(\mathfrak{gl}_\infty)$

2.1. Quantized universal enveloping algebras and its reduced q-analogues. We shall recall the quantized universal enveloping algebra $U_q(\mathfrak{g})$. Let I be an index set (for simple roots), and Q the free $\mathbb{Z}$-module with a basis $\{\alpha_i\}_{i \in I}$. Let $(\bullet, \bullet) : Q \times Q \to \mathbb{Z}$ be a symmetric bilinear form such that $(\alpha_i, \alpha_i)/2 \in \mathbb{Z}_{>0}$ for any i and $(\alpha_i^\vee, \alpha_j) \in \mathbb{Z}_{\leqslant 0}$ for $i \neq j$ where $\alpha_i^\vee := 2\alpha_i/(\alpha_i, \alpha_i)$. Let q be an indeterminate and set $\mathbb{K} := \mathbb{Q}(q)$. We define its subrings $\mathbb{A}_0$, $\mathbb{A}_\infty$ and $\mathbb{A}$ as follows.

$$\begin{aligned} \mathbb{A}_0 &= \{f \in \mathbb{K} \mid f \text{ is regular at } q = 0\}, \\ \mathbb{A}_\infty &= \{f \in \mathbb{K} \mid f \text{ is regular at } q = \infty\}, \\ \mathbb{A} &= \mathbb{Q}[q, q^{-1}]. \end{aligned}$$

Definition 2.1. *The quantized universal enveloping algebra $U_q(\mathfrak{g})$ is the $\mathbb{K}$-algebra generated by elements e_i, f_i and invertible elements t_i $(i \in I)$ with the following defining relations.*

(1) *The t_i's commute with each other.*
(2) $t_j e_i t_j^{-1} = q^{(\alpha_j, \alpha_i)} e_i$ *and* $t_j f_i t_j^{-1} = q^{-(\alpha_j, \alpha_i)} f_i$ *for any $i, j \in I$.*
(3) $[e_i, f_j] = \delta_{ij} \dfrac{t_i - t_i^{-1}}{q_i - q_i^{-1}}$ *for i, $j \in I$. Here $q_i := q^{(\alpha_i, \alpha_i)/2}$.*
(4) (Serre relation) *For $i \neq j$,*

$$\sum_{k=0}^{b} (-1)^k e_i^{(k)} e_j e_i^{(b-k)} = 0, \quad \sum_{k=0}^{b} (-1)^k f_i^{(k)} f_j f_i^{(b-k)} = 0.$$

Here $b = 1 - (\alpha_i^\vee, \alpha_j)$ *and*

$e_i^{(k)} = e_i^k/[k]_i!\,,\ f_i^{(k)} = f_i^k/[k]_i!\,,\ [k]_i = (q_i^k - q_i^{-k})/(q_i - q_i^{-1})\,,\ [k]_i! = [1]_i \cdots [k]_i\,.$

Let us denote by $U_q^-(\mathfrak{g})$ the subalgebra of $U_q(\mathfrak{g})$ generated by the f_i's.

Let e_i' and e_i^* be the operators on $U_q^-(\mathfrak{g})$ defined by

$$[e_i, a] = \frac{(e_i^* a)t_i - t_i^{-1} e_i' a}{q_i - q_i^{-1}} \quad (a \in U_q^-(\mathfrak{g})).$$

These operators satisfy the following formulas similar to derivations:

$$e_i'(ab) = e_i'(a)b + (\mathrm{Ad}(t_i)a)e_i'b, \tag{2.1}$$
$$e_i^*(ab) = ae_i^*b + (e_i^*a)(\mathrm{Ad}(t_i)b).$$

The algebra $U_q^-(\mathfrak{g})$ has a unique symmetric bilinear form $(\bullet, \circ)$ such that $(1,1) = 1$ and

$$(e_i'a, b) = (a, f_i b) \quad \text{for any } a, b \in U_q^-(\mathfrak{g}).$$

It is non-degenerate and satisfies $(e_i^* a, b) = (a, bf_i)$. Let $\mathcal{B}(\mathfrak{g})$ be the algebra generated by the e_i''s and the f_i's. The left multiplication of f_j, e_i' and e_i^* have the commutation relations

$$e_i' f_j = q^{-(\alpha_i, \alpha_j)} f_j e_i' + \delta_{ij},\ e_i^* f_j = f_j e_i^* + \delta_{ij}\,\mathrm{Ad}(t_i),$$

and both the e_i''s and the e_i^*'s satisfy the Serre relations.

Definition 2.2. *The reduced* q*-analogue* $\mathcal{B}(\mathfrak{g})$ *of* $\mathfrak{g}$ *is the* $\mathbb{Q}(q)$*-algebra generated by* e_i' *and* f_i.

2.2. Review on crystal bases and global bases. Since e_i' and f_i satisfy the q-boson relation, any element $a \in U_q^-(\mathfrak{g})$ can be written uniquely as

$$a = \sum_{n \geqslant 0} f_i^{(n)} a_n \quad \text{with } e_i' a_n = 0.$$

Here $f_i^{(n)} = \dfrac{f_i^n}{[n]_i!}$.

Definition 2.3. *We define the modified root operators* $\tilde{e}_i$ *and* $\tilde{f}_i$ *on* $U_q^-(\mathfrak{g})$ *by*

$$\tilde{e}_i a = \sum_{n \geqslant 1} f_i^{(n-1)} a_n, \quad \tilde{f}_i a = \sum_{n \geqslant 0} f_i^{(n+1)} a_n.$$

Theorem 2.4 ([Kas]). *We define*

$$L(\infty) = \sum_{\ell \geqslant 0,\, i_1, \dots, i_\ell \in I} \mathbf{A}_0 \tilde{f}_{i_1} \cdots \tilde{f}_{i_\ell} \cdot 1 \subset U_q^-(\mathfrak{g}),$$
$$B(\infty) = \left\{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_\ell} \cdot 1 \bmod qL(\infty) \mid \ell \geqslant 0, i_1, \cdots, i_\ell \in I \right\} \subset L(\infty)/qL(\infty).$$

Then we have

(i) $\tilde{e}_i L(\infty) \subset L(\infty)$ *and* $\tilde{f}_i L(\infty) \subset L(\infty)$,
(ii) $B(\infty)$ *is a basis of* $L(\infty)/qL(\infty)$,
(iii) $\tilde{f}_i B(\infty) \subset B(\infty)$ *and* $\tilde{e}_i B(\infty) \subset B(\infty) \cup \{0\}$.

We call $(L(\infty), B(\infty))$ *the crystal basis of* $U_q^-(\mathfrak{g})$.

Let $-$ be the automorphism of $\mathbf{K}$ sending q to q^{-1}. Then $\overline{\mathbf{A}_0}$ coincides with $\mathbf{A}_\infty$.

Let V be a vector space over $\mathbf{K}$, L_0 an A-submodule of V, L_∞ an $\mathbf{A}_\infty$- submodule, and $V_\mathbf{A}$ an $\mathbf{A}$-submodule. Set $E := L_0 \cap L_\infty \cap V_\mathbf{A}$.

Definition 2.5 ([Kas]). *We say that* $(L_0, L_\infty, V_{\mathbb{A}})$ *is* balanced *if each of* L_0, L_∞ *and* $V_{\mathbb{A}}$ *generates* V *as a* $\mathbb{K}$*-vector space, and if one of the following equivalent conditions is satisfied.*

(i) $E \to L_0/qL_0$ *is an isomorphism,*
(ii) $E \to L_\infty/q^{-1}L_\infty$ *is an isomorphism,*
(iii) $(L_0 \cap V_{\mathbb{A}}) \oplus (q^{-1}L_\infty \cap V_{\mathbb{A}}) \to V_{\mathbb{A}}$ *is an isomorphism.*
(iv) $\mathbb{A}_0 \otimes_{\mathbb{Q}} E \to L_0$, $\mathbb{A}_\infty \otimes_{\mathbb{Q}} E \to L_\infty$, $\mathbb{A} \otimes_{\mathbb{Q}} E \to V_{\mathbb{A}}$ *and* $\mathbb{K} \otimes_{\mathbb{Q}} E \to V$ *are isomorphisms.*

Let $-$ be the ring automorphism of $U_q(\mathfrak{g})$ sending q, t_i, e_i, f_i to q^{-1}, t_i^{-1}, e_i, f_i.

Let $U_q(\mathfrak{g})_{\mathbb{A}}$ be the $\mathbb{A}$-subalgebra of $U_q(\mathfrak{g})$ generated by $e_i^{(n)}$, $f_i^{(n)}$ and t_i. Similarly we define $U_q^-(\mathfrak{g})_{\mathbb{A}}$.

Theorem 2.6. $(L(\infty), \overline{L(\infty)}, U_q^-(\mathfrak{g})_{\mathbb{A}})$ *is balanced.*

Let

$$G^{\mathrm{low}} \colon L(\infty)/qL(\infty) \xrightarrow{\sim} E := L(\infty) \cap \overline{L(\infty)} \cap U_q^-(\mathfrak{g})_{\mathbb{A}}$$

be the inverse of $E \xrightarrow{\sim} L(\infty)/qL(\infty)$. Then $\{G^{\mathrm{low}}(b) \mid b \in B(\infty)\}$ forms a basis of $U_q^-(\mathfrak{g})$. We call it a (lower) *global basis.* It is first introduced by G. Lusztig ([L]) under the name of "canonical basis" for the A, D, E cases.

Definition 2.7. *Let*

$$\{G^{\mathrm{up}}(b) \mid b \in B(\infty)\}$$

be the dual basis of $\{G^{low}(b) \mid b \in B(\infty)\}$ *with respect to the inner product* $(\circ, \circ)$. *We call it the upper global basis of* $U_q^-(\mathfrak{g})$.

2.3. Review on the PBW basis. In the sequel, we set $I = \mathbb{Z}_{\mathrm{odd}}$ and

$$(\alpha_i, \alpha_j) = \begin{cases} 2 & \text{for } i = j, \\ -1 & \text{for } j = i \pm 2, \\ 0 & \text{otherwise,} \end{cases}$$

and we consider the corresponding quantum group $U_q(\mathfrak{gl}_\infty)$. In this case, we can parametrize the crystal basis $B(\infty)$ by the multisegments. We shall recall this parametrization and the PBW basis.

