On stability of equilibrium figures of a uniformly rotating liquid drop in n-dimensional space

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Abstract

In this note we study nonlinear stability of rigid rotations of a liquid drop in R^n, with n arbitrary. Even though the case $n > 3$ has no physical sense still it appears interesting from the mathematical point of view. Moreover we prove a non linear instability theorem by direct Lyapunov method.

1 Introduction

Problem of rotating drop has attracted the attention of researchers in several different fields as mathematics (minimal surface, finite perimeter), astrophysics (motion of stars, planets and Saturnus rings), engineering (bubbles in a liquid), nano-technology (hydrophobic, hydrophilic walls and capillary effects). It is out of purposes of this notes to consider all aspects above mentioned. We wish to give an idea only of some mathematical problems, in this regard we quote above others, the papers [2], [14], [15], [16], where existence, uniqueness and regularity of equilibrium figures of capillary fluids is studied in physical three-dimensional case, and the papers [18], [17] where non steady case is first analyzed. It is also worth mentioning the mathematical papers by [3-7] where it is studied the well posedness problem in R^n with dimension n greater than 3 for the steady case. The interest in this field is surprisingly increasing and we quote e.g., [1], [10-13], [17], [19-23] as papers related to stability of equilibrium configurations of a rotating drop. The enclosed bibliography is not at all exhaustive and doesn't give the idea of the number of different mathematical and physical problems one encounters in dealing with rotating drops, however it is enough to explain the scopes of this note.

In the present paper we consider the free boundary problem for the Navier-Stokes equations governing non-stationary motions of an isolated mass of a viscous incompressible capillary liquid in n-dimensional space. We analyze a stationary solution of this problem related to the motion of the liquid as a rigid body in the domain $\mathcal{F}$ independent of time. As in three-dimensional case, a vector field $\mathbf{U}$ in R^n represents the velocity field of a rigid motion if

$$\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i} = 0, \quad i, j = 1, ..., n,$$

holds for the components of the velocity vector field $\boldsymbol{U}(x)$ depending on the Eulerian coordinates $x \in \mathcal{F}$. The solution of this system has a form $\boldsymbol{U}(x) = \mathcal{C}x$ where $\mathcal{C}$ is a constant antisymmetric matrix. This gives the expression $|\mathcal{C}x|^2/2$ for the energy of centrifugal forces slightly different from that used in [3], [5-7] (see Remark 2.1 below). In Sec. 2 we study the kinematics of n-dimensional rigid motion, and we show that it is much more rich than in the case $n = 3$ where it essentially reduces to the rotation about a fixed axis. In Sec. 3 we pass to the *"rotating"* reference frame, and we reduce the analysis of stability of the above mentioned stationary solution to the analysis of stability of the rest state $\boldsymbol{v}(x,t) = 0$, $x \in \mathcal{F}$. We then introduce a quadratic form corresponding to a certain self-adjoint elliptic operator B_1 given on $\mathcal{G}$. In case $n = 3$ this form coincides with the classical second variation of the energy functional. We also give several technical Lemmas. In Sec 4 we prove that if the above quadratic form is positive definite, the rest state is asymptotically stable in the class of global solutions possessing regularity that permits our calculations, and unstable if it can take negative values. We emphasize that the smallness assumption is made only on the distance between the boundaries Γ_t and $\mathcal{G}$.

The proof of stability and instability is achieved by constructing a special functional playing a role of the Lyapunov function that guarantees stability or instability of the rest state in relatively weak norms. The construction goes back to the free work identity introduced in [8-10]. In the proof of instability we make the assumption $\mathrm{Ker}B_1 = \emptyset$. For $n = 3$ the problem of instability (without this additional assumption) is solved in [21,22] by means of much harder technics.

It should be observed that the construction of our Lyapunov functional requires the existence of global solutions $\boldsymbol{v}$, Γ_t to the free boundary problem, satisfying suitable estimates only on the distance between Γ_t and $\mathcal{G}$. The proof of existence of such a solution and of its estimates is outside the scope of the present paper. For $n = 3$ it was carried out in the papers of the authors cited above, when the initial data are close to the regime of a rigid rotation (i.e. the velocity at the initial moment is close to $\boldsymbol{U}(x)$ and it is defined in a domain close to $\mathcal{F}$). In the case $n > 3$ it can be done in the same way.

It becomes clear from the proofs that also self-gravitating forces can be taken into account, [17].

2 Rigid rotation of a fluid drop in R^n

We consider the evolution free boundary problem

$$\begin{aligned} \boldsymbol{v}_t + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} - \nu\nabla^2\boldsymbol{v} + \nabla p = 0, \\ \nabla\cdot\boldsymbol{v} = 0, \qquad x \in \Omega_t,\ \ t > 0, \\ \boldsymbol{v}(x,0) = \boldsymbol{v}_0, \qquad x \in \Omega_0, \\ T(\boldsymbol{v},p)\boldsymbol{n} = \sigma H(x,t)\boldsymbol{n}, \quad W = \boldsymbol{v}\cdot n, \qquad x \in \Gamma_t \equiv \partial\Omega_t. \end{aligned} \tag{2.1}$$

where unknown are a bounded domain $\Omega_t \in R^n$, the vector field $\boldsymbol{v}(x,t) = (v_1, ..., v_n)$ and the function $p(x,t)$ given in Ω_t and satisfying (2.1). Here ν and σ are positive constant coefficients of viscosity and of the surface tension, respectively, $T(\boldsymbol{v},p) = -pI + \nu S(\boldsymbol{v})$ is the stress tensor, $S(\boldsymbol{v}) = \left(\frac{\partial v_j}{\partial x_k} + \frac{\partial v_k}{\partial v_j}\right)_{j,k=1,\dots,n}$ is the rate-of-strain tensor, H is $n-1$ times mean curvature of Γ_t negative for convex domains, and W is the velocity of evolution of Γ_t in the direction of exterior normal $\boldsymbol{n}$. The density of a liquid is assumed to be equal to one. The domain Ω_0 is given. For $n = 3$ this problem was studied in [10]- [13], [17]-[23], and other papers.

We observe that the solution of (2.1) is subjected to the same "conservation laws" as in 3-dimensional case, namely,

$$|\Omega_t| = |\Omega_0|,$$

$$\int_{\Omega_t} \boldsymbol{v}(x,t)dx = \int_{\Omega_0} \boldsymbol{v}_0(x)dx, \tag{2.2}$$

$$\int_{\Omega_t} \boldsymbol{v}(x,t)\cdot\boldsymbol{\eta}_{ij}(x)dx = \int_{\Omega_0} \boldsymbol{v}_0(x)\cdot\boldsymbol{\eta}_{ij}(x)dx \equiv m_{ij}, \quad i \neq j, \tag{2.3}$$

where $\boldsymbol{\eta}_{ij}(x) = \boldsymbol{e}_j x_i - \boldsymbol{e}_i x_j$ and $\boldsymbol{e}_j$ is a unit vector in the direction of the x_j-axis. Indeed, (2.2) is easily obtained by integration of the first equation in (2.1) over Ω_t. We remind the Reynolds transport theorem

$$\frac{d}{dt}\int_{\Omega_t} f(x,t)dx = \int_{\Omega_t} [f_t + (\boldsymbol{v}\cdot\nabla)f](x,t)dx$$

that holds for domains satisfying kinematic boundary condition

$$W(x,t) = (\boldsymbol{v}\cdot\boldsymbol{n})(x,t), \qquad x \in \Gamma_t. \tag{2.4}$$

Hence we obtain

$$0 = \frac{d}{dt}\int_{\Omega_t} \boldsymbol{v}(x,t)dx - \sigma\int_{\Gamma_t} H(x,t)\boldsymbol{n}dS = \frac{d}{dt}\int_{\Omega_t} \boldsymbol{v}(x,t)dx. \tag{2.5}$$

Since $H(x,t)\boldsymbol{n} = \Delta_{\Gamma_t}\boldsymbol{x}$, and Γ_t is closed, the surface integral vanishes. In the same way equations (2.3) are obtained - see [12], [17].

