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RESONANT DECOMPOSITION AND THE I-METHOD FOR THE TWO-DIMENSIONAL ZAKHAROV SYSTEM

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Abstract. The initial value problem of the Zakharov system on a two-dimensional torus with general period is considered in this paper. We apply the I-method with some 'resonant decomposition' to show global well-posedness results for small-in-L^2 initial data belonging to some spaces weaker than the energy class. We also consider an application of our ideas to the initial value problem on $\mathbb{R}^2$ and give an improvement of the best known result by Pecher (2012).

1. **Introduction.** We consider the initial value problem of the Zakharov system:

$$\begin{cases} i\partial_t u + \Delta u = nu, & u : [-T,T] \times Z \to \mathbb{C}, \\ \partial_t^2 n - \Delta n = \Delta(|u|^2), & n : [-T,T] \times Z \to \mathbb{R}, \\ (u,n,\partial_t n)\big|_{t=0} = (u_0,n_0,n_1) \in H^s \times H^r \times |\nabla| H^r. \end{cases} \tag{1}$$

Here, $Z = \mathbb{R}^2$ or $\mathbb{T}^2_\gamma := \mathbb{R}^2/(2\pi\gamma_1\mathbb{Z}) \times (2\pi\gamma_2\mathbb{Z})$ (two-dimensional torus of general period $\gamma = (\gamma_1,\gamma_2) \in \mathbb{R}^2_+$). $|\nabla|H^r$ denotes the space of all functions f such that $|\nabla|^{-1} f \in H^r$. The Zakharov system was introduced in [16] for a model of the Langmuir turbulence in unmagnetized ionized plasma; u represents the slowly varying envelope of a rapidly oscillating electric field, and n is the deviation of ion density from its mean value.

(1) is described as a Hamiltonian PDE with the Hamiltonian given by

$$H(u,n)(t) := \|\nabla u(t)\|_{L^2}^2 + \frac{1}{2}(\|n(t)\|_{L^2}^2 + \||\nabla|^{-1}\partial_t n(t)\|_{L^2}^2) + \int_Z n(t,x)|u(t,x)|^2\,dx.$$

Local well-posedness in the energy space $H^1 \times L^2 \times |\nabla| L^2$ was obtained in [4] for $Z = \mathbb{R}^2$ and in [11] for $Z = \mathbb{T}^2_\gamma$. In particular, using conservation of the mass and the Hamiltonian and the sharp Gagliardo-Nirenberg inequality

$$\|u\|_{L^4(Z)}^4 \le \frac{2}{\|Q\|_{L^2(\mathbb{R}^2)}^2}\|u\|_{L^2(Z)}^2\|\nabla u\|_{L^2(Z)}^2 + C\|u\|_{L^2(\mathbb{T}^2_\gamma)}^4$$

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(the last term in the right hand side is required only in the periodic case; see [15, 6]), we have the a priori control of the energy norm of solutions in the energy class if $\|u_0\|_{L^2} < \|Q\|_{L^2(\mathbb{R}^2)}$, where Q is the ground state of the cubic NLS on $\mathbb{R}^2$. More precisely, if $\eta := 1 - \|u_0\|_{L^2}^2/\|Q\|_{L^2(\mathbb{R}^2)}^2 > 0$, then we have

$$\Big|\int n(t)|u(t)|^2\Big| \le \big\|n(t)\big\|_{L^2}\big\|u(t)\big\|_{L^4}^2 \le \frac{1-\eta/2}{2}\big\|n(t)\big\|_{L^2}^2 + \frac{1}{2(1-\eta/2)}\big\|u(t)\big\|_{L^4}^4$$
$$\le \frac{1-\eta/2}{2}\big\|n(t)\big\|_{L^2}^2 + \frac{1-\eta}{1-\eta/2}\big\|\nabla u(t)\big\|_{L^2}^2 + C\big\|u(t)\big\|_{L^2}^4.$$

Therefore, we have the following a priori estimate

$$\frac{\eta/2}{1-\eta/2}\big\|\nabla u(t)\big\|_{L^2}^2 + \frac{\eta}{4}\big\|n(t)\big\|_{L^2}^2 + \frac{1}{2}\big\||\nabla|^{-1}\partial_t n(t)\big\|_{L^2}^2$$
$$\le H(u,n)(t) + C\big\|u(t)\big\|_{L^2}^4 = H(u,n)(0) + C\big\|u_0\big\|_{L^2}^4$$

as long as the solution $(u(t), n(t))$ exists in the energy class. Consequently, (1) is globally well-posed for initial data in the energy space with $\|u_0\|_{L^2} < \|Q\|_{L^2(\mathbb{R}^2)}$. In fact, the solution also exists globally for initial data in $H^1 \times L^2 \times H^{-1}$ with $\|u_0\|_{L^2} \le \|Q\|_{L^2(\mathbb{R}^2)}$ (see [10] for $Z = \mathbb{R}^2$ and [12] for $Z = \mathbb{T}^2_\gamma$).

The present article addresses the global well-posedness of (1) for some initial data without finite energy. The proof will rely on the I-method, which was originally introduced by Colliander, Keel, Staffilani, Takaoka, and Tao to deal with nonlinear Schrödinger equations and has been applied to a wide variety of nonlinear dispersive equations. For the details of the I-method, we refer to [7, 14, 8] and references therein.

The I-method for the Zakharov system was initiated by Fang, Pecher, and Zhong [9] for the $\mathbb{R}^2$ case, who established the global well-posedness in $H^s \times L^2 \times |\nabla|L^2$ with $1 > s > \frac{3}{4}$. Their estimate of the modified energy was mainly based on the Strichartz estimate for the Schrödinger equation and its bilinear refinement, as well as some crude estimates with the Hölder inequality and the Sobolev embedding. It is worth noting that they did not use the scaling argument in the I-method; thus it was quite important for global well-posedness under the minimal regularity assumptions to obtain the best estimate for the lower bound of local existence time in terms of the size of initial data.

Our principal aim is to apply the I-method in the periodic case $Z = \mathbb{T}^2_\gamma$, where the local well-posedness of (1) below the energy space is known for $\frac{1}{2} \le s \le 1$, $r = 0$ ([11]). However, it turns out not to be trivial at all to adjust their argument to the periodic setting. In fact, since the dispersive effect is limited on torus, the same estimate as for $\mathbb{R}^2$ cannot be expected in general. For example, the L^4 Strichartz estimate for the Schrödinger equation on $\mathbb{T}^2_\gamma$ cannot hold without some loss of derivative (see [2, 5]). To obtain the best decay order in the almost conservation law, we will use the sharp trilinear estimates established in [11] which control various interactions between two Schrödinger solutions and a wave solution.

We remark that, in [9], the trilinear terms have the biggest contribution in the increment of the modified energy and force them to assume $s > \frac{3}{4}$. To improve further, we shall introduce a new modified energy based on the concept of 'resonant decomposition' (see [8], for instance). The trilinear terms then become harmless; in fact, we find that these terms are acceptable for the wider regularity range $s > \frac{1}{2}$. However, some portion of the quadrilinear terms in the modified energy increment still has a large contribution, which will require the regularity $s > \frac{2}{3}$ even for the case of $\mathbb{R}^2$ if we estimate it in the same manner as [9]. To control these quadrilinear terms, we carry out a more refined analysis with the Strichartz estimate for the wave equation. At the end, we will push down the threshold to $s > \frac{9}{14}$.

Theorem 1.1. *Let $1 > s > \frac{9}{14}$ and $r = 0$. Then, for any spatial period γ, (1) on $\mathbb{T}^2_\gamma$ is globally well-posed for initial data with $\|u_0\|_{L^2(\mathbb{T}^2_\gamma)} < \|Q\|_{L^2(\mathbb{R}^2)}$. Moreover, the global solutions satisfy*

$$\sup_{-T\le t\le T}\Big(\big\|u(t)\big\|_{H^s} + \big\|n(t)\big\|_{L^2} + \big\||\nabla|^{-1}\partial_t n(t)\big\|_{L^2}\Big) \le C(1+T)^{\max\{\frac{1-s}{2s-1},\, \frac{4(1-s)}{14s-9}\}+}$$

for any $T > 0$, where the constant $C > 0$ depends on s, γ, the implicit constant in the exponent, and the size of initial data.

Remark 1. (i) The period γ has nothing to do with the regularity range in the above theorem, as in the local theory [11].

(ii) In contrast to the nonperiodic problem, we know ([11]) that the data-to-solution map for (1) on $\mathbb{T}^2_\gamma$ cannot be smooth (nor C^2) for $r < 0$. That is why we restrict our attention to the case $r = 0$ in the above theorem. Compare this to Theorem 1.2 below.

Of course, these approaches are also effective for the $\mathbb{R}^2$ case. Recently, Pecher [13] extended the previous result [9] for global well-posedness on $\mathbb{R}^2$ to a wider regularity range, in $H^s \times H^r \times |\nabla|H^r$ with

$$r \le 0, \quad s < r+1, \quad s(r+\tfrac{3}{2}) > 1.$$

The new ingredient was the global well-posedness with regularity for the wave data below L^2. Note that even local well-posedness was not known in these regularities before. He first established the local well-posedness of (1) with the operator I, and then applied the argument in [9] to obtain an almost conservation law of the modified energy. Even for the case $r = 0$ he could improve the previous threshold $s > \frac{3}{4}$ to $s > \frac{2}{3}$ by refining the analysis of the worst trilinear terms in the increment of the modified energy. However, since he used the same modified energy as [9], the trilinear terms still require the regularity $s > \frac{2}{3}$. Therefore, it is strongly expected that his result, combined with our approaches, can be improved further. We carry out this and obtain the following result.

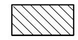
Theorem 1.2. *Let $s < 1$, $r \le 0$ be such that $r \ge s-1$ and $s > \frac{9+3r}{14+8r}$. Then, (1) on $\mathbb{R}^2$ is globally well-posed for initial data with $\|u_0\|_{L^2(\mathbb{R}^2)} < \|Q\|_{L^2(\mathbb{R}^2)}$. Moreover, the global solutions satisfy*

$$\sup_{-T\le t\le T}\Big(\big\|u(t)\big\|_{H^s} + \big\|n(t)\big\|_{H^r} + \big\||\nabla|^{-1}\partial_t n(t)\big\|_{H^r}\Big)$$
$$\le C(1+T)^{\max\{\frac{(1-s)(1+r)}{(2+r)s-1},\, \frac{4(1-s)(1+r)}{(14+8r)s-(9+3r)}\}+}$$

for any $T > 0$, where the constant $C > 0$ depends on s, r, the implicit constant in the exponent, and the size of initial data.

Remark 2. (i) If we consider the particular case $r = 0$, then the above result shows the global well-posedness for $1 > s > \frac{9}{14}$ just as the periodic case.

(ii) See Figure 1 for the range of regularity in the theorem. The previous result of Pecher [13] is indicated by ▧, and the optimal corner is $A = (\frac{1}{4}(\sqrt{17} -$

1), $\frac{1}{4}(\sqrt{17}-5)) \approx (0.781, -0.219)$. We extend it to the range , and the optimal corner is $B = (\frac{1}{16}(\sqrt{201}-3), \frac{1}{16}(\sqrt{201}-19)) \approx (0.699, -0.301)$.

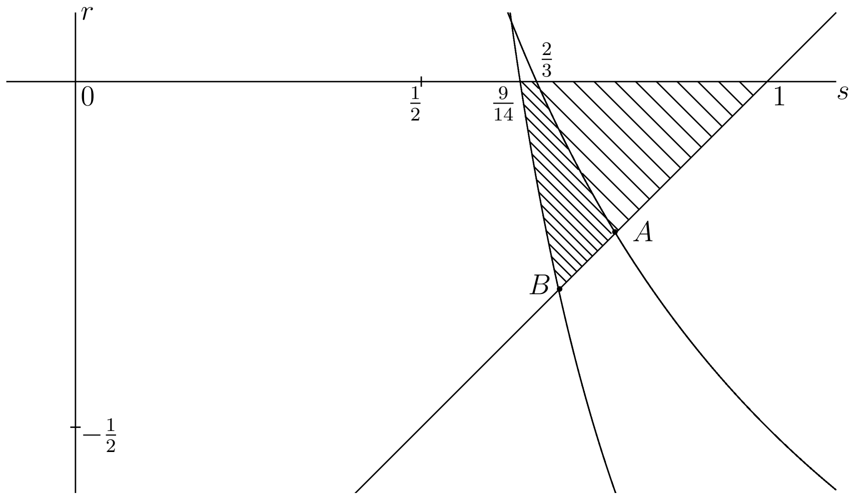


FIGURE 1. Range of regularity for global well-posedness in the nonperiodic case.

The plan of this article is as follows. In Section 2, we recall some definitions and multilinear estimates given in the previous local results on $\mathbb{R}^2$ [1] and on $\mathbb{T}^2_\gamma$ [11]. In Section 3, we construct our modified energy. A proof of the almost conservation law for the periodic case and Theorem 1.1 will be given in Section 4. We indicate in Section 5 how to apply our ideas to the nonperiodic case, obtaining Theorem 1.2. In Appendix A we give an elementary proof of the Strichartz estimate for the periodic wave equation, which is used in Section 4.

2. **Function spaces and preliminary lemmas.** First of all, we need to define dyadic decomposition operators and the $X^{s,b}$-type norms. We will use the same notations as used in [11].

Definition 2.1 (Littlewood-Paley decomposition)**.** Let $\eta \in C^\infty(\mathbb{R})$ be an even function with the properties

$$\eta \equiv 1 \quad \text{on} \quad [-1,1], \quad \operatorname{supp} \eta \subset (-2,2), \quad 0 \le \eta \le 1.$$

Define a partition of unity on $\mathbb{R}$, η_N for dyadic $N \ge 1$, by

$$\eta_1 := \eta, \qquad \eta_N(r) := \eta(\tfrac{r}{N}) - \eta(\tfrac{2r}{N}), \quad N \ge 2.$$

Define the frequency localization operator P_N on functions $f : Z \to \mathbb{C}$ by

$$\mathcal{F}_x(P_N\phi)(\xi) := \eta_N(|\xi|)\widehat{\phi}(\xi),$$

where $\mathcal{F}_x(\cdot) = \widehat{\cdot}$ denotes the spatial Fourier transform on $Z = \mathbb{R}^2$ or $\mathbb{T}^2_\gamma$. We also use the notation P_N to denote the operator on functions in (t,x),

$$\mathcal{F}_x(P_N u)(t,\xi) := \eta_N(|\xi|)\widehat{u}(t,\xi).$$

Also, define the operators Q_L^S, $Q_L^{W_\pm}$ on spacetime functions by

$$\mathcal{F}_{t,x}(Q_L^S u)(\tau,\xi) := \eta_L(\tau+|\xi|^2)\widetilde{u}(\tau,\xi), \quad \mathcal{F}_{t,x}(Q_L^{W_\pm} w)(\tau,\xi) := \eta_L(\tau\pm|\xi|)\widetilde{w}(\tau,\xi)$$

for dyadic numbers $L \geq 1$, where $\mathcal{F}_{t,x}(\cdot) = \widetilde{\cdot}$ denotes the spacetime Fourier transform on $\mathbb{R}\times Z$. We will write $P_{N,L}^S = P_N Q_L^S$, $P_{N,L}^{W_\pm} = P_N Q_L^{W_\pm}$ for brevity. Finally, we define several dyadic frequency regions:

$$\begin{aligned}
\mathcal{P}_1 &:= \left\{ (\tau,\xi) \,\middle|\, |\xi| \lesssim 2 \right\}, \quad \mathcal{P}_N := \left\{ (\tau,\xi) \,\middle|\, \tfrac{N}{2} \leq |\xi| \leq 2N \right\}, \quad N \geq 2,\\
\mathcal{S}_1 &:= \left\{ (\tau,\xi) \,\middle|\, |\tau+|\xi|^2| \lesssim 2 \right\}, \quad \mathcal{S}_L := \left\{ (\tau,\xi) \,\middle|\, \tfrac{L}{2} \leq |\tau+|\xi|^2| \leq 2L \right\}, \quad L \geq 2,\\
\mathcal{W}_1^\pm &:= \left\{ (\tau,\xi) \,\middle|\, |\tau\pm|\xi|| \lesssim 2 \right\}, \quad \mathcal{W}_L^\pm := \left\{ (\tau,\xi) \,\middle|\, \tfrac{L}{2} \leq |\tau\pm|\xi|| \leq 2L \right\}, \quad L \geq 2.
\end{aligned}$$

In what follows, capital letters N and L with various subscripts are used to denote dyadic numbers ≥ 1. We will always use N for the frequency and L for the modulation. It is convenient to introduce the notations

$$\overline{N}_{ij\ldots} := \max\{N_i, N_j, \ldots\}, \qquad \underline{N}_{ij\ldots} := \min\{N_i, N_j, \ldots\}.$$

The following will be used for the specific indices;

$$N_{\max} := \overline{N}_{012}, \quad N_{\min} := \underline{N}_{012}, \qquad L_{\max} := \overline{L}_{012}, \quad L_{\min} := \underline{L}_{012},$$

and we denote by L_{med} the median of L_0, L_1, L_2.