Definition 2.8. *For* $i, j \in I$ *such that* $i \leqslant j$, *we define a segment* $\langle i, j\rangle$ *as the interval* $[i, j] \subset \mathbb{Z}_{odd}$. *A multisegment is a formal finite sum of segments:*

$$\mathbf{m} = \sum_{i \leqslant j} m_{ij} \langle i, j\rangle$$

with $m_{i,j} \in \mathbb{Z}_{\geqslant 0}$. *If* $m_{i,j} > 0$, *we sometimes say that* $\langle i, j\rangle$ *appears in* $\mathbf{m}$. *We denote sometimes* $\langle i\rangle$ *for* $\langle i, i\rangle$. *We denote by* $\mathcal{M}$ *the set of multisegments. We denote by* $\emptyset$ *the zero element (or the empty multisegment) of* $\mathcal{M}$.

Definition 2.9. *For two segments* $\langle i_1, j_1\rangle$ *and* $\langle i_2, j_2\rangle$, *we define the ordering* $\geqslant_{PBW}$ *by the following:*

$$\langle i_1, j_1\rangle \geqslant_{PBW} \langle i_2, j_2\rangle \iff \begin{cases} j_1 > j_2 \\ \text{or} \\ j_1 = j_2 \text{ and } i_1 \geqslant i_2. \end{cases}$$

We call this ordering the PBW ordering.

Example 2.10. *We have* $\langle 1, 1\rangle >_{PBW} \langle -1, 1\rangle >_{PBW} \langle -1, -1\rangle$.

Definition 2.11. *We define the element* $P(\mathbf{m}) \in U_q^-(\mathfrak{gl}_\infty)$ *indexed by a multisegment* $\mathbf{m}$ *as follows:*

(1) *for a segment* $\langle i,j\rangle$, *we define the element* $\langle i,j\rangle \in U_q^-(\mathfrak{gl}_\infty)$ *inductively by*

$$\begin{aligned}\langle i,i\rangle &= f_i,\\ \langle i,j\rangle &= \langle i,j-2\rangle\langle j,j\rangle - q\langle j,j\rangle\langle i,j-2\rangle,\end{aligned}$$

(2) *for a multisegment* $\mathbf{m} = \sum_{i\leqslant j} m_{ij}\langle i,j\rangle$, *we define*

$$P(\mathbf{m}) = \overrightarrow{\prod} \langle i,j\rangle^{(m_{ij})}.$$

Here the product $\overrightarrow{\prod}$ *is taken over segments appearing in* $\mathbf{m}$ *from large to small with respect to the PBW ordering. The element* $\langle i,j\rangle^{(m_{ij})}$ *is the divided power of* $\langle i,j\rangle$ *i.e.*

$$\langle i,j\rangle^{(m_{ij})} = \frac{1}{[m_{ij}]!}\langle i,j\rangle^{m_{ij}}.$$

Set $\operatorname{wt} P(\mathbf{m}) = -\sum_{i\leqslant j} m_{ij}\alpha_{ij}$.

Theorem 2.12 ([L]). *The set of elements* $\{P(\mathbf{m}) \mid \mathbf{m} \in \mathcal{M}\}$ *is a basis of the* $\mathbb{K}$*-vector space* $U_q^-(\mathfrak{gl}_\infty)$. *Moreover this is a basis of the* $\mathbb{A}$*-module* $U_q^-(\mathfrak{gl}_\infty)_{\mathbb{A}}$. *We call this basis the PBW basis of* $U_q^-(\mathfrak{gl}_\infty)$.

Definition 2.13. *For two segments* $\langle i_1,j_1\rangle$ *and* $\langle i_2,j_2\rangle$, *we define the ordering* $\geqslant_{cry}$ *by the following:*

$$\langle i_1,j_1\rangle \geqslant_{cry} \langle i_2,j_2\rangle \Leftrightarrow \begin{cases} j_1 > j_2 \\ \text{or} \\ j_1 = j_2 \text{ and } i_1 \leqslant i_2. \end{cases}$$

We call this ordering the crystal ordering. For $\mathbf{m} = \sum_{i\leqslant j} m_{i,j}\langle i,j\rangle \in \mathcal{M}$ *and and* $\mathbf{m}' = \sum_{i\leqslant j} m'_{i,j}\langle i,j\rangle \in \mathcal{M}$, *we define* $\mathbf{m}' \underset{\text{cry}}{<} \mathbf{m}$ *if there exists a segment* $\langle i_0,j_0\rangle$ *such that* $m'_{i_0,j_0} < m_{i_0,j_0}$ *and* $m'_{i,j} = m_{i,j}$ *for any* $\langle i,j\rangle >_{cry} \langle i_0,j_0\rangle$.

Example 2.14. *The crystal ordering is different from the PBW ordering. For example, we have* $\langle -1,1\rangle >_{cry} \langle 1,1\rangle >_{cry} \langle -1,-1\rangle$, *while we have* $\langle 1,1\rangle >_{PBW} \langle -1,1\rangle >_{PBW} \langle -1,-1\rangle$.

Definition 2.15. *We define the crystal structure on* $\mathcal{M}$ *as follows: for* $\mathbf{m} = \sum m_{i,j}\langle i,j\rangle \in \mathcal{M}$ *and* $i \in I$, *set* $A_k^{(i)}(\mathbf{m}) = \sum_{k'\geqslant k}(m_{i,k'} - m_{i+2,k'+2})$ *for* $k \geqslant i$. *Define* $\varepsilon_i(\mathbf{m})$ *as* $\max\left\{A_k^{(i)}(\mathbf{m}) \mid k \geqslant i\right\} \geqslant 0$.

(i) *If* $\varepsilon_i(\mathbf{m}) = 0$, *then define* $\tilde{e}_i(\mathbf{m}) = 0$. *If* $\varepsilon_i(\mathbf{m}) > 0$, *let* k_e *be the largest* $k \geqslant i$ *such that* $\varepsilon_i(\mathbf{m}) = A_k^{(i)}(\mathbf{m})$ *and define* $\tilde{e}_i(\mathbf{m}) = \mathbf{m} - \langle i,k_e\rangle + \delta_{k_e\neq i}\langle i+2,k_e\rangle$.

(ii) *Let* k_f *be the smallest* $k \geqslant i$ *such that* $\varepsilon_i(\mathbf{m}) = A_k^{(i)}(\mathbf{m})$ *and define* $\tilde{f}_i(\mathbf{m}) = \mathbf{m} - \delta_{k_f\neq i}\langle i+2,k_f\rangle + \langle i,k_f\rangle$.

Remark 2.16. For $i \in I$, the actions of the operators $\tilde{e}_i$ and $\tilde{f}_i$ on $\mathbf{m} \in \mathcal{M}$ are also described by the following algorithm:

Step 1. Arrange the segments in $\mathbf{m}$ in the crystal ordering.

Step 2. For each segment $\langle i,j\rangle$, write $-$, and for each segment $\langle i+2,j\rangle$, write $+$.

Step 3. In the resulting sequence of $+$ and $-$, delete a subsequence of the form $+-$ and keep on deleting until no such subsequence remains.

Then we obtain a sequence of the form $--\cdots-++\cdots+$.

(1) $\varepsilon_i(\mathrm{m})$ is the total number of $-$ in the resulting sequence.

(2) $\widetilde{f}_i(\mathrm{m})$ is given as follows:

  (a) If the leftmost $+$ corresponds to a segment $\langle i+2,j\rangle$, then replace it with $\langle i,j\rangle$.

  (b) If no $+$ exists, add a segment $\langle i,i\rangle$ to m.

(3) $\widetilde{e}_i(\mathrm{m})$ is given as follows:

  (a) If the rightmost $-$ corresponds to a segment $\langle i,j\rangle$, then replace it with $\langle i+2,j\rangle$.

  (b) If no $-$ exists, then $\widetilde{e}_i(\mathrm{m})=0$.

Theorem 2.17. (i) $L(\infty)=\bigoplus_{\mathrm{m}\in\mathcal{M}} \mathbf{A}_0P(\mathrm{m})$.

(ii) $B(\infty)=\{P(\mathrm{m}) \bmod qL(\infty)\mid \mathrm{m}\in\mathcal{M}\}$.

(iii) *We have*

$$\begin{aligned}\widetilde{e}_iP(\mathrm{m}) &\equiv P(\widetilde{e}_i(\mathrm{m})) \mod qL(\infty),\\ \widetilde{f}_iP(\mathrm{m}) &\equiv P(\widetilde{f}_i(\mathrm{m})) \mod qL(\infty).\end{aligned}$$

Note that $\widetilde{e}_i$ and $\widetilde{f}_i$ in the left-hand-side is the modified root operators.

(iv) *We have the expansion*

$$\overline{P(\mathrm{m})}\in P(\mathrm{m})+\sum_{\mathrm{m}'\underset{\mathrm{cry}}{<}\mathrm{m}} \mathbf{A}P(\mathrm{m}').$$

Therefore we can index the crystal basis by multisegments. By this theorem we can easily see by a standard argument that $(L(\infty),\overline{L(\infty)},U_q^-(\mathfrak{g})_\mathbf{A})$ is balanced, and there exists a unique $G^{\mathrm{low}}(\mathrm{m})\in L(\infty)\cap U_q^-(\mathfrak{g})_\mathbf{A}$ such that $\overline{G^{\mathrm{low}}(\mathrm{m})}=G^{\mathrm{low}}(\mathrm{m})$ and $G^{\mathrm{low}}(\mathrm{m})\equiv P(\mathrm{m}) \bmod qL(\infty)$. The basis $\{G^{\mathrm{low}}(\mathrm{m})\}_{\mathrm{m}\in\mathcal{M}}$ is a lower global basis.

3. Representation Theory of $\mathcal{H}_n^A$ and the Lascoux-Leclerc-Thibon-Ariki Theory

3.1. The affine Hecke algebra of type A.

Definition 3.1. *For $p\in\mathbb{C}^*$, the affine Hecke algebra $\mathcal{H}_n^A$ of type A is a $\mathbb{C}$-algebra generated by*

$$T_1,\cdots,T_{n-1},X_1^{\pm1},\cdots,X_n^{\pm1}$$

satisfying the following defining relations:

(1) $X_iX_j=X_jX_i$ *for any* $1\leqslant i,j\leqslant n$.