We would like to study the stability of solutions corresponding to a rigid motion of the liquid. We say that the motion is rigid if the vector field of velocity $\boldsymbol{U}$ given as a function of Eulerian coordinates x satisfies the relations

$$\frac{\partial U_i(x)}{\partial x_j} + \frac{\partial U_j(x)}{\partial x_i} = 0, \quad i,j = 1,...,n.$$

It is easily seen that this is the case if and only if

$$\boldsymbol{U} = \mathcal{C}\boldsymbol{x} + \boldsymbol{h} \tag{2.6}$$

where $\mathcal{C}(t)$ is an antisymmetric matrix and $\boldsymbol{h}(t)$ is a vector, constants in space and functions of t only. If $\mathcal{C}$, and $\mathbf{h}$ are constant in time, then the motion will be called *uniform*. In the sequel we take $\boldsymbol{h} = 0$, and $\mathcal{C}$ constant in time. The functions

$$\boldsymbol{U}(x) = \mathcal{C}\boldsymbol{x}, \quad P(x) = \frac{1}{2}|\mathcal{C}\boldsymbol{x}|^2 + p_0, \quad p_0 = const, \tag{2.7}$$

satisfy the system of the Navier-Stokes equations. Substituting $\boldsymbol{U}$ and P into the boundary conditions we obtain the equation for the equilibrium figure $\mathcal{F}$ filled with a rotating liquid:

$$\sigma\mathcal{H}(x) + \frac{1}{2}|\mathcal{C}\boldsymbol{x}|^2 + p_0 = 0, \quad x \in \mathcal{G} \equiv \partial\mathcal{F}. \tag{2.8}$$

where $\mathcal{H}$ is $n-1$ times mean curvature of $\mathcal{G}$.

Without loss of generality we can assume that the matrix $\mathcal{C}$ has a canonical form:

$$\mathcal{C} = \mathrm{diag}\Big(C_1, ..., C_l, O\Big), \tag{2.9}$$

where $l \leq n/2$, O is $n-2l \times n-2l$ matrix whose elements are zeros and C_k are 2×2 antisymmetric matrices of the form

$$C_k = \begin{pmatrix} 0 & -\omega_k \\ \omega_k & 0 \end{pmatrix}. \tag{2.10}$$

In particular, if $n = 3$, then $l = 1$ and $\boldsymbol{U}$ is the velocity of the liquid rotating as a rigid body about the x_3-axis with the angular velocity ω_1. In the n-dimensional case there are l "angular velocities" ω_k.

Remark 2.1 *If we write explicitly the term $|\mathcal{C}\boldsymbol{x}|^2$ in* (2.8) *we find*

$$|\mathcal{C}\boldsymbol{x}|^2 = \sum_1^l \omega_k^2 (x_{2(k-1)+1}^2 + x_{2k}^2).$$

This term differs from the term $F = \omega^2 \sum_{k=1}^{n-1} x_k^2$ used in [3], [5], [6], [7], in particular F cannot represent centrifugal force if n is even. Furthermore, in case n odd F may represent a centrifugal force only for $\omega_k = \omega$ for all $k = 1, \ldots, n-1$.

Passing to the Lagrangean coordinates, it is easy to calculate the trajectories of particles, whose velocity as a function of the Eulerian coordinates is $\boldsymbol{U}(x)$. If $\boldsymbol{x}(0) = \boldsymbol{\xi}$, then

$$x_{2k-1}(t) = \xi_{2k-1} \cos\omega_k t - \xi_{2k} \sin\omega_k t,$$
$$x_{2k}(t) = \xi_{2k-1} \sin\omega_k t + \xi_{2k} \cos\omega_k t, \quad k = 1, ..., l,$$
$$x_m = \xi_m, \quad m = 2l+1, ..., n,$$

i.e., the projection of the trajectory onto the x_k, x_{k+1}-plane is a circle with the center at the origin, along which the motion proceeds with a constant velocity proportional to ω_k. This complicated motion is in general non-periodic.

We say that the figure $\mathcal{F}$ is symmetric, if it is invariant under transformation

$$x = \mathcal{Z}y$$

where

$$\mathcal{Z} = \mathrm{diag}\Big(Z_1, ..., Z_l, I_{n-2l}\Big),$$

I_{n-2l} is a unit $n-2l \times n-2l$ matrix and

$$Z_k = \begin{pmatrix} cos\varphi_k & -sin\varphi_k \\ sin\varphi_k & cos\varphi_k \end{pmatrix}, \qquad k = 1, \ldots, l.$$

It is easily seen that the velocity of liquid particles located at the boundary $\mathcal{G}$ of a symmetric $\mathcal{F}$ is tangential to $\mathcal{G}$, i.e.,

$$\mathcal{C}\boldsymbol{x} \cdot \boldsymbol{N}(x)|_{\mathcal{G}} = 0,$$

where $\boldsymbol{N}$ is the exterior normal to $\mathcal{G}$. This means that the functions (2.7) given in the symmetric domain $\mathcal{F}$ solution to (2.7) represent a stationary solution of (2.1). We consider here only symmetric $\mathcal{F}$.

It follows from the symmetry that

$$\int_{\mathcal{F}} x_j dx = 0, \quad j = 1, ..., 2l, \tag{2.11}$$

$$\int_{\mathcal{F}} x_j x_q dx = 0, \quad j = 1, ..., 2l, \;\; q = 1, ..., n, \;\; q \neq j$$

(some of these relations can be also deduced from equation (2.8), as in the three-dimensional case, see [23]). Without loss of generality we can fix the origin of coordinate system at center of mass, and we can assume that

$$\int_{\mathcal{F}} x_j dx = 0, \quad j = 1, ..., n. \tag{2.12}$$

Let

$$\boldsymbol{\eta}_{ij}(x) = x_i \boldsymbol{e}_j - x_j \boldsymbol{e}_i, \qquad i < j.$$

If the matrix $\mathcal{C}$ has a canonic form (2.9) and the figure is symmetric, then the corresponding matrix of momenta

$$m_{ij} = \int_{\mathcal{F}} \mathcal{C}\boldsymbol{x} \cdot \boldsymbol{\eta}_{ij}(x) dx$$

also has a canonic form. Indeed, since

$$\mathcal{C}\boldsymbol{x} = \sum_{q=1}^{l} \omega_q \boldsymbol{\eta}_q(x)$$

where $\boldsymbol{\eta}_q(x) = \boldsymbol{\eta}_{2q-1,2q}(x)$, it is easily verified , using (2.11), that m_{ij} can be different from zero if and only if $i = 2k-1, j = 2k$, $k \leq l$, in which case

$$m_{2k-1,2k} = \omega_k \|\boldsymbol{\eta}_k\|^2_{L_2(\mathcal{F})}.$$

We do not consider the problem of existence and uniqueness of equilibrium figures, as well as of their geometry, but we can prove the existence of a symmetric equilibrium figure of a given volume in the case of small velocities (i.e., of small C_{ik}). For $n = 3$ this result was obtained in [17].