Definition 2.2 (Function spaces $X^{s,b,p}$)**.** For $s, b \in \mathbb{R}$ and $1 \leq p < \infty$, define the spaces $X_S^{s,b,p}$ and $X_{W_\pm}^{s,b,p}$ by the completion of Schwartz functions on $\mathbb{R}\times Z$, $Z = \mathbb{R}^2$ or $\mathbb{T}_\gamma^2$, with respect to the following norms

$$\begin{aligned}
\|u\|_{X_S^{s,b,p}} &:= \Big\| \big\| N^s L^b \|P_{N,L}^S u\|_{L^2_{t,x}(\mathbb{R}\times Z)} \big\|_{\ell^p_L} \Big\|_{\ell^2_N},\\
\|u\|_{X_{W_\pm}^{s,b,p}} &:= \Big\| \big\| N^s L^b \|P_{N,L}^{W_\pm} u\|_{L^2_{t,x}(\mathbb{R}\times Z)} \big\|_{\ell^p_L} \Big\|_{\ell^2_N}.
\end{aligned}$$

For $T > 0$, define the restricted space $X_*^{s,b,p}(T)$ ($* = S$ or $W_\pm$) by the restrictions of distributions in $X_*^{s,b,p}$ to $(-T,T)\times Z$, with the norm

$$\|u\|_{X_*^{s,b,p}(T)} := \inf\left\{ \|U\|_{X_*^{s,b,p}} \,\middle|\, U \in X_*^{s,b,p} \text{ is an extension of } u \text{ to } \mathbb{R}\times Z \right\}.$$

With $n_+ := n + i|\nabla|^{-1}\partial_t n$ and $n_{+0} := n_0 + i|\nabla|^{-1} n_1$, (1) is transformed into

$$\begin{cases}
i\partial_t u + \Delta u = \frac{1}{2}(n_+ + n_-)u, \qquad u : [-T,T]\times Z \to \mathbb{C},\\
i\partial_t n_+ - |\nabla| n_+ = |\nabla|(|u|^2), \qquad n_+ : [-T,T]\times Z \to \mathbb{C},\\
(u, n_+)\big|_{t=0} = (u_0, n_{+0}) \in H^s \times H^r,
\end{cases} \tag{2}$$

where $n_- := \overline{n_+}$, which conserves (formally) the L^2 norm of $u(t)$ and

$$H(u,n_+)(t) := \|\nabla u(t)\|_{L^2}^2 + \frac{1}{2}\|n_+(t)\|_{L^2}^2 + \frac{1}{2}\int_Z (n_+(t,x) + n_-(t,x))|u(t,x)|^2 dx,$$

although $H(u,n_+)$ cannot be in general defined for $(u(t), n_+(t)) \in H^s \times H^r$ with $s < 1$ or $r < 0$. We can recover (1) from (2) by putting $n := \Re n_+$ since n is real valued.

In [1, 11], the local well-posedness of (a slightly different version of) (2) was established by means of an iteration argument in the spaces $X_S^{s,\frac{1}{2},1}(T) \times X_{W_+}^{r,\frac{1}{2},1}(T)$. For later use we recall some of the estimates used for the local theory.

The next lemma contains standard linear estimates and equally holds for $\mathbb{R}^2$ and for $\mathbb{T}^2_\gamma$. Define the Duhamel operators

$$\mathcal{I}_S F(t) := -i\int_0^t e^{i(t-t')\Delta}F(t')\,dt', \qquad \mathcal{I}_{W_\pm} G(t) := i\int_0^t e^{\mp i(t-t')|\nabla|}G(t')\,dt',$$

and a bump function $\psi_\delta(t) := \psi(t/\delta)$, where $\psi \in C_0^\infty(\mathbb{R})$ is a function with the same property as η given in Definition 2.1.

Lemma 2.3 ([11], Lemma 4.1)**.** *Let $s\in\mathbb{R}$. For any $0<\delta\le 1$ and $0<b\le\frac12$, the following estimates hold. The implicit constants do not depend on s, δ.*

$$\big\|\psi_\delta e^{it\Delta}u_0\big\|_{X_S^{s,\frac12,1}} \lesssim \big\|u_0\big\|_{H^s}, \qquad \big\|\psi_\delta e^{-it|\nabla|}w_0\big\|_{X_{W_+}^{s,\frac12,1}} \lesssim \big\|w_0\big\|_{H^s}, \tag{3}$$

$$\big\|\psi_\delta u\big\|_{X_S^{s,b,1}} \lesssim \delta^{\frac12-b}\big\|u\big\|_{X_S^{s,\frac12,1}}, \qquad \big\|\psi_\delta w\big\|_{X_{W_\pm}^{s,b,1}} \lesssim \delta^{\frac12-b}\big\|w\big\|_{X_{W_\pm}^{s,\frac12,1}}, \tag{4}$$

$$\big\|\psi_\delta \mathcal{I}_S F\big\|_{X_S^{s,\frac12,1}} \lesssim \delta^{\frac12-b}\big\|F\big\|_{X_S^{s,-b,1}}, \qquad \big\|\psi_\delta \mathcal{I}_{W_\pm} G\big\|_{X_{W_\pm}^{s,\frac12,1}} \lesssim \delta^{\frac12-b}\big\|G\big\|_{X_{W_\pm}^{s,-b,1}}. \tag{5}$$

In the periodic case, the local well-posedness for $\frac12\le s\le 1$, $r=0$ follows from the next bilinear estimates of nonlinearities, together with the above estimate (3) of linear solutions.

Lemma 2.4 ([11], Proposition 4.3)**.** *Let $\frac12\le s\le 1$ and u,v,w be smooth functions on $\mathbb{R}\times\mathbb{T}^2_\gamma$. Then, we have*

$$\big\|\mathcal{I}_S(uw)\big\|_{X_S^{s,\frac12,1}(\delta)} + \big\|\mathcal{I}_S(u\bar w)\big\|_{X_S^{s,\frac12,1}(\delta)} \lesssim \delta^{\frac12-}\big\|u\big\|_{X_S^{s,\frac12,1}(\delta)}\big\|w\big\|_{X_{W_\pm}^{0,\frac12,1}(\delta)}, \tag{6}$$

$$\big\|\mathcal{I}_{W_+}(|\nabla|(u\bar v))\big\|_{X_{W_+}^{0,\frac12,1}(\delta)} \lesssim \delta^{\frac12-}\big\|u\big\|_{X_S^{s,\frac12,1}(\delta)}\big\|v\big\|_{X_S^{s,\frac12,1}(\delta)}. \tag{7}$$

We easily see that Lemma 2.4 is also verified for functions on $\mathbb{R}\times\mathbb{R}^2$ by the same proof. In fact, similar bilinear estimates are valid at the lower regularity $s=0$, $r=-\frac12$ in the nonperiodic case. Here, we only state the estimate for the Schrödinger part, which will be used in Section 5.

Lemma 2.5 ([1], (5.11))**.** *For smooth functions u,w on $\mathbb{R}\times\mathbb{R}^2$, we have*

$$\big\|\mathcal{I}_S(uw)\big\|_{X_S^{0,\frac12,1}(\delta)} \lesssim \delta^{\frac14}\big\|u\big\|_{X_S^{0,\frac12,1}(\delta)}\big\|w\big\|_{X_{W_\pm}^{-\frac12,\frac12,1}(\delta)}. \tag{8}$$

We will also need various estimates of functions restricted in frequency over dyadic regions $\mathcal{P}_N$, $\mathcal{S}_L$, and $\mathcal{W}_L^\pm$. The next one is a periodic analog of a bilinear refinement of the linear L^4-Strichartz estimate in the $\mathbb{R}^2$ case as well as similar estimates for the Schrödinger-wave interactions.

Lemma 2.6 ([11], Lemma 2.5 with Remark 2.8)**.** *Let $N_j, L_j\ge 1$ ($j=0,1,2$) be dyadic numbers.*

(i) Suppose that $u_1,u_2\in L^2(\mathbb{R}\times\mathbb{T}^2_\gamma)$ satisfy

$$\operatorname{supp}\widetilde{u_1}\subset\mathcal{P}_{N_1}\cap\mathcal{S}_{L_1}, \qquad \operatorname{supp}\widetilde{u_2}\subset\mathcal{P}_{N_2}\cap\mathcal{S}_{L_2}.$$

We also assume $N_0\ge 2$. Then we have

$$\big\|P_{N_0}(u_1\overline{u_2})\big\|_{L^2_{t,x}} \lesssim \underline{L}_{12}^{\frac12}\Big(\frac{\overline{L}_{12}}{N_0}+1\Big)^{\frac12}N_{\min}^{\frac12}\big\|u_1\big\|_{L^2_{t,x}}\big\|u_2\big\|_{L^2_{t,x}}.$$

(ii) Suppose that $u, w \in L^2(\mathbb{R} \times \mathbb{T}^2_\gamma)$ satisfy

$$\text{supp } \widetilde{w} \subset \mathcal{P}_{N_0} \cap \mathcal{W}^{\pm}_{L_0}, \qquad \text{supp } \widetilde{u} \subset \mathcal{P}_{N_1} \cap \mathcal{S}_{L_1}.$$

Then we have

$$\|wu\|_{L^2_{t,x}} + \|\bar{w}u\|_{L^2_{t,x}} \lesssim \underline{L}_{01}^{\frac{1}{2}} \Big(\frac{\overline{L}_{01}}{N_1} + 1 \Big)^{\frac{1}{2}} \underline{N}_{01}^{\frac{1}{2}} \|w\|_{L^2_{t,x}} \|u\|_{L^2_{t,x}}.$$

In the nonperiodic case, similar estimates are true without the '+1' factor on the right hand side, which clearly shows the restricted smoothing effect in the periodic case. We will use the following for the $\mathbb{R}^2$ case as a refinement of Lemma 2.6 (ii).

Lemma 2.7 ([1], Proposition 4.3 (ii))**.** *Let $N_j, L_j \geq 1$ $(j = 0, 1)$ be dyadic numbers. Suppose that $u, w \in L^2(\mathbb{R} \times \mathbb{R}^2)$ satisfy*

$$\text{supp } \widetilde{w} \subset \mathcal{P}_{N_0} \cap \mathcal{W}^{\pm}_{L_0}, \qquad \text{supp } \widetilde{u} \subset \mathcal{P}_{N_1} \cap \mathcal{S}_{L_1}.$$

Then we have

$$\|wu\|_{L^2_{t,x}} + \|\bar{w}u\|_{L^2_{t,x}} \lesssim L_0^{\frac{1}{2}} L_1^{\frac{1}{2}} \Big(\frac{\underline{N}_{01}}{N_1} \Big)^{\frac{1}{2}} \|w\|_{L^2_{t,x}} \|u\|_{L^2_{t,x}}.$$

Here and in the sequel we write $\zeta = (\tau, \xi)$. When $Z = \mathbb{T}^2_\gamma$ we use k instead of ξ as the discrete Fourier variable with respect to x and write

$$\int_\zeta f(\zeta) = \int_{\tau \in \mathbb{R}} \frac{1}{\gamma_1 \gamma_2} \sum_{k \in \mathbb{Z}^2_\gamma} f(\tau, k), \qquad \mathbb{Z}^2_\gamma := \gamma_1^{-1}\mathbb{Z} \times \gamma_2^{-1}\mathbb{Z}.$$

Then, the bilinear estimates (6) and (7) (and corresponding estimates for the non-periodic functions) are, after applying (5), dyadic decompositions and a duality argument, reduced to trilinear estimates of

$$\iint_{\zeta_0 = \zeta_1 - \zeta_2} \Big[\widetilde{P^{W_\pm}_{N_0, L_0} w} \Big](\zeta_0) \Big[\widetilde{P^S_{N_1, L_1} u} \Big](\zeta_1) \Big[\widetilde{P^S_{N_2, L_2} v} \Big](\zeta_2)$$

with dyadic numbers $N_j, L_j \geq 1$. Therefore, we evaluate the integral

$$\iint_{\zeta_0 = \zeta_1 - \zeta_2} f(\zeta_0) g_1(\zeta_1) g_2(\zeta_2)$$

for real-valued nonnegative functions f, g_1, g_2 with the support properties

$$\text{supp } f \subset \mathcal{P}_{N_0} \cap \mathcal{W}^{\pm}_{L_0}, \quad \text{supp } g_j \subset \mathcal{P}_{N_j} \cap \mathcal{S}_{L_j}, \quad j = 1, 2, \tag{9}$$

under various assumptions on N_j, L_j; see Lemmas 2.8–2.12 below. Again, these lemmas are stated for spatially periodic functions but equally hold for functions on the whole space (in this case, however, some of them are rougher than known estimates).

Note that an application of the Cauchy-Schwarz inequality yields a bound with loss of one derivative;

$$\iint_{\zeta_0 = \zeta_1 - \zeta_2} f(\zeta_0) g_1(\zeta_1) g_2(\zeta_2) \lesssim L_{\min}^{\frac{1}{2}} N_{\min} \|f\|_{L^2_\zeta} \|g_1\|_{L^2_\zeta} \|g_2\|_{L^2_\zeta}.$$

This bound immediately implies the trilinear estimates for the high modulation case ($L_{\max} \gtrsim N_{\max}^2$) and the very low wave frequency case ($N_0 \lesssim 1$). In particular, loss of one derivative can be recovered (at the cost of $L^{\frac{1}{2}}$).

Lemma 2.8 ([11], Proposition 3.1)**.** *Let $f, g_1, g_2 \in L^2_\zeta(\mathbb{R}\times\mathbb{Z}^2_\gamma)$ be real-valued nonnegative functions satisfying* (9)*. Assume $L_{\max} \gtrsim N_{\max}^2$. Then, we have*

$$\iint_{\zeta_0=\zeta_1-\zeta_2} f(\zeta_0)g_1(\zeta_1)g_2(\zeta_2) \lesssim L_{\max}^{\frac12}L_{\mathrm{med}}^{\frac14}L_{\min}^{\frac14}N_{\min}N_{\max}^{-1}\|f\|_{L^2}\|g_1\|_{L^2}\|g_2\|_{L^2}.$$

Lemma 2.9 ([11], Corollary 3.4)**.** *Let $f, g_1, g_2 \in L^2_\zeta(\mathbb{R}\times\mathbb{Z}^2_\gamma)$ be real-valued nonnegative functions satisfying* (9)*, and assume that $N_0 \lesssim 1$. Then, we have*

$$\iint_{\zeta_0=\zeta_1-\zeta_2} f(\zeta_0)g_1(\zeta_1)g_2(\zeta_2) \lesssim (L_0L_1L_2)^{\frac16}\|f\|_{L^2}\|g_1\|_{L^2}\|g_2\|_{L^2}.$$

In the high-low interaction case ($N_1 \not\sim N_2$) we may also assume high modulation $L_{\max} \gtrsim N_{\max}^2$, because (9) and the relation $\zeta_0 = \zeta_1 - \zeta_2$ imply

$$L_{\max} \gtrsim \big|\tau_0 \pm |k_0|\big| + \big|\tau_1 + |k_1|^2\big| + \big|\tau_2 + |k_2|^2\big| \geq \big||k_1|^2 - |k_2|^2 \mp |k_0|\big| \sim N_{\max}^2.$$

A refined analysis actually yields the following estimate with gain of $\frac12$ derivative.