(2) [The braid relations of type A]

$$\begin{aligned}&T_iT_{i+1}T_i=T_{i+1}T_iT_{i+1} && (1\leqslant i\leqslant n-2),\\ &T_iT_j=T_jT_i && (|i-j|>1).\end{aligned}$$

(3) [The Hecke relations]

$$(T_i-p)(T_i+p^{-1})=0 \quad (1\leqslant i\leqslant n-1).$$

(4) [The Bernstein-Lusztig relations]

$$\begin{aligned}&T_iX_iT_i=X_{i+1} && (1\leqslant i\leqslant n-1),\\ &T_iX_j=X_jT_i && (j\neq i,i+1).\end{aligned}$$

Since we can enbed $\mathcal{H}_n^\mathrm{A}$ into $\mathcal{H}_{n+m}^\mathrm{A}$ by $T_i\mapsto T_{i+m}$ $(1\leqslant i\leqslant n-1)$, $X_j\mapsto X_{m+j}$ $(1\leqslant j\leqslant m)$, we consider $\mathcal{H}_m^\mathrm{A}\otimes\mathcal{H}_n^\mathrm{A}$ as a subalgebra of $\mathcal{H}_{n+m}^\mathrm{A}$.

Definition 3.2. *For a finite-dimensional $\mathcal{H}_n^A$-module M, let*

$$M = \bigoplus_{a\in(\mathbb{C}^*)^n} M_a$$

be the generalized eigenspace decomposition with respect to $X_1,\dots,X_n$. Here

$$M_a := \{u \in M \mid (X_i - a_i)^N u = 0 \text{ for any } 1 \leqslant i \leqslant n \text{ and } N \gg 0\}$$

for $a = (a_1,\dots,a_n) \in (\mathbb{C}^*)^n$.

(1) *We say that M is of type J if all the eigenvalues of $X_1,\dots,X_n$ belong to $J \subset \mathbb{C}^*$.*
(2) *Put*

$$K_J^A := \bigoplus_{n\geqslant 0} K_{J,n}^A.$$

Here $K_{J,n}^A$ is the Grothendieck group of the abelian category of finite-dimensional $\mathcal{H}_n^A$-modules of type J.
(3) *The group $\mathbb{Z}$ acts on $\mathbb{C}^*$ by $\mathbb{Z} \ni n\colon a \mapsto ap^{2n}$.*

Lemma 3.3. *Let J_1 and J_2 be $\mathbb{Z}$-invariant subsets in $\mathbb{C}^*$ such that $J_1 \cap J_2 = \emptyset$.*

(1) *If M is an irreducible $\mathcal{H}_m^A$-module of type J_1 and N is an irreducible $\mathcal{H}_n^A$-module of type J_2, then $\mathrm{Ind}_{\mathcal{H}_m^A\otimes\mathcal{H}_n^A}^{\mathcal{H}_{m+n}^A}(M\otimes N)$ is irreducible of type $J_1 \cup J_2$.*
(2) *Conversely, if L is an irreducible $\mathcal{H}_n^A$-module of type $J_1 \cup J_2$, then there exist m ($0 \leqslant m \leqslant n$), an irreducible $\mathcal{H}_m^A$-module M of type J_1 and an irreducible $\mathcal{H}_{n-m}^A$-module N of type J_2 such that L is isomorphic to $\mathrm{Ind}_{\mathcal{H}_m^A\otimes\mathcal{H}_{n-m}^A}^{\mathcal{H}_n^A}(M\otimes N)$.*

Hence in order to study the irreducible modules over the affine Hecke algebras of type A, it is enough to treat the irreducible modules of type J for an orbit J with respect to the $\mathbb{Z}$-action on $\mathbb{C}^*$.

3.2. The a-restriction and the a-induction.

For a $\mathbb{C}$-algebra A, let us denote by $A\text{-}\mathrm{mod}^{\mathrm{fd}}$ the abelian category of finite-dimensional A-modules.

Definition 3.4. *For $a \in \mathbb{C}^*$, let us define the functors*

$$e_a : \mathcal{H}_n^A\text{-}\mathrm{mod}^{\mathrm{fd}} \to \mathcal{H}_{n-1}^A\text{-}\mathrm{mod}^{\mathrm{fd}}, \quad f_a : \mathcal{H}_n^A\text{-}\mathrm{mod}^{\mathrm{fd}} \to \mathcal{H}_{n+1}^A\text{-}\mathrm{mod}^{\mathrm{fd}}$$

by: e_aM is the generalized a-eigenspace of M with respect to the action of X_n, and

$$f_aM := \mathrm{Ind}_{\mathcal{H}_n^A\otimes\mathbb{C}[X_{n+1}^{\pm1}]}^{\mathcal{H}_{n+1}^A} M\otimes\langle a\rangle,$$

where $\langle a\rangle$ is the 1-dimensional representation of $\mathbb{C}[X_{n+1}^{\pm1}]$ defined by $X_{n+1}\mapsto a$.
Moreover, put

$$\widetilde{e}_aM := \mathrm{soc}\, e_aM, \quad \widetilde{f}_aM := \mathrm{cosoc}\, f_aM$$

for $a \in \mathbb{C}^*$. Here the socle is the maximal semisimple submodule and the cosocle is the maximal semisimple quotient module.

Theorem 3.5 (Grojnowski-Vazirani [GV]). *Suppose M is irreducible. Then $\widetilde{f}_aM$ is irreducible, and $\widetilde{e}_aM$ is irreducible or 0 for any $a \in \mathbb{C}^*$.*

3.3. LLTA type theorems for the affine Hecke algebra of type A. In this subsection, we consider the case

$$J = \{p^k \mid k \in \mathbb{Z}_{\mathrm{odd}}\},$$

and suppose p is not a root of unity. For short, we shall write $e_i, \widetilde{e}_i, f_i$ and $\widetilde{f}_i$ for $e_{p^i}, \widetilde{e}_{p^i}, f_{p^i}$ and $\widetilde{f}_{p^i}$, respectively.

The LLTA type theorem for the affine Hecke algebra of type A consists of two parts. First is a labeling of finite-dimensional irreducible $\mathcal{H}^A$-modules by the crystal $B(\infty)$. Second is a description of some composition multiplicities by using the upper global basis.

Theorem 3.6 (Vazirani [V]). *There are complete representatives*

$$\{L_b \mid b \in B(\infty)\}$$

of the finite-dimensional irreducible $\mathcal{H}^A$-modules of type J such that

$$\widetilde{e}_i L_b = L_{\widetilde{e}_i b}, \quad \widetilde{f}_i L_b = L_{\widetilde{f}_i b}$$

for any $i \in I$.

Theorem 3.7 (Ariki [A]). *For $i \in I = \mathbb{Z}_{odd}$, let us define $e'_{i,b,b'}, f_{i,b,b'} \in \mathbb{C}[q,q^{-1}]$ by the coefficients of the expansions:*

$$e'_i G^{up}(b) = \sum_{b' \in B(\infty)} e'_{i,b,b'} G^{up}(b'), \quad f_i G^{up}(b) = \sum_{b' \in B(\infty)} f_{i,b,b'} G^{up}(b').$$

Then

$$[e_i L_b : L_{b'}] = e'_{i,b,b'}|_{q=1}, \quad [f_i L_b : L_{b'}] = f_{i,b,b'}|_{q=1}.$$

Here $[M : N]$ is the composition multiplicity of N in M on K_J^A.

Part II. The Symmetric Crystals and some LLTA Type Conjectures for Affine Hecke Algebra of Type B

4. General Definitions and Conjectures for Symmetric Crystals

We follow the notations in subsection 2.1. Let θ be an automorphism of I such that $\theta^2 = \mathrm{id}$ and $(\alpha_{\theta(i)}, \alpha_{\theta(j)}) = (\alpha_i, \alpha_j)$. Hence it extends to an automorphism of the root lattice Q by $\theta(\alpha_i) = \alpha_{\theta(i)}$, and induces an automorphism of $U_q(\mathfrak{g})$.

Definition 4.1. *Let $\mathcal{B}_\theta(\mathfrak{g})$ be the $\mathbb{K}$-algebra generated by E_i, F_i, and invertible elements T_i $(i \in I)$ satisfying the following defining relations:*

(i) *the T_i's commute with each other,*
(ii) *$T_{\theta(i)} = T_i$ for any i,*
(iii) *$T_i E_j T_i^{-1} = q^{(\alpha_i + \alpha_{\theta(i)}, \alpha_j)} E_j$ and $T_i F_j T_i^{-1} = q^{(\alpha_i + \alpha_{\theta(i)}, -\alpha_j)} F_j$ for $i, j \in I$,*
(iv) *$E_i F_j = q^{-(\alpha_i, \alpha_j)} F_j E_i + (\delta_{i,j} + \delta_{\theta(i),j} T_i)$ for $i, j \in I$,*
(v) *the E_i's and the F_i's satisfy the q-Serre relations.*

We set $E_i^{(n)} = E_i^n/[n]_i!$ and $F_i^{(n)} = F_i^n/[n]_i!$.

Proposition 4.2. *Let $\lambda \in P_+ := \{\lambda \in \mathrm{Hom}(Q, \mathbb{Q}) \mid \langle \alpha_i^\vee, \lambda \rangle \in \mathbb{Z}_{\geqslant 0}$ for any $i \in I\}$ be a dominant integral weight such that $\theta(\lambda) = \lambda$.*

(i) *There exists a $\mathcal{B}_\theta(\mathfrak{g})$-module $V_\theta(\lambda)$ generated by a non-zero vector ϕ_λ such that*
  (a) *$E_i \phi_\lambda = 0$ for any $i \in I$,*
  (b) *$T_i \phi_\lambda = q^{(\alpha_i, \lambda)} \phi_\lambda$ for any $i \in I$,*
  (c) *$\{u \in V_\theta(\lambda) \mid E_i u = 0$ for any $i \in I\} = \mathbb{K}\phi_\lambda$.*

Moreover such a $V_\theta(\lambda)$ is irreducible and unique up to an isomorphism.