3 Preliminary lemmas

Let us return to problem (2.1). We assume that $\mathcal{F}$ is a given bounded domain with a smooth boundary and that

$$|\Omega_t| = |\Omega_0| = |\mathcal{F}|, \quad \int_{\Omega_t} x_j dx = 0, \quad j = 1, ..., n, \tag{3.1}$$

$$\int_{\Omega_t} \boldsymbol{v}(x,t) dx = \int_{\Omega_0} \boldsymbol{v}_0(x) dx = 0,$$

$$\int_{\Omega_t} \boldsymbol{v}(x,t) \cdot \boldsymbol{\eta}_{ij}(x) dx = \int_{\Omega_0} \boldsymbol{v}_0(x) \cdot \boldsymbol{\eta}_{ij}(x) dx = \int_{\mathcal{F}} \mathcal{C}\boldsymbol{x} \cdot \boldsymbol{\eta}_{ij}(x) dx. \tag{3.2}$$

We shall work with the evolution problem for the perturbations

$$\boldsymbol{v}_r = \boldsymbol{v} - \boldsymbol{U}, \;\; p_r = p - P$$

written in the coordinate system rigidly connected with the liquid whose velocity is given by (2.7). We make the change of variables

$$x = \mathcal{Z}(t)y$$

and the corresponding transformation of unknown functions

$$\boldsymbol{w}(y,t) = \mathcal{Z}^{-1}(t)\boldsymbol{v}_r(\mathcal{Z}(t)y,t), \quad q(y,t) = p_r(\mathcal{Z}(t)y,t),$$

where

$$\mathcal{Z}(t) = \mathrm{diag}\Big(Z_1(t), ..., Z_l(t), I_{n-2l}\Big),$$

I_{n-2l} is a unit $n-2l \times n-2l$ matrix and

$$Z_k(t) = \begin{pmatrix} cos\omega_k t & -sin\omega_k t \\ sin\omega_k t & cos\omega_k t \end{pmatrix}.$$

This leads to the problem

$$\begin{aligned} \boldsymbol{w}_t + (\boldsymbol{w}\cdot\nabla)\boldsymbol{w} + 2\mathcal{C}\boldsymbol{w} - \nu\nabla^2\boldsymbol{w} + \nabla q = 0, \\ \nabla\cdot\boldsymbol{w} = 0, \qquad y\in\Omega_t, \qquad t>0, \\ T(\boldsymbol{w},q)\boldsymbol{n} = \Big(\sigma H + \frac{1}{2}|\mathcal{C}\boldsymbol{y}|^2 + p_0\Big)\boldsymbol{n}, \\ W = \boldsymbol{w}\cdot\boldsymbol{n}, \qquad y\in\Gamma_t, \\ \boldsymbol{w}(y,0) = \boldsymbol{v}_0(y), \qquad y\in\Omega_0, \end{aligned} \tag{3.3}$$

in a transformed domain denoted again by Ω_t. Conditions (3.1), (3.2) take the form

$$|\Omega_t| = |\mathcal{F}|, \quad \int_{\Omega_t} x_j dx = 0, \quad j = 1,...,n, \tag{3.4}$$

$$\begin{aligned} \int_{\Omega_t} \boldsymbol{w}(x,t)dx = 0, \\ \int_{\Omega_t} \boldsymbol{w}(x,t)\cdot\boldsymbol{\eta}_{ij}(x)dx + \int_{\Omega_t} \mathcal{C}\boldsymbol{x}\cdot\boldsymbol{\eta}_{ij}(x)dx = \int_{\mathcal{F}} \mathcal{C}\boldsymbol{x}\cdot\boldsymbol{\eta}_{ij}(x)dx. \end{aligned} \tag{3.5}$$

We assume that Γ_t is close to $\mathcal{G}$ and is given by equation

$$x = y + \boldsymbol{N}(y)\rho(y,t), \quad y\in\mathcal{G} \tag{3.6}$$

with a small function $\rho(y,t)$ defined on $\mathcal{G}$. Let $\boldsymbol{N}^*$ and ρ^* be extensions of $\boldsymbol{N}$ and ρ from $\mathcal{G}$ into $\mathcal{F}$ made in such a way that

$$\begin{aligned} \frac{\partial}{\partial N}\boldsymbol{N}^*(x,t)|_{\mathcal{G}} = 0, \\ \frac{\partial}{\partial N}\rho^*(x,t)|_{\mathcal{G}} = 0, \\ |\rho^*(\cdot,t)|_{C^1(\mathcal{F})} \le \delta \ll 1. \end{aligned} \tag{3.7}$$

The transformation

$$x = y + \boldsymbol{N}^*(y)\rho^*(y,t) \equiv e_\rho(y), \qquad y \in \mathcal{F} \tag{3.8}$$

is invertible, if δ is small enough, and it maps $\mathcal{F}$ onto Ω_t. Let $\mathcal{L} = \frac{\partial e_\rho}{\partial y}$ be the Jacobi matrix of this transformation with the elements

$$l_{ij} = \delta_{ij} + \rho^*(y,t)\frac{\partial}{\partial y_j}N_i^*(y) + N_i^*(y)\frac{\partial}{\partial y_j}\rho^*(y,t), \tag{3.9}$$

and with the determinant L. By l^{ij} and $\widehat{L}_{ij}$, $i,j = 1,2,...,n$ we denote the elements of the inverse matrix $\mathcal{L}^{-1}$ and of the cofactors matrix $\widehat{\mathcal{L}} = L\mathcal{L}^{-1}$, respectively. Set

$$\Lambda(y,\rho) = \boldsymbol{N}(y) \cdot \widehat{\mathcal{L}}\boldsymbol{N}(y),$$

$$\varphi(y,\rho) = \int_0^1 \rho(y)\Lambda(y,s\rho)ds,$$

$$\psi(y,\rho) = \int_0^1 (y_i + sN_i(y)\rho(y))\rho(y)\Lambda(y,s\rho)ds.$$

From formula (2.9) in [20]

$$\int_{\Omega_t} f(x)dx - \int_{\mathcal{F}} f(y)dy = \int_0^1 ds \int_{\mathcal{G}} f(e_{s\rho}(y))\rho\Lambda(y,s\rho)dS_y$$

it follows that the restrictions (3.1) can be written in terms of ρ as

$$\int_{\mathcal{G}} \varphi(y,\rho)dS = 0, \quad \int_{\mathcal{G}} \psi_i(y,\rho)dS = 0, \quad i = 1,2,3...,n, \tag{3.10}$$

We remind that the $(-1)^{i+j}\widehat{L}_{ij}$ are the determinants of $\mathcal{L}$ with row j and column i deleted. From (3.9) we notice that

$$l_{km} = a_{km}(y,\rho^*) + N_k^*(y)\frac{\partial\rho^*}{\partial y_m},$$

hence, $\widehat{\mathcal{L}}_{ij}$ does not contain products of two or more derivatives of ρ^*. This means that $\widehat{\mathcal{L}}$ is a linear function of $\nabla\rho^*$. Furthermore, the calculation of the first variation of Λ with respect to ρ (see [20], formula (2.10)) shows that $\delta\Lambda$ is independent of $\nabla\rho$. Thus, Λ, φ and ψ_i are functions of y and ρ.

