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On "M-functions" closely related to the distribution of L'/L-values*

By

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0.1 — When the ground field is the rational number field $\mathbb{Q}$, the functions $M_\sigma(z)$ and $\tilde{M}_\sigma(z)$ of $\sigma > 1/2$ and $z \in \mathbb{C}$ that we are going to construct and study can be uniquely characterized by the following properties (i)∼ (iii);

(i) as functions of z, they are Fourier duals of each other,
(ii) $\tilde{M}_\sigma(z)$ is *real analytic in* (σ, z),
(iii) (at least) when $\sigma > 1$, $M_\sigma(z)|dz|$ gives the *density measure* for the distribution of values of the logarithmic derivative

$$L'(\chi, \sigma + \tau i)/L(\chi, \sigma + \tau i) \tag{0.1.1}$$

of L-functions on $\mathbb{C}$. Here, $\tau \in \mathbb{R}$ is also fixed and χ runs over all Dirichlet characters with prime conductors. (The density measure turns out to be independent of τ; for the other notations, see §0.2 below.)

We shall work over any global field K, i.e., either an algebraic number field of finite degree, or an algebraic function field of one variable over a finite field. These "M-functions" shall depend on K.

The main purpose of this article is (I) to construct and study these functions, with more weight on the study of $\tilde{M}_\sigma(z)$, which seems to be of independent interest, and (II) to establish the above relation (iii) including other cases of K and some other families of χ (Dirichlet characters, or Hecke Grössencharacters on K); in particular, for some range of σ including some to the left of $\sigma = 1$ in the function field case. The motivation of this work came from our previous study related to $L'(\chi, 1)/L(\chi, 1)$ [2][3][4]. The present paper is for the first stage. Even unconditional results for $\sigma = 1$ in the number field case require further substantial work. As for the connections with, and differences from the Bohr-Jessen type value distribution theories, where $\chi = 1$ and τ varies, see §0.5 below.

*This is a revised version of the Introduction (§1) from the author's preprint RIMS-1574 (December 2006; the same title as above).A revised version of the preprint containing details will be submitted for publication elsewhere.

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0.2 – The function $M_\sigma(z)$ to be constructed is real valued, ≥ 0, and belongs to C^∞, while $\tilde{M}_\sigma(z)$ is complex-valued, $|\tilde{M}_\sigma(z)| \leq 1$, and real-analytic. They are the Fourier transforms of each other in the sense that

$$\tilde{M}_\sigma(z) = \int_{\mathbb{C}} M_\sigma(w)\psi_z(w)|dw|, \quad M_\sigma(z) = \int_{\mathbb{C}} \tilde{M}_\sigma(w)\psi_{-z}(w)|dw|. \tag{0.2.1}$$

Here, $\psi_z : \mathbb{C} \mapsto \mathbb{C}^1$ is the additive character

$$\psi_z(w) = \exp(i.\mathrm{Re}(\bar{z}w)) \tag{0.2.2}$$

parametrized by $z \in \mathbb{C}$, and $|dw|$ denotes the self-dual Haar measure on $\mathbb{C}$ with respect to the self-dual pairing $\psi_z(w)$ of $\mathbb{C}$; namely, $|dw| = (2\pi)^{-1}dxdy$ for $w = x + iy$.

Both are continuous also in σ, and $\tilde{M}_\sigma(z)$ is even real-analytic in σ. They have quite interesting arithmetic and analytic properties. $\tilde{M}_\sigma(z)$ has a convergent *Euler product* expansion each of whose $\wp$-factor can be expressed explicitly in terms of Bessel functions, and correspondingly, $M_\sigma(z)$ has a *convolution Euler product* expansion, each of whose $\wp$-factor being a certain hyperfunction. Also, $\tilde{M}_\sigma(z)$ has an everywhere convergent power series expansion in $z, \bar{z}$ whose coefficients are some arithmetic Dirichlet series in σ, which may be regarded also as a Dirichlet series expansion in $\sigma > 1/2$ whose coefficients are arithmetic polynomials of $z, \bar{z}$. Both decay rapidly as $|z| \mapsto \infty$. Thus, even when $1/2 < \sigma < 1$ in the number field case, where we do not know much about the zeros of $L(\chi, s)$ and hence about the poles of $L'(\chi, s)/L(\chi, s)$, and hence about the distribution of $L'(\chi, s)/L(\chi, s)$ near $z = \infty$, still, the corresponding function $M_\sigma(z)$ can be constructed independently and can be proved to be rapidly decreasing with $|z|$.