(ii) *there exists a unique symmetric bilinear form $(\circ, \circ)$ on $V_\theta(\lambda)$ such that $(\phi_\lambda, \phi_\lambda) = 1$ and $(E_i u, v) = (u, F_i v)$ for any $i \in I$ and $u, v \in V_\theta(\lambda)$, and it is non-degenerate.*

(iii) *There exists an endomorphism $-$ of $V_\theta(\lambda)$ such that $\overline{\phi_\lambda} = \phi_\lambda$ and $\overline{av} = \bar{a}\bar{v}$, $\overline{F_i v} = F_i \bar{v}$ for any $a \in \mathbb{K}$ and $v \in V_\theta(\lambda)$.*

The pair $(B_\theta(\mathfrak{g}), V_\theta(\lambda))$ is an analogue of $(\mathcal{B}(\mathfrak{g}), U_q^-(\mathfrak{g}))$. Such a $V_\theta(\lambda)$ is constructed as follows. Let $U_q^-(\mathfrak{g})\phi'_\lambda$ and $U_q^-(\mathfrak{g})\phi''_\lambda$ be a copy of a free $U_q^-(\mathfrak{g})$-module. We give the structure of a $B_\theta(\mathfrak{g})$-module on them as follows: for any $i \in I$ and $a \in U_q^-(\mathfrak{g})$

$$(4.1)\qquad \begin{cases} T_i(a\phi'_\lambda) &= q^{(\alpha_i,\lambda)}(\mathrm{Ad}(t_i t_{\theta(i)})a)\phi'_\lambda, \\ E_i(a\phi'_\lambda) &= (e'_i a + q^{(\alpha_i,\lambda)}\,\mathrm{Ad}(t_i)(e^*_{\theta(i)}a))\phi'_\lambda, \\ F_i(a\phi'_\lambda) &= (f_i a)\phi'_\lambda \end{cases}$$

and

$$(4.2)\qquad \begin{cases} T_i(a\phi''_\lambda) &= q^{(\alpha_i,\lambda)}(\mathrm{Ad}(t_i t_{\theta(i)})a)\phi''_\lambda, \\ E_i(a\phi''_\lambda) &= (e'_i a)\phi''_\lambda, \\ F_i(a\phi''_\lambda) &= (f_i a + q^{(\alpha_i,\lambda)}(\mathrm{Ad}(t_i)a)f_{\theta(i)})\phi''_\lambda. \end{cases}$$

Then there exists a unique $B_\theta(\mathfrak{g})$-linear morphism $\psi\colon U_q^-(\mathfrak{g})\phi'_\lambda \to U_q^-(\mathfrak{g})\phi''_\lambda$ sending ϕ'_λ to ϕ''_λ. Its image $\psi(U_q^-(\mathfrak{g})\phi'_\lambda)$ is $V_\theta(\lambda)$.

Hereafter we assume further that

$$\textit{there is no } i \in I \textit{ such that } \theta(i) = i.$$

We conjecture that $V_\theta(\lambda)$ has a crystal basis. This means the following. Since E_i and F_i satisfy the q-boson relation $E_i F_i = q^{-(\alpha_i,\alpha_i)} F_i E_i + 1$, we define the modified root operators:

$$\widetilde{E}_i(u) = \sum_{n\geqslant 1} F_i^{(n-1)} u_n \text{ and } \widetilde{F}_i(u) = \sum_{n \geqslant 0} F_i^{(n+1)} u_n,$$

when writing $u = \sum_{n\geqslant 0} F_i^{(n)} u_n$ with $E_i u_n = 0$. Let $L_\theta(\lambda)$ be the $\mathbb{A}_0$-submodule of $V_\theta(\lambda)$ generated by $\widetilde{F}_{i_1} \cdots \widetilde{F}_{i_\ell}\phi_\lambda$ ($\ell \geqslant 0$ and $i_1, \ldots, i_\ell \in I$), and let $B_\theta(\lambda)$ be the subset

$$\left\{ \widetilde{F}_{i_1} \cdots \widetilde{F}_{i_\ell}\phi_\lambda \bmod qL_\theta(\lambda) \mid \ell \geqslant 0,\, i_1, \ldots, i_\ell \in I \right\}$$

of $L_\theta(\lambda)/qL_\theta(\lambda)$.

Conjecture 4.3. Let λ be a dominant integral weight such that $\theta(\lambda) = \lambda$.

(1) $\widetilde{F}_i L_\theta(\lambda) \subset L_\theta(\lambda)$ and $\widetilde{E}_i L_\theta(\lambda) \subset L_\theta(\lambda)$,
(2) $B_\theta(\lambda)$ is a basis of $L_\theta(\lambda)/qL_\theta(\lambda)$,
(3) $\widetilde{F}_i B_\theta(\lambda) \subset B_\theta(\lambda)$, and $\widetilde{E}_i B_\theta(\lambda) \subset B_\theta(\lambda) \sqcup \{0\}$,
(4) $\widetilde{F}_i \widetilde{E}_i(b) = b$ for any $b \in B_\theta(\lambda)$ such that $\widetilde{E}_i b \neq 0$, and $\widetilde{E}_i \widetilde{F}_i(b) = b$ for any $b \in B_\theta(\lambda)$.

Moreover we conjecture that $V_\theta(\lambda)$ has a global crystal basis. Namely we have

Conjecture 4.4. $(L_\theta(\lambda), \overline{L_\theta(\lambda)}, V_\theta(\lambda)^{\mathrm{low}}_{\mathbb{A}})$ is balanced. Here $V_\theta(\lambda)^{\mathrm{low}}_{\mathbb{A}} := U_q^-(\mathfrak{g})_{\mathbb{A}}\phi_\lambda$.

The dual version is as follows. As in [Kas], we have

Lemma 4.5. *Assume Conjecture 4.3. Then we have*

(i) $L_\theta(\lambda) = \{v \in V_\theta(\lambda) \mid (L_\theta(\lambda), v) \subset \mathbb{A}_0\}$,
(ii) *Let $(\circ, \circ)_0$ be the $\mathbb{C}$-valued symmetric bilinear form on $L_\theta(\lambda)/qL_\theta(\lambda)$ induced by $(\circ, \circ)$. Then $B_\theta(\lambda)$ is an orthonormal basis with respect to $(\circ, \circ)_0$.*

Let us denote by $V_\theta(\lambda)^{\mathrm{up}}_{\mathbf{A}}$ the dual space $\{v \in V_\theta(\lambda) \mid (V_\theta(\lambda)^{\mathrm{low}}_{\mathbf{A}}, v) \in \mathbf{A}\}$. Then Conjecture 4.4 is equivalent to the following conjecture.

Conjecture 4.6. $(L_\theta(\lambda), c(L_\theta(\lambda)), V_\theta(\lambda)^{\mathrm{up}}_{\mathbf{A}})$ is balanced.

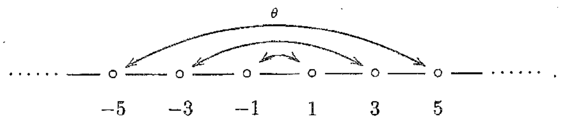
Here c is a unique endomorphism of $V_\theta(\lambda)$ such that $c(\phi_\lambda) = \phi_\lambda$ and $c(av) = \bar{a}c(v)$, $c(E_i v) = E_i c(v)$ for any $a \in \mathbb{K}$ and $v \in V_\theta(\lambda)$. We have $(c(v'), v) = \overline{(v', \bar{v})}$ for any $v, v' \in V_\theta(\lambda)$.

Note that $V_\theta(\lambda)^{\mathrm{up}}_{\mathbf{A}}$ is the largest $\mathbf{A}$-submodule M of $V_\theta(\lambda)$ such that M is invariant by the $E_i^{(n)}$'s and $M \cap \mathbb{K}\phi_\lambda = \mathbf{A}\phi_\lambda$.

By Conjecture 4.6, $L_\theta(\lambda) \cap c(L_\theta(\lambda)) \cap V_\theta(0)^{\mathrm{up}} \to L_\theta(\lambda)/qL_\theta(\lambda)$ is an isomorphism. Let G_θ^{up} be its inverse. Then $\{G_\theta^{\mathrm{up}}(b)\}_{b \in B_\theta(\lambda)}$ is a basis of $V_\theta(\lambda)$, which we call the *upper global basis* of $V_\theta(\lambda)$. Note that $\{G_\theta^{\mathrm{up}}(b)\}_{b \in B_\theta(\lambda)}$ is the dual basis to $\{G_\theta^{\mathrm{low}}(b)\}_{b \in B_\theta(\lambda)}$ with respect to the inner product of $V_\theta(\lambda)$.

5. Symmetric Crystals for $\mathfrak{gl}_\infty$

In this section, we consider the case $\mathfrak{g} = \mathfrak{gl}_\infty$ and the Dynkin involution θ of I defined by $\theta(i) = -i$ for $i \in I = \mathbb{Z}_{\mathrm{odd}}$.



We shall prove in this case Conjectures 4.3 and 4.4 for $\lambda = 0$.

We set

$$\widetilde{V}_\theta(0) := B_\theta(\mathfrak{g})/(\sum_i B_\theta(\mathfrak{g})E_i + \sum_i B_\theta(\mathfrak{g})(F_i - F_{\theta(i)})) \simeq U_q^-(\mathfrak{gl}_\infty)/\sum_i U_q^-(\mathfrak{gl}_\infty)(f_i - f_{\theta(i)}).$$

Since $F_i\phi_0'' = (f_i + f_{\theta(i)})\phi_0'' = F_{\theta(i)}\phi_0''$, we have an epimorphism

$$\widetilde{V}_\theta(0) \twoheadrightarrow V_\theta(0). \tag{5.1}$$

It is in fact an isomorphism (see Theorem 5.9).

5.1. θ-restricted multisegments.

Definition 5.1. *If a multisegment* m *has the form*

$$\mathrm{m} = \sum_{-j \leqslant i \leqslant j} m_{ij}\langle i, j\rangle,$$

we call m *a* θ-restricted *multisegment. We denote by* $\mathcal{M}_\theta$ *the set of* θ*-restricted multisegments.*

Definition 5.2. *For a* θ*-restricted segment* $\langle i, j\rangle$*, we define its modified divided power by*

$$\langle i, j\rangle^{[m]} = \begin{cases} \langle i, j\rangle^{(m)} = \dfrac{1}{[m]!}\langle i, j\rangle^m & (i \neq -j), \\ \dfrac{1}{\prod_{\nu=1}^m [2\nu]}\langle -j, j\rangle^m & (i = -j). \end{cases}$$

Definition 5.3. *For* $\mathrm{m} \in \mathcal{M}_\theta$*, we define the elements* $P_\theta(\mathrm{m}) \in U_q^-(\mathfrak{g}) \subset B_\theta(\mathfrak{g})$ *by*

$$P_\theta(\mathrm{m}) = \overrightarrow{\prod_{\langle i,j\rangle \in \mathrm{m}}} \langle i, j\rangle^{[m_{ij}]}.$$

Here the product $\overrightarrow{\prod}$ is taken over the segments appearing in $\mathbf{m}$ from large to small with respect to the PBW-ordering.