In order to use Korn's inequality, we need to introduce the part $\boldsymbol{w}^\perp$ of $\boldsymbol{w}$ orthogonal to all $\boldsymbol{\eta}_{km}$:

$$\boldsymbol{w}^\perp = \boldsymbol{w} - \boldsymbol{w}', \quad \boldsymbol{w}' = \sum_{k<m} \gamma_{km}(t)\boldsymbol{\eta}_{km}(x). \tag{3.11}$$

Since $\boldsymbol{\eta}_{km}(x)$ are linearly independent, the matrix $\mathcal{A}(t)$ with the elements

$$A_{km,ij}(t) = \int_{\Omega_t} \boldsymbol{\eta}_{km}(x) \cdot \boldsymbol{\eta}_{ij}(x)dx \tag{3.12}$$

is non-degenerate, moreover, it is positive definite. By virtue of (3.11) and (3.5), the functions $\gamma_{km}(t)$ are defined by

$$\gamma_{km}(t) = \sum_{i<j} A^{km,ij}(t) \int_{\Omega_t} \boldsymbol{w}'(x,t)\cdot\boldsymbol{\eta}_{ij}(x)dx \tag{3.13}$$
$$= -\sum_{i<j} A^{km,ij}(t)\Big(\int_{\Omega_t} \mathcal{C}\boldsymbol{x}\cdot\boldsymbol{\eta}_{ij}(x)dx - \int_{\mathcal{F}} \mathcal{C}\boldsymbol{y}\cdot\boldsymbol{\eta}_{ij}(y)dy\Big)$$

where $A^{km,ij}(t)$ are elements of $\mathcal{A}^{-1}(t)$. We need the following auxiliary proposition

Lemma 3.1 *For arbitrary $k,m \le n$, $k<m$ the vector field $\mathcal{C}\boldsymbol{\eta}_{km}(x)$ can be represented in the form*

$$2\mathcal{C}\boldsymbol{\eta}_{km}(x) = -\nabla(\mathcal{C}\boldsymbol{x}\cdot\boldsymbol{\eta}_{km}(x)) + \boldsymbol{R}_{km}(x) \tag{3.14}$$

where $\boldsymbol{R}_{km}$ is a linear combination of $\boldsymbol{\eta}_{ij}$.

Proof We consider the left hand side, and we have

$$2\mathcal{C}\boldsymbol{\eta}_{km}(x) = 2\sum_{i=1}^{n}(C^{im}x_k - C^{ik}x_m)\mathbf{e}_i = 2\sum_{i=1}^{n}(C^{im}x_k - C^{ik}x_m)\nabla x_i = \tag{3.15}$$
$$\sum_{i=1}^{n} C^{im}x_k\nabla x_i + \sum_{i=1}^{n} C^{im}[\nabla(x_ix_k) - x_i\nabla x_k] -$$
$$\sum_{i=1}^{n} C^{ik}x_m\nabla x_i - \sum_{i=1}^{n} C^{ik}[\nabla(x_ix_m) - x_i\nabla x_m] =$$
$$\sum_{i=1}^{n} C^{im}\boldsymbol{\eta}_{ki}(x) - \sum_{i=1}^{n} C^{ik}\boldsymbol{\eta}_{mi}(x) + \sum_{i=1}^{n}\nabla(C^{im}x_kx_i - C^{ik}x_mx_i) =$$
$$\sum_{i=1}^{n} C^{im}\boldsymbol{\eta}_{ki}(x) - \sum_{i=1}^{n} C^{ik}\boldsymbol{\eta}_{mi}(x) + \sum_{i=1}^{n}\nabla(\mathcal{C}\boldsymbol{x}\cdot(x_m\boldsymbol{e}_k(x) - x_k\boldsymbol{e}_m(x) =$$
$$\sum_{i=1}^{n} C^{im}\boldsymbol{\eta}_{ki}(x) - \sum_{i=1}^{n} C^{ik}\boldsymbol{\eta}_{mi}(x) - \sum_{i=1}^{n}\nabla\mathcal{C}\boldsymbol{x}\cdot\boldsymbol{\eta}_{km}(x).$$

The proposition is proved. □

From lemma 3.1 it follows that

$$2\mathcal{C}\boldsymbol{w}' = -\nabla(\mathcal{C}\boldsymbol{x}\cdot\boldsymbol{w}') + \boldsymbol{R}$$

where $\boldsymbol{w}'$ is defined in (3.11) and $\boldsymbol{R}$ is a linear combination of $\boldsymbol{\eta}_{ij}$.

Let us introduce the operators

$$B_0\rho = -\sigma\delta(H(x) - \mathcal{H}(y))[\rho] - \frac{1}{2}\delta(|\mathcal{C}\boldsymbol{x}|^2 - |\mathcal{C}\boldsymbol{y}|^2)[\rho] = -\sigma\Delta_{\mathcal{G}}\rho - b(y)\rho$$

and

$$B_1\rho = B_0\rho - \delta\mathcal{C}\boldsymbol{x}\cdot\boldsymbol{w}'$$
$$= B_0\rho + \sum_{k<m,i<j} A_0^{km,ij}\mathcal{C}\boldsymbol{x}\cdot\boldsymbol{\eta}_{km}(x)\int_{\mathcal{G}}\rho\mathcal{C}\boldsymbol{y}\cdot\boldsymbol{\eta}_{ij}(y)dS.$$

Here $A_0^{km,ij}$ are elements of $\mathcal{A}_0^{-1}$ and $\mathcal{A}_0$ is the matrix with the elements (3.12) calculated for Ω_t replaced with $\mathcal{F}$. By δ we mean the first variation with respect to ρ:

$$\delta(g(e_\rho(y)) - g(y)) = \frac{d}{ds}(g(y + s\rho(y)) - g(y))\Big|_{s=0};$$

hence, B_0 is a linear operator and $\delta\mathcal{C}\boldsymbol{x}\cdot\boldsymbol{w}'$ is a linear functional of ρ. It follows from well known formula for the variation of the mean curvature that $B_0\rho$ does not contain the first derivatives of ρ, and it holds

$$b(y) = \sigma c^2(y) + \mathcal{C}\boldsymbol{y}\cdot\mathcal{C}\boldsymbol{N}(y),$$

where $c^2(y)$ is the sum of squares of the principal curvatures of $\mathcal{G}$ at point y.

Let P be an orthogonal in $L_2(\mathcal{G})$ projector onto the subspace H of the functions $r\in L_2(\mathcal{G})$ satisfying the orthogonality conditions

$$\int_{\mathcal{G}} r(y)\chi_p(y)dS = 0, \qquad p = 0, ..., n,$$

where

$$\chi_0(y) = 1, \quad \chi_i(y) = y_i, \quad i = 1, ..., n.$$

Lemma 3.2 *Assume that $\rho(y)$ satisfies (3.10) and that δ in (3.7) is sufficiently small. Then*

$$c_1\|\rho\|_{W_2^1(\mathcal{G})} \le \|r\|_{W_2^1(\mathcal{G})} \le c_2\|\rho\|_{W_2^1(\mathcal{G})}$$

where c_i are constants independent of ρ, $W_2^1(\mathcal{G})$ is the Sobolev space (see for instance [24]).

