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| Author(s) | Ramis, J.-P.; Sauloy, J. |
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# THE $q$-ANALOGUE OF THE WILD FUNDAMENTAL GROUP (I)

J.-P. Ramis* J. Sauloy †

**Abstract**

We describe an explicit construction of galoisian Stokes operators for irregular linear q-difference equations. These operators are parameterized by the points of an elliptic curve minus a finite set of singularities. Taking residues at these singularities, one gets q-analogues of alien derivations which "freely" generate the Lie algebra of the Stokes subgroup of the Galois group.

## 1 Introduction

In this paper we return to the local analytic classification of $q$-difference modules. In [23] we gave such a classification in Birkhoff style [2, 3], using normal forms and index theorems. The classification of [23] is complete in the "integral slope case". (One could extend it to the general case using some results of [12].) In [29] (cf. also [22], [24]) appears another version of our classification, using non abelian cohomology of sheaves on an elliptic curve.

Here our aim is to give a new version of our classification, based upon a "fundamental group" and its finite dimensional representations, in the style of the Riemann-Hilbert correspondence for linear differential equations. At some abstract level, such a classification exists: the fundamental group is the tannakian Galois group of the tannakian category of our $q$-modules. But we want more information: our essential aim is to get a *smaller* fundamental group (as small as possible !) which is Zariski dense in the tannakian Galois group and to describe it *explicitly*. (As a byproduct, we shall get finally a complete description of the tannakian Galois group itself.) It is important to notice that the tannakian Galois group is an algebraic object, but that the construction of the smaller

*Institut de France (Académie des Sciences) and Laboratoire Emile Picard.

†Laboratoire Emile Picard, CNRS UMR 5580, U.F.R. M.I.G., 118, route de Narbonne, 31062 Toulouse CEDEX 4, France.

group is based upon *transcendental techniques* (topology and complex analysis).

As in the differential case, the construction of the fundamental group is a Russian-dolls construction using semi-direct products (heuristically, going from interior to exterior, each new "slice of infinitesimal neighborhood" of the origin (each "scale") corresponds to an invariant subgroup in a new semi-direct product). At the end there is a fascinating parallel between the differential and the $q$-difference case. However, it has been impossible (for us) to mimick the differential approach (essentially based upon the concept of *solutions*); instead, we shall follow a new path. In order to understand our approach and our results in the $q$-difference case, it can be useful (even if *not indispensable*) to have some ideas about what happens in the differential case (in this case the "fundamental group" is the *wild fundamental group* introduced by the first author). In this introduction, we shall detail only the simplest case, the local case of regular singular linear differential equations, it will be our basic model. For the convenience of the reader, we shall recall the general differential situation in the next section (without proof, but with precise references); we shall insist on the underlying geometric ideas. The reader can choose to skip this section if he prefers (we shall not use the corresponding results in our paper, only their flavour).

We shall use *tannakian categories* as an essential tool (cf. [5], [31], §6, page 67). First recall some basic facts. Assume $\mathcal{E}$ is a *neutral tannakian category*, with fiber functor $\omega$ (to the category of $\mathbf{C}$-vector spaces). Then $Aut^{\otimes}(\omega)$ has a structure of *complex pro-algebraic affine group scheme*; we shall call it the tannakian group of the tannakian category $\mathcal{E}$. The category $\mathcal{E}$ is isomorphic to the category of finite dimensional representations [1] of $Aut^{\otimes}(\omega)$ (by *definition*, such a representation factors through a representation of one of the algebraic quotients). Conversely, if $G$ is a complex pro-algebraic group, its category of complex representations $Rep_{\mathbf{C}}(G)$ is a neutral tannakian category with a natural fiber functor $\omega_G$ (the obvious forgetful functor) and $G = Aut^{\otimes}(\omega_G, Rep_{\mathbf{C}}(G))$; the complex space $\mathbf{aut}^{\otimes}(\omega_G, Rep_{\mathbf{C}}(G))$ of Lie-like $\otimes$-endomorphisms of the fiber functor $\omega_G$ is the Lie-algebra of $Aut^{\otimes}(\omega_G, Rep_{\mathbf{C}}(G))$.

Assuming that $\Gamma$ is a *finitely generated* group, a *pro-algebraic completion* of $\Gamma$ is, by definition, a *universal* pair $(\iota_{al}, \Gamma^{al})$ where $\iota_{al} : \Gamma \to \Gamma^{al}$ is a group homomorphism from $\Gamma$ to a *pro-algebraic group* $\Gamma^{al}$. It is unique up to an isomorphism of pro-algebraic groups. A finite dimensional representation of $\Gamma$ clearly factors through a pro-algebraic completion of $\Gamma$. We can get a pro-algebraic completion of $\Gamma$ from the tannakian mechanism: $Rep_{\mathbf{C}}(\Gamma)$ is a neutral tannakian category with a natural fiber functor (the obvious forgetful functor) $\omega$, and the group $G = Aut^{\otimes}(\omega, Rep_{\mathbf{C}}(\Gamma))$ is a pro-algebraic completion of $\Gamma$. The groups $\Gamma$ and $G$ have the same representations, more precisely the natural homomorphism of groups $\Gamma \to G$ induces an isomorphism of tannakian categories: $Rep_{\mathbf{C}}(G) \to Rep_{\mathbf{C}}(\Gamma)$. We shall encounter below similar situations associated

[1] Each time we speak of representations of a pro-algebraic group, they are tacitly assumed to be morphisms for the pro-algebraic structure (i.e. *rational* representations).

to different classification problems (in a little more general setting: $\Gamma$ will not be in general a finitely generated group).

The first example (our baby example) is the category of local meromorphic regular-singular connections, or equivalently the category $\mathcal{D}_f^{(0)}$ of regular singular $\mathcal{D}$-modules, where $\mathcal{D} = \mathbf{C}(\{z\})[d/dz]$ ($\mathbf{C}(\{z\})$ is the field of fractions of $\mathbf{C}\{z\}$). A meromorphic connection is equivalent to an equivalence class of differential systems $\Delta_A : \frac{dY}{dx} = AY$ up to the gauge-equivalence: $\Delta_A \sim \Delta_B$ if and only if there exists $P \in Gl_n(\mathbf{C}(\{z\}))$ such that $B = P^{-1}AP - P^{-1}dP/dx$. We consider the fundamental group $\pi_1(D^*, d)$ of a germ at zero of punctured disk, pointed on a germ of direction $d$. We choose a generator $\gamma$ (a one turn loop in the positive sense) and we get an isomorphism $\mathbf{Z} \to \pi_1(D^*, d)$, $n \mapsto \gamma^n$. Then, by a very simple application of the Riemann-Hilbert correspondance, our category $\mathcal{D}_f^{(0)}$ is equivalent (via the *monodromy representation*) to the category of finite dimensional representations of the fundamental group $\pi_1(D^*, d)$. A regular singular $\mathcal{D}$-module $M$ corresponds to a representation $\rho_M$ of $\pi_1(D^*, d)$.

We can apply the tannakian machinery to the group $\Gamma = \mathbf{Z}$ (or equivalently to $\Gamma = \pi_1(D^*, d)$). Then our category $\mathcal{D}_f^{(0)}$ is equivalent to the category of representations of $\pi_1(D^*, d)$: a regular singular $\mathcal{D}$-module $M$ "is" a representation $\rho_M$ of the topological fundamental group $\pi_1(D^*, d)$, it is also equivalent to the category of representations of the pro-algebraic completion $\pi_1^{\otimes}(D^*, d)$ of $\pi_1(D^*, d)$: a regular singular $\mathcal{D}$-module "is" a representation $\rho_M^{\otimes}$ of the tannakian fundamental group $\pi_1^{\otimes}(D^*, d)$. The "small fundamental group" is the topological group $\pi_1(D^*, d)$, the "big fundamental group" is the pro-algebraic group $\pi_1^{\otimes}(D^*, d)$. The small group is *Zariski-dense* in the big group: the image of $\rho_M$ is the monodromy group of $M$, it is Zariski-dense in the image of $\rho_M^{\otimes}$ which "is" the differential galois group of $M$.

It is not difficult but important to understand the classification mechanism on this baby example: all the information is hidden in the group $\mathbf{Z}$ and we must extract it. The essential point is to understand the structure of the pro-algebraic completion of $\mathbf{Z}$. We can use the tannakian machinery (this is "folklore knowledge", a reference is [27]): the pro-algebraic hull of $\mathbf{Z}$ is $\mathbf{Z}^{al} = Aut^{\otimes}(\omega)$, it is commutative and the product of its semi-simple part $\mathbf{Z}_s^{al}$ and its unipotent part $\mathbf{Z}_u^{al}$: $\mathbf{Z}_s^{al} = Hom_{gr}(\mathbf{C}^*, \mathbf{C}^*)$, $\mathbf{Z}_u^{al} = \mathbf{C}$ (the additive group) and $\iota_{al} : \mathbf{Z} \to \mathbf{Z}^{al}$ is defined by $1 \mapsto (id_{\mathbf{C}^*}, 1)$ ($n \mapsto ((z \mapsto z^n), n)$). In order to understand what will happen in more difficult situations, it is interesting to understand the pro-algebraic completion of $\mathbf{Z}$ using regular singular differential equations and differential galois theory. We shall start from $\pi_1(D^*, d)$ and shall "compute" its pro-algebraic completion using Riemann-Hilbert correspondance. The main tool is a universal Picard-Vessiot algebra $\mathcal{U}_f$ ([14]). We consider some holomorphic functions on the Riemann surface of the logarithm: $\log x$ and $x^{\alpha} = e^{\alpha \log x}$ ($\alpha \in \mathbf{C}$). They generate over $\mathbf{C}(\{z\})$ a *differential algebra*: $\mathcal{U}_f = \mathbf{C}(\{z\})\{(x^{\alpha})_{\alpha \in \mathbf{C}}, \log\ x\}$ (it is a *simple* differential algebra; the brackets $\{\cdots\}$ mean "differential algebra generated by" ...). For each object $M$ of $\mathcal{D}_f^{(0)}$, the algebra $\mathcal{U}_f$

contains one and only one Picard-Vessiot algebra for $M$. Equivalently we can solve any regular-singular system $\Delta_A : \frac{dY}{dx} = AY$ using $\mathcal{U}_f$ (that is we can find a fundamental matrix solution with entries in $\mathcal{U}_f$). The differential Galois group $G_f$ of $\mathcal{U}_f$ (or equivalently of its field of fractions) is a pro-algebraic group ($\mathcal{U}_f$ is an inductive limit of finite type differential extensions). The monodromy, that is the action of the loop $\gamma$ is galois, therefore we can identify $\gamma$ with an element of $G_f$ ($\gamma(x^\alpha) = e^{2i\pi\alpha}x^\alpha$ and $\gamma(\log\, x) = \log\, x + 2i\pi$), and we get an injective homomorphism of groups: $\iota : \pi_1(D^*, d) \to G_f$. It is not difficult to prove that $(G_f, \iota)$ is a pro-algebraic completion of $\pi_1(D^*, d)$ and to compute $G_f$ (we shall admit these results here): its semi-simple part $G_{f,s} = \check{\mathbf{C}/\mathbf{Z}}$ is the topological dual group of $\mathbf{C}/\mathbf{Z}$ considered as the inductive limit of its finitely generated subgroups; its unipotent part $G_{f,u}$ is the differential Galois group of the extension $\mathbf{C}(\{z\})\{\log\, z\}$, that is the additive group $\mathbf{C}$. We have an exact sequence of groups:

$$0 \to \mathbf{Q}/\mathbf{Z} \to \mathbf{C}/\mathbf{Z} \to \mathbf{C}/\mathbf{Q} \to 0$$

and an exact sequence of dual groups:

$$1 \to \mathbf{T}_f \to G_{f,s} \to \hat{\mathbf{Z}}(1) \to 1.$$

(The proalgebraic group $\mathbf{T}_f$ is the topological dual group of the group of "monodromy exponents"). Here, $G_{f,s} = Hom_{gr}(\mathbf{C}/\mathbf{Z}, \mathbf{C}^*) \approx Hom_{gr}(\mathbf{C}^*, \mathbf{C}^*)$. The respective images of $\gamma$ in $G_{f,s}$ and in $G_{f,u}$ are $z \mapsto z$ and $2i\pi$.

The aim of this paper is to describe $q$-analogues of the differential fundamental groups. The construction is independant of the construction of the differential case; yet, like in that case it is done in three steps: (1) regular-singular or fuchsian equations, (2) formal or pure equations, (3) arbitrary equations meromorphic at the origin. We shall limit ourselves to the integral slopes case (cf. some comments below). (The reader of section 2 will recognise the main actors of the differential case under various disguises.) The first two steps are already well known and the new and difficult part is the last one.