5.2. Crystal structure on $\mathcal{M}_\theta$.

Definition 5.4. *Suppose $k > 0$. For a θ-restricted multisegment $\mathbf{m} = \sum_{-j \leqslant i \leqslant j} m_{i,j}\langle i, j\rangle$, we set*

$$\varepsilon_{-k}(\mathbf{m}) = \max\left\{A_\ell^{(-k)}(\mathbf{m}) \mid \ell \geqslant -k\right\},$$

where

$$\begin{aligned}
A_\ell^{(-k)}(\mathbf{m}) &= \sum_{\ell' \geqslant \ell}(m_{-k,\ell} - m_{-k+2,\ell+2}) \quad \text{for } \ell > k,\\
A_k^{(-k)}(\mathbf{m}) &= \sum_{\ell > k}(m_{-k,\ell} - m_{-k+2,\ell}) + 2m_{-k,k} + \delta(m_{-k+2,k} \text{ is odd}),\\
A_j^{(-k)}(\mathbf{m}) &= \sum_{\ell > k}(m_{-k,\ell} - m_{-k+2,\ell}) + 2m_{-k,k} - 2m_{-k+2,k-2} + \sum_{-k+2 < i \leqslant j+2} m_{i,k} - \sum_{-k+2 < i \leqslant j} m_{i,k-2}\\
&\qquad \text{for } -k+2 \leqslant j \leqslant k-2.
\end{aligned}$$

(i) *Let n_f be the smallest $\ell \geqslant -k+2$, with respect to the ordering $\cdots > k+2 > k > -k+2 > \cdots > k-2$, such that $\varepsilon_{-k}(\mathbf{m}) = A_\ell^{(-k)}(\mathbf{m})$. We define*

$$\widetilde{F}_{-k}(\mathbf{m}) = \begin{cases}
\mathbf{m} - \langle -k+2, n_f\rangle + \langle -k, n_f\rangle & \text{if } n_f > k,\\
\mathbf{m} - \langle -k+2, k\rangle + \langle -k, k\rangle & \text{if } n_f = k \text{ and } m_{-k+2,k} \text{ is odd},\\
\mathbf{m} - \delta_{k\neq 1}\langle -k+2, k-2\rangle + \langle -k+2, k\rangle & \text{if } n_f = k \text{ and } m_{-k+2,k} \text{ is even},\\
\mathbf{m} - \delta_{n_f \neq k-2}\langle n_f+2, k-2\rangle + \langle n_f+2, k\rangle & \text{if } -k+2 \leqslant n_f \leqslant k-2.
\end{cases}$$

(ii) *If $\varepsilon_{-k}(\mathbf{m}) = 0$, then $\widetilde{E}_{-k}(\mathbf{m}) = 0$. If $\varepsilon_{-k}(\mathbf{m}) > 0$, then let n_e be the largest $\ell \geqslant -k+2$, with respect to the above ordering, such that $\varepsilon_{-k}(\mathbf{m}) = A_\ell^{(-k)}(\mathbf{m})$. We define*

$$\widetilde{E}_{-k}(\mathbf{m}) = \begin{cases}
\mathbf{m} - \langle -k, n_e\rangle + \langle -k+2, n_e\rangle & \text{if } n_e > k,\\
\mathbf{m} - \langle -k, k\rangle + \langle -k+2, k\rangle & \text{if } n_e = k \text{ and } m_{-k+2,k} \text{ is even},\\
\mathbf{m} - \langle -k+2, k\rangle + \delta_{k\neq 1}\langle -k+2, k-2\rangle & \text{if } n_e = k \text{ and } m_{-k+2,k} \text{ is odd},\\
\mathbf{m} - \langle n_e+2, k\rangle + \delta_{n_e \neq k-2}\langle n_e+2, k-2\rangle & \text{if } -k+2 \leqslant n_e \leqslant k-2.
\end{cases}$$

Remark 5.5. For $0 < k \in I$, the actions of $\widetilde{E}_{-k}$ and $\widetilde{F}_{-k}$ on $\mathbf{m} \in \mathcal{M}_\theta$ are described by the following algorithm.

Step 1. Arrange segments in $\mathbf{m}$ of the form $\langle -k, j\rangle$ ($j \geqslant$k), $\langle -k+2, j\rangle$ ($j \geqslant k-2, 0$), $\langle i, k\rangle$ ($-k \leqslant i \leqslant$k), $\langle i, k-2\rangle$ ($-k+2 \leqslant i \leqslant k-2$) in the order

$$\cdots, \langle -k, k+2\rangle, \langle -k+2, k+2\rangle, \langle -k, k\rangle, \langle -k+2, k\rangle, \langle -k+2, k-2\rangle,$$
$$\langle -k+4, k\rangle, \langle -k+4, k-2\rangle, \cdots, \langle k-2, k\rangle, \langle k-2, k-2\rangle, \langle k\rangle.$$

Step 2. Write signatures for each segment appearing in $\mathbf{m}$ by the following rules.

(i) If a segment is not $\langle -k+2, k\rangle$, then

- For $\langle -k, k\rangle$, write $--$,
- For $\langle -k, j\rangle$ with $j > k$, write $-$,
- For $\langle -k+2, k-2\rangle$ with $k > 1$, write $++$,
- For $\langle -k+2, j\rangle$ with $j > k$, write $+$,
- For $\langle j, k\rangle$ if $-k < j \leqslant k$, write $-$,
- For $\langle j, k-2\rangle$ if $-k+2 < j \leqslant k-2$, write $+$,

- If otherwise, write no signature.

(ii) For segments $m_{-k+2,k}\langle -k+2,k\rangle$, if $m_{-k+2,k}$ is even, then write no signature, and if $m_{-k+2,k}$ is odd, then write a sequence $-+$.

Step 3. In the resulting sequence of $+$ and $-$, delete a subsequence of the form $+-$ and keep on deleting until no such subsequence remains.

Then we obtain a sequence of the form $--\cdots-++\cdots+$.

(1) $\varepsilon_{-k}(\mathrm{m})$ is given as the total number of $-$ in the resulting sequence.

(2) $\widetilde{F}_{-k}(\mathrm{m})$ is given as follows:
  (i) if the leftmost $+$ corresponds to a segment $\langle -k+2,j\rangle$ $(j>k)$, then replace the segment with $\langle -k,j\rangle$,
  (ii) if the leftmost $+$ corresponds to a segment $\langle j,k-2\rangle$, then replace the segment with $\langle j,k\rangle$,
  (iii) f the leftmost $+$ corresponds to segment $\langle -k+2,k\rangle^{m_{-k+2,k}}$, then replace one of the segments with $\langle -k,k\rangle$,
  (iv) if no $+$ exists, add a segment $\langle k,k\rangle$ to m.

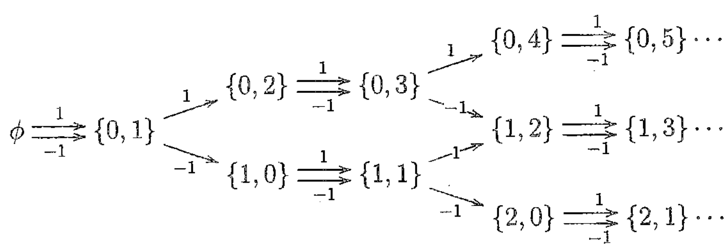
(3) $\widetilde{E}_{-k}(\mathrm{m})$ is given as follows:
  (i) if the rightmost $-$ corresponds to a segment $\langle -k,j\rangle$, then replace the segment with $\langle -k+2,j\rangle$,
  (ii) if the rightmost $-$ corresponds to a segment $\langle j,k\rangle$ $(j\neq -k+2)$, then replace the segment with $\langle j,k-2\rangle$,
  (iii) if the rightmost $-$ corresponds to segments $m_{-k+2,k}\langle -k+2,k\rangle$, then replace one of the segment with $\langle -k+2,k-2\rangle$,
  (iv) if no $-$ exists, then $\widetilde{E}_{-k}(\mathrm{m})=0$.

Definition 5.6. *For $k\in I_{>0}$, we define $\widetilde{F}_k$, $\widetilde{E}_k$ and ε_k by the same rule as in* Definition 2.15 *for $\tilde{f}_k$ and $\tilde{e}_k$.*

Theorem 5.7. *By $\widetilde{F}_k$, $\widetilde{E}_k$, ε_k $(k\in I)$, $\mathcal{M}_\theta$ is a crystal, in the sense that, for any $k\in I$, we have*

(i) $\widetilde{F}_k\mathcal{M}_\theta\subset\mathcal{M}_\theta$ *and* $\widetilde{E}_k\mathcal{M}_\theta\subset\mathcal{M}_\theta\sqcup\{0\}$,
(ii) $\widetilde{F}_k\widetilde{E}_k(\mathrm{m})=\mathrm{m}$ *if* $\widetilde{E}_k(\mathrm{m})\neq 0$, *and* $\widetilde{E}_k\circ\widetilde{F}_k=\mathrm{id}$,
(iii) $\varepsilon_k(\mathrm{m})=\max\left\{n\geqslant 0\mid \widetilde{E}^n(\mathrm{m})\neq 0\right\}<\infty$ *for any* $\mathrm{m}\in\mathcal{M}_\theta$.