Lemma 2.10 ([11], Proposition 3.2)**.** *Let $f, g_1, g_2 \in L^2_\zeta(\mathbb{R}\times\mathbb{Z}^2_\gamma)$ be real-valued nonnegative functions satisfying* (9)*, and assume $N_1 \gg N_2$ or $N_2 \gg N_1$. Then, we have*

$$\iint_{\zeta_0=\zeta_1-\zeta_2} f(\zeta_0)g_1(\zeta_1)g_2(\zeta_2) \lesssim L_{\max}^{\frac12}L_{\mathrm{med}}^{\frac38}L_{\min}^{\frac38}\underline{N}_{12}^{\frac12}\overline{N}_{12}^{-1}\|f\|_{L^2}\|g_1\|_{L^2}\|g_2\|_{L^2}.$$

For the lower modulation cases ($L_{\max} \ll N_{\max}^2$) it turns out that the frequencies (k_0, k_1, k_2) should be confined to a rather small region, so we can still have the trilinear estimates with no derivative loss. However, the proof is much more involved.

Lemma 2.11 ([11], Proposition 3.5)**.** *Let $f, g_1, g_2 \in L^2_\zeta(\mathbb{R}\times\mathbb{Z}^2_\gamma)$ be real-valued nonnegative functions satisfying* (9)*, and assume that $1 \ll N_0 \lesssim N_1 \sim N_2 \lesssim L_{\max} \ll N_1^2$. Then, we have*

$$\iint_{\zeta_0=\zeta_1-\zeta_2} f(\zeta_0)g_1(\zeta_1)g_2(\zeta_2) \lesssim L_{\max}^{\frac38+}L_{\mathrm{med}}^{\frac38+}L_{\min}^{\frac14}\Big(\frac{N_0}{N_1}\Big)^{0+}\|f\|_{L^2}\|g_1\|_{L^2}\|g_2\|_{L^2}.$$

Lemma 2.12 ([11], Proposition 3.8)**.** *Let $f, g_1, g_2 \in L^2_\zeta(\mathbb{R}\times\mathbb{Z}^2_\gamma)$ be real-valued nonnegative functions with the support properties*

$$\operatorname{supp} f \subset \{|k| \gg 1\} \cap \mathcal{W}^{\pm}_{L_0}, \quad \operatorname{supp} g_j \subset \mathcal{P}_{N_j} \cap \mathcal{S}_{L_j}, \quad j = 1, 2.$$

Assume that $1 \ll N_1 \sim N_2$ and $L_{\max} \ll N_1$. Then, we have

$$\iint_{\zeta_0=\zeta_1-\zeta_2} f(\zeta_0)g_1(\zeta_1)g_2(\zeta_2) \lesssim L_{\max}^{\frac38}L_{\mathrm{med}}^{\frac38}\|f\|_{L^2}\|g_1\|_{L^2}\|g_2\|_{L^2}.$$

Finally, we give a Strichartz-type estimate for the periodic (reduced) wave equation. It seems that the Strichartz estimates in periodic setting do not follow immediately from that on the whole space, because the finite speed of propagation does not hold for the reduced wave linear propagator $e^{\mp it|\nabla|}$. An elementary proof of it will be given in Appendix A. This lemma is also true for spatially nonperiodic functions.

Lemma 2.13. *Let $N, L \geq 1$ be dyadic numbers, and suppose that $u \in L^2(\mathbb{R}\times\mathbb{T}^2_\gamma)$ satisfies* $\operatorname{supp}\, \widetilde{u} \subset \mathcal{P}_N \cap \mathcal{W}^{\pm}_L$*. Then we have*

$$\|u\|_{L^4_{t,x}} \lesssim L^{\frac38}N^{\frac38}\|u\|_{L^2_{t,x}}.$$

3. **Modified energy and resonant decomposition.** In this section we introduce our almost conservation quantity and prepare some basic lemmas in the I-method, treating $Z = \mathbb{R}^2$ and $Z = \mathbb{T}^2_\gamma$ simultaneously.

For $s < 1$, $r \le 0$, and $N \gg 1$, we define the operator $I^S_{s,N}$ for the Schrödinger equation and the operator $I^{W_+}_{r,N}$ for the reduced wave equation as

$$I^S_{s,N} := \mathcal{F}^{-1}_\xi m_{1-s,N}(\xi)\mathcal{F}_x, \qquad I^{W_+}_{r,N} := \mathcal{F}^{-1}_\xi m_{-r,N}(\xi)\mathcal{F}_x$$

with a radial function $m_{q,N} \in C^\infty(\mathbb{R}^2)$ $(q \ge 0)$, non-increasing in $|\xi|$, such that

$$m_{q,N} = \begin{cases} 1 & \text{for} \quad |\xi| < N, \\ (N/|\xi|)^q & \text{for} \quad |\xi| > 2N. \end{cases}$$

Note that $I^S_{s,N} \in \mathcal{B}(H^s, H^1)$, $I^{W_+}_{r,N} \in \mathcal{B}(H^r, L^2)$, and $I^{W_+}_{0,N}$ is the identity operator.

Define the modified energy of (u, n_+) by

$$\begin{aligned} H(I^S_{s,N}u, I^{W_+}_{r,N}n_+)(t) := {}& \big\|\nabla I^S_{s,N}u(t)\big\|^2_{L^2} + \frac{1}{2}\big\|I^{W_+}_{r,N}n_+(t)\big\|^2_{L^2} \\ &+ \frac{1}{2}\int_Z I^{W_+}_{r,N}(n_+(t,x) + n_-(t,x))|I^S_{s,N}u(t,x)|^2 dx. \end{aligned}$$

The operators $I^S_{s,N}$ and $I^{W_+}_{r,N}$ only act on u or $\bar{u}$ and on $n_\pm$, respectively, so in what follows we abbreviate as

$$\begin{aligned} H(Iu, In_+)(t) := {}& \big\|\nabla Iu(t)\big\|^2_{L^2} + \frac{1}{2}\big\|In_+(t)\big\|^2_{L^2} \\ &+ \frac{1}{2}\int_Z I(n_+(t,x) + n_-(t,x))|Iu(t,x)|^2 dx. \end{aligned}$$

For an integer $p \ge 2$, we write $\int_{\Sigma_p}$ to denote

$$\begin{aligned} &\int_{\Sigma_p} f(\xi_1, \dots, \xi_p) \\ &:= (2\pi)^{-(p-2)} \int_{\mathbb{R}^2} \cdots \int_{\mathbb{R}^2} f(\xi_1, \dots, \xi_p)\delta(\xi_1 + \cdots + \xi_p = 0)\, d\xi_1 \cdots d\xi_p \end{aligned}$$

for the case $Z = \mathbb{R}^2$ and

$$\int_{\Sigma_p} f(k_1, \dots, k_p) := (2\pi)^{-(p-2)} \cdot \frac{1}{(\gamma_1\gamma_2)^{p-1}} \sum_{\substack{k_1, \dots, k_p \in \mathbb{Z}^2_\gamma \\ k_1 + \cdots + k_p = 0}} f(k_1, \dots, k_p)$$

for the case $Z = \mathbb{T}^2_\gamma$. Also, we use the notations $\xi_{ij} := \xi_i + \xi_j$, $m_{q,j} := m_{q,N}(\xi_j)$. Note that

$$\begin{aligned} H(Iu, In_+) = {}& \int_{\Sigma_2} |\xi_1|^2 m^2_{1-s,1}\widehat{u}(\xi_1)\widehat{\bar{u}}(\xi_2) + \frac{1}{2}\int_{\Sigma_2} m^2_{-r,1}\widehat{n}_+(\xi_1)\widehat{n}_-(\xi_2) \\ &+ \frac{1}{2}\int_{\Sigma_3} m_{1-s,1}m_{1-s,2}m_{-r,3}\widehat{u}(\xi_1)\widehat{\bar{u}}(\xi_2)\big(\widehat{n}_+ + \widehat{n}_-\big)(\xi_3). \end{aligned}$$

If $\|u_0\|_{L^2} < \|Q\|_{L^2(\mathbb{R}^2)}$, then $\|Iu(t)\|_{L^2} \le \|u(t)\|_{L^2} = \|u_0\|_{L^2} < \|Q\|_{L^2(\mathbb{R}^2)}$ and we have

$$\big\|Iu(t)\big\|^2_{H^1} + \big\|In_+(t)\big\|^2_{L^2} \sim \big\|Iu(t)\big\|^2_{L^2} + H(Iu, In_+)(t).$$

Hence, we need an almost conservation law for the modified energy, as well as the local well-posedness with the existence time written in terms of $\|Iu_0\|_{H^1} + \|In_{+0}\|_{L^2}$.

For better decay of the increment of the modified energy, we introduce another quantity

$$\widetilde{H}(u,n_+) := \int_{\Sigma_2} |\xi_1|^2 m_{1-s,1}^2 \widehat{u}(\xi_1)\widehat{\bar{u}}(\xi_2) + \frac{1}{2}\int_{\Sigma_2} m_{-r,1}^2 \widehat{n}_+(\xi_1)\widehat{n}_-(\xi_2) + \frac{1}{2}\int_{\Sigma_3} \widehat{u}(\xi_1)\widehat{\bar{u}}(\xi_2)\big(\sigma_+(\xi_1,\xi_2)\widehat{n}_+(\xi_3) + \sigma_-(\xi_1,\xi_2)\widehat{n}_-(\xi_3)\big),$$

where the multipliers $\sigma_\pm$ will be defined soon. A direct calculation using (2) shows that

$$\begin{aligned}
&\frac{d}{dt}\widetilde{H}(u,n_+) \\
&= \frac{i}{2}\int_{\Sigma_3} \Big(|\xi_1|^2 m_{1-s,1}^2 - |\xi_2|^2 m_{1-s,2}^2 + |\xi_3| m_{-r,3}^2 \\
&\qquad - \big(|\xi_1|^2 - |\xi_2|^2 + |\xi_3|\big)\sigma_+(\xi_1,\xi_2)\Big)\widehat{u}(\xi_1)\widehat{\bar{u}}(\xi_2)\widehat{n}_+(\xi_3) \\
&\quad + \frac{i}{2}\int_{\Sigma_3} \Big(|\xi_1|^2 m_{1-s,1}^2 - |\xi_2|^2 m_{1-s,2}^2 - |\xi_3| m_{-r,3}^2 \\
&\qquad - \big(|\xi_1|^2 - |\xi_2|^2 - |\xi_3|\big)\sigma_-(\xi_1,\xi_2)\Big)\widehat{u}(\xi_1)\widehat{\bar{u}}(\xi_2)\widehat{n}_-(\xi_3) \\
&\quad - \frac{i}{4}\int_{\Sigma_4} \Big(\big(\sigma_+(\xi_{13},\xi_2) - \sigma_+(\xi_1,\xi_{23})\big)\widehat{n}_+(\xi_4) \\
&\qquad + \big(\sigma_-(\xi_{13},\xi_2) - \sigma_-(\xi_1,\xi_{23})\big)\widehat{n}_-(\xi_4)\Big)\widehat{u}(\xi_1)\widehat{\bar{u}}(\xi_2)\big(\widehat{n}_+ + \widehat{n}_-\big)(\xi_3) \\
&\quad - \frac{i}{2}\int_{\Sigma_4} |\xi_{12}|\big(\sigma_+ - \sigma_-\big)(\xi_1,\xi_2)\widehat{u}(\xi_1)\widehat{\bar{u}}(\xi_2)\widehat{u}(\xi_3)\widehat{\bar{u}}(\xi_4).
\end{aligned}$$

An initial guess for $\sigma_\pm$ would be $\sigma_\pm = \sigma_\pm^Z$ defined by

$$\sigma_\pm^Z(\xi_1,\xi_2) := \frac{|\xi_1|^2 m_{1-s,1}^2 - |\xi_2|^2 m_{1-s,2}^2 \pm |\xi_{12}| m_{-r,12}^2}{|\xi_1|^2 - |\xi_2|^2 \pm |\xi_{12}|}, \tag{10}$$

which remove all the trilinear terms. Under this definition, however, $\sigma_\pm^Z$ have singularities and we will fail to estimate the quadrilinear terms. Here arises an essential difficulty in applying the I-method to the Zakharov system.

In [9, 13], they did not distinguish σ_+ and σ_- and used $\sigma_\pm = \sigma^S$ defined as

$$\sigma^S(\xi_1,\xi_2) := \frac{|\xi_1|^2 m_{1-s,1}^2 - |\xi_2|^2 m_{1-s,2}^2}{|\xi_1|^2 - |\xi_2|^2} \tag{11}$$

so that the worst terms including two derivatives would disappear in the trilinear terms. It is easy to check that σ^S is bounded. However, the remaining trilinear terms are still much more massive than the quadrilinear terms. In fact, it was exactly these terms that determined the regularity threshold for global well-posedness, both in [9] ($s > \frac{3}{4}$) and in [13] ($s > \frac{2}{3}$).

We will use both (10) and (11) to obtain a slightly better estimate. It turns out that when we use σ^S the biggest contribution in the remaining trilinear terms comes from the frequency region for high-low interactions ($|\xi_1| \not\sim |\xi_2|$), which has no intersection with the region $\big||\xi_1|^2 - |\xi_2|^2\big| \sim |\xi_{12}|$, where $\sigma_\pm^Z$ become unbounded. Motivated by this fact, we shall employ the following definition.

$$\sigma_\pm(\xi_1,\xi_2) := \begin{cases} \sigma_\pm^Z(\xi_1,\xi_2) & \text{if} \quad \big||\xi_1|^2 - |\xi_2|^2\big| > 2|\xi_{12}|, \\ \sigma^S(\xi_1,\xi_2) & \text{if} \quad \big||\xi_1|^2 - |\xi_2|^2\big| \le 2|\xi_{12}|. \end{cases} \tag{12}$$

The above definition can be regarded as a variant of 'resonant decomposition' introduced in [8] in the context of two-dimensional cubic NLS, since we consider resonant and non-resonant frequencies separately to prevent the multipliers from becoming singular. Observe that $\sigma_\pm(\xi_1,\xi_2)=\sigma_\pm(-\xi_1,-\xi_2)=\sigma_\mp(\xi_2,\xi_1)$, and that $\sigma_\pm(\xi_1,\xi_2)\equiv 1$ when $\max\{|\xi_1|,|\xi_2|\}\le N/2$. Moreover, we can easily show the following lemma. In particular, $\sigma_\pm(\xi_1,\xi_2)$ are bounded.

Lemma 3.1. *The multipliers $\sigma_\pm(\xi_1,\xi_2)$ given in (12) obey the following estimates.*
(i) If $|\xi_1|\gg|\xi_2|$, then $|\sigma_\pm(\xi_1,\xi_2)-m_{1-s,1}^2|\lesssim\frac{|\xi_2|^2}{|\xi_1|^2}+\frac{1}{|\xi_1|}$.
(ii) If $|\xi_1|\sim|\xi_2|$, then $|\sigma_\pm(\xi_1,\xi_2)|\lesssim 1$.