Notations. We fix $q \in \mathbf{C}$ such that $|q| > 1$ and write $q = e^{-2i\pi\tau}$, Im $\tau > 0$.

(1) We begin with the *regular singular case*: a germ of meromorphic system at the origin $\sigma_q Y = AY$ is regular singular if and only if it is meromophically equivalent to a *fuchsian* system $\sigma_q Y = BY$ ($B(0) \in Gl_n(\mathbf{C})$). We call the corresponding category $\mathcal{E}_f^{(0)}$ the category of *fuchsian modules*, its tannakian Galois group is isomorphic to $Hom_{gr}(\mathbf{E}_q, \mathbf{C}^*) \times \mathbf{C}$, where $\mathbf{E}_q = \mathbf{C}^*/q^{\mathbf{Z}}$ is (the underlying abstract group of) the elliptic curve associated to $q$ (cf. [27] 2.2.2). There exists also a "small group" Zariski dense in the tannakian Galois group, and one can guess it using a $q$-analogy: the image of $\mathbf{Z}$ in $Hom_{gr}(\mathbf{C}^*, \mathbf{C}^*)$ is the subgroup of group homomorphisms which are algebraic group homomorphisms, therefore it is natural to consider the subgroup $\Pi$ of the elements of $Hom_{gr}(\mathbf{E}_q, \mathbf{C}^*)$ which are *continuous*. We use the decomposition $\mathbf{C}^* = \mathbf{U} \times q^{\mathbf{R}}$ ($\mathbf{U} \subset \mathbf{C}^*$ is the unit circle) and we denote $\gamma_1, \gamma_2 \in Hom_{gr}(\mathbf{E}_q, \mathbf{C}^*)$ the continuous group

homomorphisms defined respectively by $uq^y \mapsto u$ and $uq^y \mapsto e^{2i\pi y}$. Then $\Pi$ is generated by $\gamma_1$ and $\gamma_2$ and is Zariski-dense in $Hom_{gr}(\mathbf{C}^*, \mathbf{C}^*)$, the "fundamental group" of the category $\mathcal{E}_f^{(0)}$ (the local fundamental group) $\pi_{1,q,f}$ is by definition the subgroup of $Hom_{gr}(E_q, \mathbf{C}^* \times \mathbf{C})$ whose semi-simple component is generated by $\gamma_1$ and $\gamma_2$ and whose unipotent component is $\mathbf{Z}$ (cf. [27] 2.2.2).

(2) The next step is the study of the category $\mathcal{E}_{form}$ of *formal* $q$-difference modules. We shall limit ourselves to the integral slope case: the category $\mathcal{E}_{form,int}$ (or equivalently of the category $\mathcal{E}_{p,1}^{(0)}$ of pure meromorphic modules with integral slopes, cf. below). It is a neutral tannakian category. As in the differential case, in order to compute the corresponding "fundamental groups", it is necessary to understand the formal classification of $q$-difference equations of order one: two such equations $\sigma_q y - \hat{a} y = 0$ and $\sigma_q y - \hat{b} y = 0$ ($\hat{a}, \hat{b} \in \mathbf{C}((z))^*$) are formally equivalent if and only if $a^{-1}b \in \sigma_{q,\log} \mathbf{C}((z))$, where $\sigma_{q,\log}\hat{f} = \sigma_q(\hat{f})/\hat{f}$. Then the order one equations are classified by the abelian group $\mathbf{C}^*/q^{\mathbf{Z}} \times (z^m)_{m\in\mathbf{Z}} \simeq \mathbf{E}_q \times \mathbf{Z}$ ($\mathbf{E}_q$ correspond to the fuchsian equations, $(z^m)_{m\in\mathbf{Z}}$ to irregular equations). The "basic" irregular equation is $\sigma_q y - zy = 0$, it admits the Jacobi theta function $\theta_q$ as a solution (cf. below) and its $q$-difference Galois group is isomorphic to $\mathbf{C}^*$. Then one can prove that the tannakian Galois group $G_{form,int}$ of the category $\mathcal{E}_{form,int}$ is isomorphic to the topological dual group of $\mathbf{E}_q \times \mathbf{Z}$ (where $\mathbf{E}_q$ is interpreted as the inductive limit of its finitely generated subgroups), that is to $\mathbf{C}^* \times (Hom_{gr}(\mathbf{E}_q, \mathbf{C}^*) \times \mathbf{C})$: $\mathbf{C}^*$ is by definition the *theta torus*, it is the $q$-analogue of the exponential torus, cf. section 2, below. (The tannakian Galois group $G_{p,1}^{(0)}$ of the category $\mathcal{E}_{p,1}^{(0)}$ of pure meromorphic modules with integral slope is isomorphic to $G_{form,int}$.) We do not know what will happen in the non integral slope case.

(3) The last step and the main purpose of this paper is the study of the category $\mathcal{E}_1^{(0)}$ of $q$-difference modules whose Newton polygon admits only integral slopes. It is a neutral tannakian category, we shall prove that there exists a *semi-direct* decomposition of its tannakian Galois group $G_1^{(0)} = \mathfrak{St} \rtimes G_{p,1}^{(0)}$, where $\mathfrak{St}$ is a unipotent pro-algebraic group, and we shall describe the Lie algebra $\mathfrak{st}$ of $\mathfrak{St}$: like in the differential case this Lie algebra is a "pro-algebraic completion" of a *free* complex Lie algebra generated by a family of "$q$-alien derivations": $\big(\dot{\Delta}_{\bar{c}}^{(\delta)}\big)_{\delta\in\mathbf{N}^*, \bar{c}\in\mathbf{E}_q}$.

These $q$-alien derivations are indexed by labels $(\delta, \mathbf{c})$ (which are the $q$-analogs of the labels $(q, \mathbf{d})$ of the differential case):$\delta$ is by definition a weight on the $\theta$-torus $\mathbf{C}^*$ (that is, an element of the topological dual group $\mathbf{Z}$; actually, only the $\delta > 0$ have a non trivial action, so that we harmlessly take $\delta \in \mathbf{N}^*$), and $\mathbf{c}$ is a pair formed by $c \in E_q$ (*i.e.* a $q$-direction, representing a germ of $q$-spiral at the origin) and an element $\xi$ of the $q$-local fundamental group $\pi_{1,q,f}$. In order to define the $q$-alien derivations, we will use, as in the differential case, some summability tools (here, an algebraic version of the $q$-multisummability due to the second author), but the approach will be different: we will no longer use solutions but replace them by fiber functors ($\otimes$-functors). We will deal with meromorphic families of Lie-like automorphisms of fiber functors (the variable being the $q$-direction of summability) and extract their singularities by a residue process. This will give birth to $q$-alien derivations $\dot{\Delta}_c$.

In this paper, we define the $q$-alien derivations in all generality and compute them in the one-level case using a $q$-Borel transform (of some convenient order). This relates alien derivations to the irregularity invariants introduced in [22] and proves that, in this case, $q$-alien derivations are *a complete set of irregularity invariants.* We shall extend these results to the general case in a forthcoming paper [21]. The principle is similar but it is necessary to introduce a double family of categories "interpolating" between $\mathcal{E}_1^{(0)}$ and respectively $\mathcal{E}_{form}$ and $\mathcal{E}_{p,1}^{(0)}$, in relation with slopes and $q$-Gevrey estimates. With these tools, we are able to prove that, in the general case also, $q$-alien derivations are a complete set of irregularity invariants and that the $q$-resurgence group is Zariski dense in $\mathfrak{St}$. (The reader can check as an exercise that, in the general case, for an isoformal family of meromorphic $q$-difference modules $M$, the dimensions of the $\mathbf{C}$-vector space of the irregular invariants of [22] and of the $\mathbf{C}$-vector space generated by the "acting" $q$-alien derivations are equal: they are equal to the area of the "closed Newton polygon" of $M$).

# 2 The differential case

As we explained in the introduction, we will give in this section, for the convenience of the reader, a description of the wild fundamental group in the differential case. We will *not use* these results later.

After the study of the category of local meromorphic regular-singular connections (cf. the introduction), the next step is the study of the category of *formal* connections, or equivalently the category $\mathcal{D}_{form}$ of $\hat{\mathcal{D}}$-modules, where $\hat{\mathcal{D}} = \mathbf{C}((z))[d/dz]$ ($\mathbf{C}((z))$ is the field of fractions of $\mathbf{C}[[z]]$). It is a neutral tannakian category (cf. for instance [14]). One associates to a $\hat{\mathcal{D}}$-module $M$ its *Newton polygon* $N(M)$. The slopes of $N(M)$ are positive rational numbers. For sake of simplicity we shall limit ourselves to the full subcategory $\mathcal{D}_{form,int}$ of modules $M$ whose Newton polygon has only *integer* slopes. In order to compute the corresponding "fundamental groups", it is necessary to understand the formal classification of differential equations of order one: two such equations $dy/dx - \hat{a}y = 0$ and $dy/dx - \hat{b}y = 0$ ($\hat{a}, \hat{b} \in \mathbf{C}((z))$) are formally equivalent if and only if $(b-a)dx \in d_{log}\ \mathbf{C}((z))$ ($d_{log}\ \mathbf{C}((z)) = \{d\hat{c}/\hat{c}|\ \hat{c} \in \mathbf{C}((z))\}$). We have $d_{log}\ \mathbf{C}((z)) = \mathbf{Z}\frac{dx}{x} \oplus \mathbf{C}[[z]]dx$ and $\mathbf{C}((z))dx/d_{log}\ \mathbf{C}((z)) = \mathbf{C}/\mathbf{Z}\frac{dx}{x} \oplus \mathbf{C}((z))/z^{-1}\mathbf{C}[[z]]dz$. By integration $a\,dx \in \mathbf{C}((z))dx$ gives $q = \int a\,dx$ and we get an isomorphism between $\mathbf{C}((z))dx/d_{log}\ \mathbf{C}((z))$ and $\mathbf{C}/\mathbf{Z}\ \log\ x \oplus \mathbf{C}((z))/\mathbf{C}[[z]] = \mathbf{C}/\mathbf{Z}\ \log\ x \oplus \frac{1}{z}\mathbf{C}[\frac{1}{z}]$. To $\alpha \log\ x \in \mathbf{C}/\mathbf{Z}\ \log\ x$ corresponds $e^{\alpha \log\ x} = x^{\alpha}$ (a solution of $dy/dx - \alpha y = 0$), to $q \in \frac{1}{z}\mathbf{C}[\frac{1}{z}]$ corresponds $e^q$ (a solution of $dy/dx - q'y = 0$). Therefore it is natural to introduce the differential algebra $\mathcal{U}_{form,int} = \mathbf{C}((z))\{(x^{\alpha})_{\alpha\in\mathbf{C}}, (e^{q})_{q\in\frac{1}{z}\mathbf{C}[\frac{1}{z}]}, \log\ x\}$. It is a universal Picard-Vessiot algebra for the formal connections whose Newton polygons have only integer slopes and the differential Galois group of $\mathcal{U}_{form,int}$ is isomorphic to the tannakian Galois group of the category $\mathcal{D}_{form,int}$. We consider $\frac{1}{z}\mathbf{C}[\frac{1}{z}]$ as a $\mathbf{Z}$-module. It has no torsion, it is an infinite dimensional lattice and we consider it as the inductive limit of its finite dimensional sublattices. The topological dual group of such a sublattice is a *torus* (an algebraic group isomorphic to some $(\mathbf{C}^*)^{\mu}$), therefore the dual of $\frac{1}{z}\mathbf{C}[\frac{1}{z}]$ is a pro-torus; by definition it is the *exponential torus* $\mathbf{T}_{exp,int}$ (integral slopes case). Then the tannakian Galois group $\pi^{\otimes}_{1,form,int}$ of the category $\mathcal{D}_{form,int}$ is isomorphic to the product of the exponential torus $\mathbf{T}_{exp,int}$ by the fuchsian group $Hom_{gr}(\mathbf{C}^*, \mathbf{C}^*) \times \mathbf{C}$: this is the "big fundamental group"; the "small fundamental group" is the product of the exponential torus $\mathbf{T}_{exp,int}$ by the topological fundamental group $\pi_1(D^*, d)$ (be careful, the product decompositions *are not canonical*). In the general case, without any restriction on the slopes, it is necessary to enlarge the universal algebra (replacing the variable $x$ by all its ramifications $t^m = x$, $m \in \mathbf{N}^*$). Then there is a *non trivial* action of $\gamma$ on the $\mathbf{Z}$-module $\bigcup_{m\in\mathbf{N}^*} \frac{1}{x^{1/m}}\mathbf{C}[\frac{1}{x^{1/m}}]$ (by monodromy) and therefore on its dual $\mathbf{T}_{exp}$, the exponential torus. Then we have in the general case *semidirect products*: $\pi^{\otimes}_{1,form} = \mathbf{T}_{exp} \rtimes (Hom_{gr}(\mathbf{C}^*, \mathbf{C}^*) \times \mathbf{C})$ and $\pi_{1,form} = \mathbf{T}_{exp} \rtimes \pi_1(D^*, d)$.