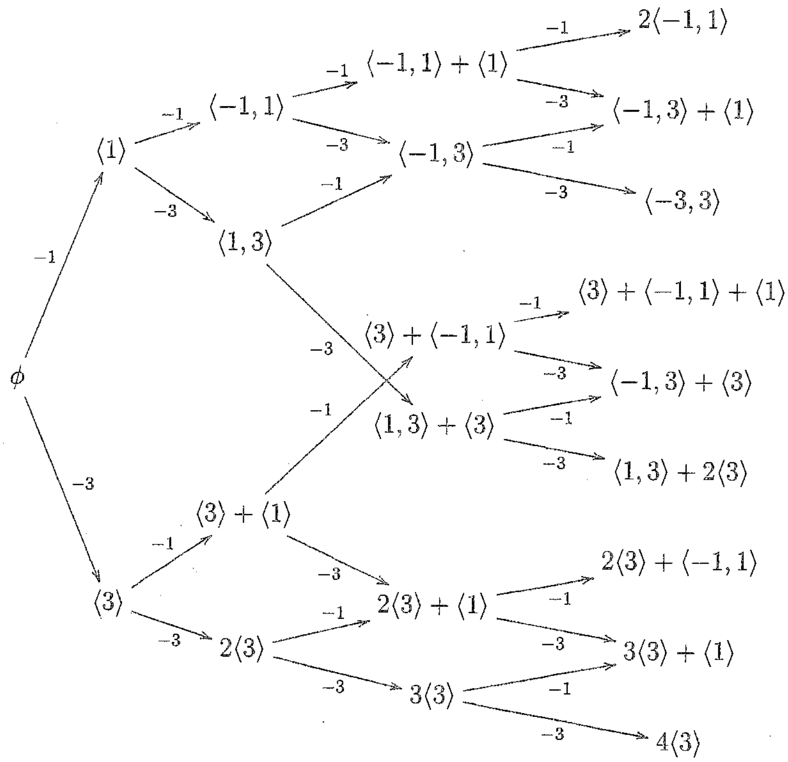
Example 5.8. (1) *We shall write $\{a,b\}$ for $a\langle -1,1\rangle+b\langle 1\rangle$. The following diagram is the part of the crystal graph of $B_\theta(0)$ that concerns only the 1-arrows and the (-1)-arrows.*



Especially the part of (-1)-arrows is the following diagram.

$$\{0,2n\}\xrightarrow{-1}\{0,2n+1\}\xrightarrow{-1}\{1,2n\}\xrightarrow{-1}\{1,2n+1\}\xrightarrow{-1}\{2,2n\}\xrightarrow{-1}\cdots$$

(2) *The following diagram is the part of the crystal graph of $B_\theta(0)$ that concerns only the (-1)-arrows and the (-3)-arrows. This diagram is isomorphic as a graph to the crystal graph of A_2.*



(3) *Here is the part of the crystal graph of $B_\theta(0)$ that concerns only the n-arrows and the $(-n)$-arrows for an odd integer $n \geqslant 3$:*

$$\phi \underset{-n}{\overset{n}{\rightleftarrows}} \langle n\rangle \underset{-n}{\overset{n}{\rightleftarrows}} 2\langle n\rangle \underset{-n}{\overset{n}{\rightleftarrows}} 3\langle n\rangle \underset{-n}{\overset{n}{\rightleftarrows}} 4\langle n\rangle \cdots$$

5.3. Main Theorem. We write ϕ for the generator ϕ_0 of $V_\theta(0)$, for short.

Theorem 5.9. (i) *The morphism*

$$\widetilde{V}_\theta(0) = U_q^-(\mathfrak{g}) / \sum_{k\in I} U_q^-(\mathfrak{g})(f_k - f_{-k}) \to V_\theta(0)$$

is an isomorphism.

(ii) $\{P_\theta(\mathbf{m})\phi\}_{\mathbf{m}\in\mathcal{M}_\theta}$ *is a basis of the* $\mathbb{K}$*-vector space* $V_\theta(0)$.

(iii) *Set*

$$L_\theta(0) := \sum_{\ell\geqslant 0, i_1,\dots,i_\ell\in I} \mathbf{A}_0 \widetilde{F}_{i_1}\cdots\widetilde{F}_{i_\ell}\phi \subset V_\theta(0),$$

$$B_\theta(0) = \left\{ \widetilde{F}_{i_1}\cdots\widetilde{F}_{i_\ell}\phi \bmod qL_\theta(0) \mid \ell \geqslant 0, i_1,\dots,i_\ell \in I \right\}.$$

Then, $B_\theta(0)$ is a basis of $L_\theta(0)/qL_\theta(0)$ and $(L_\theta(0), B_\theta(0))$ is a crystal basis of $V_\theta(0)$, and the crystal structure coincide with the one of $\mathcal{M}_\theta$.

(iv) *More precisely, we have*

(a) $L_\theta(0)=\sum_{\mathrm{m}\in\mathcal{M}_\theta} \mathrm{A}_0 P_\theta(\mathrm{m})\phi$,
(b) $B_\theta(0)=\{P_\theta(\mathrm{m})\phi \bmod qL_\theta(0) \mid \mathrm{m}\in\mathcal{M}_\theta\}$,
(c) *for any* $k\in I$ *and* $\mathrm{m}\in\mathcal{M}_\theta$, *we have*
  (1) $\widetilde{F}_k P_\theta(\mathrm{m})\phi \equiv P_\theta(\widetilde{F}_k\mathrm{m})\phi \mod qL_\theta(0)$,
  (2) $\widetilde{E}_k P_\theta(\mathrm{m})\phi \equiv P_\theta(\widetilde{E}_k\mathrm{m})\phi \mod qL_\theta(0)$, *where we understand* $P_\theta(0)=0$,
  (3) $\widetilde{E}_k^n P_\theta(\mathrm{m})\phi \in qL_\theta(0)$ *if and only if* $n>\varepsilon_k(\mathrm{m})$.

5.3.1. *Global Basis of* $V_\theta(0)$. Recall that $\mathrm{A}=\mathbb{Q}[q,q^{-1}]$, and $V_\theta(0)_\mathrm{A}=U_q^-(\mathfrak{gl}_\infty)_\mathrm{A}\phi$.

Lemma 5.10. (i) $V_\theta(0)_\mathrm{A}=\bigoplus_{\mathrm{m}\in\mathcal{M}_\theta} \mathrm{A}P_\theta(\mathrm{m})\phi$.

(ii) *For* $\mathrm{m}\in\mathcal{M}$,

$$\overline{P_\theta(\mathrm{m})\phi}\in P_\theta(\mathrm{m})\phi+\sum_{\substack{\mathrm{n}<\mathrm{m}\\ \mathrm{cry}}} \mathrm{A}P_\theta(\mathrm{n})\phi.$$

By the above lemma, we obtain the following theorem.

Theorem 5.11. (i) $(L_\theta(0),\overline{L_\theta(0)},V_\theta(0)_\mathrm{A})$ *is balanced.*
(ii) *For any* $\mathrm{m}\in\mathcal{M}_\theta$, *there exists a unique* $G_\theta^{\mathrm{low}}(\mathrm{m})\in L_\theta(0)\cap V_\theta(0)_\mathrm{A}$ *such that* $\overline{G_\theta^{\mathrm{low}}(\mathrm{m})}=G_\theta^{\mathrm{low}}(\mathrm{m})$ *and* $G_\theta^{\mathrm{low}}(\mathrm{m})\equiv P_\theta(\mathrm{m})\phi \mod qL_\theta(0)$.
(iii) $G_\theta^{\mathrm{low}}(\mathrm{m})\in P_\theta(\mathrm{m})\phi+\sum_{\mathrm{n}\underset{\mathrm{cry}}{<}\mathrm{m}} q\,\mathbb{C}[q]P_\theta(\mathrm{n})\phi$ *for any* $\mathrm{m}\in\mathcal{M}_\theta$.

6. Representation Theory of $\mathcal{H}_n^B$ and LLTA Type Conjectures

6.1. The affine Hecke algebra of type B.

Definition 6.1. *For* p_0, $p_1\in\mathbb{C}^*$, *the affine Hecke algebra* $\mathcal{H}_n^B$ *of type* B *is a* $\mathbb{C}$*-algebra generated by*

$$T_0,T_1,\cdots,T_{n-1},X_1^{\pm1},\cdots,X_n^{\pm1}$$

satisfying the following defining relations:
(i) $X_iX_j=X_jX_i$ *for any* $1\leqslant i,j\leqslant n$.
(ii) [The braid relations of type B]

$$\begin{aligned}&T_0T_1T_0T_1=T_1T_0T_1T_0,\\ &T_iT_{i+1}T_i=T_{i+1}T_iT_{i+1} && (1\leqslant i\leqslant n-2),\\ &T_iT_j=T_jT_i && (|i-j|>1).\end{aligned}$$

(iii) [The Hecke relations]

$$(T_0-p_0)(T_0+p_0^{-1})=0,\quad (T_i-p_1)(T_i+p_1^{-1})=0\quad (1\leqslant i\leqslant n-1).$$

(iv) [The Bernstein-Lusztig relations]

$$\begin{aligned}&T_0X_1^{-1}T_0=X_1,\\ &T_iX_iT_i=X_{i+1} && (1\leqslant i\leqslant n-1),\\ &T_iX_j=X_jT_i && (j\neq i,i+1).\end{aligned}$$

Note that the subalgebra generated by T_i $(1\leqslant i\leqslant n-1)$ and $X_j^{\pm1}$ $(1\leqslant j\leqslant n)$ is isomorphic to the affine Hecke algebra $\mathcal{H}_n^A$.

We assume that p_0, $p_1\in\mathbb{C}^*$ satisfy

$$p_0^2\neq1,\ p_1^2\neq1.$$

Let us denote by $\mathbb{P}\mathrm{ol}_n$ the Laurent polynomial ring $\mathbb{C}[X_1^{\pm1},\dots,X_n^{\pm1}]$, and by $\widetilde{\mathbb{P}\mathrm{ol}}_n$ its quotient field $\mathbb{C}(X_1,\dots,X_n)$. Then $\mathcal{H}_n^B$ is isomorphic to the tensor product of $\mathbb{P}\mathrm{ol}_n$ and the subalgebra generated by the T_i's that is isomorphic to the Hecke algebra of type B_n. We have

$$T_i a = (s_i a)T_i + (p_i - p_i^{-1})\frac{a - s_i a}{1 - X^{-\alpha_i^\vee}} \quad \text{for } a \in \mathbb{P}\mathrm{ol}_n.$$

Here $p_i = p_1$ $(1 < i < n)$, and $X^{-\alpha_i^\vee} = X_1^{-2}$ $(i = 0)$ and $X^{-\alpha_i^\vee} = X_i X_{i+1}^{-1}$ $(1 \leqslant i < n)$. The s_i's are the Weyl group action on $\mathbb{P}\mathrm{ol}_n$: $(s_i a)(X_1, \dots, X_n) = a(X_1^{-1}, X_2, \dots, X_n)$ for $i = 0$ and $(s_i a)(X_1, \dots, X_n) = a(X_1, \dots, X_{i+1}, X_i, \dots, X_n)$ for $1 \leqslant i < n$.