Proof. It was shown in [9], Lemma 3.4, that $|\sigma^S(\xi_1,\xi_2)|\lesssim 1$ (this is true for any $s\le 1$). Therefore, we restrict our attention to the case $\big||\xi_1|^2-|\xi_2|^2\big|>2|\xi_{12}|$, where $\sigma_\pm=\sigma_\pm^Z$ and $\big||\xi_1|^2-|\xi_2|^2\pm|\xi_{12}|\big|\ge\frac12\big||\xi_1|^2-|\xi_2|^2\big|$. If $|\xi_1|\gg|\xi_2|$, then

$$\begin{aligned}|\sigma_\pm(\xi_1,\xi_2)-m_{1-s,1}^2|&=\Big|\frac{|\xi_2|^2(m_{1-s,1}^2-m_{1-s,2}^2)\pm|\xi_{12}|(m_{-r,12}^2-m_{1-s,1}^2)}{|\xi_1|^2-|\xi_2|^2\pm|\xi_{12}|}\Big|\\&\lesssim\frac{|\xi_2|^2+|\xi_{12}|}{\big||\xi_1|^2-|\xi_2|^2\big|}\sim\frac{|\xi_2|^2+|\xi_1|}{|\xi_1|^2},\end{aligned}$$

which implies (i). On the other hand, it holds that

$$\begin{aligned}|\sigma_\pm^Z(\xi_1,\xi_2)|&\lesssim\frac{\big||\xi_1|^2m_{1-s,1}^2-|\xi_2|^2m_{1-s,2}^2\big|+|\xi_{12}|m_{-r,12}^2}{\big||\xi_1|^2-|\xi_2|^2\big|}\\&\le|\sigma^S|+\frac{|\xi_{12}|}{\big||\xi_1|^2-|\xi_2|^2\big|}\lesssim 1,\end{aligned}$$

which shows (ii). □

We next show that the new quantity $\widetilde{H}(u,n_+)$, which is our almost conserved quantity, is always close to the (first generation) modified energy $H(Iu,In_+)$.

Proposition 1 (Fixed-time difference)**.** *Let $1>s>\frac12$, $0\ge r>-\frac12$. Suppose that $r>1-2s$. Then, for any $t\in\mathbb{R}$, we have*

$$\big|H(Iu,In_+)(t)-\widetilde{H}(u,n_+)(t)\big|\lesssim N^{-1+}\big\|Iu(t)\big\|_{H^1}^2\big\|In_+(t)\big\|_{L^2}.$$

Proof. From the definition and boundedness of multipliers, we have

$$\begin{aligned}&\big|H(Iu,In_+)(t)-\widetilde{H}(u,n_+)(t)\big|\\&\le\frac12\int_{\Sigma_3}|\widehat{u}(\xi_1)||\widehat{\bar{u}}(\xi_2)|\Big|\big(m_{1-s,1}m_{1-s,2}m_{-r,3}-\sigma_+(\xi_1,\xi_2)\big)\widehat{n}_+(\xi_3)\\&\qquad\qquad+\big(m_{1-s,1}m_{1-s,2}m_{-r,3}-\sigma_-(\xi_1,\xi_2)\big)\widehat{n}_-(\xi_3)\Big|\\&\le\frac12\int_{\Sigma_3}\mathbf{1}_{\{|\xi_1|>N/2\text{ or }|\xi_2|>N/2\}}(\xi_1,\xi_2)|\widehat{u}(\xi_1)||\widehat{\bar{u}}(\xi_2)|\big(|\widehat{n}_+(\xi_3)|+|\widehat{n}_-(\xi_3)|\big).\end{aligned}$$

We may assume that all of $\widehat{u}$, $\widehat{\bar{u}}$, $\widehat{n}_\pm$ are real-valued and non-negative. Symmetry allows us to assume $|\xi_1|\ge|\xi_2|$. Also, it suffices to consider the case of n_+. Then the above is bounded by

$$\sum_{N_1\gtrsim N}\sum_{N_2\le N_1}\sum_{N_0\lesssim N_1}(\frac{N_1}{N})^{1-s}\Big((\frac{N_2}{N})^{1-s}+1\Big)\Big((\frac{N_0}{N})^{-r}+1\Big)$$
$$\times\big\|P_{N_1}Iu\big\|_{L^2}\big\|P_{N_2}Iu\big\|_{L^\infty}\big\|P_{N_0}In_+\big\|_{L^2}$$
$$\lesssim\sum_{N_1\gtrsim N}\sum_{N_2\le N_1}\sum_{N_0\lesssim N_1}(\frac{N_1}{N})^{1-s}\Big((\frac{N_2}{N})^{1-s}+1\Big)\Big((\frac{N_0}{N})^{-r}+1\Big)$$
$$\times\frac{1}{N_1}\big\|P_{N_1}Iu\big\|_{H^1}\big\|P_{N_2}Iu\big\|_{H^1}\big\|P_{N_0}In_+\big\|_{L^2}.$$

Since $2(1-s)-r<1$, the prefactor is exceeded by $N^{-1+}N_1^{0-}$. Applying the Cauchy-Schwarz inequality to each summation we reach the claim. □

4. **Global solutions for the periodic case.** In this section we consider the periodic case and prove Theorem 1.1. Since we always assume the wave data to be in L^2, the operator I is only applied to the Schrödinger equation, so we use the notation $m(k)$ to denote $m_{1-s,N}(k)$ for simplicity.

Now we shall establish an almost conservation law for $\widetilde{H}(u,n_+)$.

Proposition 2 (Almost conservation law). *Let $1>s>\frac{1}{2}$, $r=0$, $0<\delta\le 1$, and let (u,n_+) be a smooth solution to (2) on $(t,x)\in[0,\delta]\times\mathbb{T}^2_\gamma$. Then, we have*

$$|\widetilde{H}(u,n_+)(\delta)-\widetilde{H}(u,n_+)(0)|\lesssim N^{-1+}\delta^{\frac{1}{2}-}\big\|Iu\big\|^2_{X_S^{1,\frac{1}{2},1}(\delta)}\big\|n_+\big\|_{X_{W_+}^{0,\frac{1}{2},1}(\delta)}$$
$$+(N^{-2+}+N^{-\frac{5}{4}+}\delta^{\frac{1}{4}-}+N^{-1+}\delta^{1-})$$
$$\times\Big(\big\|Iu\big\|^2_{X_S^{1,\frac{1}{2},1}(\delta)}\big\|n_+\big\|^2_{X_{W_+}^{0,\frac{1}{2},1}(\delta)}+\big\|Iu\big\|^4_{X_S^{1,\frac{1}{2},1}(\delta)}\Big).$$

Proof. From the definition,

$$\widetilde{H}(u,n_+)(\delta)-\widetilde{H}(u,n_+)(0)=\int_0^\delta\frac{d}{dt}\widetilde{H}(u,n_+)(t)\,dt$$
$$=\frac{i}{2}\int_0^\delta\int_{\Sigma_3}\mathbf{1}_{\{||k_1|^2-|k_2|^2|\le 2|k_{12}|\}}(k_1,k_2)|k_{12}|\widehat{u}(t,k_1)\widehat{\bar{u}}(t,k_2)\tag{13}$$
$$\times\Big(\big(1-\sigma_+(k_1,k_2)\big)\widehat{n}_+(t,k_3)-\big(1-\sigma_-(k_1,k_2)\big)\widehat{n}_-(t,k_3)\Big)\,dt$$
$$-\frac{i}{4}\int_0^\delta\int_{\Sigma_4}\widehat{u}(t,k_1)\widehat{\bar{u}}(t,k_2)\big(\widehat{n}_++\widehat{n}_-\big)(t,k_3)$$
$$\times\Big(\big(\sigma_+(k_{13},k_2)-\sigma_+(k_1,k_{23})\big)\widehat{n}_+(t,k_4)\tag{14}$$
$$+\big(\sigma_-(k_{13},k_2)-\sigma_-(k_1,k_{23})\big)\widehat{n}_-(t,k_4)\Big)\,dt$$
$$-\frac{i}{2}\int_0^\delta\int_{\Sigma_4}|k_{12}|\big(\sigma_+-\sigma_-\big)(k_1,k_2)\widehat{u}(t,k_1)\widehat{\bar{u}}(t,k_2)\widehat{u}(t,k_3)\widehat{\bar{u}}(t,k_4)\,dt.\tag{15}$$

Estimate of (13). We may assume $\max\{|k_1|,|k_2|\}>N$; otherwise (13) $=0$. Note that $\big||k_1|^2-|k_2|^2\big|\le 2|k_{12}|$ implies $\big||k_1|-|k_2|\big|\le 2$. Therefore, we may assume $|k_1|\sim|k_2|\gtrsim N$. We shall see only the first term in (13), since the second one is exactly the complex conjugate of the first one. Thus, we need to estimate

$$\begin{aligned}&\Big|\int_{\mathbb{R}}\int_{\Sigma_3}\mathbf{1}_{\{||k_1|^2-|k_2|^2|\le 2|k_{12}|\}}(k_1,k_2)|k_{12}|\\&\qquad\times|\psi_\delta\widehat{u}(t,k_1)\psi_\delta\widehat{\bar{u}}(t,k_2)\big(1-\sigma_+(k_1,k_2)\big)\chi_\delta\widehat{n}_+(t,k_3)\,dt\Big|\\&\lesssim\int_{\zeta_0=\zeta_1-\zeta_2}\mathbf{1}_{\{||k_1|^2-|k_2|^2-|k_0||\lesssim|k_0|\}}|k_0||\widetilde{\psi_\delta u}(\zeta_1)\widetilde{\psi_\delta u}(\zeta_2)\widetilde{\chi_\delta n_+}(\zeta_0)|\\&\le\sum_{N_1\sim N_2\gtrsim N}\sum_{N_0\lesssim N_1}\sum_{L_0,L_1,L_2}N_0\int_{\zeta_0=\zeta_1-\zeta_2}\Big|\big[\widetilde{P^S_{N_1,L_1}\psi_\delta u}\big](\zeta_1)\big[\widetilde{P^S_{N_2,L_2}\psi_\delta u}\big](\zeta_2)\\&\qquad\times\big[\widetilde{P^{W_+}_{N_0,L_0}\chi_\delta n_+}\big](\zeta_0)\Big|,\end{aligned}$$

where $\chi_\delta:=\mathbf{1}_{[0,\delta]}$. We remark that in the above summation, since $\big||k_1|^2-|k_2|^2-|k_0|\big|\lesssim|k_0|$, either $L_{\max}\lesssim N_0$ or $L_{\max}\sim L_{\text{med}}$ holds. Then, from Lemmas 2.8, 2.9, 2.11, and 2.12, this is bounded by

$$\begin{aligned}&\sum_{N_1\sim N_2\gtrsim N}\sum_{N_0\lesssim N_1}\sum_{L_0,L_1,L_2}N_0(L_{\max}L_{\text{med}})^{\frac{3}{8}+}L_{\min}^{\frac{1}{4}}\\&\qquad\times\big\|P^S_{N_1,L_1}\psi_\delta u\big\|_{L^2_{t,x}}\big\|P^S_{N_2,L_2}\psi_\delta u\big\|_{L^2_{t,x}}\big\|P^{W_+}_{N_0,L_0}\chi_\delta n_+\big\|_{L^2_{t,x}}\\&\lesssim\sum_{N_1\sim N_2\gtrsim N}N_1\big\|P_{N_1}\psi_\delta u\big\|_{X_S^{0,\frac{3}{8}+,1}}\big\|P_{N_2}\psi_\delta u\big\|_{X_S^{0,\frac{3}{8}+,1}}\big\|\chi_\delta n_+\big\|_{X_{W_+}^{0,\frac{1}{4},1}}+\text{ similar terms}\\&\lesssim\sum_{N_1\sim N_2\gtrsim N}\frac{1}{N_1}\Big(\frac{N_1}{N}\Big)^{2(1-s)}\big\|P_{N_1}\psi_\delta Iu\big\|_{X_S^{1,\frac{3}{8}+,1}}\big\|P_{N_2}\psi_\delta Iu\big\|_{X_S^{1,\frac{3}{8}+,1}}\big\|\chi_\delta n_+\big\|_{X_{W_+}^{0,\frac{1}{4},1}}\\&\qquad+\text{ similar terms}\qquad(16)\\&\lesssim N^{-1}\big\|\psi_\delta Iu\big\|_{X_S^{1,\frac{3}{8}+,1}}\big\|\psi_\delta Iu\big\|_{X_S^{1,\frac{3}{8}+,1}}\big\|\chi_\delta n_+\big\|_{X_{W_+}^{0,\frac{1}{4},1}}+\text{ similar terms}\\&\lesssim N^{-1}\delta^{\frac{1}{2}-}\big\|Iu\big\|^2_{X_S^{1,\frac{1}{2},1}}\big\|n_+\big\|_{X_{W_+}^{0,\frac{1}{2},1}}.\end{aligned}$$

In the last inequality we have used (4) and

$$\big\|\chi_\delta n\big\|_{X^{s,b,1}}\lesssim\delta^{\frac{1}{2}-b}\big\|n\big\|_{X^{s,\frac{1}{2},1}},\quad 0<b<\frac{1}{2},\tag{17}$$

which can be verified similarly to (4).

Estimate of (14). Motivated by the argument in [9], we add

$$\frac{i}{4}\int_0^\delta\int_{\Sigma_4}\widehat{u}(t,k_1)\widehat{\bar{u}}(t,k_2)\big(\widehat{n}_++\widehat{n}_-\big)(t,k_3)\big(\widehat{n}_++\widehat{n}_-\big)(t,k_4)\cdot\big(m_{13}^2-m_{23}^2\big)\,dt=0$$

to (14) and consider the estimate of

$$\begin{aligned}&\frac{i}{4}\int_0^\delta\int_{\Sigma_4}\widehat{u}(t,k_1)\widehat{\bar{u}}(t,k_2)\big(\widehat{n}_++\widehat{n}_-\big)(t,k_3)\\&\qquad\times\big(\sigma_+(k_{13},k_2)-m_{13}^2-\sigma_-(k_{23},k_1)+m_{23}^2\big)\widehat{n}_+(t,k_4)\,dt,\\&\frac{i}{4}\int_0^\delta\int_{\Sigma_4}\widehat{u}(t,k_1)\widehat{\bar{u}}(t,k_2)\big(\widehat{n}_++\widehat{n}_-\big)(t,k_3)\\&\qquad\times\big(\sigma_-(k_{13},k_2)-m_{13}^2-\sigma_+(k_{23},k_1)+m_{23}^2\big)\widehat{n}_-(t,k_4)\,dt.\end{aligned}$$

It is then sufficient to estimate

$$\begin{aligned}&\Big|\int_0^\delta\int_{\Sigma_4}\big(\sigma_\pm(k_{13},k_2)-m_{13}^2\big)\widehat{u}(t,k_1)\widehat{\bar{u}}(t,k_2)\widehat{n}_\pm(t,k_3)\widehat{n}_\pm(t,k_4)\,dt\Big|\\ &\lesssim\int_{\zeta_1+\zeta_2+\zeta_3+\zeta_4=0}\big|\sigma_\pm(k_{13},k_2)-m_{13}^2\big|\big|\widetilde{\psi_\delta u}(\zeta_1)\widetilde{\psi_\delta\bar{u}}(\zeta_2)\widetilde{\chi_\delta n_\pm}(\zeta_3)\widetilde{\chi_\delta n_\pm}(\zeta_4)\big|\\ &\lesssim\sum_{N_1,\dots,N_4\ge 1}\int_{\zeta_1+\zeta_2+\zeta_3+\zeta_4=0}\big|\sigma_\pm(k_{13},k_2)-m_{13}^2\big|\\ &\qquad\qquad\times\big|\widetilde{\psi_\delta P_{N_1}u}(\zeta_1)\widetilde{\psi_\delta P_{N_2}\bar{u}}(\zeta_2)\widetilde{\chi_\delta P_{N_3}n_\pm}(\zeta_3)\widetilde{\chi_\delta P_{N_4}n_\pm}(\zeta_4)\big|\end{aligned}$$

with an arbitrary choice of $\pm$. However, since the choice of $n_\pm$ plays no role in the following, we consider the case n_+ only, and write

$$\widetilde{u_1}:=|\widetilde{\psi_\delta P_{N_1}u}|,\quad \widetilde{\widetilde{u_2}}:=|\widetilde{\psi_\delta P_{N_2}\bar{u}}|,\quad \widetilde{n_3}:=|\widetilde{\chi_\delta P_{N_3}n_+}|,\quad \widetilde{n_4}:=|\widetilde{\chi_\delta P_{N_4}n_+}|$$

for simplicity. We thus need to estimate

$$\sum_{N_1,\dots,N_4\ge 1}\int_{\zeta_1+\zeta_2+\zeta_3+\zeta_4=0}\big|\sigma_\pm(k_{13},k_2)-m_{13}^2\big|\widetilde{u_1}(\zeta_1)\widetilde{\widetilde{u_2}}(\zeta_2)\widetilde{n_3}(\zeta_3)\widetilde{n_4}(\zeta_4). \tag{18}$$

First, we state an estimate which will be frequently used later.