Proof We have

$$P\rho = \rho(y) - \sum_{p=0}^n c_p\chi_p(y).$$

The constants c_p are found from the equations

$$\int_{\mathcal{G}}\rho\chi_q dS = \sum_{p=0}^n c_p\int_{\mathcal{G}}\chi_p\chi_q dS \equiv \sum_{p=0}^n X_{pq}c_p.$$

Since χ_p are linearly independent functions on $\mathcal{G}$, the matrix with the elements X_{pq} is non-degenerate, and

$$c_q = \sum_{p=0}^n X^{qp}\int_{\mathcal{G}}\rho\chi_p dS,$$

where X^{pq} are elements of the inverse matrix. Since

$$\int_{\mathcal{G}}\rho\chi_0 dS = \int_{\mathcal{G}}\rho dS = \int_{\mathcal{G}}(\rho - \varphi(y,\rho))dS,$$

$$\int_{\mathcal{G}}\rho\chi_i dS = \int_{\mathcal{G}}\rho y_i dS = \int_{\mathcal{G}}(\rho y_i - \psi_i(y,\rho))dS, \quad i = 1, ..., n,$$

we have
$$\sum_{p=0}^{n} |c_p| \le c \sum_{p=0}^{n} \left| \int_{\mathcal{G}} \rho \chi_p dS \right| \le c\delta \|\rho\|_{L_2(\mathcal{G})},$$
which proves the lemma.

The following lemma is a modification of problem (23) solved in [13], see also Lemma 4.1 of [20], and [12] for the case $n = 3$, the proof given in [13], and [20] may be extended to the n-dimensional case. It concerns the construction of a special auxiliary vector field satisfying estimates in some Sobolev-Slobodevski norms.

Lemma 3.3 *Assume that $\Gamma_t = \partial\Omega_t$ is given by equation (3.6) with ρ satisfying (3.7) and having bounded first derivatives with respect to t and second derivatives with respect to x_i. Let $f_0(y,t), \quad y \in \mathcal{G}$ be an arbitrary function with a finite norm*
$$\|f_0(\cdot,t)\|_{W_2^{1/2}(\mathcal{G})} + \|f_0(\cdot,t))\|_{L_q(\mathcal{G})} + \|f_{0t}(\cdot,t))\|_{L_2(\mathcal{G})}, \quad q > 1,$$
that satisfies the condition
$$\int_{\mathcal{G}} f_0(y,t) dS = 0.$$
Then there exists a vector field $\boldsymbol{V}(x,t), \quad x \in \Omega_t$ such that
$$\nabla \cdot \boldsymbol{V}(x,t) = 0,$$
$$\boldsymbol{V} \cdot \boldsymbol{\tau}_i|_{x=e_\rho(y)} = 0, \quad \boldsymbol{V} \cdot \boldsymbol{n}|_{x=e_\rho(y)} = f_0(y)/|\widehat{\mathcal{L}}^T \boldsymbol{N}(y)|, \qquad x \in \Gamma_t,$$
with $\boldsymbol{\tau}_i$, $i = 1, \ldots, n-1$ tangential unit vectors, and
$$\int_{\Omega_t} \mathbf{V}(x,t) \cdot \boldsymbol{\eta}_{ij}(x) dx = 0$$
Finally, the estimates
$$\|\boldsymbol{V}(\cdot,t)\|_{W_2^1(\Omega_t)} \le c\|f_0(\cdot,t)\|_{W_2^{1/2}(\mathcal{G})},$$
$$\|\boldsymbol{V}(\cdot,t)\|_{L_q(\Omega_t)} \le c\|f_0(\cdot,t)\|_{L_q(\mathcal{G})}, \qquad q > 1$$
$$\|\boldsymbol{V}_t(\cdot,t)\|_{L_2(\Omega_t)} \le c\Big(\|f_{0t}(\cdot,t)\|_{L_2(\mathcal{G})} + \|f_0(\cdot,t)\|_{W_2^{1/2}(\mathcal{G})}\Big),$$
hold with constants independent of t.

4 Non linear stability and instability of rigid rotations of a fluid drop in R^n

Now, we obtain the main result of the paper.

Theorem 4.1 *Assume that problem* (3.3) *has a classical solution defined for* $t \in [0, T]$, $T \le \infty$, *and that* Ω_t *satisfies the assumptions of Lemma 3.3, in particular,* Γ_t *is given by equation* (3.6), *and*

$$|\rho(\cdot, t)|_{C^1(\mathcal{G})} \le \delta \tag{4.1}$$

with a small (but fixed) $\delta > 0$.

1. If

$$\int_{\mathcal{G}} r(y) B_1 r(y) dS \ge c \|r\|^2_{W^1_2(\mathcal{G})} \tag{4.2}$$

for all r *satisfying*

$$\int_{\mathcal{G}} r(y) dS = 0, \quad \int_{\mathcal{G}} r(y) y_i dS = 0, \quad i = 1, ..., n, \tag{4.3}$$

then

$$\|\boldsymbol{w}(\cdot, t)\|^2_{L_2(\Omega_t)} + \|\rho(\cdot, t)\|^2_{W^1_2(\mathcal{G})} \le c e^{-bt} \Big(\|\boldsymbol{w}_0\|^2_{L_2(\Omega_t)} + \|\rho_0\|^2_{W^1_2(\mathcal{G})} \Big) \tag{4.4}$$

with $b, c > 0$ *independent of* T.

2. Assume that the form $\int_{\mathcal{G}} r(y) B_1\, r(y) dS$ *can take negative values for some* r *satisfying* (4.3), *and that* $\mathrm{Ker} B_1 = \emptyset$. *Then there exist arbitrarily small initial values* $(\boldsymbol{w}_0, \rho_0)$ *such that the solution of* (3.3) *leaves sooner or later a certain neighborhood of zero, i.e. for a certain* $t > 0$ *it holds the inequality*

$$\|\boldsymbol{w}(\cdot, t)\|^2_{L_2(\Omega_t)} + \|\rho(\cdot, t)\|^2_{W^1_2(\mathcal{G})} \ge \epsilon > 0. \tag{4.5}$$

In particular, condition (4.1) *cannot be verified for all* $t > 0$.

Proof We observe first of all that if inequality (4.2) holds for all r satisfying (4.3), then it is true also for ρ sufficiently small and satisfying (3.10) (this follows from Lemma 3.2). When we multiply the first equation in (3.3) by $\boldsymbol{w}$, integrate over Ω_t and take account of the Reynolds transport formula (see Sec. 1), we obtain the energy relation

$$\frac{d}{dt} \Big(\frac{1}{2} \|\boldsymbol{w}(\cdot, t)\|^2_{L_2(\Omega_t)} + \sigma |\Gamma_t| - \frac{1}{2} \int_{\Omega_t} |\mathcal{C} \boldsymbol{x}|^2 dx \Big) + \frac{\nu}{2} \|S(\boldsymbol{w})\|^2_{L_2(\Omega_t)} = 0$$

that can be written in the form

$$\frac{d}{dt} \Big(\frac{1}{2} \|\boldsymbol{w}^\perp(\cdot, t)\|^2_{L_2(\Omega_t)} + \frac{1}{2} \|\boldsymbol{w}'(\cdot, t)\|^2_{L_2(\Omega_t)} + G(t) - G^{(0)} \Big) + \frac{\nu}{2} \|S(\boldsymbol{w})\|^2_{L_2(\Omega_t)} = 0. \tag{4.6}$$