Note that $\mathcal{H}_n^B = \mathbb{C}$ for $n = 0$.

The algebra $\mathcal{H}_n^B$ acts faithfully on $\mathcal{H}_n^B / \sum_{i=0}^{n-1} \mathcal{H}_n^B (T_i - p_i) \simeq \mathbb{P}\mathrm{ol}_n$. Set

$$\varphi_i = (1 - X^{-\alpha_i^\vee})T_i - (p_i - p_i^{-1}) \in \mathcal{H}_n^B$$

and

$$\tilde{\varphi}_i = (p_i^{-1} - p_i X^{-\alpha_i^\vee})^{-1} \varphi_i \in \widetilde{\mathbb{P}\mathrm{ol}}_n \otimes_{\mathbb{P}\mathrm{ol}_n} \mathcal{H}_n^B.$$

Then the action of $\tilde{\varphi}_i$ on $\mathbb{P}\mathrm{ol}_n$ coincides with s_i. They are called *intertwiners*.

6.2. Block decomposition of $\mathcal{H}_n^B$-mod$^{\mathrm{fd}}$. For $n, m \geqslant 0$, set

$$\mathbb{F}_{n,m} := \mathbb{C}[X_1^{\pm 1}, \dots, X_{n+m}^{\pm 1}, D^{-1}],$$

where

$$D := \prod_{1 \leqslant i \leqslant n < j \leqslant n+m} (X_i - p_1^2 X_j)(X_i - p_1^{-2} X_j)(X_i - p_1^2 X_j^{-1})(X_i - p_1^{-2} X_j^{-1})(X_i - X_j)(X_i - X_j^{-1}).$$

Then we can embed $\mathcal{H}_n^B$ into $\mathcal{H}_{n+m}^B \otimes_{\mathbb{P}\mathrm{ol}_{n+m}} \mathbb{F}_{n,m}$ by

$$T_0 \mapsto \tilde{\varphi}_n \cdots \tilde{\varphi}_1 T_0 \tilde{\varphi}_1 \cdots \tilde{\varphi}_n, \quad T_i \mapsto T_{i+n} \ (1 \leqslant i < m), \quad X_i \mapsto X_{i+n} \ (1 \leqslant i \leqslant m).$$

Its image commute with $\mathcal{H}_n^B \subset \mathcal{H}_{n+m}^B$. Hence $\mathcal{H}_{n+m}^B \otimes_{\mathbb{P}\mathrm{ol}_{n+m}} \mathbb{F}_{n,m}$ is a right $\mathcal{H}_n^B \otimes \mathcal{H}_m^B$-module. Note that $(\mathcal{H}_n^B \otimes \mathcal{H}_m^B) \otimes_{\mathbb{P}\mathrm{ol}_{n+m}} \mathbb{F}_{n,m} = \mathbb{F}_{n,m} \otimes_{\mathbb{P}\mathrm{ol}_{n+m}} (\mathcal{H}_n^B \otimes \mathcal{H}_m^B)$ is an algebra.

Lemma 6.2. $\mathcal{H}_{n+m}^A \underset{\mathcal{H}_n^A \otimes \mathcal{H}_m^A}{\otimes} (\mathcal{H}_n^B \otimes \mathcal{H}_m^B) \otimes_{\mathbb{P}\mathrm{ol}_{n+m}} \mathbb{F}_{n,m} \xrightarrow{\sim} \mathcal{H}_{n+m}^B \otimes_{\mathbb{P}\mathrm{ol}_{n+m}} \mathbb{F}_{n,m}$.

Proof. Let W_n^A and W_n^B be the finite Weyl group of type A and B. Note that $|W_{n+m}^A| \cdot |W_n^B| \cdot |W_m^B| / (|W_n^A| \cdot |W_m^A|) = |W_{n+m}^B|$. Hence the both sides are free modules of rank $|W_{n+m}^B|$ over $\mathbb{F}_{n,m}$. We prove that the map is surjective.

For short, we denote the image of $\mathcal{H}_{n+m}^A \underset{\mathcal{H}_n^A \otimes \mathcal{H}_m^A}{\otimes} (\mathcal{H}_n^B \otimes \mathcal{H}_m^B) \otimes_{\mathbb{P}\mathrm{ol}_{n+m}} \mathbb{F}_{n,m}$ by $\mathcal{H}_{n,m}^{\mathrm{loc}} \subset \mathcal{H}_{n+m}^{\mathrm{B}} \otimes_{\mathbb{P}\mathrm{ol}_{n+m}} \mathbb{F}_{n,m}$. Note that $\tilde{\varphi}_i \cdots \tilde{\varphi}_n \in \mathcal{H}_{n+m}^{\mathrm{A}} \otimes_{\mathbb{P}\mathrm{ol}_{n+m}} \mathbb{F}_{n,m}$ for $1 \leqslant i \leqslant n$.

First, we have $\tilde{\varphi}_n \cdots \tilde{\varphi}_1 T_0 \tilde{\varphi}_1 \cdots \tilde{\varphi}_n \in \mathcal{H}_m^B \otimes_{\mathbb{P}\mathrm{ol}_n} \mathbb{F}_{n,m}$. Since $(\tilde{\varphi}_n \cdots \tilde{\varphi}_1)^{-1} = \tilde{\varphi}_1 \cdots \tilde{\varphi}_n \in \mathcal{H}_{n+m}^A \otimes_{\mathbb{P}\mathrm{ol}_n} \mathbb{F}_{n,m}$, we have $T_0 \tilde{\varphi}_1 \cdots \tilde{\varphi}_n \in \mathcal{H}_{n,m}^{\mathrm{loc}}$.

Second, note that

$$T_i = \left(\tilde{\varphi}_i (p_i^{-1} - p_i X_i^{-1} X_{i+1}) - (p_i - p_i^{-1}) X_i^{-1} X_{i+1}\right) (1 - X_i^{-1} X_{i+1})^{-1} \ (1 \leqslant i < n).$$

If $T_0 T_1 \cdots T_{i-1} \tilde{\varphi}_i \cdots \tilde{\varphi}_n \in \mathcal{H}_{n,m}^{\mathrm{loc}}$, then $T_0 T_1 \cdots T_i \tilde{\varphi}_{i+1} \cdots \tilde{\varphi}_n \in \mathcal{H}_{n,m}^{\mathrm{loc}}$ for $1 \leqslant i < n$. Indeed, we have

$$\begin{aligned} T_0 \cdots T_i \tilde{\varphi}_{i+1} \cdots \tilde{\varphi}_n &= T_0 \cdots T_{i-1} \tilde{\varphi}_i \cdots \tilde{\varphi}_n (p_i^{-1} - p_i X_i^{-1} X_{n+1})(1 - X_i^{-1} X_{n+1})^{-1} \\ &\quad - (p_i - p_i^{-1}) T_0 \cdots T_{i-1} \tilde{\varphi}_{i+1} \cdots \tilde{\varphi}_n X_i^{-1} X_{n+1} (1 - X_i^{-1} X_{n+1})^{-1} \end{aligned}$$

and

$$T_0 \cdots T_{i-1} \tilde{\varphi}_{i+1} \cdots \tilde{\varphi}_n = \tilde{\varphi}_{i+1} \cdots \tilde{\varphi}_n T_0 \cdots T_{i-1} \in \mathcal{H}_{n+m}^{\mathrm{A}} \mathbb{F}_{n,m} \mathcal{H}_n^{\mathrm{B}}.$$

Therefore $T_0 T_1 \cdots T_n \in \mathcal{H}_{n,m}^{\mathrm{loc}}$. Hence $T_0 T_1 \cdots T_i \in \mathcal{H}_{n,m}^{\mathrm{loc}}$ $(1 \leqslant i < n+m)$. Indeed, if $i < n$, then $T_0 T_1 \cdots T_i \in \mathcal{H}_n^B$. If $n \leqslant i$, then $T_0 T_1 \cdots T_n \in \mathcal{H}_{n,m}^{\mathrm{loc}}$ and $T_{n+1} \cdots T_i \in \mathcal{H}_m^B$.

Finally, we prove the surjectivity by the induction on m. Note that

$$\mathcal{H}^B_{n+m}=\sum_{i=1}^{n+m} T_iT_{i+1}\cdots T_{n+m-1}\mathcal{H}^B_{n+m-1}+\sum_{i=0}^{n+m-1} T_i\cdots T_1T_0T_1\cdots T_{n+m-1}\mathcal{H}^{\mathrm{B}}_{n+m-1}$$

and $T_iT_{i+1}\cdots T_{n+m-1}\in\mathcal{H}^A_{n+m-1}$. Furthermore, $\mathcal{H}^B_{n+m-1}\subset\mathcal{H}^{\mathrm{loc}}_{n,m-1}$ by the induction hypothesis. Thus it is sufficient to prove that $T_0\mathcal{H}^{A,\mathrm{fin}}_{n+m}\subset\mathcal{H}^{\mathrm{loc}}_{n,m}$. Here, $\mathcal{H}^{A,\mathrm{fin}}_{n+m}$ is the subalgebra of $\mathcal{H}^{\mathrm{A}}_{n+m}$ generated by $T_1,\dots,T_{n+m-1}$. This follows from

$$\mathcal{H}^{\mathrm{A,fin}}_{n+m}=\sum_{i=0}^{n+m-1}\langle T_2,\cdots,T_{n+m-1}\rangle T_1T_2\cdots T_i$$

and $T_0T_1\cdots T_i\in\mathcal{H}^{\mathrm{loc}}_{n,m}$. □

Definition 6.3. *For a finite-dimensional $\mathcal{H}^B_n$-module M, let*

$$M=\bigoplus_{a\in(\mathbb{C}^*)^n} M_a$$

be the generalized eigenspace decomposition with respect to $X_1,\dots,X_n$:

$$M_a:=\{u\in M\mid (X_i-a_i)^Nu=0 \text{ for any } 1\leqslant i\leqslant n \text{ and } N\gg 0\}$$

for $a=(a_1,\dots,a_n)\in(\mathbb{C}^*)^n$.