Lemma 4.1. *Suppose that u and n satisfy*

$$\operatorname{supp}\widetilde{u}\subset\mathcal{P}_{N_1},\qquad \operatorname{supp}\widetilde{n}\subset\mathcal{P}_N$$

for some dyadic $N_1,N\ge 1$. Then, for any $0<\varepsilon\ll 1$, we have

$$\|un\|_{L^2_{t,x}}\lesssim\|u\|_{X_S^{2\varepsilon,\frac12,1}}\|n\|_{X_{W_\pm}^{0,\frac12-\varepsilon,1}}+\|u\|_{X_S^{\frac12+\varepsilon,\frac12,1}}\|n\|_{X_{W_\pm}^{0,0,1}}. \tag{19}$$

Here, the $\pm$ signs are allowed to be chosen as $(+,+)$ or $(-,-)$ only.

Proof. From Lemma 2.6, we have

$$\|un\|_{L^2_{t,x}}\lesssim\|u\|_{X_S^{0,\frac12,1}}\|n\|_{X_{W_\pm}^{0,\frac12,1}}+\|u\|_{X_S^{\frac12,\frac12,1}}\|n\|_{X_{W_\pm}^{0,0,1}}.$$

On the other hand, an application of the Hölder inequality shows that

$$\|un\|_{L^2_{t,x}}\lesssim\|u\|_{L^\infty_{t,x}}\|n\|_{L^2_{t,x}}\lesssim\|u\|_{X_S^{1,\frac12,1}}\|n\|_{L^2_{t,x}}.$$

The required estimate is obtained from an interpolation between them. $\square$

Let us begin to estimate (18). First of all we note that the multiplier $\sigma_\pm(k_{13},k_2)-m_{13}^2$ vanishes if $N_2,N_4\ll N$. We consider some cases separately.

Case 1. $N_2\gtrsim N_4$. In this case we can assume $N_2\gtrsim N$ and bound the multiplier by 1. Also, we see that either N_1 or N_2 has to be comparable to the biggest one among N_j's.

(i) Consider the case $N_1 \gtrsim N$. We use (19) twice to have

$$\begin{aligned}
(18) &\lesssim \sum_{N_1,\dots,N_4} \|u_1 n_3\|_{L^2} \|\overline{u_2} n_4\|_{L^2} \\
&\lesssim \sum_{N_1,\dots,N_4} \Big(\frac{N_1}{N}\Big)^{1-s} \Big(\frac{N_2}{N}\Big)^{1-s} \frac{1}{N_1 N_2} \\
&\qquad \times \Big(N_1^{2\varepsilon} \|Iu_1\|_{X_S^{1,\frac12,1}} \|n_3\|_{X_{W_+}^{0,\frac12-\varepsilon,1}} + N_1^{\frac12+\varepsilon} \|Iu_1\|_{X_S^{1,\frac12,1}} \|n_3\|_{X_{W_+}^{0,0,1}} \Big) \\
&\qquad \times \Big(N_2^{2\varepsilon} \|Iu_2\|_{X_S^{1,\frac12,1}} \|n_4\|_{X_{W_+}^{0,\frac12-\varepsilon,1}} + N_2^{\frac12+\varepsilon} \|Iu_2\|_{X_S^{1,\frac12,1}} \|n_4\|_{X_{W_+}^{0,0,1}} \Big).
\end{aligned}$$

Since $s > \frac12$, there remains $N_1^{0-} N_2^{0-}$ if we choose $\varepsilon > 0$ sufficiently small. Summing over N_j's and then applying (4) and (17), we obtain a bound of

$$(N^{-2+} + N^{-1+}\delta^{1-}) \|Iu\|^2_{X_S^{1,\frac12,1}} \|n_+\|^2_{X_{W_+}^{0,\frac12,1}}.$$

(ii) Consider the case $N_1 \ll N$, where we may assume $N_2 \gg N_1$ and N_2 is comparable to the max. We further decompose the integral as

$$\sum_{N_2 \gtrsim N} \sum_{N_1 \ll N} \sum_{N_3, N_4 \lesssim N_2} \sum_{L_1,\dots,L_4 \ge 1} \int_{\zeta_1+\dots+\zeta_4=0} \widetilde{Q_{L_1}^S u_1}(\zeta_1) \overline{\widetilde{Q_{L_2}^S u_2}}(\zeta_2) \times \widetilde{Q_{L_3}^{W_+} n_3}(\zeta_3) \widetilde{Q_{L_4}^{W_+} n_4}(\zeta_4). \tag{20}$$

Observe that if $\zeta_1 + \dots + \zeta_4 = 0$, then

$$\begin{aligned}
\overline{L}_{1234} &\gtrsim \big| (\tau_1 + |k_1|^2) + (\tau_2 - |k_2|^2) + (\tau_3 + |k_3|) + (\tau_4 + |k_4|) \big| \\
&= \big| |k_1|^2 - |k_2|^2 + |k_3| + |k_4| \big| \gtrsim N_2^2.
\end{aligned}$$

We begin with the case $\overline{L}_{34} = \overline{L}_{1234}$. Without loss of generality we assume L_3 is the biggest one. We apply the Hölder inequality and Lemma 2.6 (ii) to obtain that

$$\begin{aligned}
&\sum_{L_1,\dots,L_4 \ge 1} \int_{\zeta_1+\dots+\zeta_4=0} \widetilde{Q_{L_1}^S u_1}(\zeta_1) \overline{\widetilde{Q_{L_2}^S u_2}}(\zeta_2) \widetilde{Q_{L_3}^{W_+} n_3}(\zeta_3) \widetilde{Q_{L_4}^{W_+} n_4}(\zeta_4) \\
&\lesssim \sum_{L_1,\dots,L_4 \ge 1} \|Q_{L_1}^S u_1\|_{L^\infty_{t,x}} \|Q_{L_3}^{W_+} n_3\|_{L^2_{t,x}} \|\overline{Q_{L_2}^S u_2} Q_{L_4}^{W_+} n_4\|_{L^2_{t,x}} \\
&\lesssim \sum_{L_2, L_4 \ge 1} \|u_1\|_{X_S^{1,\frac12,1}} N_2^{-1+} \|n_3\|_{X_{W_+}^{0,\frac12-,1}} \\
&\qquad \times \underline{L}_{24}^{\frac12} \Big(\frac{\overline{L}_{24}}{N_2} + 1 \Big)^{\frac12} N_4^{\frac12} \|Q_{L_2}^S u_2\|_{L^2_{t,x}} \|Q_{L_4}^{W_+} n_4\|_{L^2_{t,x}} \\
&\lesssim \Big(\frac{N_2}{N} \Big)^{1-s} N_2^{-2+} N_4^{\frac12} \|Iu_1\|_{X_S^{1,\frac12,1}} \|Iu_2\|_{X_S^{1,\frac12,1}} \|n_3\|_{X_{W_+}^{0,\frac12-,1}} \\
&\qquad \times \Big(N_2^{-\frac12} \|n_4\|_{X_{W_+}^{0,\frac12-,1}} + \|n_4\|_{X_{W_+}^{0,0,1}} \Big).
\end{aligned}$$

At the last inequality we have used $\overline{L}_{24}^{0+} \le L_3^{0+}$. We perform the summation in N_j's and use (4) and (17), concluding

$$(20) \lesssim (N^{-2+} + N^{-\frac32+}\delta^{\frac12-}) \|Iu\|^2_{X_S^{1,\frac12,1}} \|n_+\|^2_{X_{W_+}^{0,\frac12,1}}.$$

We next treat $\overline{L}_{12} = \overline{L}_{1234} \gg \overline{L}_{34}$, which is actually the worst case. (When L_1 is the max, however, we can have some better bound than obtained below.) If L_2 is the max, (20) is bounded by

$$\sum_{N_2 \gtrsim N} \sum_{N_1 \ll N} \sum_{N_3, N_4 \lesssim N_2} \sum_{L_1,\dots,L_4 \geq 1} \big\|Q^S_{L_1} u_1\big\|_{L^\infty_{t,x}} \big\|Q^S_{L_2} u_2\big\|_{L^2_{t,x}} \times \big\|Q^{W_+}_{L_3} n_3\big\|_{L^4_{t,x}} \big\|Q^{W_+}_{L_4} n_4\big\|_{L^4_{t,x}}.$$

Now, we use the L^4 Strichartz estimate for wave (Lemma 2.13) to bound this by

$$\begin{aligned}
&\sum_{N_2 \gtrsim N} \sum_{N_1 \ll N} \sum_{N_3, N_4 \lesssim N_2} \big\|u_1\big\|_{X^{1,\frac12,1}_S} N_2^{-1} \big\|u_2\big\|_{X^{0,\frac12,1}_S} (N_3N_4)^{\frac38} \big\|n_3\big\|_{X^{0,\frac38,1}_{W_+}} \big\|n_4\big\|_{X^{0,\frac38,1}_{W_+}} \\
&\lesssim \sum_{N_2 \gtrsim N} \sum_{N_1 \ll N} \sum_{N_3, N_4 \lesssim N_2} \Big(\frac{N_2}{N}\Big)^{1-s} N_2^{-\frac54} \big\|Iu_1\big\|_{X^{1,\frac12,1}_S} \big\|Iu_2\big\|_{X^{1,\frac12,1}_S} \\
&\qquad \times \big\|n_3\big\|_{X^{0,\frac38,1}_{W_+}} \big\|n_4\big\|_{X^{0,\frac38,1}_{W_+}} \\
&\lesssim N^{-\frac54+} \delta^{\frac14-} \big\|Iu\big\|^2_{X^{1,\frac12,1}_S} \big\|n_+\big\|^2_{X^{0,\frac12,1}_{W_+}}.
\end{aligned}$$

If L_1 is the max, we first apply the Hölder inequality as $L^2_t L^\infty_x \cdot L^\infty_t L^2_x \cdot L^4_{t,x} \cdot L^4_{t,x}$ and then make a similar argument, concluding the same bound.

Case 2. $N_2 \ll N_4$. In this case $|k_{13}| = |k_{24}| \gg |k_2|$ in the integral (18), so we use Lemma 3.1 (1) to replace the multiplier with $\frac{N_2^2}{N_4^2} + \frac{1}{N_4}$. We may also assume $N_4 \gtrsim N$.

(i) The case $N_1 \gtrsim N$. We follow the argument in Case 1 (i). Applying (19) twice, we have

$$\begin{aligned}
(18) \lesssim \sum_{N_1,\dots,N_4} &\Big(\frac{N_2^2}{N_4^2} + \frac{1}{N_4}\Big)\Big(\frac{N_1}{N}\Big)^{1-s}\Big(\Big(\frac{N_2}{N}\Big)^{1-s} + 1\Big)\frac{1}{N_1N_2} \\
&\times \Big(N_1^{2\varepsilon}\big\|Iu_1\big\|_{X^{1,\frac12,1}_S}\big\|n_3\big\|_{X^{0,\frac12-\varepsilon,1}_{W_+}} + N_1^{\frac12+\varepsilon}\big\|Iu_1\big\|_{X^{1,\frac12,1}_S}\big\|n_3\big\|_{X^{0,0,1}_{W_+}}\Big) \\
&\times \Big(N_2^{2\varepsilon}\big\|Iu_2\big\|_{X^{1,\frac12,1}_S}\big\|n_4\big\|_{X^{0,\frac12-\varepsilon,1}_{W_+}} + N_2^{\frac12+\varepsilon}\big\|Iu_2\big\|_{X^{1,\frac12,1}_S}\big\|n_4\big\|_{X^{0,0,1}_{W_+}}\Big).
\end{aligned}$$

After some calculation we reach the bound with prefactor $N^{-2+} + N^{-1+}\delta^{1-}$.

(ii) The case $N_1 \ll N$, where $N_3 \sim N_4$ is the max. If N_2 is so small that $N_2^2 \lesssim N_4$, the multiplier is bounded by $\frac{1}{N_4}$ and we obtain

$$\begin{aligned}
(18) &\lesssim \sum_{N_1,\dots,N_4} \frac{1}{N_4}\Big(\Big(\frac{N_2}{N}\Big)^{1-s} + 1\Big)\big\|Iu_1\big\|_{L^{2+}_t L^\infty_x}\big\|Iu_2\big\|_{L^{2+}_t L^\infty_x}\big\|n_3\big\|_{L^{\infty-}_t L^2_x}\big\|n_4\big\|_{L^{\infty-}_t L^2_x} \\
&\lesssim \sum_{N_1,\dots,N_4} \frac{1}{N_4}\Big(\Big(\frac{N_2}{N}\Big)^{1-s} + 1\Big)\big\|Iu_1\big\|_{X^{1,0+,1}_S}\big\|Iu_2\big\|_{X^{1,0+,1}_S}\big\|n_3\big\|_{X^{0,\frac12-,1}_{W_+}}\big\|n_4\big\|_{X^{0,\frac12-,1}_{W_+}} \\
&\lesssim N^{-1+}\delta^{1-}\big\|Iu\big\|^2_{X^{1,\frac12,1}_S}\big\|n_+\big\|^2_{X^{0,\frac12,1}_{W_+}}.
\end{aligned}$$

We thus assume $N_2^2 \gg N_4$. Now, we can employ the same argument as Case 1 (ii) with a minor modification exploiting the term $\frac{N_2^2}{N_4^2}$. The bound will be $N^{-\frac54+}\delta^{\frac14-}$.

Estimate of (15). We bound the multiplier $\sigma_+ - \sigma_-$ by 1, and decompose each function dyadically in k, obtaining a bound on (15) of

$$\sum_{N_1,\dots,N_4} (N_1+N_2)\int_{\mathbb{R}}\int_{\mathbb{T}^2_\gamma} u_1\overline{u_2}u_3\overline{u_4}\,dx\,dt, \tag{21}$$

where

$$\widetilde{u_1} := |\widetilde{\psi_\delta P_{N_1} u}|, \quad \widetilde{u_2} := |\widetilde{\psi_\delta P_{N_2} \bar{u}}|, \quad \widetilde{u_3} := |\widetilde{\chi_\delta P_{N_3} u}|, \quad \widetilde{u_4} := |\widetilde{\chi_\delta P_{N_4} \bar{u}}|.$$

Without loss of generality we assume $N_1 \geq N_2$, which implies $N_1 \gtrsim N$; otherwise the multiplier vanishes. We may also assume that at least two of N_j's are $\gtrsim N$

Case 1. Two of N_j's $\ll N$. It will be sufficient to consider the particular case $N_1, N_2 \gtrsim N \gg N_3, N_4$, where $N_1 \sim N_2$ is the max. From a Hölder argument,

$$\begin{aligned}
(21) &\lesssim \sum_{N_1\sim N_2 \gtrsim N \gg N_3,N_4} N_1 \|u_1\|_{L_t^{2+}L_x^2}\|u_2\|_{L_t^{2+}L_x^2}\|u_3\|_{L_t^{\infty-}L_x^\infty}\|u_4\|_{L_t^{\infty-}L_x^\infty}\\
&\lesssim \sum_{N_1,\dots,N_4} \Big(\frac{N_1}{N}\Big)^{1-s}\Big(\frac{N_2}{N}\Big)^{1-s}\frac{1}{N_1}\|Iu_1\|_{X_S^{1,0+,1}}\|Iu_2\|_{X_S^{1,0+,1}}\\
&\qquad\qquad \times \|Iu_3\|_{X_S^{1,\frac12-,1}}\|Iu_4\|_{X_S^{1,\frac12-,1}}\\
&\lesssim N^{-1+}\delta^{1-}\|Iu\|^4_{X_S^{1,\frac12,1}}.
\end{aligned}$$

Case 2. More than two of N_j's $\gtrsim N$. Prepare the following lemma.