The last step is the study of the category of *meromorphic* connections, or equivalently the category $\mathcal{D}_{an}$ of $\mathcal{D}$-modules, where $\mathcal{D} = \mathbf{C}(\{z\})[d/dz]$. This step is *very difficult* and involves a lot of delicate and deep analysis. Here we shall only describe roughly the results (for more information one can read the original papers [10, 18, 19, 20], and for more details [14]). Heuristically the origin 0 in $\mathbf{C}$ has an "analytic infinitesimal neighborhood" and an "algebraic infinitesimal neighborhood", the algebraic neighborhood lying in the heart of the analytic neighborhood and being "very small" (cf. [6] and [8] for a detailed and precise presentation). The algebraic neighborhood corresponds to $\hat{\mathbf{Z}}(1) = \varprojlim_{n\in\mathbf{N}^*} \mu_n$ ($\mu_n$ is the group of complex $n$-th roots of the unity) considered as a *quotient* of $Hom_{gr}(\mathbf{C}^*, \mathbf{C}^*)$ (the unipotent component $\mathbf{C}$ corresponds to a "very very small" neighborhood of 0 in the heart of the algebraic neighborhood). The fuchsian torus $\mathbf{T}_f$ corresponds to a "part very near of the algebraic neighborhood". It remains to understand what happens in the "huge" region in the analytic neighborhood located between the algebraic neighborhood and the exterior, the "actual world" $\mathbf{C}^*$: one must imagine it as filled by "points" (that we shall label $(q, d)$ below, $d$ being a direction and $q$ a "parameter" of scale). Each point will be responsible for a "monodromy", the semi-simple part of this monodromy will be related to the exponential torus and its unipotent part will have an *infinitesimal generator*, which is a Galois derivation: we call it an *alien derivation* (and denote it $\dot{\Delta}_{q,d}$). It is possible to give a rigorous meaning to this heuristic description. There are various approaches, the more interesting for the study of $q$-analogues is the tannakian one (cf. [6]): one thinks to *fiber functors* as "points" and to *isomorphisms between fiber functors* as "paths" (*automorphisms* of fiber functors corresponding to "loops"). Here the paths are made of classical paths (analytic continuation) and new paths corresponding to *multisummability* of formal (divergent) power series (it is worth noticing that at the algorithmic level these two families of paths are in fact very similar: [7]). Heuristically when you have "sufficiently many points and loops", then the loops "fill" the tannakian Galois group (topologically in Zariski sense): this situation will correspond to the "fundamental group" (the small one).

Let us describe now the "points" of the "annulus of the infinitesimal neighborhood" between the algebraic neighborhood and the "exterior real world" $\mathbf{C}^*$. We shall first give the description and justify it later: the points will appear naturally from the analysis of the Stokes phenomena, that is from the construction of the paths. We remark that the infinite dimensional lattice $\frac{1}{z}\mathbf{C}[\frac{1}{z}]$ is the topological dual of its topological dual, the exponential torus $\mathbf{T}_{exp,int}$. Then each polynomial $q \in \frac{1}{z}\mathbf{C}[\frac{1}{z}]$ can be interpreted as a *weight* on the exponential torus: if $\tau \in \mathbf{T}_{exp,int}$, $\tau(e^q) = q(\tau)e^q$, $q : \mathbf{T}_{exp,int} \to \mathbf{C}^*$ is a morphism of pro-algebraic groups. The set of directions $d$ issued from the origin is parametrized by the unit circle $S^1$ (which we can identify with the boundary of $\mathbf{C}^*$, the real blow up of the origin in $\mathbf{C}$ corresponding to $r = 0$ in polar coordinates $(r, \theta)$). We shall call *degree of* $q$ its degree in $1/x$. If $a/x^k$ ($a \in \mathbf{C}^*, k \in \mathbf{N}^*$) is the monomial of highest degree of $q$, then it controls the growth or the decay of $e^q$ near the origin (except perhaps on the family of $2k$ "oscillating lines": $\Re(a/x^k) = 0$, classically named Stokes lines or, better..., anti-Stokes lines), we have $k$ open sectors of *exponential decay* (of order $k$) of $e^q$ and $k$ open sectors of *exponential growth* (of order $k$) of $e^q$. To each pair $(q,d) \in \frac{1}{z}\mathbf{C}[\frac{1}{z}] \times S^1$ such that the direction $d$ bisects a sector of decay of $e^q$ we associate a label $(q,d)$: the labels will correspond to the points in the "terra incognita", our mysterious annulus. We introduce on $\frac{1}{z}\mathbf{C}[\frac{1}{z}] = \check{\mathbf{T}}_{exp,int}$ the filtration by the degree $k$ (it corresponds to the slope filtration associated to the Newton polygon in the formal category). Heuristically, if $k = \deg q$, then the corresponding point $(q,d)$ "belongs" to the direction $d$ and if $k$ is "big" this point is far from the algebraic neighborhood and near of the exterior world $\mathbf{C}^*$. (To each $k \in \mathbf{N}^*$ corresponds a "slice" isomorphic to $\mathbf{C}^*$, an annulus. If $k > k'$, then the $k'$-annulus is "surrounded" by the $k$ annulus, and "very small" compared to it [17]). We shall actually need points on the "universal covering" of our annulus. They are labelled by the $(q, \mathbf{d})$, where $\mathbf{d}$ is a direction above $d$ on the Riemann surface of the logarithm.

In order to describe the "paths", we need the notion of *multisummability* ([9], [1], [14], [16]). Let $\hat{f} \in \mathbf{C}((z))$; we shall say that it is *holonomic* if there exists $D \in \mathcal{D} = \mathbf{C}(\{z\})[d/dx]$ such that $Df = 0$. The set of holonomic power series expansions is a sub-differential *algebra* $\mathcal{K}$ of $\mathbf{C}((z))$ (containing $\mathbf{C}(\{z\})$) and there is a family of summation operators $(S_d^{\pm})_{d \in S^1}$ ($-$ is for "before $d$" and $+$ is for "after $d$" when one turns on $S^1$ in the positive sense): $S_d^{\pm} : \mathcal{K} \to O_d$ (where $O_d$ is the algebra of germs of holomorphic functions on sectors bisected by $d$), these operators are *injective homomorphisms of differential algebras*, their restriction to $\mathbf{C}(\{z\})$ is the classical sum of a convergent power series and $S^{\pm}(\hat{f})$ admits $\hat{f}$ as an *asymptotic expansion*; moreover, for a fixed $\hat{f}$, the two summations $S_d^+$ and $S_d^-$ coïncide, except perhaps for a *finite* set of *singular directions*; when $d$ moves between two singular directions the sums $S_d^+(\hat{f}) = S_d^-(\hat{f})$ glue together by *analytic continuation*. When $d$ crosses a singular line, there is a jump in the sum: this is the *Stokes phenomenon*. We consider now the differential algebra $\mathcal{U}_{an} = \mathcal{K}\{(x^\alpha)_{\alpha\in\mathbf{C}}, (e^q)_{q\in\frac{1}{z}\mathbf{C}[\frac{1}{z}]}, \log x\}$, it is the *universal differential algebra* associated to the family of germs of meromorphic connections. There are natural extensions of the operators $S_d^\pm$ to $\mathcal{U}_{an}$, but we have to be careful: we must define $S_d^\pm(\log x)$ and $S_d^\pm(x^\alpha)$. In order to do that we need to choose a branch of the logarithm in a germ of sector bisected by $d$ ($x^\alpha = e^{\alpha \log x}$). This corresponds to the choice of a direction $\mathbf{d}$ above $d$ on the Riemann surface of the logarithm ($\mathbf{d} \in (\mathbf{R},0)$, which is the universal covering of $(S^1,1)$). In the end, we get a family of summation operators $(S_\mathbf{d}^\pm)_{\mathbf{d}\in\mathbf{R}} : \mathcal{U}_{an} \to O_d$; they are *injective homomorphisms of differential algebras*.

Let $\Delta : \frac{dY}{dx} = AY$ be a germ of meromorphic system at the origin (integral slopes case). It admits a *formal fundamental matrix solution* $\hat{F} : \frac{d\hat{F}}{dx} = A\hat{F}$. The entries of $\hat{F}$ belongs to the universal algebra $\mathcal{U}_{an}$, therefore $F_\mathbf{d}^+ = S_\mathbf{d}^+(\hat{F})$ and $F_\mathbf{d}^- = S_\mathbf{d}^-(\hat{F})$ are germs of *actual* fundamental solutions on germs of sectors bisected by $d$. We have $F_\mathbf{d}^+ = F_\mathbf{d}^- C_d$, where the *constant* matrix $C_d \in Gl_n(\mathbf{C})$ is a *Stokes matrix* (it is unipotent). The map $St_d = (S_\mathbf{d}^+)^{-1} S_\mathbf{d}^+$ induces an automorphism of the differential algebra $\mathbf{C}(\{z\})\{\hat{F}\}$, therefore it defines an element of the differential Galois group of the system $\Delta$. More generally $St_d = (S_\mathbf{d}^+)^{-1} S_\mathbf{d}^+$ is an automorphism of the simple differential algebra $\mathcal{U}_{an}$ and defines an element of the differential Galois group of this algebra. This element is pro-unipotent and we can define a *Galois derivation* $\dot{\Delta}_\mathbf{d}$ of $\mathcal{U}_{an}$ by $St_d = e^{\dot{\Delta}_\mathbf{d}}$; by definition, $\dot{\Delta}_\mathbf{d}$ is the *alien derivation* in the direction $\mathbf{d}$. Now there is a quite *subtle point* in our analysis: from the germ of meromorphic system $\Delta : dY/dx = AY$ we get a representation $\rho_{\Delta,form} : \pi_{1,form} \to Gl(V)$ and a family of Stokes automorphisms $(St_\mathbf{d}(\Delta) \in Gl(V))_{\mathbf{d}\in\mathbf{R}}$. This last datum is equivalent to the knowledge of the corresponding family of alien derivations $(\dot{\Delta}_\mathbf{d}(\Delta) \in End(V))_{\mathbf{d}\in\mathbf{R}}$. There is a natural action of the topological fundamental group on the family of alien derivations: $\gamma\dot{\Delta}_\mathbf{d}\gamma^{-1} = \dot{\Delta}_{\gamma(\mathbf{d})}$ ($\gamma(\mathbf{d})$ is a translation of $-2\pi$ of $\mathbf{d}$), therefore it is natural to introduce the semi-direct product $exp(*_{\mathbf{d}\in\mathbf{R}}\mathbf{C}\dot{\Delta}_\mathbf{d}) \rtimes (\gamma)$ (where $*_{\mathbf{d}\in\mathbf{R}}\mathbf{C}\dot{\Delta}_\mathbf{d})$ is the *free Lie-algebra* generated by the symbols $\dot{\Delta}_\mathbf{d}$ and $exp(*_{\mathbf{d}\in\mathbf{R}}\mathbf{C}\dot{\Delta}_\mathbf{d})$ its exponential group in a "good sense") and to observe that the connection defined by $\Delta$ "is" the representation of this group. We could stop here and be happy: why not decide that $exp(*_{\mathbf{d}\in\mathbf{R}}\mathbf{C}\dot{\Delta}_\mathbf{d}) \rtimes (\gamma)$ is the fundamental group for the meromorphic category? This *does not work*. Of course we have all the knowledge but in a bad form: to a connection we can associate a representation of our group, but conversely there are representations which do not come from a connection, the admissible representations are *conditionned*. The geometric meaning of the problem is clear: $St_d$ corresponds to a loop around a whole bunch of points: all the $(q,\mathbf{d})$ corresponding to all the $q \in \frac{1}{z}\mathbf{C}[\frac{1}{z}]$ admitting $d$ as a line of maximal decay for $e^q$ (we shall say in that case that $q$ is supported by $d$ and note $(q,\mathbf{d}) \in d$), but a "good" fundamental group must allow loops around each *individual* point $(q,\mathbf{d})$. It is not difficult to solve the problem; we know *a priori* that our representation must contain in some sense the answer, it remains "only" to extract it. The idea is quite natural: using the exponential torus we shall "vibrate" the alien derivation $\dot{\Delta}_{\mathbf{d}}$ and extracts the "Fourier coefficients" $\dot{\Delta}_{q,\mathbf{d}}$ (for $q$ supported by $d$). We introduce, in the Lie algebra of the differential Galois group $G_{an}$ of $\mathcal{U}_{an}$, the family $(\tau\dot{\Delta}_{\mathbf{d}}\tau^{-1})_{\tau\in\mathbf{T}_{exp,int}}$ (it is a family of Galois derivations), then we consider the "Fourier expansion" $\tau\dot{\Delta}_{\mathbf{d}}\tau^{-1} = \sum_{(q,\mathbf{d})\in d} q(\tau)\dot{\Delta}_{q,\mathbf{d}}$ (it makes sense because for each connection the sum is finite). The coefficients are also in the Lie algebra of the $G_{an}$, they are Galois derivations. Now we have won: we consider the free Lie algebra *Lie* $\mathcal{R} = *_{\mathbf{d}\in\mathbf{R},(q,\mathbf{d})\in d}\mathbf{C}\dot{\Delta}_{(q,\mathbf{d})}$ (it is, by definition, the *resurgence algebra*), and the corresponding exponential group $\mathcal{R}$ (it makes sense [9], it is by definition the *resurgence group*). We have an action of the formal fundamental group on the resurgence Lie algebra: $\gamma\dot{\Delta}_{(q,\mathbf{d})}\gamma^{-1} = \dot{\Delta}_{(q,\gamma(\mathbf{d}))}$, $\tau\dot{\Delta}_{(q,\mathbf{d})}\tau^{-1} = q(\tau)\dot{\Delta}_{(q,\mathbf{d})}$ ($\tau \in \mathbf{T}_{exp,int}$) and we get a semi-direct product $\mathcal{R} \rtimes \pi_{1,form,int} = exp(*_{\mathbf{d}\in\mathbf{R},(q,\mathbf{d})\in d}\mathbf{C}\dot{\Delta}_{(q,\mathbf{d})}) \rtimes (\mathbf{T}_{exp,int} \times (\gamma))$. The knowledge of a representation is equivalent to the knowledge of its restriction to the formal part and its "infinitesimal restriction" to the free Lie algebra. Now the objects of our category (the meromorphic connections) correspond to *unconditioned* representations (by representation we mean, of course, finite dimensional representation whose restriction to the exponential torus is a morphism). We have now a fundamental group (the small one), it is the *wild fundamental group* (this is in the integral slope case, but with small adaptations it is easy to build the wild fundamental group in the general case). What about the big fundamental group (that is the tannakian Galois group)? We can easily derive its description from the knowledge of the wild fundamental group. The first step is to build some sort of pro-algebraic completion of the resurgent Lie algebra *Lie* $\mathcal{R}$ (cf. [14]): if $\rho$ is a representation of our wild fundamental group, we can suppose that $V = \mathbf{C}^n$ and that the image of the exponential torus is *diagonal*, it follows that the corresponding "infinitesimal restriction" $\psi = L\rho$ to the resurgent Lie algebra satisfies *automatically* the two conditions:

1. $\psi(\dot{\Delta}_{(q,\mathbf{d})})$ is *nilpotent* for every $\dot{\Delta}_{(q,\mathbf{d})}$.

2. There are only finitely many $\dot{\Delta}_{(q,\mathbf{d})}$ such that $\psi(\dot{\Delta}_{(q,\mathbf{d})}) \neq 0$.

By definition, the pro-algebraic completion $(Lie\ \mathcal{R})^{alg}$ of the free Lie algebra *Lie* $\mathcal{R}$ is a projective limit of *algebraic* Lie algebras: $(Lie\ \mathcal{R})^{alg} = \varprojlim_{\psi} Lie\ \mathcal{R}/\mathrm{Ker}\ \psi$, where the projective limit is taken over all homomorphisms of $\mathbf{C}$-algebras $\psi : Lie\ \mathcal{R} \to End(V)$ (where $V$ is an arbitrary finite dimensional complex space) satisfying conditions (1) and (2). Each algebraic Lie algebra $Lie\ \mathcal{R}/\mathrm{Ker}\ \psi$ is the Lie algebra of a connected algebraic subgroup of $Gl(V)$. We can consider the projective limit of these subgroups, it is a pro-algebraic group (a kind of pro-algebraic completion of the resurgent group $\mathcal{R}$). We shall call it the resurgent pro-algebraic group and denote it $\mathcal{R}^{alg}$, its Lie algebra is $(Lie\ \mathcal{R})^{alg}$: $Lie\ \mathcal{R}^{alg} = (Lie\ \mathcal{R})^{alg}$. The action of $(\gamma)$ on *Lie* $\mathcal{R}$ gives an action on $Lie\ \mathcal{R}^{alg}$, this action can be extended "by continuity" to an action of $\pi_{1,f}^{\otimes} = Hom_{gr}(\mathbf{C}^*, \mathbf{C}^*) \times \mathbf{C}$, and, using the exponential, we get an action of $\pi_{1,f}$ on $\mathcal{R}^{alg}$; there is also clearly an action of the exponential torus $\mathbf{T}_{exp,int}$ on $\mathcal{R}^{alg}$. Finally, we get a semi-direct product $\mathcal{R}^{alg} \rtimes \pi^{\otimes}_{1,form,int}$, which is isomorphic to the tannakian group $\pi^{\otimes}_{1,an,int}$.

Let us end this section with a short comparison of the three parallel steps in the differential and in the $q$-difference cases.

The first step is a little bit more complicated in the $q$-difference case than in the differential case. The second step is a lot simpler (the exponential torus is replaced by a *theta torus* and, in the integral slope case, this theta torus is isomorphic to $\mathbf{C}^*$, therefore radically simpler than the exponential torus). For the last step, the proofs, that we will expose below, are less intuitive in the $q$-difference case, than in the differential case; but, in the end, the results are in some sense simpler: one of the essential simplifications is due to the fact that the $q$-resurgent group is *unipotent* (the differential resurgent group contains, on the contrary, a lot of $Sl_2$ pairs, because one can play with $q$ and $-q$, which exchange the sectors of growth and decay of $e^q, e^{-q}$).

# 3 Prerequisites (mostly from [27], [28], [22] and [29])

## 3.1 General facts

**Notations, general conventions.** We fix $q \in \mathbf{C}$ such that $|q| > 1$. We then define the automorphism $\sigma_q$ on various rings, fields or spaces of functions by putting $\sigma_q f(z) = f(qz)$. This holds in particular for the ring $\mathbf{C}\{z\}$ of convergent power series and its field of fractions $\mathbf{C}(\{z\})$, the ring $\mathbf{C}[[z]]$ of formal power series and its field of fractions $\mathbf{C}((z))$, the ring $\mathcal{O}(\mathbf{C}^*, 0)$ of holomorphic germs and the field $\mathcal{M}(\mathbf{C}^*, 0)$ of meromorphic germs in the punctured neighborhood of 0, the ring $\mathcal{O}(\mathbf{C}^*)$ of holomorphic functions and the field $\mathcal{M}(\mathbf{C}^*)$ of meromorphic functions on $\mathbf{C}^*$; this also holds for all modules or spaces of vectors or matrices over these rings and fields. For any such ring (resp. field) $R$, the $\sigma_q$-invariants elements make up the subring (resp. subfield) $R^{\sigma_q}$ of constants. The field of constants of $\mathcal{M}(\mathbf{C}^*, 0)$ and that of $\mathcal{M}(\mathbf{C}^*)$ can be identified with a field of elliptic functions, the field $\mathcal{M}(\mathbf{E}_q)$ of meromorphic functions over the complex torus $\mathbf{E}_q = \mathbf{C}^*/q^{\mathbf{Z}}$. We shall write $\overline{a} = \pi(a) \in \mathbf{E}_q$ for the image of $a \in \mathbf{C}^*$ by the natural projection $\pi : \mathbf{C}^* \to \mathbf{E}_q$, and $[a;q] = aq^{\mathbf{Z}} = \pi^{-1}(\overline{a}) \subset \mathbf{C}^*$ for the preimage of $\overline{a}$ in $\mathbf{C}^*$, a discrete $q$-spiral. These notations extend to subsets $A \subset \mathbf{C}^*$: $\overline{A} = \pi(A) \subset \mathbf{E}_q$ and $[A;q] = Aq^{\mathbf{Z}} = \pi^{-1}(\overline{A}) \subset \mathbf{C}^*$.

**Categories.** Let $K$ denote any one of the forementioned fields of functions. Then, we write $\mathcal{D}_{q,K} = K\left\langle \sigma, \sigma^{-1} \right\rangle$ for the Öre algebra of non commutative Laurent polynomials characterized by the relation $\sigma.f = \sigma_q(f).\sigma$. We now define the category of $q$-difference modules in three clearly equivalent ways:

$$\begin{aligned} & DiffMod(K,\sigma_q) \\ = \ & \{(E,\Phi) \ / \ E \text{ a } K\text{-vector space of finite rank }, \Phi : E \to E \ \text{ a } \sigma_q\text{-linear map}\} \\ = \ & \{(K^n,\Phi_A) \ / \ A \in GL_n(K), \Phi_A(X) = A^{-1}\sigma_q X\} \\ = \ & \{ \text{ finite length left } \mathcal{D}_{q,K}\text{-modules}\}. \end{aligned}$$

For instance, a morphism from $M_A = (K^n,\Phi_A)$ to $M_B = (K^n,\Phi_B)$, where $A \in GL_n(K)$ and $B \in GL_p(K)$, is a $F \in M_{p,n}(K)$ such that $(\sigma_q)FA = BF$. Then, $DiffMod(K,\sigma_q)$ is a **C**-linear abelian rigid tensor category, hence a tannakian category. Moreover, all objects in $DiffMod(K,\sigma_q)$ have the form $\mathcal{D}_{q,K}/\mathcal{D}_{q,K}P$. In the case of $K = \mathbf{C}(\{z\})$, the category $DiffMod(K,\sigma_q)$ will be written $\mathcal{E}^{(0)}$ (for "equations near 0").

**Vector bundles and fiber functors.** To any module $M_A$ in $\mathcal{E}^{(0)}$, one can associate a holomorphic vector bundle $\mathcal{F}_A$ over $\mathbf{E}_q$:

$$\mathcal{F}_A = \frac{(\mathbf{C}^*,0)\times \mathbf{C}^n}{(z,X)\sim(qz,A(z)X)} \to \frac{(\mathbf{C}^*,0)}{z\sim qz} = \mathbf{E}_q.$$

This is the usual construction from equivariant bundles except that the germ $(\mathbf{C}^*,0)$ is only endowed with the action of the semigroup $q^{-\mathbf{N}}$ instead of a group; correspondingly, the projection map is not a covering. The pullback $\tilde{\mathcal{F}_A} = \pi^*(\mathcal{F}_A)$ over the open Riemann surface $\mathbf{C}^*$ is the trivial bundle $\mathbf{C}^* \times \mathbf{C}^n$, but with an equivariant action by $q^{\mathbf{Z}}$. The $\mathcal{O}_{\mathbf{E}_q}$-module of sections of $\mathcal{F}_A$ (also written $\mathcal{F}_A$) is the sheaf over $\mathbf{E}_q$ defined by: $\mathcal{F}_A(V) = \{\text{solutions of } \sigma_q X = AX \text{ holomorphic over } \pi^{-1}(V)\}$. From these two descriptions, the following is immediate:

**Proposition 3.1** *This gives an exact faithful $\otimes$-functor $M_A \rightsquigarrow \mathcal{F}_A$ from $\mathcal{E}^{(0)}$ to the category $Fib(\mathbf{E}_q)$ of holomorphic vector bundles. Taking the fiber of $\tilde{\mathcal{F}_A}$ at $a \in \mathbf{C}^*$ yields a fiber functor $\omega_a^{(0)}$ on $\mathcal{E}^{(0)}$ over* **C**.