The functional $G(t) = G(\rho)$ is given by

$$G(t) = \sigma |\Gamma_t| - \frac{1}{2} \int_{\Omega_t} |\mathcal{C} \boldsymbol{x}|^2 dx - p_0 |\Omega_t|$$

and $G^{(0)}$ is the value of this functional with Ω_t replaced by $\mathcal{F}$. As in the three-dimensional case, it can be shown that

$$\delta(G(t) - G^{(0)})[\rho] = 0,$$

by virtue of (2.8), and

$$\delta^2(G(t)-G^{(0)})[\rho]=\int_{\mathcal{G}}\rho B_0\rho dS.$$

Now, we write the first equation in (3.3) in the form

$$\boldsymbol{w}^\perp_t+(\boldsymbol{w}\cdot\nabla)\boldsymbol{w}^\perp+(\boldsymbol{w}\cdot\nabla)\boldsymbol{w}'+2\mathcal{C}\boldsymbol{w}^\perp-\nu\nabla^2\boldsymbol{w}^\perp$$

$$+\nabla(p-\mathcal{C}\boldsymbol{x}\cdot\boldsymbol{w}')=-\boldsymbol{w}'_t-\boldsymbol{R}$$

multiply it by the vector field $\boldsymbol{V}$ constructed in proposition 3.2 and integrate over Ω_t. For the moment we leave the function $f_0(y)$ indefinite. After integration by parts we obtain

$$\frac{d}{dt}\int_{\Omega_t}\boldsymbol{w}^\perp\cdot\boldsymbol{V}dx-\int_{\Omega_t}\boldsymbol{w}^\perp\cdot(\boldsymbol{V}_t+(\boldsymbol{w}\cdot\nabla)\boldsymbol{V})dx+\int_{\Omega_t}(\boldsymbol{w}\cdot\nabla)\boldsymbol{w}'\cdot\boldsymbol{V}dx$$

$$+2\int_{\Omega_t}\mathcal{C}\boldsymbol{w}^\perp\cdot\boldsymbol{V}dx-\int_{\mathcal{G}}\Big(\sigma H+\frac{1}{2}|\mathcal{C}x|^2+p_0+\mathcal{C}\boldsymbol{x}\cdot\boldsymbol{w}'\Big)\Big|_{x=y+\boldsymbol{N}\rho}f_0dS=0. \tag{4.7}$$

Next, we add (4.6) and (4.7) multiplied by a small number $\gamma>0$, and we set $f_0=P\rho\equiv r$. This leads to

$$\frac{dE(t)}{dt}+E_1(t)=0 \tag{4.8}$$

with

$$E(t)=\frac{1}{2}\|\boldsymbol{w}^\perp\|^2_{L_2(\Omega_t)}+\frac{1}{2}\|\boldsymbol{w}'\|^2_{L_2(\Omega_t)}+(G(t)-G^{(0)})+\gamma\int_{\Omega_t}\boldsymbol{w}^\perp\cdot\boldsymbol{V}dx,$$

$$E_1(t)=\frac{\nu}{2}\|S(\boldsymbol{w}^\perp)\|^2_{L_2(\Omega_t)}-\gamma\int_{\mathcal{G}}\Big(\sigma H+\frac{1}{2}|\mathcal{C}x|^2+p_0+\mathcal{C}\boldsymbol{x}\cdot\boldsymbol{w}'\Big)\Big|_{x=y+\boldsymbol{N}\rho}rdS$$

$$-\gamma\int_{\Omega_t}\boldsymbol{w}^\perp\cdot(\boldsymbol{V}_t+(\boldsymbol{w}^\perp+\boldsymbol{w}')\cdot\nabla\,\boldsymbol{V})dx+$$

$$\gamma\int_{\Omega_t}(\boldsymbol{w}^\perp+\boldsymbol{w}')\cdot\nabla\,\boldsymbol{w}'\cdot\boldsymbol{V}dx+2\gamma\int_{\Omega_t}\mathcal{C}\boldsymbol{w}^\perp\cdot\boldsymbol{V}dx. \tag{4.9}$$

Now, we show that if γ and δ are small enough, the following estimates hold, with constants independent of t:

$$c_1(\|\boldsymbol{w}(\cdot,t)\|^2_{L_2(\Omega_t)}+\|\rho(\cdot,t)\|^2_{W^1_2(\mathcal{G})})\le E(t)\le c_2(\|\boldsymbol{w}(\cdot,t)\|^2_{L_2(\Omega_t)}+\|\rho(\cdot,t)\|^2_{W^1_2(\mathcal{G})}), \tag{4.10a}$$

$$\mathcal{D}=\frac{\nu}{2}\|S(\boldsymbol{w}^\perp)\|^2_{L_2(\Omega_t)}-\gamma\int_{\mathcal{G}}\Big(\sigma H+\frac{1}{2}|\mathcal{C}x|^2+p_0-\mathcal{C}\boldsymbol{x}\cdot\boldsymbol{w}'\Big)\Big|_{x=y+\boldsymbol{N}\rho}rdS\ge \tag{4.10b}$$

$$c_3(\|\boldsymbol{w}^\perp(\cdot,t)\|^2_{W^1_2(\Omega_t)}+\gamma\|\rho(\cdot,t)\|^2_{W^1_2(\mathcal{G})}),$$

$$-\gamma\int_{\Omega_t}\Big[\boldsymbol{w}^\perp\cdot(\boldsymbol{V}_t+(\boldsymbol{w}^\perp+\boldsymbol{w}')\cdot\nabla\boldsymbol{V})-(\boldsymbol{w}^\perp+\boldsymbol{w}')\cdot\nabla\boldsymbol{w}'\cdot\boldsymbol{V}-2\mathcal{C}\boldsymbol{w}^\perp\cdot\boldsymbol{V}\Big]dx\geq \tag{4.10c}$$

$$-\frac{\nu}{4}\|S(\boldsymbol{w}^\perp)\|^2_{L_2(\Omega)}-c\gamma^2\|\rho(\cdot,t)\|^2_{W^1_2(\mathcal{G})}.$$

We prove (4.10a) observing that

$$G[\rho]-G^{(0)}=\int_0^1\frac{d}{ds}G[s\rho]ds=\int_0^1\Big(\frac{d}{ds}G[s\rho]-\frac{d}{ds}G[s\rho]\Big|_{s=0}\Big)ds$$

$$=\frac{1}{2}\delta^2(G[\rho]-G^{(0)})+\int_0^1\Big(\frac{d^2}{ds^2}G[s\rho]-\frac{d^2}{ds^2}G[s\rho]\Big|_{s=0}\Big)ds$$

$$=\frac{1}{2}\delta^2(G[\rho]-G^{(0)})+q_1(\rho),$$

where $q_1(\rho)$ is a small remainder, and

$$\|\boldsymbol{w}(\cdot,t)\|^2_{L_2(\Omega_t)}=\|\boldsymbol{w}^\perp(\cdot,t)\|^2_{L_2(\Omega_t)}+\|\boldsymbol{w}'(\cdot,t)\|^2_{L_2(\Omega_t)}.$$

By (3.11) and (3.13),

$$\|\boldsymbol{w}'(\cdot,t)\|^2_{L_2(\Omega_t)}=\sum_{k<m,l<q}\gamma_{km}\gamma_{lq}\int_{\Omega_t}\boldsymbol{\eta}_{km}\cdot\boldsymbol{\eta}_{lq}dx$$

$$=\sum_{k<m,l<q}\sum_{i<j,r<s}A^{km,ij}(t)A^{lq,rs}(t)A_{lq,km}(t)I_{ij}(t)I_{rs}(t)$$

$$=\sum_{i<j,r<s}A^{ij,rs}(t)I_{ij}(t)I_{rs}(t),$$

where

$$I_{ij}(t)=\int_{\Omega_t}\mathcal{C}\boldsymbol{x}\cdot\boldsymbol{\eta}_{ij}(x)dx-\int_{\mathcal{F}}\mathcal{C}\boldsymbol{x}\cdot\boldsymbol{\eta}_{ij}(x)dx. \tag{4.11}$$