Lemma 4.2. *Suppose that u_1 and u_2 satisfy*

$$\operatorname{supp}\,\widetilde{u_1} \subset \mathcal{P}_{N_1}, \qquad \operatorname{supp}\,\widetilde{u_2} \subset \mathcal{P}_{N_2}$$

for some dyadic $N_1, N_2 \geq 1$. Then, for any $0<\varepsilon\ll 1$, we have

$$\|u_1u_2\|_{L^2_{t,x}} \lesssim \overline{N}_{12}^{\varepsilon}\big(\|u_1\|_{X_S^{0,\frac12-\varepsilon,1}}\|u_2\|_{X_S^{4\varepsilon,\frac12-\varepsilon,1}} + \|u_1\|_{X_S^{0,\frac12-\varepsilon,1}}\|u_2\|_{X_S^{\frac12+2\varepsilon,\varepsilon,1}}\big). \tag{22}$$

Proof. Making dyadic decompositions, we have

$$\|u_1u_2\|_{L^2} = \|u_1\overline{u_2}\|_{L^2_{t,x}} \lesssim \sum_{N_0\leq \overline{N}_{12}}\sum_{L_1,L_2\geq 1}\|P_{N_0}(Q^S_{L_1}u_1\cdot\overline{Q^S_{L_2}u_2})\|_{L^2}. \tag{23}$$

We use Lemma 2.6 (i) for $N_0 \geq 2$ and Lemma 2.9 for $N_0 = 1$,

$$(23) \lesssim \sum_{N_0\leq\overline{N}_{12}}\sum_{L_1,L_2\geq 1} \underline{L}_{12}^{\frac12}(\overline{L}_{12}^{\frac12}+N_2^{\frac12})\|Q^S_{L_1}u_1\|_{L^2}\|Q^S_{L_2}u_2\|_{L^2}.$$

On the other hand, we apply the Hölder inequality to obtain

$$(23) \lesssim \sum_{N_0\leq\overline{N}_{12}}\sum_{L_1,L_2\geq 1} \underline{L}_{12}^{\frac12}N_2\|Q^S_{L_1}u_1\|_{L^2}\|Q^S_{L_2}u_2\|_{L^2}.$$

The required estimate is obtained from an interpolation between them. ☐

We go back to the estimate of (21). Define the biggest, the second biggest and the smallest one among N_2, N_3, N_4 as N_a, N_b and N_c, respectively. Then, we may assume that $N_a \gtrsim N_1, N_b, N_c$. From (22), we obtain

$$\begin{aligned}
(21) \lesssim & \sum_{N_1, N_a, N_b \gtrsim N, N_c} N_1 \big\| u_1 u_b \big\|_{L^2} \big\| u_a u_c \big\|_{L^2} \\
\lesssim & \sum_{N_1, \dots, N_4} \big(\frac{N_1}{N}\big)^{1-s} \big(\frac{N_b}{N}\big)^{1-s} N_a^{\varepsilon} \big(\frac{N_a}{N}\big)^{1-s} \Big(\big(\frac{N_c}{N}\big)^{1-s} + 1 \Big) \frac{N_a^{\varepsilon}}{N_a} \\
& \times \Big(N_b^{-1+4\varepsilon} \big\| I u_1 \big\|_{X_S^{1,\frac{1}{2}-\varepsilon,1}} \big\| I u_b \big\|_{X_S^{1,\frac{1}{2}-\varepsilon,1}} + N_b^{-\frac{1}{2}+2\varepsilon} \big\| I u_1 \big\|_{X_S^{1,\frac{1}{2}-\varepsilon,1}} \big\| I u_b \big\|_{X_S^{1,\varepsilon,1}} \Big) \\
& \times \Big(N_c^{-1+4\varepsilon} \big\| I u_a \big\|_{X_S^{1,\frac{1}{2}-\varepsilon,1}} \big\| I u_c \big\|_{X_S^{1,\frac{1}{2}-\varepsilon,1}} + N_c^{-\frac{1}{2}+2\varepsilon} \big\| I u_a \big\|_{X_S^{1,\frac{1}{2}-\varepsilon,1}} \big\| I u_c \big\|_{X_S^{1,\varepsilon,1}} \Big).
\end{aligned}$$

We observe

$$\begin{gathered}
\big(\frac{N_1}{N}\big)^{1-s} \big(\frac{N_a}{N}\big)^{1-s} \frac{N_a^{2\varepsilon}}{N_a} \lesssim N^{-1+} N_a^{0-}, \\
\big(\frac{N_b}{N}\big)^{1-s} N_b^{-1+4\varepsilon} \lesssim N^{-1+}, \quad \big(\frac{N_b}{N}\big)^{1-s} N_b^{-\frac{1}{2}+2\varepsilon} \lesssim N^{-\frac{1}{2}+}, \\
\Big(\big(\frac{N_c}{N}\big)^{1-s} + 1 \Big) N_c^{-1+4\varepsilon} + \Big(\big(\frac{N_c}{N}\big)^{1-s} + 1 \Big) N_c^{-\frac{1}{2}+2\varepsilon} \lesssim 1
\end{gathered}$$

if $s > \frac{1}{2}$, and $\varepsilon > 0$ sufficiently small. Consequently, we obtain a bound of

$$(N^{-2+} + N^{-\frac{3}{2}+} \delta^{\frac{1}{2}-}) \big\| I u \big\|^4_{X_S^{1,\frac{1}{2},1}}.$$

Now, the proof of Proposition 2 is completed. □

As we have seen, the (first generation) modified energy $H(Iu, In_+)(t)$, which is close to the almost conserved quantity $\widetilde{H}(u, n_+)(t)$, controls the norm $\|Iu(t)\|_{H^1} + \|In_+(t)\|_{L^2}$ with the help of the L^2 conservation. Therefore, we need the local well-posedness of (2) with the existence time written in terms of $\|Iu_0\|_{H^1} + \|In_{+0}\|_{L^2}$. For this purpose we upgrade the bilinear estimates given in Lemma 2.4 to the following.

Lemma 4.3. *Let $1 > s \geq \frac{1}{2}$. Then, we have*

$$\begin{gathered}
\big\| \mathcal{I}_S (I(n_\pm u)) \big\|_{X_S^{1,\frac{1}{2},1}(\delta)} \lesssim \delta^{\frac{1}{2}-} \big\| I u \big\|_{X_S^{1,\frac{1}{2},1}(\delta)} \big\| n_\pm \big\|_{X_{W_\pm}^{0,\frac{1}{2},1}(\delta)}, \\
\big\| \mathcal{I}_{W_+} (|\nabla| (u_1 \overline{u_2})) \big\|_{X_{W_+}^{0,\frac{1}{2},1}(\delta)} \lesssim \delta^{\frac{1}{2}-} \big\| I u_1 \big\|_{X_S^{1,\frac{1}{2},1}(\delta)} \big\| I u_2 \big\|_{X_S^{1,\frac{1}{2},1}(\delta)}.
\end{gathered}$$

Proof. Since $\|u\|_{X_S^{s,\frac{1}{2},1}} \leq \|Iu\|_{X_S^{1,\frac{1}{2},1}}$, the second estimate immediately follows from (7). For the first estimate, we decompose u into two parts. For the low frequency part, supp $\widetilde{u} \subset \{|k| \lesssim N\}$, the claim follows from $I \leq 1$ and (6) with $s = 1$. For high frequency supp $\widetilde{u} \subset \{|k| \gtrsim N\}$, we observe that

$$m(k_1) \langle k_1 \rangle^{1-s} \lesssim N^{1-s} \sim m(k_2) \langle k_2 \rangle^{1-s}$$

for $|k_2| \gtrsim N$, where k_1 and k_2 denote the frequency variables for $n_\pm u$ and u, respectively. Then the estimate follows from (6). □

The standard iteration argument using Lemma 4.3 and (3) yields the modified local well-posedness adapted to the I-method.

Proposition 3. *Let $1 > s \geq \frac{1}{2}$. Then, for any $(u_0, n_{+0}) \in H^s \times L^2$, there exists a unique solution to (2), $(u, n_+) \in X_S^{s,\frac{1}{2},1}(\delta) \times X_{W_+}^{0,\frac{1}{2},1}(\delta)$, with the existence time*

$$\delta \sim (\|Iu_0\|_{H^1} + \|n_{+0}\|_{L^2})^{-2-},$$

such that the following estimate holds:

$$\|Iu\|_{X_S^{1,\frac{1}{2},1}(\delta)} + \|n_+\|_{X_{W_+}^{0,\frac{1}{2},1}(\delta)} \lesssim \|Iu_0\|_{H^1} + \|n_{+0}\|_{L^2}.$$

In particular, we have

$$\sup_{-\delta \leq t \leq \delta} \big(\|Iu(t)\|_{H^1} + \|n_+(t)\|_{L^2}\big) \lesssim \|Iu_0\|_{H^1} + \|n_{+0}\|_{L^2}.$$

We are now in a position to prove the main theorem.

Proof of Theorem 1.1. Let $(u_0, n_{0+}) \in H^s \times L^2$ be an initial datum with $\|u_0\|_{L^2} < \|Q\|_{L^2(\mathbb{R}^2)}$. The datum then satisfies

$$\|Iu_0\|_{H^1} + \|n_{+0}\|_{L^2} \lesssim N^{1-s}, \qquad \|Iu_0\|_{L^2} \leq \|u_0\|_{L^2} < \|Q\|_{L^2(\mathbb{R}^2)},$$

and its modified energy obeys

$$H(Iu_0, n_{+0}) \leq C_0 N^{2(1-s)}.$$

Since $H(Iu, n_+)(t)$ and the (a priori bounded) L^2 norm of $Iu(t)$ control $\|Iu(t)\|_{H^1} + \|n_+(t)\|_{L^2}$, we see from Proposition 3 that the solution to the initial value problem on $[0, t_0]$ can be extended up to $t = t_0 + \delta$ with a uniform time $\delta \sim N^{-2(1-s)-}$ and satisfies

$$\|Iu(\cdot - t_0)\|_{X_S^{1,\frac{1}{2},1}(\delta)} + \|n_+(\cdot - t_0)\|_{X_{W_+}^{0,\frac{1}{2},1}(\delta)} \lesssim N^{1-s},$$

as long as

$$H(Iu, n_+)(t_0) \leq 2C_0 N^{2(1-s)}.$$

If we could iterate the local theory M times, then Propositions 1 and 2 imply that the increment of the modified energy would be bounded by

$$\begin{aligned}
&|H(Iu, n_+)(M\delta) - H(Iu, n_+)(0)| \\
&\leq |H(Iu, n_+)(M\delta) - \widetilde{H}(u, n_+)(M\delta)| + \sum_{j=0}^{M-1} |\widetilde{H}(u, n_+)((j+1)\delta) - \widetilde{H}(u, n_+)(j\delta)| \\
&\quad + |\widetilde{H}(u, n_+)(0) - H(Iu, n_+)(0)| \\
&\lesssim N^{-1+}(N^{1-s})^3 \\
&\quad + M\Big\{N^{-1+}\delta^{\frac{1}{2}-}(N^{1-s})^3 + (N^{-2+} + N^{-\frac{5}{4}+}\delta^{\frac{1}{4}-} + N^{-1+}\delta^{1-})(N^{1-s})^4\Big\} \\
&\sim \Big\{N^{-s+} + M\big(N^{-1+} + N^{\frac{1}{4}-\frac{3}{2}s+}\big)\Big\}N^{2(1-s)},
\end{aligned}$$

which means that we can repeat $O(N^{\min\{1, \frac{3}{2}s-\frac{1}{4}\}-})$ times, obtaining the solution up to some time $\sim \delta N^{\min\{1, \frac{3}{2}s-\frac{1}{4}\}-} \sim N^{\min\{2s-1, \frac{7}{2}s-\frac{9}{4}\}-}$. Hence, we can solve the equation up to the arbitrarily large given time T by setting a large parameter N to be $\sim T^{\max\{\frac{1}{2s-1}, \frac{4}{14s-9}\}+}$, whenever $s > \frac{9}{14}$.

Moreover, we have

$$\sup_{-T\le t\le T}\big(\|u(t)\|_{H^s}+\|n_+(t)\|_{L^2}\big)\lesssim \sup_{-T\le t\le T}\big(\|Iu(t)\|_{H^1}+\|n_+(t)\|_{L^2}\big)$$
$$\lesssim N^{1-s}\sim T^{\max\{\frac{1-s}{2s-1},\,\frac{4(1-s)}{14s-9}\}+}.$$

Going back to the original Zakharov system (1), we obtain the a priori estimate

$$\sup_{-T\le t\le T}\big(\|u(t)\|_{H^s}+\|n(t)\|_{L^2}+\||\nabla|^{-1}\partial_t n(t)\|_{L^2}\big)\lesssim T^{\max\{\frac{1-s}{2s-1},\,\frac{4(1-s)}{14s-9}\}+},$$

concluding the proof of Theorem 1.1. □

5. **Global solutions for the nonperiodic case.** In this section we treat the $\mathbb{R}^2$ case and also put the operator I on the wave equation. An adaptation of the argument for periodic problem easily implies the following almost conservation law.