**Newton polygon.** Any $q$-difference module $M$ over $\mathbf{C}(\{z\})$ or $\mathbf{C}((z))$, can be given a Newton polygon $N(M)$ at 0, or, equivalently, a Newton function $r_M$ sending the slope [2] $\mu \in S(M) \subset \mathbf{Q}$ to its multiplicity $r_M(\mu) \in \mathbf{N}^*$ (and the $\mu$ out of the support $S(M)$ to 0). For instance, the $q$-difference operator $L = qz\sigma^2 - (1+z)\sigma + 1 \in \mathcal{D}_{q,K}$ gives rise to the $q$-difference equation $qz\sigma_q^2 f - (1+z)\sigma_q f + f = 0$, of which the so-called Tschakaloff series $\sum_{n\geq 0} q^{n(n-1)/2} z^n$ is a solution (it is a natural $q$-analogue of the Euler series). By vectorisation, this equation gives rise to the system $\sigma_q X = AX$, where $A = \begin{pmatrix} z^{-1} & z^{-1} \\ 0 & 1 \end{pmatrix}$, and to the module $M = M_A$. The latter is isomorphic to $\mathcal{D}_{q,K}/\mathcal{D}_{q,K}\hat{L}$, where $\hat{L} = \sigma^2 - (z+1)\sigma + z = (\sigma - z)(\sigma - 1)$ is the dual operator of $L$. We respectively attach to $\sigma - z$ and $\sigma - 1$ the slopes $-1$ and $0$ and take $S(M) = \{-1,0\}$, with multiplicities $r_M(-1) = r_M(0) = 1$.

[2] It should be noted that the slopes defined and used in the present paper are the *opposites* of the slopes defined in previous papers.

### 3.2 Filtration by the slopes

**The filtration and the associated graded module.** The module $M$ is said to be pure isoclinic of slope $\mu$ if $S(M) = \{\mu\}$ and fuchsian if moreover $\mu = 0$. Direct sums of pure isoclinic modules are called pure modules [3]: they are irregular objects without wild monodromy, as follows from [22] and [29]. The tannakian subcategory of $\mathcal{E}^{(0)}$ made up of pure modules is called $\mathcal{E}_p^{(0)}$. Modules with integral slopes also form tannakian subcategories, which we write $\mathcal{E}_1^{(0)}$ and $\mathcal{E}_{p,1}^{(0)}$.

It was proved in [27] that the category $\mathcal{E}_f^{(0)}$ of fuchsian modules is equivalent to the category of *flat* holomorphic vector bundles over $\mathbf{E}_q$ and that its Galois group $G_f^{(0)}$ is isomorphic to $Hom_{gr}(\mathbf{C}^*/q^{\mathbf{Z}}, \mathbf{C}^*) \times \mathbf{C}$ (here, as in the introduction, $Hom_{gr}$ means "morphisms of *abstract* groups"). Since objects of $\mathcal{E}_{p,1}^{(0)}$ are essentially $\mathbf{Z}$-graded objects with fuchsian components, the Galois group of $\mathcal{E}_{p,1}^{(0)}$ is $G_{p,1}^{(0)} = \mathbf{C}^* \times G_f^{(0)}$.

**Theorem 3.2** *[28] Let the letter $K$ stand for the field $\mathbf{C}(\{z\})$ (convergent case) or the field $\mathbf{C}((z))$ (formal case). In any case, any object $M$ of $DiffMod(K, \sigma_q)$ admits a unique filtration $(F_{\leq\mu}(M))_{\mu\in\mathbf{Q}}$ by subobjects such that each $F_{(\mu)}(M) = \frac{F_{\leq\mu}(M)}{F_{<\mu}(M)}$ is pure of slope $\mu$ (thus of rank $r_M(\mu)$). The $F_{(\mu)}$ are endofunctors of $DiffMod(K, \sigma_q)$ and $gr = \bigoplus F_{(\mu)}$ is a faithful exact $\mathbf{C}$-linear $\otimes$-compatible functor and a retraction of the inclusion of $\mathcal{E}_p^{(0)}$ into $\mathcal{E}^{(0)}$. In particular, the functor $gr$ retracts $\mathcal{E}_1^{(0)}$ to $\mathcal{E}_{p,1}^{(0)}$.In the formal case, $gr$ is isomorphic to the identity functor.*

**Corollary 3.3** *For each $a \in \mathbf{C}^*$, the functor $\hat{\omega}_a^{(0)} = \omega_a^{(0)} \circ gr$ is a fiber functor.*

We shall consistently select an arbitrary basepoint $a \in \mathbf{C}^*$ and identify the Galois group $G^{(0)}$ as $Aut^{\otimes}\left(\hat{\omega}_a^{(0)}\right)$.

**Corollary 3.4** *The Galois group $G^{(0)}$ of $\mathcal{E}^{(0)}$ is the semi-direct product $\mathfrak{St} \rtimes G_p^{(0)}$ of the Galois group $G_p^{(0)}$ of $\mathcal{E}_p^{(0)}$ by a prounipotent group, the Stokes group $\mathfrak{St}$.*

***From now on, we only consider modules with integral slopes***.
Further studies would have to be based on the work [12] by van der Put and Reversat.

**Description in matrix terms.** We now introduce notational conventions which will be used all along this paper for a module $M$ in $\mathcal{E}_1^{(0)}$ and its associated graded module $M_0 = gr(M)$, an object of $\mathcal{E}_{p,1}^{(0)}$. The module $M$ may be given the shape $M = (\mathbf{C}(\{z\})^n, \Phi_A)$,

[3] It should be noted that we call in the present paper a pure isoclinic (resp. pure) module what was called a pure (resp. tamely irregular) module in previous papers.

with:

$$A = A_U \underset{def}{=} \begin{pmatrix} z^{\mu_1}A_1 & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & U_{i,j} & \dots \\ 0 & \dots & \dots & \dots & \dots \\ \dots & 0 & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & z^{\mu_k}A_k \end{pmatrix}, \tag{1}$$

where the slopes $\mu_1 < \cdots < \mu_k$ are integers, $r_i \in \mathbf{N}^*$, $A_i \in GL_{r_i}(\mathbf{C})$ $(i = 1,\dots,k)$ and

$$U = (U_{i,j})_{1\leq i<j\leq k} \in \prod_{1\leq i<j\leq k} \mathrm{Mat}_{r_i,r_j}(\mathbf{C}(\{z\})).$$

The associated graded module is then a direct sum $M_0 = P_1 \oplus \cdots \oplus P_k$, where, for $1 \leq i < j \leq k$, the module $P_i$ is pure of rank $r_i$ and slope $\mu_i$ and can be put into the form $P_i = (\mathbf{C}(\{z\})^{r_i}, \Phi_{z^{\mu_i}A_i})$. Therefore, one has $M_0 = (\mathbf{C}(\{z\})^n, \Phi_{A_0})$, where the matrix $A_0$ is block-diagonal (it is the same as $A_U$, with all $U_{i,j} = 0$).
We write $\mathfrak{G} \subset GL_n$ for the algebraic subgroup and $\mathfrak{g}$ for its Lie algebra, made up of matrices of the form:

$$F = \begin{pmatrix} I_{r_1} & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & F_{i,j} & \dots \\ 0 & \dots & \dots & \dots & \dots \\ \dots & 0 & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & I_{r_k} \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} 0_{r_1} & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & F_{i,j} & \dots \\ 0 & \dots & \dots & \dots & \dots \\ \dots & 0 & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & 0_{r_k} \end{pmatrix}. \tag{2}$$

For $F$ in $\mathfrak{G}$, we shall write $F[A] = \left(\sigma_q F\right) A F^{-1}$ the result of the gauge transformation $F$ on the matrix $A$. Theorem 3.2 entails:

$$\forall (U_{i,j})_{1\leq i<j\leq k} \in \prod_{1\leq i<j\leq k} \mathrm{Mat}_{r_i,r_j}(\mathbf{C}(\{z\}))\ ,\ \exists! \hat{F} \in \mathfrak{G}(\mathbf{C}((z)))\ :\ \hat{F}[A_0] = A_U.$$

This $\hat{F}$ will be written $\hat{F}_A$ (where $A = A_U$). The blocks $\hat{F}_{i,j}$ are recursively computed as follows. For $j < i$, $\hat{F}_{i,j} = 0$. For $j = i$, $\hat{F}_{i,j} = I_{r_i}$. Then, for $j > i$, one must solve the non homogeneous first order equation:

$$\sigma_q \hat{F}_{i,j} z^{\mu_j} A_j - z^{\mu_i} A_i \hat{F}_{i,j} = \sum_{i<k<j} U_{i,k} \hat{F}_{k,j} + U_{i,j}. \tag{3}$$

**Description of the Stokes group.** To go further, we choose to fix an arbitrary basepoint $a \in \mathbf{C}^*$ (see the corollary to theorem 3.2) and we identify the Galois groups accordingly:

$$G_1^{(0)} \underset{def}{=} Gal(\mathcal{E}_1^{(0)}) = Aut^{\otimes}(\hat{\omega}_a^{(0)}).$$

Recall from the quoted papers the action of the pure component $G_{p,1}^{(0)} = \mathbf{C}^* \times Hom_{gr}(\mathbf{C}^*/q^{\mathbf{Z}}, \mathbf{C}^*) \times \mathbf{C}$. We keep the notations above. For any $A$ with graded part $A_0$, an element $(\alpha,\gamma,\lambda) \in G_{p,1}^{(0)}$ yields the automorphism of $\hat{\omega}_a^{(0)}(A) = \omega_a^{(0)}(A_0) = \mathbf{C}^n$ given by the matrix:

$$\begin{pmatrix} \alpha^{\mu_1}\gamma(A_{s,1})A_{u,1}^{\lambda} & \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & 0 & \ldots \\ 0 & \ldots & \ldots & \ldots & \ldots \\ \ldots & 0 & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & \ldots & \alpha^{\mu_k}\gamma(A_{s,k})A_{u,k}^{\lambda} \end{pmatrix},$$

where we have written $A_i = A_{s,i}A_{u,i}$ the multiplicative Dunford decomposition (into a semi-simple and a unipotent factor that commute) and $\gamma$ acts on a semi-simple matrix through its eigenvalues. These matrices generate the group $G_{p,1}^{(0)}(A) \subset GL_n(\mathbf{C})$.

We then have a semi-direct decomposition: $G_1^{(0)} = \mathfrak{St} \rtimes G_{p,1}^{(0)}$, where the Stokes group $\mathfrak{St}$ is the kernel of the morphism $G_1^{(0)} \to G_{p,1}^{(0)}$. The group $\mathfrak{St}(A)$ is an algebraic subgroup of $\mathfrak{G}(\mathbf{C})$. The above matrix of $G_{p,1}^{(0)}(A)$ acts by conjugation on the matrix described by (2): the $F_{i,j}$ block is sent to $\alpha^{\mu_i-\mu_j}\gamma(A_{s,i})A_{u,i}^{\lambda}F_{i,j}\left(\gamma(A_{s,j})A_{u,j}^{\lambda}\right)^{-1}$. In particular, the group $\mathbf{C}^*$ acts on the "level $\delta$" upper diagonal $\mu_j - \mu_i = \delta$ (where $\delta \in \mathbf{N}$) by multiplication by $\alpha^{-d}$. The group $\mathfrak{St}(A)$ is *filtered* by the *normal* subgroups $\mathfrak{St}_\delta(A)$ defined by: $\mu_j - \mu_i \geq \delta$ (meaning that all blocks such that $0 < \mu_j - \mu_i < \delta$ vanish).

Likewise, the Lie algebra $\mathfrak{st}(A) = Lie\left(\mathfrak{St}(A)\right)$, which is a subalgebra of $\mathfrak{g}(\mathbf{C})$, admits an adjoint action described by the same formulas (this is because $\log PFP^{-1} = P \log FP^{-1}$). The algebra $\mathfrak{st}(A)$ is *graded* by its "level $\delta$" upper diagonals $\mathfrak{st}_\delta(A)$, defined by $\mu_j - \mu_i = \delta$. As noted in [29], the algebra $\mathfrak{st}_\delta(A)$ can be identified with the (group) kernel of the central extension $\mathfrak{St}(A)/\mathfrak{St}_{\delta+1}(A) \to \mathfrak{St}(A)/\mathfrak{St}_\delta(A)$.

## 3.3 Stokes operators

**Algebraic summation.** The following computations are extracted from [29]. We need the following theta function of Jacobi: $\theta_q(z) = \sum_{n\in\mathbf{Z}} q^{-n(n+1)/2} z^n$. It is holomorphic in $\mathbf{C}^*$ with simple zeroes, all located on the discrete $q$-spiral $[-1;q]$. It satisfies the functional equation: $\sigma_q\theta_q = z\theta_q$. We then define $\theta_{q,c}(z) = \theta_q(z/c)$ (for $c \in \mathbf{C}^*$); it is holomorphic in $\mathbf{C}^*$ with simple zeroes, all located on the discrete $q$-spiral $[-c;q]$ and satisfies the functional equation: $\sigma_q\theta_{q,c} = \frac{z}{c}\theta_{q,c}$.