0.3 – The symbolical relations among $M_\sigma(z)$,$\tilde{M}_\sigma(z)$ and $L'(\chi, s)/L(\chi, s)$, under optimal circumstances are,

$$M_\sigma(z) = \mathrm{Avg}_\chi \delta_z \left(\frac{L'(\chi, s)}{L(\chi, s)}\right), \quad \tilde{M}_\sigma(z) = \mathrm{Avg}_\chi \psi_z \left(\frac{L'(\chi, s)}{L(\chi, s)}\right), \tag{0.3.1}$$

where Avg_χ means a certain weighted average, ψ_z is the additive character of $\mathbb{C}$ defined above (0.2.2), and $\delta_z(w)|dw|$ is the Dirac delta measure on $\mathbb{C}$ with support at z. In other words, the first formula of (0.3.1) means that

$$\int_{\mathbb{C}} M_\sigma(w)\Phi(w)|dw| = \mathrm{Avg}_\chi \Phi \left(\frac{L'(\chi, s)}{L(\chi, s)}\right) \tag{0.3.2}$$

holds for any test function Φ on $\mathbb{C}$, and the second formula is its special case where $\Phi = \psi_z$. When $\Phi(w) = P^{(a,b)}(w) = \bar{w}^a.w^b$ (a, b non-negative integers), again under optimal circumstances,

$$\text{(0.3.3)} \qquad \mathrm{Avg}_\chi P^{(a,b)}\left(\frac{L'(\chi,s)}{L(\chi,s)}\right) = \int_{\mathbb{C}} M_\sigma(w)P^{(a,b)}(w)|dw|$$
$$= \left(\frac{2}{i}\right)^{a+b} \frac{\partial^{a+b}}{\partial z^a \partial \bar{z}^b}\tilde{M}_\sigma(z) \mid_{z=0}= (-1)^{a+b}\mu_\sigma^{(a,b)},$$

where $\mu_\sigma^{(a,b)}$ is as defined by (0.4.10) below. The present work is, though logically independent, in a sense, a continuation of [4] where this value was studied in the special case $K = \mathbb{Q}$ and $s = 1$.

0.4 – Our main results may be summarized as follows.

Theorem $\tilde{M}$ (i) *For each non-archimedean prime $\wp$ of K, consider the function of $\sigma > 0$ and $z \in \mathbb{C}$ defined by the convergent series*

$$\text{(0.4.1)} \qquad \tilde{M}_{\sigma,\wp}(z) = 1 + \sum_{n=1}^{\infty} \frac{G_n(-\frac{i}{2}z\log N(\wp))G_n(-\frac{i}{2}\bar{z}\log N(\wp))}{N(\wp)^{2\sigma n}},$$

where $i = \sqrt{-1}$ and

$$\text{(0.4.2)} \qquad G_n(w) = \sum_{k=1}^{n} \frac{1}{k!}\binom{n-1}{k-1} w^k.$$

Then when $\sigma > 1/2$, the Euler product

$$\text{(0.4.3)} \qquad \tilde{M}_\sigma(z) = \prod_\wp \tilde{M}_{\sigma,\wp}(z)$$

converges in the following sense. For any $R > 0$ there exists $y = y(\sigma, R)$ such that $\tilde{M}_{\sigma,\wp}(z)$ for $N(\wp) > y$ has no zeros on $|z| \leq R$ and that their product over all these $\wp$ converges absolutely to a nowhere vanishing function on this disk. This function $\tilde{M}_\sigma(z)$ is real analytic in (σ, z), and as a function of z, belongs to L^p for all $1 \leq p \leq \infty$.