(1) *We say that M is of type J if all the eigenvalues of $X_1,\dots,X_n$ belong to $J\subset\mathbb{C}^*$. Put*

$$K^B_J:=\bigoplus_{n\geqslant 0} K^B_{J,n}.$$

Here $K^B_{J,n}$ is the Grothendieck group of the abelian category of finite-dimensional $\mathcal{H}^B_n$-modules of type J.

(2) *The semi-direct product group $\mathbb{Z}\rtimes\mathbb{Z}_2=\mathbb{Z}\times\{1,-1\}$ acts on $\mathbb{C}^*$ by $(n,\epsilon)\colon a\mapsto a^\epsilon p_1^{2n}$.*

(3) *Let J_1 and J_2 be $\mathbb{Z}\rtimes\mathbb{Z}_2$-invariant subsets of $\mathbb{C}^*$ such that $J_1\cap J_2=\emptyset$. Then for an $\mathcal{H}^B_n$-module N of type J_1 and an $\mathcal{H}^B_m$-module M of type J_2, the action of $\mathbb{P}\mathrm{ol}_{n+m}$ on $N\otimes M$ extends to an action of $\mathbb{F}_{n,m}$. We set*

$$N\diamond M:=(\mathcal{H}^B_{n+m}\otimes_{\mathbb{P}\mathrm{ol}_{n+m}}\mathbb{F}_{n,m})\otimes_{(\mathcal{H}^B_n\otimes\mathcal{H}^B_m)\otimes_{\mathbb{P}\mathrm{ol}_{n+m}}\mathbb{F}_{n,m}}(N\otimes M).$$

By the lemma above, $N\diamond M$ is isomorphic to $\mathrm{Ind}^{\mathcal{H}^A_{n+m}}_{\mathcal{H}^A_n\otimes\mathcal{H}^A_m}(N\otimes M)$ as an $\mathcal{H}^A_{n+m}$-module.

Proposition 6.4. *Let J_1 and J_2 be $\mathbb{Z}\rtimes\mathbb{Z}_2$-invariant subsets of $\mathbb{C}^*$ such that $J_1\cap J_2=\emptyset$.*

(1) *Let N be an irreducible $\mathcal{H}^B_n$-module of type J_1 and M an irreducible $\mathcal{H}^B_m$-module of type J_2. Then $N\diamond M$ is an irreducible $\mathcal{H}^B_{n+m}$-module of type $J_1\cup J_2$.*

(2) *Conversely if L is an irreducible $\mathcal{H}^B_n$-module of type $J_1\cup J_2$, then there exist an integer m $(0\leqslant m\leqslant n)$, an irreducible $\mathcal{H}^B_m$-module N of type J_1 and an irreducible $\mathcal{H}^B_{n-m}$-module M of type J_2 such that $L\simeq N\diamond M$.*

(3) *Assume that a $\mathbb{Z}\rtimes\mathbb{Z}_2$-orbit J decomposes into $J=J_+\sqcup J_-$ where $J_\pm$ are $\mathbb{Z}$-orbits and $J_-=(J_+)^{-1}$. Assume that $\pm1,\pm p_0\notin J$. Then for any irreducible $\mathcal{H}^B_n$-module L of type J, there exists an irreducible $\mathcal{H}^A_n$-module M such that $L\simeq\mathrm{Ind}^{\mathcal{H}^B_n}_{\mathcal{H}^A_n}M$.*

Proof. (1) Let $(N\diamond M)_{J_1,J_2}$ be the generalized eigenspace, where the eigenvalues of X_i $(1\leqslant i\leqslant n)$ are in J_1 and the eigenvalues of X_j $(n<j\leqslant n+m)$ are in J_2. Then $(N\diamond M)_{J_1,J_2}=N\otimes M$ by $J_1\cap J_2=\emptyset$ by the above lemma and the shuffle lemma (e.g. [G, Lemma 5.5]). Suppose there exists non-zero $\mathcal{H}^B_{n+m}$-submodule S in $N\diamond M$. Then $S_{J_1,J_2}\neq 0$

as an $\mathcal{H}_n^B \otimes \mathcal{H}_m^B$-module. Hence $S_{J_1,J_2} = N \otimes M$ by the irreducibility of $N \otimes M$ as an $\mathcal{H}_n^B \otimes \mathcal{H}_m^B$-module. We obtain $S = N \diamond M$.
(2) For an irreducible $\mathcal{H}_n^B$-module L, the $\mathcal{H}_m^B \otimes \mathcal{H}_{n-m}^B$-module L_{J_1,J_2} does not vanish for some m. Take an irreducible $\mathcal{H}_m^B \otimes \mathcal{H}_{n-m}^B$-submodule S in L. Then there exist an irreducible $\mathcal{H}_m^B$-module N of type J_1 and an irreducible $\mathcal{H}_{n-m}^B$-module M of type J_2 such that $S = N \otimes M$. Hence there exists a surjective homomorphism $\mathrm{Ind}(N \otimes M) = N \diamond M \to L$. Since $N \diamond M$ is irreducible, this is an isomorphism.
(3) See [M, Section 6]. □

Hence in order to study $\mathcal{H}^B$-modules, it is enough to study irreducible modules of type J for a $\mathbb{Z} \rtimes \mathbb{Z}_2$-orbit J in $\mathbb{C}^*$ such that J is a $\mathbb{Z}$-orbit or J contains one of $\pm 1, \pm p_0$.

6.3. The a-restriction and a-induction.

Definition 6.5. *For $a \in \mathbb{C}^*$ and a finite-dimensional $\mathcal{H}_n^B$-module M, let us define the functors*

$$E_a : \mathcal{H}_n^B\text{-mod}^{\mathrm{fd}} \to \mathcal{H}_{n-1}^B\text{-mod}^{\mathrm{fd}}, \quad F_a : \mathcal{H}_n^B\text{-mod}^{\mathrm{fd}} \to \mathcal{H}_{n+1}^B\text{-mod}^{\mathrm{fd}}$$

by: $E_a M$ is the generalized a-eigenspace of M with respect to the action of X_n, and

$$F_a M := \mathrm{Ind}_{\mathcal{H}_n^B \otimes \mathbb{C}[X_{n+1}^{\pm 1}]}^{\mathcal{H}_{n+1}^B} M \otimes \langle a \rangle,$$

where $\langle a \rangle$ is the 1-dimensional representation of $\mathbb{C}[X_{n+1}^{\pm 1}]$ defined by $X_{n+1} \mapsto a$.
Define

$$\widetilde{E}_a M := \mathrm{soc}\, E_a M, \quad \widetilde{F}_a M := \mathrm{cosoc}\, F_a M$$

for $a \in \mathbb{C}^*$.

Theorem 6.6 (Miemietz [M]). *Suppose M is irreducible. Then $\widetilde{F}_a M$ is irreducible and $\widetilde{E}_a M$ is irreducible or 0 for any $a \in \mathbb{C}^* \setminus \{\pm 1\}$.*

6.4. LLTA type conjectures for type B. Now we take the case

$$J = \left\{ p_1^k \mid k \in \mathbb{Z}_{\mathrm{odd}} \right\}.$$

Assume that any of ± 1 and $\pm p_0$ is not contained in J. For short, we shall write E_i, $\widetilde{E}_i$, F_i and $\widetilde{F}_i$ for E_{p^i}, $\widetilde{E}_{p^i}$, F_{p^i} and $\widetilde{F}_{p^i}$, respectively.

Conjecture 6.7. (1) There are complete representatives

$$\{L_b \mid b \in B_\theta(0)\}$$

of the finite-dimensional irreducible $\mathcal{H}^B$-modules of type J such that

$$\widetilde{E}_i L_b = L_{\widetilde{E}_i b}, \quad \widetilde{F}_i L_b = L_{\widetilde{F}_i b}$$

for any $i \in I := \mathbb{Z}_{\mathrm{odd}}$.
(2) For any $i \in \mathbb{Z}_{\mathrm{odd}}$, let us define $E_{i,b,b'}, F_{i,b,b'} \in \mathbb{C}[q, q^{-1}]$ by the coefficients of the following expansions:

$$E_i G_\theta^{\mathrm{up}}(b) = \sum_{b' \in B_\theta(0)} E_{i,b,b'} G_\theta^{\mathrm{up}}(b'), \quad F_i G_\theta^{\mathrm{up}}(b) = \sum_{b' \in B_\theta(0)} F_{i,b,b'} G_\theta^{\mathrm{up}}(b').$$

Then

$$[E_i L_b : L_{b'}] = E_{i,b,b'}|_{q=1}, \quad [F_i L_b : L_{b'}] = F_{i,b,b'}|_{q=1}.$$

Here $[M : N]$ is the composition multiplicity of N in M on K_J^B.

Remark 6.8. There is a one-to-one correspondence between the above index set $B_\theta(0)$ and Syu Kato's parametrization ([Kat]) of irreducible representations of $\mathcal{H}_n^B$ of type J.

Remark 6.9. (i) For conjectures for other $\mathbb{Z} \rtimes \mathbb{Z}_2$-orbits J, see [EK1].

(ii) Similar conjectures for type D are presented by the second author and Vanessa Miemietz ([KM]).

Errata to "Symmetric crystals and affine Hecke algebras of type B, Proc. Japan Acad., 82, no. 8, 2006, 131–136" :

(i) In Conjecture 3.8, $\lambda = \Lambda_{p_0} + \Lambda_{p_0^{-1}}$ should be read as $\lambda = \sum_{a \in A} \Lambda_a$, where $A = I \cap \{p_0, p_0^{-1}, -p_0, -p_0^{-1}\}$. We thank S. Ariki who informed us that the original conjecture is false.

(ii) In the two diagrams of $B_\theta(\lambda)$ at the end of § 2, λ should be 0.

(iii) Throughout the paper, $A_\ell^{(1)}$ should be read as $A_{\ell-1}^{(1)}$.

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RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES, KYOTO UNIVERSITY, KYOTO 606–8502, JAPAN

E-mail address: henon@kurims.kyoto-u.ac.jp

RESEARCH INSTITUTE FOR MATHEMATICAL SCIENCES, KYOTO UNIVERSITY, KYOTO 606–8502, JAPAN

E-mail address: masaki@kurims.kyoto-u.ac.jp