For a given formal class described by $A_0$, $\mu_1,\ldots,\mu_k$ and $r_1,\ldots,r_k$ as above, and for any $c \in \mathbf{C}^*$, we introduce the matrix $T_{c,A_0} \in \mathrm{Mat}_n(\mathcal{M}(\mathbf{C}^*))$ which is block-diagonal with blocks $\theta_c^{-\mu_i} I_{r_i}$. Moreover, we shall assume the following normalisation due to Birkhoff and Guenther (see [22], [29]):

$$\forall i < j\ ,\ \text{all coefficients of } U_{i,j} \text{ belong to } \sum_{\mu_i \leq d < \mu_j} \mathbf{C} z^d.$$

Then, putting $A'_i = c^{\mu_i} A_i \in GL_{r_i}(\mathbf{C})$ and $U'_{i,j} = (z/c)^{-\mu_i} \theta_c^{\mu_j - \mu_i} U_{i,j} \in \mathrm{Mat}_{r_i,r_j}(O(\mathbf{C}^*))$, we have:

$$A'_{U'} \underset{def}{=} T_{c,A_0}[A_U] = \begin{pmatrix} A'_1 & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & U'_{i,j} & \dots \\ 0 & \dots & \dots & \dots & \dots \\ \dots & 0 & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & A'_k \end{pmatrix}.$$

Now, if the images in $\mathbf{E}_q$ of the spectra $Sp(A'_i)$ are pairwise disjoint, there is a unique $F' \in \mathfrak{G}(O(\mathbf{C}^*))$ such that $F'[A'_0] = A'_{U'}$. Its coefficients are recursively defined by the equations:

$$\sigma_q F'_{i,j} A'_j - A'_i F'_{i,j} = \sum_{i<k<j} U'_{i,k} F'_{k,j} + U'_{i,j}.$$

The unique solution of this equation in $O(\mathbf{C}^*)$ is obtained by taking the Laurent series:

$$F'_{i,j} = \sum_{p \in \mathbf{Z}} \Phi^{-1}_{q^p A'_j, A'_i}(V_p) z^p, \quad \Big(\sum_{p \in \mathbf{Z}} V_p z^p = \sum_{i<k<j} U'_{i,k} F'_{k,j} + U'_{i,j}\Big),$$

where one writes $\Phi_{B,C}(M) = MB - CM$ (that map is one to one if and only if $Sp(B) \cap Sp(C) = \emptyset$). Note for further use that the condition we have to impose on the spectra is the following:

$$(4) \qquad \forall i < j \;,\; q^{\mathbf{Z}} c^{\mu_i} Sp(A_i) \cap q^{\mathbf{Z}} c^{\mu_j} Sp(A_j) = \emptyset.$$

This is equivalent to requiring that $\overline{c} \notin \Sigma_{A_0}$, where $\Sigma_{A_0}$ is some explicit finite subset of $\mathbf{E}_q$. From the equalities $A'_{U'} = T_{c,A_0}[A_U]$, $A'_0 = T_{c,A_0}[A_0]$ and $F'[A'_0] = A'_{U'}$, we get at last $F[A_0] = A_U$, where $F = T^{-1}_{c,A_0} F' T_{c,A_0}$ can be easily computed: it belongs to $\mathfrak{G}(\mathcal{M}(\mathbf{C}^*))$ and $F_{i,j} = \theta_c^{\mu_i - \mu_j} F'_{i,j}$. The condition that the $F'_{i,j}$ are holomorphic over $\mathbf{C}^*$ is equivalent to the following condition:

(5) $\quad \forall i < j$ , $F_{i,j}$ has poles only on $[-c;q]$, and with multiplicities $\leq \mu_j - \mu_i$.

Then, we get the following conclusion: there is a *unique* $F \in \mathfrak{G}(\mathcal{M}(\mathbf{C}^*))$ such that condition (5) holds and $F[A_0] = A_U$. Note that the condition depends on $\overline{c} \in \mathbf{E}_q$ rather than $c$. To summarize the discussion:

**Proposition 3.5** *For every $c \in \mathbf{C}^*$ satisfying condition (4)* (i.e., $\overline{c} \notin \Sigma_{A_0}$*), there is a unique $F \in \mathfrak{G}(\mathcal{M}(\mathbf{C}^*))$ satisfying condition (5) and such that $F[A_0] = A_U$. We consider this $F$ as obtained by* summation of $\hat{F}_A$ in the direction $\overline{c} \in \mathbf{E}_q$ *and, accordingly, write it $S_{\overline{c}} \hat{F}_A$.*

If we now choose two *q-directions of summation* $\overline{c}, \overline{d} \in \mathbf{E}_q$, the ambiguity of summation is expressed by:

$$(6) \qquad S_{\overline{c},\overline{d}} \hat{F}_A \underset{def}{=} (S_{\overline{c}} \hat{F}_A)^{-1} S_{\overline{d}} \hat{F}_A.$$

This is a meromorphic automorphism of $A_0$. As explained in [22] and [29], it is a *Stokes operator*.

**Case of one level.** For further use, we now specialize some of the previous results to the case of two (integral) slopes $\mu < \nu$, and only one "level" $\delta = \nu - \mu \in \mathbf{N}^*$. For simplicity, we write our matrices:

$$M_0 = \begin{pmatrix} z^\mu A & 0 \\ 0 & z^\nu B \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} z^\mu A & z^\mu UB \\ 0 & z^\nu B \end{pmatrix}, \tag{7}$$

where $A \in GLr(\mathbf{C})$, $B \in GL_s(\mathbf{C})$ and $U \in \text{Mat}_{r,s}(\mathbf{C}_{\delta-1}[z])$ (polynomials with degree $< \delta$). It is clear that the upper right block can indeed be written in such a way. Then the unique element of $\mathfrak{G}(\mathbf{C}((z)))$ which sends $M_0$ to $M$ is the matrix $\begin{pmatrix} I_r & F \\ 0 & I_s \end{pmatrix}$, where $F$ is the unique element of $\text{Mat}_{r,s}(\mathbf{C}((z)))$ such that:

$$z^\delta \sigma_q F - \Lambda(F) = U, \tag{8}$$

Here, we have written $\Lambda(F) = AFB^{-1}$ (thus, an endomorphism of $\text{Mat}_{r,s}(\mathbf{C})$ and similar spaces). The formal solution $F$ can be computed by identification of coefficients, *i.e.* by solving:

$$\forall n \in \mathbf{Z}\ ,\ q^{n-\delta} F_{n-\delta} - AF_n B^{-1} = U_n. \tag{9}$$

Similarly, the unique element of $\mathfrak{G}(\mathcal{M}(\mathbf{C}^*))$ such that condition (5) holds which sends $M_0$ to $M$ is the matrix $\begin{pmatrix} I_r & F \\ 0 & I_s \end{pmatrix}$, where $F$ is the unique of $\text{Mat}_{r,s}(\mathcal{M}(\mathbf{C}^*))$ with poles only on $[-c;q]$ and with multiplicities $\leq d$ which is solution of equation (8). This is solved by putting $F = \theta_c^{-\delta} G$, so that $G$ is a solution holomorphic on $\mathbf{C}^*$ of the following equation:

$$c^\delta \sigma_q G - \Lambda(G) = V, \text{ where } G = \theta_c^\delta F \text{ and } V = \theta_c^\delta U. \tag{10}$$

This can be solved by identification of coefficients of the corresponding Laurent series, *i.e.* by solving:

$$\forall n \in \mathbf{Z}\ ,\ c^\delta q^n G_n - AG_n B^{-1} = V_n. \tag{11}$$

This is possible if $c^\delta q^{\mathbf{Z}} \cap Sp(A)/Sp(B) = \emptyset$, which is precisely condition (4) specialized to the present setting. Then we can take $G_n = (c^\delta q^n \text{Id} - \Lambda)^{-1} V_n$.

# 4 Stokes operators and alien derivations

## 4.1 Stokes operators are galois

We take on the notations of section 3.2 and consider moreover another object $B$, to which we apply similar notations: graded $B_0$, diagonal blocks $B_j$ corresponding to slopes $\nu_j$ with multiplicities $s_j$, etc.

**Lemma 4.1** *(i) Assume the following condition:*

(12) $$\forall i < i', j < j' \ , \ q^{\mathbf{Z}} c^{\mu_i + \nu_j} Sp(A_i) Sp(B_j) \cap q^{\mathbf{Z}} c^{\mu_{i'} + \nu_{j'}} Sp(A_{i'}) Sp(B_{j'}) = \emptyset.$$

*Then:*

(13) $$S_{\overline{c}} \hat{F}_{A \otimes B} = (S_{\overline{c}} \hat{F}_A) \otimes (S_{\overline{c}} \hat{F}_B).$$

*(ii) Assume the following condition:*

(14) $$\forall i, j \text{ such that } \mu_i < \nu_j \ , \ q^{\mathbf{Z}} c^{\mu_i} Sp(A_i) \cap q^{\mathbf{Z}} c^{\nu_j} Sp(B_j) = \emptyset.$$

*Then, for any morphism $F : A \to B$, writing $F_0 = grF$, we have:*

(15) $$F \ S_{\overline{c}} \hat{F}_A = S_{\overline{c}} \hat{F}_B \ F_0.$$

*Proof.* - (i) From elementary properties of the tensor product, we draw that the diagonal blocks of $A \otimes B$ are the $z^{\mu_i + \nu_j} A_i \otimes B_j$ and that $Sp(A_i \otimes B_j) = Sp(A_i) Sp(B_j)$; thus the right hand side of the equality is a morphism from $\mathrm{gr}(A \otimes B) = A_0 \otimes B_0$ to $A \otimes B$ satisfying condition (5) on poles: it has to be $S_{\overline{c}} \hat{F}_{A \otimes B}$.
(ii) From the functoriality of the filtration, we know that $F$ only has rectangular blocks relating slopes $\mu_i \leq \nu_j$, and that $F_0$ is made up of those such that $\mu_i = \nu_j$. It is sensible to call the latter "diagonal blocks". Then, the compositum $(S_{\overline{c}} \hat{F}_B)^{-1} F \ S_{\overline{c}} \hat{F}_A$ is a (meromorphic) morphism from $A_0$ to $B_0$, with diagonal $F_0$ (since $S_{\overline{c}} \hat{F}_A$ and $S_{\overline{c}} \hat{F}_B$ are in $\mathfrak{G}$) and with 0 under the diagonal. Any block $F_{i,j}$ such that $\mu_i < \nu_j$ has all its poles on $[-c; q]$, and with multiplicities $\leq \nu_j - \mu_i$; thus, $F_{i,j} = \theta_c^{\mu_i - \nu_j} F'_{i,j}$, where $F'_{i,j}$ is holomorphic on $\mathbf{C}^*$ and satisfies: $\sigma_q F'_{i,j} c^{\mu_j} A_j = c^{\mu_i} A_i F'_{i,j}$ The same computation (with the Laurent series) as in section 3.3 shows that, under condition (14), this implies $F'_{i,j} = 0$. Therefore $(S_{\overline{c}} \hat{F}_B)^{-1} F \ S_{\overline{c}} \hat{F}_A = F_0$ and (15) holds. □

In terms of the fiber functors introduced after theorem 3.2, the meaning of the above lemma is that, under proper restrictions to ensure that $S_{\overline{c}} \hat{F}_A$ is well defined at $a$,and that the nonresonancy conditions (12) and (14) hold for any pair of objects, $A \rightsquigarrow S_{\overline{c}} \hat{F}_A(a)$ is an $\otimes$-isomorphism from $\hat{\omega}_a^{(0)}$ to $\omega_a^{(0)}$. For any pure $A_0$, taking up the previous notations, we therefore define, first its "weighted spectrum" and singular locus:

$$\begin{aligned} \overline{WSp}(A_0) &= \text{ the subgroup of } \mathbf{E}_q \times \mathbf{Z} \text{ generated by } \bigcup_i \left( \overline{Sp(A_i)} \times \{\mu_i\} \right), \\ \tilde{\Sigma}(A_0) &= \bigcup_{\mu \neq 0} \{ \overline{c} \in \mathbf{E}_q \ / \ (\mu \overline{c}, \mu) \in \overline{WSp}(A_0) \}. \end{aligned}$$

**Proposition 4.2** *Let $< A >$ be the tannakian subcategory of $\mathcal{E}_1^{(0)}$ generated by A. Fix $\overline{c} \notin \tilde{\Sigma}(A_0)$ and $a \notin [-c; q]$. Then $B \rightsquigarrow S_{\overline{c}} \hat{F}_B(a)$ is an $\otimes$-isomorphism from $\hat{\omega}_a^{(0)}$ to $\omega_a^{(0)}$, both being restricted to $< A >$.*

*Proof.* - Apply the lemma and the formulas giving the slopes of linear constructions in [28]. □

Now we recall, from [29] that, for a pure object $A = A_0$, all the $S_{\overline{c}}\hat{F}_A$ are equal (they are indeed equal to the formal Stokes operator $\hat{F}_A$ which is actually analytic).