(There are other expressions of each local factor in terms of Bessel functions. As for the zeros, one can at least show that each local factor has an infinite discrete set of zeros on the imaginary axis.)

(ii) *$\tilde{M}_\sigma(z)$ has an everywhere convergent power series expansion*

$$\text{(0.4.4)} \qquad \tilde{M}_\sigma(z) = 1 + \sum_{a,b=1}^{\infty} (-i/2)^{a+b}\mu_\sigma^{(a,b)}\frac{z^a\bar{z}^b}{a!b!},$$

and a convergent Dirichlet series expansion on $\sigma > 1/2$

$$\tilde{M}_\sigma(z) = \sum_{D:integral} \frac{\lambda_D(z)\lambda_D(\bar{z})}{N(D)^{2\sigma}}, \tag{0.4.5}$$

with positive real constants $\mu_\sigma^{(a,b)}$ *and polynomials* $\lambda_D(z)$ *defined below. Here,* D *runs over all "integral" divisors of* K*, i.e., the products of non-negative powers of non-archimedean primes.*

The author's initial definition of $\tilde{M}_\sigma(z)$ was by the (formal) power series (0.4.4), because the only information on the "would-be" density measure function $M_\sigma(z)$ was (0.3.3), by [4] (when $s = 1$). Then the Euler product decomposition was found by a different route, and then the more natural explanation described below (§0.5) was recognized. Incidentally, our proof of the fact that the series (0.4.4) converges everywhere requires an argument where $z, \bar{z}$ are treated as two independent *complex* variables.

The coefficients $\mu_\sigma^{(a,b)}$ and $\lambda_D(z)$ are defined as follows. First, for any integral ideal D of K, set

$$\begin{aligned} \Lambda(D) &= \log N(\wp) && \cdots \textit{if } D = \wp^r,\ r \geq 1, \textit{ for some prime divisor } \wp, \\ &= 0 && \cdots \textit{otherwise.} \end{aligned} \tag{0.4.6}$$

Then define $\Lambda_k(D)$ ($k \geq 0$, $k \in \mathbb{Z}$) by

$$\begin{aligned} \Lambda_0(D) &= 1 && \cdots \textit{if } D = (1) \\ &= 0 && \cdots \textit{otherwise,} \end{aligned} \tag{0.4.7}$$

$$\Lambda_k(D) = \sum_{D = D_1 \cdots D_k} \Lambda(D_1) \cdots \Lambda(D_k) \qquad (k \geq 1). \tag{0.4.8}$$

For each D, the following $\lambda_D(z)$ is a *polynomial* of z.

$$\lambda_D(z) = \sum_{k=0}^{\infty} (-i/2)^k \frac{\Lambda_k(D)}{k!} z^k. \tag{0.4.9}$$

Finally, for each pair (a, b) of non-negative integers and Re$(s) > 1/2$, we define the invariant $\mu_s^{(a,b)}$ by the absolutely convergent Dirichlet series

$$\mu_s^{(a,b)} = \sum_D \frac{\Lambda_a(D)\Lambda_b(D)}{N(D)^{2s}}. \tag{0.4.10}$$

When $s = \sigma > 1/2$ and $a, b \geq 1$, this takes a positive real value.

Theorem M *There exists a unique continuous function* $M_\sigma(z)$ *of* $\sigma > 1/2$ *and* z *such that*

$$\tilde{M}_\sigma(z) = \int_{\mathbb{C}} M_\sigma(w)\psi_z(w)|dw|, \quad M_\sigma(z) = \int_{\mathbb{C}} \tilde{M}_\sigma(w)\psi_{-z}(w)|dw|. \tag{0.4.11}$$

It is non-negative real valued, C^∞ *in* z, *and satisfies*

$$\int_{\mathbb{C}} M_\sigma(z)|dz| = 1. \tag{0.4.12}$$

As for the connections with the L'/L-values, presently, we focus our attention to the following three families of characters;