Proposition 4 (Almost conservation law)**.** *Let $1>s>\frac{1}{2}$, $0\ge r\ge s-1$ be such that $r>1-2s$ and $r>-\frac{1}{2}s$. Let $0<\delta\le 1$ and (u,n_+) be a smooth solution to (2) on $(t,x)\in[0,\delta]\times\mathbb{R}^2$. Then, we have*

$$|\widetilde{H}(u,n_+)(\delta)-\widetilde{H}(u,n_+)(0)|\lesssim N^{-1+}\delta^{\frac{1}{2}-}\|Iu\|^2_{X_S^{1,\frac{1}{2},1}(\delta)}\|In_+\|_{X_{W_+}^{0,\frac{1}{2},1}(\delta)}$$
$$+\,(N^{-2+}+N^{-\frac{5}{4}+}\delta^{\frac{1}{4}-}+N^{-1+}\delta^{1-})$$
$$\times\big(\|Iu\|^2_{X_S^{1,\frac{1}{2},1}(\delta)}\|In_+\|^2_{X_{W_+}^{0,\frac{1}{2},1}(\delta)}+\|Iu\|^4_{X_S^{1,\frac{1}{2},1}(\delta)}\big).$$

Proof. We follow the proof of Proposition 2 and only indicate the difference from it. We have to consider the following three terms:

$$\frac{i}{2}\int_0^\delta\int_{\Sigma_3}\mathbf{1}_{\{||\xi_1|^2-|\xi_2|^2|\le 2|\xi_{12}|\}}(\xi_1,\xi_2)|\xi_{12}|\widehat{u}(t,\xi_1)\widehat{\overline{u}}(t,\xi_2)\tag{24}$$
$$\times\Big(\big(m_{-r,12}^2-\sigma_+(\xi_1,\xi_2)\big)\widehat{n}_+(t,\xi_3)-\big(m_{-r,12}^2-\sigma_-(\xi_1,\xi_2)\big)\widehat{n}_-(t,\xi_3)\Big)\,dt,$$

$$-\frac{i}{4}\int_0^\delta\int_{\Sigma_4}\widehat{u}(t,\xi_1)\widehat{\overline{u}}(t,\xi_2)\big(\widehat{n}_++\widehat{n}_-\big)(t,\xi_3)\tag{25}$$
$$\times\Big(\big(\sigma_+(\xi_{13},\xi_2)-\sigma_+(\xi_1,\xi_{23})\big)\widehat{n}_+(t,\xi_4)+\big(\sigma_-(\xi_{13},\xi_2)-\sigma_-(\xi_1,\xi_{23})\big)\widehat{n}_-(t,\xi_4)\Big)\,dt,$$

$$-\frac{i}{2}\int_0^\delta\int_{\Sigma_4}|\xi_{12}|\big(\sigma_+-\sigma_-\big)(\xi_1,\xi_2)\widehat{u}(t,\xi_1)\widehat{\overline{u}}(t,\xi_2)\widehat{u}(t,\xi_3)\widehat{\overline{u}}(t,\xi_4)\,dt.\tag{26}$$

Estimate of (24). We bound the multiplier by 1 as in the periodic case. We should consider

$$\sum_{N_1\sim N_2\gtrsim N}\sum_{N_0\lesssim N_1}\frac{N_0}{N_1^2}\Big(\frac{N_1}{N}\Big)^{2(1-s)}\Big(\Big(\frac{N_0}{N}\Big)^{-r}+1\Big)$$
$$\times\|P_{N_1}\psi_\delta Iu\|_{X_S^{1,\frac{3}{8}+,1}}\|P_{N_2}\psi_\delta Iu\|_{X_S^{1,\frac{3}{8}+,1}}\|P_{N_0}\chi_\delta In_+\|_{X_{W_+}^{0,\frac{1}{4},1}}$$

instead of (16). This is bounded by $N^{-1+}\delta^{\frac{1}{2}-}\|Iu\|^2_{X_S^{1,\frac{1}{2},1}}\|In_+\|_{X_{W_+}^{0,\frac{1}{2},1}}$ in the same manner, provided $2(1-s)-r<1$.

Estimate of (25). We can obtain simpler estimate

$$\|un\|_{L^2_{t,x}}\lesssim\|u\|_{X_S^{2\varepsilon,\frac{1}{2},1}}\|n\|_{X_{W_\pm}^{0,\frac{1}{2}-\varepsilon,1}}$$

instead of (19) by using Lemma 2.7 instead of Lemma 2.6.

Case 1 ($N_2 \gtrsim N_4$).

(i) $N_1 \gtrsim N$. In this case we need to consider the quantity

$$\sum_{N_1,N_2\gtrsim N}\sum_{N_3\lesssim \overline{N}_{12}}\sum_{N_4\lesssim N_2}\Big(\frac{N_1}{N}\Big)^{1-s}\Big(\frac{N_2}{N}\Big)^{1-s}\Big(\Big(\frac{N_3}{N}\Big)^{-r}+1\Big)\Big(\Big(\frac{N_4}{N}\Big)^{-r}+1\Big)\frac{1}{N_1N_2}$$
$$\times\Big(N_1^{2\varepsilon}\|Iu_1\|_{X_S^{1,\frac12,1}}\|In_3\|_{X_{W_+}^{0,\frac12-\varepsilon,1}}\Big)\Big(N_2^{2\varepsilon}\|Iu_2\|_{X_S^{1,\frac12,1}}\|In_4\|_{X_{W_+}^{0,\frac12-\varepsilon,1}}\Big).$$

Considering the worst case $N \lesssim N_1 \ll N_3 \sim N_4 \sim N_2$, we can bound the above by $N^{-2+}\|Iu\|^2_{X_S^{1,\frac12,1}}\|In_+\|^2_{X_{W_+}^{0,\frac12,1}}$ provided $1-s-2r<1$.

(ii) $N_1 \ll N$. Make the same decomposition as (20). When $\overline{L}_{34}=\overline{L}_{1234}$, we use Lemma 2.7 instead of Lemma 2.6 to obtain the following bound,

$$\sum_{N_1\ll N}\sum_{N_2\gtrsim N}\sum_{N_3,N_4\lesssim N_2}\Big(\frac{N_2}{N}\Big)^{1-s}\Big(\Big(\frac{N_3}{N}\Big)^{-r}+1\Big)\Big(\Big(\frac{N_4}{N}\Big)^{-r}+1\Big)N_2^{-2+}$$
$$\times\|Iu_1\|_{X_S^{1,\frac12,1}}\|Iu_2\|_{X_S^{1,\frac12,1}}\|In_3\|_{X_{W_+}^{0,\frac12-,1}}\|In_4\|_{X_{W_+}^{0,\frac12-,1}}.$$

Even the worst case $N_2 \sim N_3 \sim N_4 \gtrsim N$ can be estimated with decay factor N^{-2+} whenever $1-s-2r<2$. When $\overline{L}_{12}=\overline{L}_{1234}\gg\overline{L}_{34}$, we follow the argument for periodic case precisely to encounter the quantity

$$\sum_{N_1\ll N}\sum_{N_2\gtrsim N}\sum_{N_3,N_4\lesssim N_2}\Big(\frac{N_2}{N}\Big)^{1-s}\Big(\Big(\frac{N_3}{N}\Big)^{-r}+1\Big)\Big(\Big(\frac{N_4}{N}\Big)^{-r}+1\Big)N_2^{-\frac54}$$
$$\times\|Iu_1\|_{X_S^{1,\frac12,1}}\|Iu_2\|_{X_S^{1,\frac12,1}}\|In_3\|_{X_{W_+}^{0,\frac38,1}}\|In_4\|_{X_{W_+}^{0,\frac38,1}}.$$

This can be treated appropriately if $1-s-2r<\frac54$. The decay $N^{-\frac54+}\delta^{\frac14-}$ is obtained.

Case 2 ($N_2 \ll N_4$).

(i) $N_1 \gtrsim N$. With a modification of the argument for periodic case similar to Case 1 (i), we estimate

$$\sum_{N_1\gtrsim N}\sum_{N_4\gtrsim N}\sum_{N_3\lesssim\overline{N}_{14}}\sum_{N_2\ll N_4}\Big(\frac{N_2^2}{N_4^2}+\frac{1}{N_4}\Big)\Big(\frac{N_1}{N}\Big)^{1-s}\Big(\Big(\frac{N_2}{N}\Big)^{1-s}+1\Big)\Big(\Big(\frac{N_3}{N}\Big)^{-r}+1\Big)$$
$$\times\Big(\frac{N_4}{N}\Big)^{-r}\frac{1}{N_1N_2}\Big(N_1^{2\varepsilon}\|Iu_1\|_{X_S^{1,\frac12,1}}\|In_3\|_{X_{W_+}^{0,\frac12-\varepsilon,1}}\Big)$$
$$\times\Big(N_2^{2\varepsilon}\|Iu_2\|_{X_S^{1,\frac12,1}}\|In_4\|_{X_{W_+}^{0,\frac12-\varepsilon,1}}\Big).$$

The worst case is $N \lesssim N_1 \ll N_2 \ll N_3 \sim N_4$, which is controlled if $1-s-2r<1$. We obtain the decay N^{-2+} in this case.

(ii) $N_1 \ll N$. If $N_2^2 \lesssim N_4$, then we have

$$\sum_{N_1\ll N}\sum_{N_4\gtrsim N}\sum_{N_3\sim N_4}\sum_{N_2\lesssim N_4^{1/2}}\frac{1}{N_4}\Big(\Big(\frac{N_2}{N}\Big)^{1-s}+1\Big)\Big(\frac{N_3}{N}\Big)^{-r}\Big(\frac{N_4}{N}\Big)^{-r}$$
$$\times\|Iu_1\|_{X_S^{1,0+,1}}\|Iu_2\|_{X_S^{1,0+,1}}\|In_3\|_{X_{W_+}^{0,\frac12-,1}}\|In_4\|_{X_{W_+}^{0,\frac12-,1}},$$

which is estimated with decay $N^{-1+}\delta^{1-}$ whenever $\frac12(1-s)-2r<1$. If $N_2^2 \gg N_4$, we can employ the same argument as Case 1 (ii) and obtain the decay $N^{-\frac54+}\delta^{\frac14-}$.

Estimate of (26). This is identical with the periodic case, because (26) includes no n_+. We have the bound $(N^{-2+}+N^{-1+}\delta^{1-})\|Iu\|^4_{X_S^{1,\frac12,1}}$. □

We also obtain the following bilinear estimates as a counterpart to Lemma 4.3.

Lemma 5.1. *Let $1>s>\frac12$, $0\geq r\geq s-1$. Then, we have*

$$\big\|\mathcal{I}_S(I_s^S(n_\pm u))\big\|_{X_S^{1,\frac12,1}(\delta)}\lesssim \delta^{\frac{1+r}{2}-}\big\|Iu\big\|_{X_S^{1,\frac12,1}(\delta)}\big\|In_\pm\big\|_{X_{W_\pm}^{0,\frac12,1}(\delta)}, \tag{27}$$

$$\big\|\mathcal{I}_{W_+}(|\nabla|I_r^{W_+}(u_1\overline{u_2}))\big\|_{X_{W_\pm}^{0,\frac12,1}(\delta)}\lesssim \delta^{\frac12-}\big\|Iu_1\big\|_{X_S^{1,\frac12,1}(\delta)}\big\|Iu_2\big\|_{X_S^{1,\frac12,1}(\delta)}. \tag{28}$$

Proof. (28) follows easily from (7), $I_r^{W_+}\leq 1$, and $\|u\|_{X_S^{s,\frac12,1}}\leq\|Iu\|_{X_S^{1,\frac12,1}}$. We thus focus on (27). First of all, we show

$$\big\|\mathcal{I}_S(n_\pm u)\big\|_{X_S^{s,\frac12,1}(\delta)}\lesssim \delta^{\frac{1+r}{2}-}\big\|u\big\|_{X_S^{s,\frac12,1}(\delta)}\big\|n_\pm\big\|_{X_{W_\pm}^{r,\frac12,1}(\delta)}. \tag{29}$$

Use ζ_0, ζ_1, ζ_2 for the Fourier variables of $n_\pm$, $n_\pm u$, u, respectively (thus $\zeta_0=\zeta_1-\zeta_2$).

(i) The case $|\xi_1|\lesssim|\xi_2|$. Since $s\geq r+\frac12$, (29) is reduced to

$$\big\|\mathcal{I}_S(n_\pm u)\big\|_{X_S^{r+\frac12,\frac12,1}(\delta)}\lesssim \delta^{\frac{1+r}{2}-}\big\|u\big\|_{X_S^{r+\frac12,\frac12,1}(\delta)}\big\|n_\pm\big\|_{X_{W_\pm}^{r,\frac12,1}(\delta)}.$$

It is not difficult to obtain this by interpolation between (8) and (6) with $s=\frac12$.

(ii) The case $|\xi_1|\gg|\xi_2|$. An interpolation between Lemmas 2.8 and 2.10 implies

$$\begin{aligned}&\iint_{\zeta_0=\zeta_1-\zeta_2} f(\zeta_0)g_1(\zeta_1)g_2(\zeta_2)\\ &\lesssim L_{\max}^{\frac12}(L_{\mathrm{med}}L_{\min})^{\frac{1-r}{4}+}N_2^{1+r-}N_1^{-1}\big\|f\big\|_{L^2}\big\|g_1\big\|_{L^2}\big\|g_2\big\|_{L^2}\end{aligned}$$

for $f,g_1,g_2\in L^2_\zeta(\mathbb{R}\times\mathbb{R}^2)$ satisfying (9) with $N_1\gg N_2$. (We can choose $1+r->\frac12$ because $r>-\frac12$. Note that $L_{\max}\gtrsim N_1^2$ is required for nonzero contribution under this assumption.) To apply this, we have to decompose $\mathcal{I}_S(n_\pm u)$ as

$$\sum_{N_1\geq1}\sum_{N_2\ll N_1}\sum_{N_0\sim N_1}\sum_{L_0,L_1,L_2\geq1}\mathcal{I}_SP^S_{N_1,L_1}(P^{W_\pm}_{N_0,L_0}n_\pm P^S_{N_2,L_2}u).$$

If $L_0=L_{\max}$ (similar for the case $L_2=L_{\max}$), we use the above estimate and Lemma 2.3 to obtain

$$\begin{aligned}&\big\|P_{N_1}\mathcal{I}_S(P_{N_0}n_\pm\cdot u)\big\|_{X_S^{s,\frac12,1}(\delta)}\\ &\lesssim \delta^{\frac{1+r}{4}-}N_1^s\sum_{N_2\ll N_1}\big\|P_{N_1}(P_{N_0}n_\pm\cdot\psi_\delta P_{N_2}u)\big\|_{X_S^{0,-\frac{1-r}{4}-,\infty}}\\ &\lesssim \delta^{\frac{1+r}{4}-}N_1^{s-1}N_0^{-r}\sum_{N_2\ll N_1}N_2^{1+r-s-}\big\|\psi_\delta P_{N_2}u\big\|_{X_S^{s,\frac{1-r}{4}+,1}}\big\|P_{N_0}n_\pm\big\|_{X_{W_\pm}^{r,\frac12,1}}\\ &\lesssim \delta^{\frac{1+r}{2}-}\big\|u\big\|_{X_S^{s,\frac12,1}}\big\|P_{N_0}n_\pm\big\|_{X_{W_\pm}^{r,\frac12,1}},\end{aligned}$$

where at the last inequality we have used the assumption $1+r-s\geq0$. Squaring and summing up the above in N_1 we obtain (29) (note that $N_0\sim N_1$). In the case

$L_1 = L_{\max}$, a similar argument yields

$$\begin{aligned}
&\big\|P_{N_1}\mathcal{I}_S(P_{N_0}n_\pm \cdot u)\big\|_{X_S^{s,\frac{1}{2},1}(\delta)} \\
&\lesssim N_1^s \sum_{L_1} L_1^{-\frac{1}{2}} \sum_{N_2 \ll N_1} \big\|P^S_{N_1,L_1}(\psi_\delta P_{N_0}n_\pm \cdot \psi_\delta P_{N_2}u)\big\|_{L^2_{t,x}} \\
&\lesssim N_1^{s-1}N_0^{-r} \sum_{L_0,L_1,L_2} (L_0L_2)^{\frac{1-r}{4}+} \sum_{N_2 \ll N_1} N_2^{1+r-s-}\, N_2^s\big\|\psi_\delta P^S_{N_2,L_2}u\big\|_{L^2_{t,x}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad \times N_0^r\big\|\psi_\delta P^{W_\pm}_{N_0,L_0}n_\pm\big\|_{L^2_{t,x}}.
\end{aligned}$$

We can carry out the sum in L_1 using the fact $L_1 \sim \max\{\overline{L}_{02},\, N_1^2\}$, and have the same bound as the previous case. This completes the proof of (29).

To upgrade (29) to (27), we only have to show

$$m_{1-s,N}(\xi_1)\langle\xi_1\rangle^{1-s} \lesssim m_{1-s,N}(\xi_2)\langle\xi_2\rangle^{1-s} \cdot m_{-r,N}(\xi_0)\langle\xi_0\rangle^{-r}$$

for ξ_0, ξ_1, ξ_2 such that $\xi_0 = \xi_1 - \xi_2$. This is true for the case $|\xi_1| \lesssim |\xi_2|$ or the case $|\xi_2| \gtrsim N$, because if $q \geq 0$ we have $m_{q,N}(\xi)\langle\xi\rangle^q \geq 1$, $m_{q,N}(\xi_1)\langle\xi_1\rangle^q \lesssim m_{q,N}(\xi_2)\langle\xi_2\rangle^q$ for $|\xi_1| \lesssim |\xi_2|$, and $m_{q,N}(\xi)\langle\xi\rangle^q \sim m_{q,N}(\xi)|\xi|^q = N^q$ for $|\xi| \geq 2N$.