**Theorem 4.3** *With the same restrictions, fix an arbitrary $\overline{c_0} \notin \tilde{\Sigma}(A_0) \cup \{\overline{-a}\}$. Then, for all $\overline{c} \notin \tilde{\Sigma}(A_0) \cup \{\overline{-a}\}$, we have, using notation (6):*

$$S_{\overline{c_0},\overline{c}}\hat{F}_A(a) \in \mathfrak{St}(A).$$

*Proof.* - By the above proposition, it is in the Galois group; by the remark above, it is killed by the functor gr. □

**Corollary 4.4** *We get a family of elements of Lie-like automorphisms: $LS_{\overline{c},a}(A) \underset{def}{=} \log(S_{\overline{c_0},\overline{c}}\hat{F}_A(a)) \in \mathfrak{st}(A)$.*

Now, although the functoriality and $\otimes$-compatibility were proved only for $\overline{c} \notin \tilde{\Sigma}(A_0) \cup \{\overline{-a}\}$, the above formula is actually well defined for all $\overline{c} \notin \Sigma_{A_0} \cup \{\overline{-a}\}$. Moreover, from the explicit computation in section 3.3 (multiplications by powers of $\theta_c$ and resolution of recursive equations by inversion of $\Phi_{q^p A'_j, A'_i}$), we see that the mapping $\overline{c} \mapsto LS_{\overline{c},a}(A)$ is meromorphic on $\mathbf{E}_q$, with poles on $\Sigma_{A_0}$. Moreover, it takes values in the vector space $\mathfrak{st}(A)$ for all $\overline{c}$ except for a denumerable subset: therefore, it takes all its values in $\mathfrak{st}(A)$. Last, taking residues at a pole is an integration process and gives values in the same vector space.

**Theorem 4.5** *Define the q-alien derivations by the formula:*

$$\dot{\Delta}_{\overline{c}}(A) = Res_{\overline{d}=\overline{c}} LS_{\overline{d},a}(A).$$

*Then, $\dot{\Delta}_{\overline{c}}(A) \in \mathfrak{st}(A)$. (In order to alleviate the notation, we do not mention the arbitrary basepoint $a \in \mathbf{C}^*$.)*

Of course, for $\overline{c} \notin \Sigma_{A_0}$, we have $\dot{\Delta}_{\overline{c}}(A) = 0$. According to the graduation of $\mathfrak{st}$ described at the end of section 3.2, each alien derivation admits a canonical decomposition:

$$\dot{\Delta} = \bigoplus_{\delta \geq 1} \dot{\Delta}_{\overline{c}}^{(\delta)}, \tag{16}$$

where $\dot{\Delta}_{\overline{c}}^{(\delta)}(A) \in \mathfrak{st}_\delta(A)$ has only non null blocks for $\mu_j - \mu_i = \delta$.

**Theorem 4.6** *The alien derivations are Lie-like $\otimes$-endomorphisms of $\hat{\omega}_a^{(0)}$ over $\mathcal{E}_1^{(0)}$.*

*Proof.* - This means first that they are functorial; for all morphisms $F : A \to B$, one has:

$$\dot{\Delta}_{\overline{c}}(B) \circ \hat{\omega}_a^{(0)}(F) = \hat{\omega}_a^{(0)}(F) \circ \dot{\Delta}_{\overline{c}}(A).$$

Note that $\hat{\omega}_a^{(0)}(F) = F_0(a)$. First assume the previous restrictions on $\overline{c}$. Then, from the lemma 4.1, we get that $S_{\overline{c_0},\overline{c}}\hat{F}_B(a) \circ F_0(a) = F_0(a) \circ S_{\overline{c_0},\overline{c}}\hat{F}_A(a)$. Now, the logarithm of a unipotent matrix $P$ being a polynomial of $P$, we have $R \circ Q = Q \circ P \Rightarrow \log R \circ Q = Q \circ \log P$, so that we have $LS_{\overline{c},a}(B) \circ F_0(a) = F_0(a) \circ LS_{\overline{c},a}(A)$ and we take the residues on both sides. Now that the equality is established outside a denumerable set of values of $\overline{c}$, we can extend it to all values by holomorphy.
The assertion means, second, Lie-like $\otimes$-compatibility:

$$\dot{\Delta}_{\overline{c}}(A \otimes B) = 1 \otimes \dot{\Delta}_{\overline{c}}(B) + \dot{\Delta}_{\overline{c}}(A) \otimes 1,$$

where the left and right 1 respectively denote the identities of $\hat{\omega}_a^{(0)}(A)$ and $\hat{\omega}_a^{(0)}(B)$. This equality makes sense because $\hat{\omega}_a^{(0)}$ is itself $\otimes$-compatible. From the lemma 4.1 we get first that $LS_{\overline{c},a}(A \otimes B) = LS_{\overline{c},a}(A) \otimes LS_{\overline{c},a}(A)$. Then, we note that, for any two unipotent matrices $P$ and $Q$, the commuting product $P \otimes Q = (P \otimes 1)(1 \otimes Q) = (1 \otimes Q)(P \otimes 1)$ entails $\log(P \otimes Q) = (\log P) \otimes 1 + 1 \otimes (\log Q)$. The proof is then finished as above. □

## 4.2 Alien derivations and $q$-Borel transform

Let $\delta \in \mathbf{N}^*$ and assume that the matrix $A$ of (1) has only null blocks $U_{i,j}$ for $\mu_j - \mu_i < \delta$. Then, in the computation (3), we find the following equation for $\mu_j - \mu_i < \delta$: $\sigma_q \hat{F}_{i,j} z^{\mu_j} A_j - z^{\mu_i} A_i \hat{F}_{i,j} = 0$. Likewise, the upper diagonal blocks of any $S_{\overline{c}}\hat{F}_A$ satisfy exactly the same equations. These have no non trivial formal solution, neither non trivial meromorphic wih less than $(\mu_j - \mu_i)$ poles modulo $q^{\mathbf{Z}}$ (this follows from 3.3). Hence, as well $\hat{F}_A$ as all the summations $S_{\overline{c}}\hat{F}_A$ have null blocks $F_{i,j}$ for $0 < \mu_j - \mu_i < \delta$.
On level $\mu_j - \mu_i = \delta$, the equations to be solved are:

$$\sigma_q F_{i,j} z^{\mu_j} A_j - z^{\mu_i} A_i F_{i,j} = U_{i,j},$$

which is of the same type as those of 3.3. The properties of this *first non trivial level* of $\hat{F}_A$ and $S_{\overline{c}}\hat{F}_A$ will play a crucial role in [21]. Indeed, the logarithm $\log F$ has, as first non trivial level the same level $\delta$, and the corresponding diagonal is equal to that of $F$. Therefore, after taking residues, on gets straightaway the $\dot{\Delta}_{\overline{c}}^{(\delta)}$. To study it in some detail, we therefore take again the light notations of 3.3.

**Solving (8) with $q$-Borel transforms.** We consider $\delta \in \mathbf{N}^*$ as fixed, to alleviate notations. Let the Laurent series expansion:

$$\theta^{\delta} = \sum_{n \in \mathbf{Z}} t_n z^n.$$

Then, from the functional equation $\sigma_q \theta^\delta = z^\delta \theta^\delta$, we draw the recurrence relations:

$$\forall n \in \mathbf{Z}\,,\; t_{n-\delta} = q^n t_n.$$

From this, we get the useful estimation:

$$t_n \approx |q|^{-n^2/2\delta}.$$

The notation $u_n \approx v_n$ for positive sequences here means "same order of magnitude up to a polynomial factor", more precisely:

$$u_n \approx v_n \Longleftrightarrow \exists\, R > 0 \,:\, u_n = O(R^n v_n) \text{ and } v_n = O(R^n u_n).$$

Now, for any Laurent series $F(z) = \sum F_n z^n \in E \otimes \mathbf{C}[[z, z^{-1}]]$ with coefficients $F_n$ in some finite dimensional $\mathbf{C}$-vector space $E$, we define its $q$-Borel transform at level $\delta$ by the formula:

$$\mathcal{B}_q^{(\delta)} F(\xi) = \sum t_{-n} F_n \xi^n \in E \otimes \mathbf{C}[[\xi, \xi^{-1}]].$$

This transformation strongly increases the convergence properties; for instance, if $F \in E \otimes \mathbf{C}\{z\}$, then $\mathcal{B}_q^{(\delta)} F \in E \otimes \mathcal{O}(\mathbf{C})$, etc. Since we are interested in analyticity of $\mathcal{B}_q^{(\delta)} F$, we introduce conditions on the order of growth of coefficients, adapted from [15]. Let $G \in E \otimes \mathbf{C}((\xi)) = E \otimes \mathbf{C}[[\xi]][\xi^{-1}]$. We say that $G := \sum G_n \xi^n \in E \otimes \mathbf{C}(\{\xi\})_{q,\delta}$ if $\parallel G_n \parallel = O(R^n q^{-n^2/2d})$ for some $R > 0$. We say that $G \in E \otimes \mathbf{C}(\{\xi\})_{q,(\delta)}$ if $\parallel G_n \parallel = O(R^n q^{-n^2/2d})$ for all $R > 0$. In the case that, moreover, $G$ has no pole at 0 ($G \in \mathbf{C}[[\xi]]$), we respectively say that $G \in E \otimes \mathbf{C}\{\xi\}_{q,\delta}$, resp. $G \in E \otimes \mathbf{C}\{\xi\}_{q,(\delta)}$. Thus, we have the obvious equivalences:

$$\begin{aligned}
F \in E \otimes \mathbf{C}\{z\} &\iff \mathcal{B}_q^{(\delta)} F \in E \otimes \mathbf{C}\{\xi\}_{q,\delta},\\
F \in E \otimes \mathbf{C}(\{z\}) &\iff \mathcal{B}_q^{(\delta)} F \in E \otimes \mathbf{C}(\{\xi\})_{q,\delta},\\
F \in E \otimes \mathcal{O}(\mathbf{C}) &\iff \mathcal{B}_q^{(\delta)} F \in E \otimes \mathbf{C}\{\xi\}_{q,(\delta)},\\
F \in E \otimes \mathcal{O}(\mathbf{C})[z^{-1}] &\iff \mathcal{B}_q^{(\delta)} F \in E \otimes \mathbf{C}(\{\xi\})_{q,(\delta)}.
\end{aligned}$$

With the notations of equation (8), write $G = \mathcal{B}_q^{(\delta)} F = \sum G_n \xi^n$ and $V = \mathcal{B}_q^{(\delta)} U = \sum G_n \xi^n$ (so that $G_n = t_{-n} F_n$ and $V_n = t_{-n} U_n$). Then, multiplying relation (9) by $t_{-n}$ and noting that $q^{n-\delta} t_{-n} = t_{-(n-\delta)}$, we get:

$$\forall n \in \mathbf{Z}\,,\; G_{n-\delta} - \Lambda(G_n) = V_n.$$

Multiplying by $\xi^n$ and summing for $n \in \mathbf{Z}$ yields:

$$(\xi^\delta \mathrm{Id} - \Lambda) \mathcal{B}_q^{(\delta)} F(\xi) = \mathcal{B}_q^{(\delta)} U(\xi).$$

Since $U$ has a positive radius of convergence, $\mathcal{B}_q^{(\delta)} U(\xi)$ is an entire function. For $F$ to be a *convergent* solution, it is necessary that $\mathcal{B}_q^{(\delta)} F$ be an *entire* function. We shall now appeal to linear algebra. We first write $A = A_s A_u$ and $B = B_s B_u$ the multiplicative Dunford decompositions. Then $A_u^{1/\delta}$ and $B_u^{1/\delta}$ are well defined. In order to define $A_s^{1/\delta}$ and $B_s^{1/\delta}$, it is enough to choose a mapping $x \mapsto x^{1/\delta}$ on $\mathbf{C}^*$ and to apply it to the eigenvalues. We then put $A^{1/\delta} = A_s^{1/\delta} A_u^{1/\delta}$, $B^{1/\delta} = B_s^{1/\delta} B_u^{1/\delta}$ and get a linear map $L : F \mapsto A^{1/\delta} F (B^{1/\delta})^{-1}$, which is a $\delta^{th}$ root of $\Lambda$. Call $\mu_\delta$ the set of $\delta^{th}$ roots of 1 in $\mathbf{C}$.