(Case A) K is either the rational number field $\mathbb{Q}$, an imaginary quadratic field, or a function field of one variable over a finite field given together with an "infinite" prime divisor $\wp_\infty$ of degree 1, which will be considered "archimedean" and excluded from the L and $\tilde{M}$, M -Euler factors. The character χ runs over all Dirichlet characters on K with *prime* conductors normalized by the condition $\chi(\wp_\infty) = 1$;

(Case B) K is a number field having more than one archimedean prime, and χ runs over all "normalized unramified Grössencharacters" of K. (This family forms a free $\mathbb{Z}$-module of rank $[K : \mathbb{Q}] - 1$);

(Case C) $K = \mathbb{Q}$ and χ runs over the characters of the form $N(\wp)^{-\tau i}$. This is the case related to Bohr-Jessen type theories (§0.5).

For each such family, the average Avg_χ will be suitably defined.

Theorem $L \sim M$ *Let* $\sigma = \mathrm{Re}(s)$, *and* χ *run over one of the above families. Assume* $\sigma > 1$ *in the number field case, and* $\sigma > 3/4$ *in the function field case. Then*

(i)

$$\mathrm{Avg}_\chi \Phi\left(\frac{L'(\chi, s)}{L(\chi, s)}\right) = \int_{\mathbb{C}} M_\sigma(z)\Phi(z)|dz| \tag{0.4.13}$$

holds for any "mild" test function Φ *on* $\mathbb{C}$.

(ii)

$$\mathrm{Avg}_\chi \psi_z\left(\frac{L'(\chi, s)}{L(\chi, s)}\right) = \tilde{M}_\sigma(z), \tag{0.4.14}$$

(iii)

$$\mathrm{Avg}_\chi P^{(a,b)}\left(\frac{L'(\chi, s)}{L(\chi, s)}\right) = (-1)^{a+b}\mu_\sigma^{(a,b)}, \tag{0.4.15}$$

for the polynomials $P^{(a,b)}(w) = \bar{w}^a w^b$ $(a, b \geq 0)$.

We expect that Theorem $L \sim M$ should hold for any $\sigma > 1/2$ (at least) in the function field case. [3] But even in the function field case where the Weil's Riemann Hypothesis is valid, the above restriction $\sigma > 3/4$ seems to be the limit of our method. On the other hand, we have an optimistic point of view for the possibility of having (unconditional) theory for $\sigma \leq 1$ also in the number field case, because $\psi_z(L'(\chi,s)/L(\chi,s))$ makes sense even at possible zeros of $L(\chi, s)$.

0.5 – We shall now explain the main line of construction of the functions $M_\sigma(z)$ and $\tilde{M}_\sigma(z)$. It is based on the Euler sum expansion of L'/L coming from the Euler product expansion of L, and the basic geometric idea goes back to a seminal work of Bohr-Jessen [1]. Fix $s \in \mathbb{C}$, with $\sigma = \mathrm{Re}(s)$.

(Local constructions) Let $\sigma > 0$, and P be a finite set of non-archimedean primes of K. Put

$$T_P = \prod_{\wp \in P} \mathbb{C}^1 \tag{0.5.1}$$

(a torus), and let $g_{\sigma,P} : T_P \longmapsto \mathbb{C}$ be defined by

$$g_{\sigma,P}(t) = \sum_{\wp \in P} g_{\sigma,\wp}(t_\wp) = \sum_{\wp \in P} \frac{t_\wp \log N(\wp)}{t_\wp - N(\wp)^\sigma} \tag{0.5.2}$$

$(t = (t_\wp) \in T_P)$. For any abelian character χ on K which is unramified over P, let

$$L_P(\chi, s) = \prod_{\wp \in P} (1 - \chi(\wp) N(\wp)^{-s})^{-1} \tag{0.5.3}$$

be the partial L-function. Then $L'_P(\chi,s)/L_P(\chi,s)$ can be regarded as a special value

$$\frac{L'_P(\chi,s)}{L_P(\chi,s)} = g_{\sigma,P}(\chi_P . N(P)^{-i.\tau}) \tag{0.5.4}$$

of the rational function $g_{\sigma,P}$ on T_P at the point $\chi_P . N(P)^{-i.\tau}$, where $\tau = \mathrm{Im}(s)$ and

$$\chi_P = (\chi(\wp))_\wp, \quad N(P)^{-i.\tau} = (N(\wp)^{-i.\tau})_\wp. \tag{0.5.5}$$

The optimal circumstance is where the family of χ satisfies the following two conditions:

[3] Added September18, 2007: As for (iii), this is now a theorem in the recently revised full paper submitted for publication.