Since

$$\delta I_{ij}=\int_{\mathcal{G}}\rho\mathcal{C}\boldsymbol{x}\cdot\boldsymbol{\eta}_{ij}(x)dS,$$

we arrive at

$$\|\boldsymbol{w}'(\cdot,t)\|^2_{L_2(\Omega_t)}=\sum_{i<j,r<s}A_0^{ij,rs}(t)\int_{\mathcal{G}}\rho\mathcal{C}\boldsymbol{x}\cdot\boldsymbol{\eta}_{ij}(x)dS\int_{\mathcal{G}}\rho\mathcal{C}\boldsymbol{x}\cdot\boldsymbol{\eta}_{rs}(x)dS+q_2(\rho)$$

and

$$E(t)=\frac{1}{2}\|\boldsymbol{w}^\perp(\cdot,t)\|^2_{L_2(\Omega_t)}+\int_{\mathcal{G}}\rho B_1\rho dS+$$

$$\frac{1}{2}\Big(\sum_{i<j,r<s}A_0^{ij,rs}(t)\int_{\mathcal{G}}\rho\mathcal{C}\boldsymbol{x}\cdot\boldsymbol{\eta}_{ij}(x)dS\int_{\mathcal{G}}\rho\mathcal{C}\boldsymbol{x}\cdot\boldsymbol{\eta}_{rs}(x)dS\Big)+\gamma\int_{\Omega_t}\boldsymbol{w}^\perp\cdot\boldsymbol{V}dx+q_3(\rho)$$

with $q_3(\rho)=q_1(\rho)+\frac{1}{2}q_2(\rho)$ satisfying

$$|q_3(\rho)|\leq c\delta\|\rho(\cdot,t)\|^2_{W^1_2(\mathcal{G})}. \tag{4.12}$$

(concerning the estimates of remainders q_i, see [20] and [21], Sec 4). We also have

$$\left|\int_{\Omega_t} \boldsymbol{w}^{\perp}\cdot\boldsymbol{V}dx\right| \le c\|\boldsymbol{w}^{\perp}(\cdot,t)\|_{L_2(\Omega_t)}\|\rho(\cdot,t)\|_{W_2^1(\mathcal{G})}.$$

hence, for γ small enough, (4.10a) holds.

We pass to the proof of (4.10b) and consider the surface integral in (4.9), we call it $-I$. Since, by (2.8),

$$\sigma H(x)+\frac{1}{2}|\mathcal{C}x|^2+p_0=\sigma(H(x)-\mathcal{H}(y))+\frac{1}{2}(|\mathcal{C}x|^2-|\mathcal{C}y|^2)$$

$$=-B_0\rho+\sigma(H(x)-\mathcal{H}(y)-\delta(H(x)-\mathcal{H}(y)))+\frac{1}{2}(|\mathcal{C}x|^2-|\mathcal{C}y|^2-\delta(|\mathcal{C}x|^2-|\mathcal{C}y|^2)),$$

$$\mathcal{C}\boldsymbol{x}\cdot\boldsymbol{w}'=-\sum_{k<m,i<j}A^{km,ij}(t)I_{ij}\mathcal{C}\boldsymbol{x}\cdot\boldsymbol{\eta}_{km}(x),$$

where $x=e_\rho(y)$. We have

$$-I=\int_{\mathcal{G}} rB_1\rho dS+q_4(\rho)=\int_{\mathcal{G}} rB_1 r dS+q_5(\rho)$$

with q_4, q_5 satisfying (4.12), hence (4.10b) holds.

Now, we obtain (4.10c). The kinematic boundary condition $W=\boldsymbol{w}\cdot\boldsymbol{n}$ in (3.3) is equivalent to

$$\rho_t(y,t)=\frac{\boldsymbol{w}(x,t)\cdot\boldsymbol{n}(x)}{\boldsymbol{N}(y)\cdot\boldsymbol{n}(x)}.$$

Also, the definition of I_{ij}

$$I_{ij}=\int_{\Gamma_t}\mathcal{C}\boldsymbol{x}\cdot\boldsymbol{\eta}_{ij}dS-\int_{\mathcal{G}}\mathcal{C}\boldsymbol{y}\cdot\boldsymbol{\eta}_{ij}(y)dS=\int_0^1 ds\int_{\mathcal{G}}\mathcal{C}\boldsymbol{z}\cdot\boldsymbol{\eta}_{ij}(z)\rho\Lambda(y,s\rho)dS$$

with $z=\mathbf{e}_{s\rho}(y)$, yields

$$|\boldsymbol{w}'|_{C^1(\Omega_t)}\le c\sum_{i,j}|I_{i,j}|\le c\|\rho\|_{L_2(\mathcal{G})}.$$

Furthermore,

$$\|\boldsymbol{V}_t\|_{L_2(\Omega_t)}\le c\|r_t(.,t)\|_{L_2(\mathcal{G})})\le c\|P(\boldsymbol{w}^{\perp}\cdot\boldsymbol{n}+\boldsymbol{w}'\cdot\boldsymbol{n})\|_{L_2(\mathcal{G})}$$

$$\le c(\|w^{\perp}\|_{W_2^1(\Omega_t)}+\|\rho(.,t)\|_{L_2(\mathcal{G})})$$

$$\left|\int_{\Omega_t}\boldsymbol{w}^{\perp}\cdot\boldsymbol{V}_t dx\right|\le c\|\boldsymbol{w}^{\perp}\|_{L_2(\Omega_t)}(\|\boldsymbol{w}^{\perp}\|_{L_2(\Omega_t)}+\|\rho\|_{L_2(\mathcal{G})}),$$

$$\left|\int_{\Omega_t}(\boldsymbol{w}'\cdot\nabla)\boldsymbol{V}\cdot\boldsymbol{w}^{\perp}dx\right|\le\|\boldsymbol{w}^{\perp}\|_{L_2(\Omega_t)}\|\nabla\boldsymbol{V}\|_{L_2(\Omega_t)}\|\boldsymbol{w}'\|_{L_\infty(\Omega_t)}\le$$

$$c\|\boldsymbol{w}^{\perp}\|_{L_2(\Omega_t)}\|\rho\|_{W_2^{1/2}(\mathcal{G})}\|\boldsymbol{w}'\|_{L_\infty(\Omega_t)},$$

$$\left|\int_{\Omega_t}(\boldsymbol{w}^{\perp}+\boldsymbol{w}')\cdot\nabla)\boldsymbol{w}'\cdot\boldsymbol{V}dx\right|\le(\|\boldsymbol{w}^{\perp}\|_{L_2(\Omega_t)}+\|\boldsymbol{w}'\|_{L_2(\Omega_t)})\|\nabla\boldsymbol{w}'\|_{L_\infty(\Omega_t)}\|\boldsymbol{V}\|_{L_2(\Omega_t)})\le$$

$$c(\|\boldsymbol{w}^\perp\|_{L_2(\Omega_t)} + \|\rho\|_{L_2(\mathcal{G})})\|\rho\|_{L_2(\mathcal{G})}\|\nabla \boldsymbol{w}'\|_{L_\infty(\Omega_t)},$$

$$\left|\int_{\Omega_t} \mathcal{C}\boldsymbol{w}^\perp \cdot \boldsymbol{V} dx\right| \le c\|\boldsymbol{w}^\perp\|_{L_2(\Omega_t)}\|\rho\|_{L_2(\mathcal{G})}.$$