**Lemma 4.7** *Let $E$ be a finite dimensional $\mathbf{C}$-vector space, $A$ an endomorphism of $E$ and $R$ be any of the following algebras of functions: $\mathcal{O}(\mathbf{C})$; $\mathcal{O}(\mathbf{C})[\xi^{-1}]$; $\mathbf{C}\{\xi\}_{q,\delta}$; $\mathbf{C}(\{\xi\})_{q,\delta}$; $\mathbf{C}\{\xi\}_{q,(\delta)}$; $\mathbf{C}(\{\xi\})_{q,(\delta)}$. Then the linear operator $(\xi^\delta - A^\delta)$ maps injectively $E \otimes R$ into itself, its image has a finite codimension $\delta \dim E$ and there is an explicit projection formula on the supplementary space $E \oplus \cdots \oplus E\xi^{\delta-1}$ of the image:*

$$V \mapsto \sum_{j \in \mu_\delta} d(jA)^{\delta-1} P_j(A,\xi) V(jA),$$

*where $P_j(A,\xi)$ and $V(jA)$ respectively are the following linear operator and vector:*

$$P_j(A,\xi) = \sum_{i=0}^{\delta-1} (jA)^i \xi^{\delta-1-i}, \quad V(jA) = \sum (jA)^n V_n \in E, \text{ (where } V = \sum V_n \xi^n \text{ is entire).}$$

*Proof.* - The algebraic part of the proof rests on the following computation:

$$1 = \sum_{j \in \mu_\delta} \delta (ja)^{\delta-1} P_j(a,X), \quad \text{where } P_j(a,X) = \frac{X^\delta - a^\delta}{X - a} = \sum_{i=0}^{\delta-1} (ja)^i X^{\delta-1-i}.$$

From this, we draw:

$$\begin{aligned} V(\xi) &= \sum_{j \in \mu_\delta} \delta (jA)^{\delta-1} P_j(A,\xi) V(\xi) \\ &= \sum_{j \in \mu_\delta} \delta (jA)^{\delta-1} P_j(A,\xi) \big( V(\xi) - V(jA) \big) + \sum_{j \in \mu_\delta} \delta (jA)^{\delta-1} P_j(A,\xi) V(jA); \end{aligned}$$

then we note that, since $P_j(A,\xi)(\xi - jA) = \xi^\delta - A^\delta$, the first term of the last right hand side is in the image of the linear operator $(\xi^\delta - A^\delta)$. The second term plainly belongs to the supplementary space $E \oplus \cdots \oplus E\xi^{\delta-1}$.
Then, there are growth conditions on the coefficients to be checked. In the case of $\mathcal{O}(\mathbf{C})$; $\mathcal{O}(\mathbf{C})[\xi^{-1}]$, they are standard. In the case of $\mathbf{C}\{\xi\}_{q,\delta}$, $\mathbf{C}(\{\xi\})_{q,\delta}$, $\mathbf{C}\{\xi\}_{q,(\delta)}$ and $\mathbf{C}(\{\xi\})_{q,(\delta)}$, they follow from the estimations given in the proof of lemma 2.9 of [29]. □

**Theorem 4.8** *With the notations of section 3.3, equation (8) has a convergent solution if, and only if, $\mathcal{B}_q^{(\delta)} U(jL) = 0$ for all $j \in \mu_\delta$. More precisely, the family $\left(\mathcal{B}_q^{(\delta)} U(jL)\right)_{j \in \mu_\delta} \in Mat_{r,s}(\mathbf{C})^{\mu_\delta} \simeq Mat_{r,s}(\mathbf{C})^\delta$ is a complete set of invariants for analytic classification within the formal class $M_0$.*

**Solving (8) with $\theta$ functions and residue invariants.** From the computations in the one level case of 3.3, we see that the only solution of (8) such that condition (5) holds is given by the explicit formula:

$$F_{\overline{c}}(z) = \frac{1}{\theta_c^\delta} \sum_{n \in \mathbf{Z}} (c^\delta q^n \mathrm{Id} - \Lambda)^{-1} V_n z^n, \text{ where } V = \theta_c^\delta U.$$

A short computation shows that $V = \sum t_p c^{-p} U_n z^{n+p}$, so that, $V_0 = \sum t_{-n} c^n U_n = \mathcal{B}_q^{(\delta)} U(c)$. In order to compute explicitly the alien derivation in the one level case, it is convenient to normalize the setting, by requiring that all eigenvalues of $A$ and $B$ lie in the fundamental annulus $1 \leq |z| < |q|$ (up to shearing transformation, this is always possible). We may further decompose the pure blocks $z^\mu A$ and $z^\nu B$ into their corresponding characteristic subspaces. In other words, we may (and shall) assume here that $A$ and $B$ are block diagonal, each block $A_\alpha$ (resp. $B_\beta$) have the unique eigenvalue $\alpha$ (resp. $\beta$), this lying in the fundamental annulus. We write $\Lambda_{\alpha,\beta}$, $L_{\alpha,\beta}$, $U_{\alpha,\beta}$, and compute the corresponding component $\dot{\Delta}_{\overline{\xi}}^{(\delta,\alpha,\beta)}(M)$ of $\dot{\Delta}_{\overline{\xi}}^{(\delta)}(M)$. Let $\xi \in \mathbf{C}^*$ be a prohibited (polar) value of $c$. This means that one of the matrices $(\xi^\delta q^n \mathrm{Id} - \Lambda_{\alpha,\beta})$ is singular, so that $\xi^\delta q^n = \alpha/\beta$. From the normalisation condition, we see that this can occur only for one value of $n$. Since residues are actually defined on $\mathbf{E}_q$, one can choose $\xi$ such that the bad value of $n$ is $n = 0$. Then, we are to compute:

$$\dot{\Delta}_{\overline{\xi}}^{(\delta,\alpha,\beta)}(M) = Res_{c=\xi} \frac{1}{\theta_c^\delta(a)} (c^\delta \mathrm{Id} - \Lambda_{\alpha,\beta})^{-1} \mathcal{B}_q^{(\delta)} U_{\alpha,\beta}(c).$$

Note that the arbitrary basepoint $a \in \mathbf{C}^*$ (which provides us with the fiber functor $\hat{\omega}_a^{(0)}$) appears only in the theta factor. As in the previous section, we introduce $L_{\alpha,\beta}$ such that $L_{\alpha,\beta}^\delta = \Lambda_{\alpha,\beta}$ and get, from the same formulas as before:

$$\dot{\Delta}_{\overline{\xi}}^{(\delta,\alpha,\beta)}(M) = Res_{c=\xi} \frac{1}{\theta_c^\delta(a)} \sum_{j \in \mu_\delta} \delta (j_{\alpha,\beta} L)^{\delta-1} (c - jL_{\alpha,\beta})^{-1} \mathcal{B}_q^{(\delta)} U_{\alpha,\beta}(c).$$

Now, $\xi$ is an eigenvalue of one and only one of the $jL_{\alpha,\beta}$, call it $L_{\xi\alpha,\beta}$. From classical "holomorphic functional calculus" (see *e.g.* [26]), we get:

$$\dot{\Delta}_{\overline{\xi}}^{(\delta,\alpha,\beta)}(M) = \theta^{-\delta}(L_{\xi\alpha,\beta}^{-1}) \, \delta L_{\xi\alpha,\beta}^{\delta-1} \, \mathcal{B}_q^{(\delta)} U_{\alpha,\beta}(L_{\xi\alpha,\beta}).$$

Recall that, as in *loc. cit.*, the theta factor is the application of a holomorphic function to a linear operator.

**Theorem 4.9** *Call $\Phi_a$ the automorphism of $Mat_{r,s}(\mathbf{C})^{\mu_\delta}$, which, on the $(\alpha,\beta)$ component, is left multiplication by $\theta^{-\delta}(L_{\xi\alpha,\beta}^{-1})$. Then $\Phi_a$ sends the q-Borel invariant $\left(\mathcal{B}_q^{(\delta)} U(jL)\right)_{j \in \mu_\delta}$ to the $\dot{\Delta}$ invariant: $\bigoplus_{\xi} \dot{\Delta}_{\overline{\xi}}^{(\delta)}(M)$.*

Here is an example similar to that at the end of section 3.1. We take $M = \begin{pmatrix} \alpha & u \\ 0 & \beta z \end{pmatrix}$, where $\alpha, \beta \in \mathbf{C}^*$ and $u \in \mathbf{C}\{z\}$. The slopes are $\mu = 0$ and $\nu = 1$ and the only level is $\delta = 1$. The associated non homogeneous equation is $\beta z \sigma_q f - \alpha f = u$, which, in the Borel plane, becomes $(\beta\xi - \alpha)\mathcal{B}_q^{(1)} f = \mathcal{B}_q^{(1)} u$, and the obstruction to finding an analytical solution is the complex number $\mathcal{B}_q^{(1)} u(\alpha/\beta)$. This is also the invariant associated to the analytical class of $M$ within its formal class.
On the side of resolution with $\theta$ and residues, we first get: $f_{\overline{c}}(z) = \frac{1}{\theta_c} \sum_{n \in \mathbf{Z}} (cq^n - \alpha\beta^{-1})^{-1} v_n z^n$, where $v = \theta_c u$, then the only non trivial alien derivation, given for $\xi = \alpha/\beta$:

$$\dot{\Delta}_{\overline{\xi}}^{(\delta)} = Res_{c=\xi} f_{\overline{c}}(a) = Res_{c=\xi} \frac{1}{\theta_c(a)} (c - \xi)^{-1} \mathcal{B}_q^{(1)} u(c) = \frac{1}{\theta(a/\xi)} \mathcal{B}_q^{(1)} u(\xi).$$

# 5 Conclusion

The construction of the alien derivations in the differential and $q$-difference case are apparently quite different. In fact, it is possible to reformulate things in the differential case to exhibit some analogy; one can mimick the constructions of the $q$-difference case: in place of a meromorphic function of a $q$-direction of summation in $E_q$, one gets in the differential case a locally constant function of a direction of summation in $S^1$ minus a finite singular set, the poles being replaced by "jump points". The jumps are evaluated by a non-abelian boundary value: one gets the Stokes operators. The alien derivation are the logarithms of these operators. There is a slight difference with the $q$-difference case: in this last case, we took the logarithm *before* evaluating the singularity by a residue. We remark that to consider locally constant functions on $S^1$ with a finite set of jumps as the differential analog of meromorphic functions in the $q$-difference case is in perfect accordance with the study of the confluence process by the second author in [30].

Our constructions suggest some interesting problems.
1. If we consider the computation of the $q$-alien derivations $\dot{\Delta}_{\xi}$ in simple cases, there appears theta factors and factors coming from a $q$-Borel transform. In the simplest cases we can define (pointed) alien derivations $\dot{\Delta}_{\xi}$ as operators acting on some $q$-holonomic power series, and, eliminating the theta factors, we can observe that this modified $q$-derivative of a power series is itself a power series: we get a new operator, an *unpointed* alien derivation $\Delta_{\xi}$. This suggests the possibility to copy the Ecalle's definition of alien derivations (cf. [4]) in the $q$-difference case: a resurgence lattice in the Borel plane is replaced by the set of singularities in the different $q$-Borel planes corresponding to the different $q$-levels $\delta \in \mathbf{N}^*$ (the $q$-direction of summation being fixed), the Ecalle's analytic continuation paths by summation paths "between the levels" and the boundary values by residues. In this program it is important to remark that the "algebraic" definition of summability used in this paper is equivalent to "analytic" definitions in Borel-Laplace style [11, 25, 32]

2. The global classification: we must put together the work of [27] and the results of the present article (at zero and infinity). It is not difficult to guess what will happen and to describe a "fundamental group" which is Zariski dense in the tannakian Galois group, but some great problems remain: even in the regular singular case, we know neither the structure of the global tannakian Galois group (except in the *abelian* case), nor if there exists a reasonable "localisation theory" for the singularities on $\mathbf{C}^*$ (between 0 and $\infty$).
3. The confluence problem. Some simple examples suggest an extension of the results of the second author (cf. [30]) to the irregular case: confluence of $q$-Stokes phenomena at 0 to Stokes phenomena. It is natural to study what will happen with the alien derivations (there is no hope with the pointed $q$-alien derivations, due to the bad properties of *theta* functions in the confluence processes, but it could work nicely with the unpointed $q$-alien derivations).

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