(UniDistr) χ_P is *uniformly distributed* on T_P,

(Conv) $L'_P(\chi,s)/L_P(\chi,s)$ converges to $L'(\chi,s)/L(\chi,s)$ fast enough, ideally, uniformly in χ.
Now, for each family of χ that we shall consider, all but finitely many χ have conductors coprime with P and the above condition (UniDistr) is satisfied. Therefore, (0.3.1), with L_P in place of L, must be given by the corresponding integrals

(0.5.6) $$M_{\sigma,P}(z) = \int_{T_P} \delta_z(g_{\sigma,P}(t))d^*t, \quad \tilde{M}_{\sigma,P}(z) = \int_{T_P} \psi_z(g_{\sigma,P}(t))d^*t.$$

(d^*t: the normalized Haar measure on T_P. Note that the contribution of $\mathrm{Im}(s)$ is "averaged away".) We thus have

(0.5.7) $$\int_{\mathbb{C}} M_{\sigma,P}(w)\Phi(w)|dw| = \mathrm{Avg}_\chi \Phi\left(\frac{L'_P(\chi,s)}{L_P(\chi,s)}\right)$$

for any continuous function Φ on $\mathbb{C}$. Note here that each $M_{\sigma,P}(z)$ is compactly supported. The summation over $\wp \in P$ in (0.5.2) is translated into "the basic product expansions"

(0.5.8) $$M_{\sigma,P}(z) = *_{\wp\in P} M_{\sigma,\wp}(z), \quad \tilde{M}_{\sigma,P}(z) = \prod_{\wp\in P} \tilde{M}_{\sigma,\wp}(z),$$

where $*$ denotes the convolution product. Using the simple fact that each $g_{\sigma,\wp}$ maps $\mathbb{C}^1$ to another small circle, we are able to compute each of $M_{\sigma,\wp}(z)$ and $\tilde{M}_{\sigma,\wp}(z)$ explicitly.

(Global constructions) Now let $\sigma > 1/2$, and set $P = P_y = \{\wp; N(\wp) \le y\}$. Then the point is that when $y \mapsto \infty$, each $M_{\sigma,P}(z)$ (resp. $\tilde{M}_{\sigma,P}(z)$) converges uniformly (and also w.r.t. some other L^p topologies) to a not-everywhere vanishing function $M_\sigma(z)$ (resp. $\tilde{M}_\sigma(z)$). Thus, these are functions obtained from $\delta_z(L'_P(\chi,s)/L_P(\chi,s))$ (resp. $\psi_z(L'_P(\chi,s)/L_P(\chi,s))$) by first fixing P and averaging over an infinite family of characters χ and then letting $y \mapsto \infty$. This way we can enter the region $1/2 < \sigma < 1$ unconditionally. The connection between the global objects, $\delta_z(L'(\chi,s)/L(\chi,s))$ with $M_\sigma(z)$, (resp. $\psi_z(L'(\chi,s)/L(\chi,s))$ with $\tilde{M}_\sigma(z)$), can be made when the condition (Conv) is satisfied at a sufficiently high level. First, when $\sigma > 1$, this convergence is uniform; hence the local relation (0.5.7) directly passes over to the global relation (0.3.2). Secondly, for the family (A) in the function field case, the convergence holds for any $\sigma > 1/2$ but (apparently) not uniformly with respect to χ. In this case, by choosing the intermediate object

(0.5.9) $$\mathrm{Avg}_{N(\mathfrak{f}_\chi)\le m}(\psi_z(L'_P(\chi,s)/L_P(\chi,s))),$$

where m and $P = P_y$ are related by $y = (\log m)^b$, with a suitable positive constant b, we are able to relate the M's with the L'/L for $\sigma > 3/4$. (If $(\sigma - 1/2)b > 1$ the convergence is fast enough, while if $(1-\sigma)b < 1$ the distribution is quantitatively uniform enough.) This is done after some Fourier analysis of the function $\psi_z(g_{\sigma,P}(t))$ on T_P.