We also have

$$\int_{\Omega_t} (\boldsymbol{w}^\perp \cdot \nabla)\boldsymbol{V} \cdot \boldsymbol{w}^\perp dx = -\int_{\Omega_t} (\boldsymbol{w}^\perp \cdot \nabla)\boldsymbol{w}^\perp \cdot \boldsymbol{V} dx + \int_{\Gamma_t} (\boldsymbol{w}^\perp \cdot \boldsymbol{N})\boldsymbol{w}^\perp \cdot \boldsymbol{V} dS,$$

$$\left| -\int_{\Omega_t} (\boldsymbol{w}^\perp \cdot \nabla)\boldsymbol{w}^\perp \cdot \boldsymbol{V} dx\right| \le \|\nabla \boldsymbol{w}^\perp\|_{L_2(\Omega_t)}\|\boldsymbol{w}^\perp\|_{L_{\frac{2n}{n-2}}(\Omega_t)}\|\boldsymbol{V}\|_{L_n(\Omega_t)}) \le$$

$$c\|\mathbf{S}(\boldsymbol{w}^\perp)\|^2_{L_2(\Omega_t)}\|\rho\|_{L_n(\mathcal{G})}$$

$$\left|\int_{\Gamma_t} (\boldsymbol{w}^\perp \cdot \boldsymbol{N})\boldsymbol{w}^\perp \cdot \boldsymbol{V} dS\right| \le \left|\int_{\Gamma_t} (\boldsymbol{w}^\perp \cdot \boldsymbol{N})^2 \frac{f_0}{|\hat{\mathcal{L}}^T\boldsymbol{N}|} dS\right| \le$$

$$c\|\rho\|_{L_\infty(\mathcal{G})}\|\mathbf{S}(\boldsymbol{w}^\perp)\|^2_{L_2(\Omega_t)} \le c\delta\|\mathbf{S}(\boldsymbol{w}^\perp)\|^2_{L_2(\Omega_t)}$$

Hence, for small γ, and δ, (4.10c) follows. From inequality (4.10c) we also deduce that

$$E_1(t) \ge bE(t). \tag{4.13}$$

Moreover, applying Gronwall's lemma we obtain $E(t) \le e^{-bt}E(0)$ and, as a consequence, (4.4). The first part of the theorem is proved.

Let us consider the case 2. Since $\mathrm{Ker}B_1 = \emptyset$, the space $H \subset L_2(\mathcal{G})$ of functions satisfying (4.3) is representable as the orthogonal sum $H = H_- \oplus H_+$ where $H_- = \mathrm{Span}(\varphi_1, ...\varphi_m)$, φ_j are eigenfunctions of B_1 corresponding to the negative eigenvalues, and H_+ is the lineal hull of eigenfunctions of B_1 corresponding to the positive eigenvalues. Let $P_\pm$ be projectors onto these spaces and let $r_\pm = P_\pm r$. For arbitrary $r \in H$ we have

$$(r, B_1 r) = (r_+, B_1 r_+) + (r_-, B_1 r_-)$$

where (r_1, r_2) is a scalar product in $L_2(\mathcal{G})$, and

$$c_1\|r\|^2_{W_2^1(\mathcal{G})} \le (r_+, B_1 r_+) - (r_-, B_1 r_-) \le c_2\|r\|^2_{W_2^1(\mathcal{G})}.$$

We assume that problem (3.3) has a solution defined for $t \ge 0$ and satisfying condition (4.1). We aim to show that this is impossible for some special (arbitrarily small) initial data and some small but fixed ε. Let $\boldsymbol{V}_\pm$ be the vector fields mentioned in Lemma 3.3, corresponding to $f_0 = P_\pm P\rho \equiv r_\pm$. When we set $\boldsymbol{V} = \boldsymbol{V}_+ - \boldsymbol{V}_-$ in (4.6), we obtain

$$\begin{aligned}\frac{d}{dt}\int_{\Omega_t} \boldsymbol{w}^\perp \cdot (\boldsymbol{V}_+ - \boldsymbol{V}_-)dx - \int_{\Omega_t} \boldsymbol{w}^\perp \cdot ((\boldsymbol{V}_+ - \boldsymbol{V}_-)_t + (\boldsymbol{w} \cdot \nabla)(\boldsymbol{V}_+ - \boldsymbol{V}_-))dx \\ + \int_{\Omega_t} (\boldsymbol{w} \cdot \nabla)\boldsymbol{w}' \cdot (\boldsymbol{V}_+ - \boldsymbol{V}_-)dx + 2\int_{\Omega_t} \mathcal{C}\boldsymbol{w}^\perp \cdot (\boldsymbol{V}_+ - \boldsymbol{V}_-)dx \\ - \int_{\mathcal{G}} \Big(\sigma H(x) + \frac{1}{2}|\mathcal{C}x|^2 + p_0 - \mathcal{C}\boldsymbol{x} \cdot \boldsymbol{w}'\Big)\Big|_{x=y+\rho}(r_+(y) - r_-(y))dS = 0.\end{aligned} \tag{4.14}$$

From (4.14) and (4.6) we deduce

$$\frac{dz(t)}{dt} = z_1(t)$$

where

$$z(t) = -\frac{1}{2}\|\boldsymbol{w}(\cdot,t)\|^2_{L_2(\Omega_t)} - (G(t) - G^{(0)}) - \gamma \int_{\Omega_t} \boldsymbol{w}^{\perp} \cdot (\boldsymbol{V}_+ - \boldsymbol{V}_-)dx,$$

$$\begin{aligned} z_1(t) = &-\gamma \int_{\mathcal{G}} \Big(\sigma H(x) + \frac{1}{2}|\mathcal{C}x|^2 + p_0 - \mathcal{C}\boldsymbol{x}\cdot\boldsymbol{w}'\Big)\Big|_{x=y+\boldsymbol{N}\rho}(r_+(y) - r_-(y))dS \\ &+\frac{\nu}{2}\|S(\boldsymbol{w}^{\perp})\|^2_{L_2(\Omega_t)} - \gamma \int_{\Omega_t} \boldsymbol{w}^{\perp}\cdot((\boldsymbol{V}_+ - \boldsymbol{V}_-)_t + (\boldsymbol{w}\cdot\nabla)(\boldsymbol{V}_+ - \boldsymbol{V}_-))dx \\ &+\gamma \int_{\Omega_t} (\boldsymbol{w}\cdot\nabla)\boldsymbol{w}'\cdot(\boldsymbol{V}_+ - \boldsymbol{V}_-)dx + 2\gamma \int_{\Omega_t} \mathcal{C}\boldsymbol{w}^{\perp}\cdot(\boldsymbol{V}_+ - \boldsymbol{V}_-)dx. \end{aligned} \tag{4.15}$$

The surface integral in (4.15) equals

$$\int_{\mathcal{G}} (r_+ - r_-)B_1\rho dS + q_6(\rho) = (r_+, B_1 r_+) - (r_-, B_1 r_-) + q_7(\rho)$$

with q_6, q_7 satisfying (4.12). Other integrals in (4.14) are estimated as above in the case 1, so we have

$$z_1(t) \geq c(\|\boldsymbol{w}(\cdot,t)\|^2_{L_2(\Omega_t)} + \|\rho(\cdot,t)\|^2_{W_2^1(\mathcal{G})}) \geq bz(t).$$

Hence, $\frac{dz(t)}{dt} \geq bz(t)$, and if we choose initial data arbitrarily small but such that $z(0) > 0$ (which is possible), we obtain

$$z(t) \geq e^{bt}z(0).$$

Hence, for t large enough (4.5) holds, and the theorem is proved.□

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