(Relations with Bohr-Jessen type theories) Bohr-Jessen studied the distribution of values of $\log\zeta(\sigma + \tau i)$, where $\sigma > 1/2$ is fixed and $\tau \in \mathbb{R}$ varies [1](cf. also [6][7][10]). Since $\zeta(\sigma+\tau i) = L(\chi_\tau, \sigma)$, where $\chi_\tau(\wp) = N(\wp)^{-\tau i}$, this is the same as the value distribution of $\log L(\chi_\tau, s)$, where s is fixed and χ_τ runs over this one parameter family of trivial characters (the Case C family). Although we started by studying the "variable χ problem" for the Case A family and found the above construction, as was kindly pointed out by A.Fujii, the basic idea for such construction was essentially the same as in [1], i.e., goes back to Bohr-Jessen. Indeed, [1] uses the Euler sum expansion of $\log\zeta(s)$ and the uniform distribution property of χ_P on T_P, to relate the problem to the distribution of sum of points on the images of $\mathbb{C}^1$ by the mappings $t \mapsto -\log(1-p^{-\sigma}t)$. Then Jessen-Wintner [5] gave more general treatments using probability measure theory including Fourier analysis, and Matsumoto [8][9] generalized this to the case of any number field K.

In spite of these similarities, there are three major differences.

(I) The directly related "$d\log\zeta(s)$-version" does not seem to have been so seriously studied. This is probably because of the difficulty in this case to get to the left of $\sigma = 1$. There are also differences in local structures; the $d\log$-version is in a sense easier, as the image $g_{\sigma,\wp}(\mathbb{C}^1)$ for this case is a circle; but on the other hand, the center-shifts and the metric twists cause some complications by which we cannot directly apply their theories, e.g. [5]. There are plenty of similarities, but it is still easier to treat this case directly.

(II) For this one parameter family $\chi = \chi_\tau$, the uniform distribution of χ_P on T_P holds *only when* $K = \mathbb{Q}$. Indeed, $\chi(\wp)$ depends only on $N(\wp)$. So, the $d\log$ analogue of Matsumoto's distribution measure for $K \neq \mathbb{Q}$ is different from ours. In the function field case, the situation is decisively different from the Case A family; all $(\chi_\tau)_P$ lie on a one-dimensional subtorus of T_P, and the image of the global $\zeta_K'(\sigma+\tau i)/\zeta_K(\sigma+\tau i)$ ($\sigma > 1/2$: fixed) is a bounded curve.

(III) The Fourier dual appears in [5][9], etc., but mainly for auxiliary purpose to prove the convergence of the local measure. Those properties of $\tilde{M}_\sigma(z)$ as in the above stated Theorem $\tilde{M}$ (ii), which are obtained by using analytic functions of three complex variables (s, z_1, z_2) extending $(\sigma, z, \bar{z})$, do not seem to have been known.
We hope that the present approach will shed some light to the variable τ theory (too).

0.6 – We have also left untouched various basic questions related to $M_\sigma(z)$, $\tilde{M}_\sigma(z)$; for example, their zeros, their values at the central points (such as $M_\sigma(0)$, $M_\sigma(\zeta_K'(2\sigma)/\zeta_K(2\sigma))$), determination of the value of

$$\int_{\mathbb{C}} M_\sigma(z)^2|dz| = \int_{\mathbb{C}} |\tilde{M}_\sigma(z)|^2|dz|. \tag{0.6.1}$$

We hope to be able to discuss these in the near future, together with more applications.

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