In the remaining case, $|\xi_2| \ll |\xi_1|$ and $|\xi_2| \ll N$, we have $|\xi_0| \sim |\xi_1|$ and then

$$m_{1-s,N}(\xi_1) \sim m_{1-s,N}(\xi_2)m_{1-s,N}(\xi_0) \lesssim m_{1-s,N}(\xi_2)m_{-r,N}(\xi_0),$$

since $1 - s \geq -r$. This and (6) with $s = 1$ imply (27). □

By a standard argument, we can deduce from Lemma 5.1 the following local well-posedness.

Proposition 5. *Let $1 > s > \frac{1}{2}$, $0 \geq r \geq s-1$. Then, for any $(u_0, n_{+0}) \in H^s \times H^r$, there exists a unique solution to (2) on $\mathbb{R}^2$, $(u, n_+) \in X_S^{s,\frac{1}{2},1}(\delta) \times X_{W_+}^{r,\frac{1}{2},1}(\delta)$, with the existence time*

$$\delta \sim (\|Iu_0\|_{H^1} + \|In_{+0}\|_{L^2})^{-\frac{2}{1+r}-},$$

such that the following estimate holds:

$$\|Iu\|_{X_S^{1,\frac{1}{2},1}(\delta)} + \|In_+\|_{X_{W_+}^{0,\frac{1}{2},1}(\delta)} \lesssim \|Iu_0\|_{H^1} + \|In_{+0}\|_{L^2}.$$

In particular, we have

$$\sup_{-\delta \leq t \leq \delta} (\|Iu(t)\|_{H^1} + \|In_+(t)\|_{L^2}) \lesssim \|Iu_0\|_{H^1} + \|In_{+0}\|_{L^2}.$$

We remark that our local existence time $\delta \sim \|\text{data}\|^{-\frac{2}{1+r}-}$ is longer than that obtained in [13], which was $\delta \sim \|\text{data}\|^{-\frac{2}{1+2r}-}$. In fact, a longer local existence time will lead to the global well-posedness for a lower regularity.

Proof of Theorem 1.2. Here we assume

$$1 > s > \tfrac{1}{2}, \qquad 0 \geq r \geq s - 1, \tag{30}$$

$$r > 1 - 2s, \qquad r > -\tfrac{1}{2}s. \tag{31}$$

Let $(u_0, n_{0+}) \in H^s \times H^r$ be an initial datum with $\|u_0\|_{L^2} < \|Q\|_{L^2(\mathbb{R}^2)}$. The modified energy $H(Iu, In_+)(t)$, satisfying the initial bound

$$H(Iu_0, In_{+0}) \leq C(N^{2(1-s)} + N^{-2r}) \leq C_0 N^{2(1-s)},$$

controls $\|Iu(t)\|_{H^1} + \|In_+(t)\|_{L^2}$. Proposition 5 shows that the solution on $[0, t_0]$ can be extended up to $t = t_0 + \delta$ with a uniform time $\delta \sim N^{-\frac{2(1-s)}{1+r}-}$ and satisfies

$$\big\|Iu(\cdot - t_0)\big\|_{X_S^{1,\frac12,1}(\delta)} + \big\|In_+(\cdot - t_0)\big\|_{X_{W_+}^{0,\frac12,1}(\delta)} \lesssim N^{1-s},$$

as long as $H(Iu, n_+)(t_0) \le 2C_0 N^{2(1-s)}$. If we could iterate the local theory M times, then from Propositions 1 and 4,

$$\begin{aligned}
&|H(Iu,n_+)(M\delta) - H(Iu,n_+)(0)|\\
&\lesssim N^{-1+}(N^{1-s})^3\\
&\quad + M\Big\{N^{-1+}\delta^{\frac12-}(N^{1-s})^3 + (N^{-2+} + N^{-\frac54+}\delta^{\frac14-} + N^{-1+}\delta^{1-})(N^{1-s})^4\Big\}\\
&\sim \Big\{N^{-s+} + MN^{-\alpha_0(s,r)+}\Big\}N^{2(1-s)}, \qquad \alpha_0(s,r) := \min\{\tfrac{1+rs}{1+r}, \tfrac{-1-3r+6s+8rs}{4(1+r)}\}.
\end{aligned}$$

Thus, we can repeat the local procedure $O(N^{\alpha_0-})$ times to reach some time $\sim \delta N^{\alpha_0-} \sim N^{\alpha_1-}$,

$$\alpha_1(s,r) := \min\{\tfrac{-1+2s+rs}{1+r}, \tfrac{-9-3r+14s+8rs}{4(1+r)}\}.$$

The required conditions for global well-posedness are

$$-1 + 2s + rs > 0, \tag{32}$$

$$-9 - 3r + 14s + 8rs > 0. \tag{33}$$

It turns out that (31) and (32) are automatically satisfied under the assumptions (30) and (33). Moreover, we have

$$\sup_{-T\le t\le T}\big(\big\|Iu(t)\big\|_{H^1} + \big\|In_+(t)\big\|_{L^2}\big) \lesssim N^{1-s} \sim T^{\frac{1-s}{\alpha_1}+} \sim T^{\alpha_2+},$$

$$\alpha_2(s,r) := \max\{\tfrac{(1-s)(1+r)}{-1+2s+rs}, \tfrac{4(1-s)(1+r)}{-9-3r+14s+8rs}\}.$$

We obtain the same a priori estimate for solutions to the original equation (1), concluding the proof of Theorem 1.2. □

Appendix A. Proof of Lemma 2.13. Here we shall give a proof of the following bilinear estimate.

Proposition 6. *We have*

$$\big\|uv\big\|_{L^2_{t,x}} \lesssim L^{\frac34}N^{\frac34}\big\|u\big\|_{L^2_{t,x}}\big\|v\big\|_{L^2_{t,x}}$$

for $u, v \in L^2(\mathbb{R}\times Z)$, $Z = \mathbb{T}^2_\gamma$ *or* $\mathbb{R}^2$, *such that* supp $\widetilde{u}$, supp $\widetilde{v} \subset \mathcal{P}_N \cap \mathcal{W}_L^+$.

Lemma 2.13 then follows by letting $v = u$. The standard argument reduces the problem to the following; for details, see e.g. the proof of Lemma 2.5 in [11].

Proposition 7. *Let* $N, L \ge 1$. *Then, for any* $k \in \mathbb{R}^2$ *and* $A \ge |k|$, *the set*

$$\big\{\, k' \in \mathbb{R}^2 \,\big|\, |k'| \le N,\ |k-k'| \le N,\ |k'| + |k-k'| \in [A, A+L] \,\big\}$$

is covered with at most $O(N^{\frac32}L^{\frac12})$ *squares of unit size.*

We begin with preparing the following lemma.

Lemma A.1. *Let $a \geq b \gg 1$. Define*

$$E_< := \{\, (x,y) \in \mathbb{R}^2 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1 \,\},$$
$$E_> := \{\, (x,y) \in \mathbb{R}^2 \mid \frac{x^2}{(a+100\frac{a}{b})^2} + \frac{y^2}{(b+100)^2} \geq 1 \,\}.$$

Then, there exists no unit square in $\mathbb{R}^2$ intersecting with both $E_<$ and $E_>$. The same holds for

$$E'_< = \{\, (x,y) \in \mathbb{R}^2 \mid \frac{x^2}{(a-100\frac{a}{b})^2} + \frac{y^2}{(b-100)^2} \leq 1 \,\},$$
$$E'_> = \{\, (x,y) \in \mathbb{R}^2 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} \geq 1 \,\}$$

instead of $E_<$, $E_>$.

Proof. We only prove the first half of the claim. The second half will be shown by a similar argument.

Assume for contradiction that there existed such a square of side length 1. Then, it would hold for some $(x,y) \in E_<$ and $(x',y') \in E_>$ that

$$(x-x')^2 + (y-y')^2 \leq 2, \tag{34}$$
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} \leq \frac{x'^2}{(a+100\frac{a}{b})^2} + \frac{y'^2}{(b+100)^2}. \tag{35}$$

Note that

$$\begin{aligned}\frac{x'^2}{(a+100\frac{a}{b})^2} - \frac{x^2}{a^2} &= \frac{x'^2 - x^2}{a^2} - x'^2\Big(\frac{1}{a^2} - \frac{1}{(a+100\frac{a}{b})^2}\Big)\\ &= \frac{x'+x}{a^2}(x'-x) - \frac{x'^2}{(a+100\frac{a}{b})^2}\Big(\frac{200}{b} + (\frac{100}{b})^2\Big),\\ \frac{y'^2}{(b+100)^2} - \frac{y^2}{b^2} &= \frac{y'^2 - y^2}{b^2} - y'^2\Big(\frac{1}{b^2} - \frac{1}{(b+100)^2}\Big)\\ &= \frac{y'+y}{b^2}(y'-y) - \frac{y'^2}{(b+100)^2}\Big(\frac{200}{b} + (\frac{100}{b})^2\Big).\end{aligned}$$

From these estimates and the fact $(x',y') \in E_>$,

$$\begin{aligned}&\frac{x'^2}{(a+100\frac{a}{b})^2} + \frac{y'^2}{(b+100)^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2}\\ &\leq \frac{|x'+x|}{a^2}|x'-x| + \frac{|y'+y|}{b^2}|y'-y| - \Big(\frac{200}{b} + (\frac{100}{b})^2\Big),\end{aligned}$$

which is, from (34) and $(x,y) \in E_<$,

$$\begin{aligned}&\leq \frac{2|x| + \sqrt{2}}{a^2}\sqrt{2} + \frac{2|y| + \sqrt{2}}{b^2}\sqrt{2} - \frac{200}{b}\\ &\leq \frac{10}{a} + \frac{10}{b} - \frac{200}{b} \leq -\frac{180}{b} < 0.\end{aligned}$$

This contradicts (35). □

Proof of Proposition 6. We may assume $|k| \le 2N$, otherwise the set is empty. Treat several cases separately.

(i) $L \gtrsim N$. In this case, we use the condition $|k'| \le N$ to estimate the number of squares by $N^2 \lesssim N^{\frac{3}{2}} L^{\frac{1}{2}}$.

(ii) $L \ll N$, $|k| \lesssim 1$. In this case we have $|k'| \le N$ and $A - C \le 2|k'| \le A + L + C$. It is easy to see that such a region, which is a disk of radius L or the intersection of a disk of radius N and an annulus of width L, can be covered with $\lesssim NL$ unit squares. $L \lesssim N$ implies the claim.

(iii) $L \ll N$, $A \le |k| + 10L$. We have

$$|k'| + |k - k'| \le |k| + 11L,$$

which shows that k' is inside an ellipse of distance between foci $|k|$, length of long axis

$$|k| + 11L \lesssim N,$$

and length of short axis

$$\sqrt{(|k| + 11L)^2 - |k|^2} = \sqrt{22|k|L + 121L^2} \lesssim \sqrt{NL}.$$

Therefore, we can cover this region with $\lesssim N \times \sqrt{NL}$ unit squares.

We remark that k' is confined to the region

$$\mathcal{R} := \left\{\, k' \in \mathbb{R}^2 \;\middle|\; |k'| + |k - k'| \in [A, A + L] \,\right\}$$

between two ellipses with common foci 0, k, longer axis A and $A + L$, respectively.

(iv) $L \ll N$, $A \ge 10N$. In this case the region is close to an annulus. In fact,

$$\begin{aligned} 2a &= A, & 2b &= \sqrt{A^2 - |k|^2} \ge \sqrt{A^2 - (A/5)^2} \ge \tfrac{9}{10} \cdot 2a, \\ 2a' &= A + L, & 2b' &= \sqrt{(A + L)^2 - |k|^2}, \end{aligned}$$

with $2a, 2a'$ (resp. $2b, 2b'$) the length of the long (resp. short) axes of inner and outer ellipses. We first change the scale in the direction of short axis to make the inner ellipse a circle. Then, the new region $\mathcal{R}'$ is included in an annulus of width $\max\{a' - a, \frac{a}{b}(b' - b)\}$. We see $a' - a = L$ and

$$\begin{aligned} 2\frac{a}{b}(b - b') &= \frac{a}{b}\left(\sqrt{(A + L)^2 - |k|^2} - \sqrt{A^2 - |k|^2}\right) \\ &= \frac{a}{b} \frac{2AL + L^2}{\sqrt{(A + L)^2 - |k|^2} + \sqrt{A^2 - |k|^2}} \sim 1 \cdot \frac{AL}{A} = L. \end{aligned}$$

Hence, the intersection of any ball of radius $2N$ and $\mathcal{R}'$ is covered with $\lesssim NL$ unit squares, which shows that the intersection of any ball of radius N and the original $\mathcal{R}$ is also covered with the same number of unit squares.

(v) $L \ll N$, $|k| \gg 1$, and $|k| + 10L \le A \le 10N$. By translation and rotation, we may consider the covering of

$$\widetilde{\mathcal{R}} := \left\{\, (x, y) \in \mathbb{R}^2 \;\middle|\; \frac{x^2}{a'^2} + \frac{y^2}{b'^2} \le 1 \le \frac{x^2}{a^2} + \frac{y^2}{b^2} \,\right\}$$

with $2a = A$, $2a' = A + L$, $2b = \sqrt{A^2 - |k|^2}$, $2b' = \sqrt{(A + L)^2 - |k|^2}$. Note also that

$$a \ge \tfrac{1}{2}|k| \gg 1, \qquad b = \tfrac{1}{2}\sqrt{A^2 - |k|^2} \ge \tfrac{1}{2}\sqrt{(|k| + 10L)^2 - |k|^2} \ge \sqrt{|k|L} \gg 1.$$

From Lemma A.1, we see that the smallest (axis-aligned) lattice polygon including the inside of outer boundary of $\widetilde{\mathcal{R}}$ is included in the inside of an ellipse with long axis $2(a'+100\frac{a'}{b'})$ and short axis $2(b'+100)$. In the same manner, the biggest (axis-aligned) lattice polygon included in the inside of inner boundary of $\widetilde{\mathcal{R}}$ includes an ellipse with long axis $2(a-100\frac{a}{b})$ and short axis $2(b-100)$. Therefore, the number of needed unit squares is estimated by

$$\begin{aligned}&\Big(a'+100\frac{a'}{b'}\Big)(b'+100)-\Big(a-100\frac{a}{b}\Big)(b-100)\\&=\Big((a'-a)+100(\frac{a'}{b'}+\frac{a}{b})\Big)(b'+100)\ +\ \Big(a-100\frac{a}{b}\Big)(b'-b+200).\end{aligned}$$

We find $b'+100\lesssim N$, $|a-100\frac{a}{b}|\lesssim N$, $a'-a\lesssim L$, and

$$\begin{aligned}\frac{a}{b}&=\frac{A}{\sqrt{A^2-|k|^2}}=\frac{1}{\sqrt{1-(\frac{|k|}{A})^2}}\le\frac{1}{\sqrt{1-(\frac{|k|}{|k|+10L})^2}}\le\frac{1}{\sqrt{1-(\frac{2N}{2N+10L})^2}}\\&=\frac{2N+10L}{\sqrt{(2N+10L)^2-(2N)^2}}\sim\frac{N}{\sqrt{NL}}=\sqrt{\frac{N}{L}}.\end{aligned}$$

We also see $a'/b'\lesssim\sqrt{N/L}$ in the same manner. Finally,

$$\begin{aligned}2(b'-b)&=\sqrt{(A+L)^2-|k|^2}-\sqrt{A^2-|k|^2}=\frac{2AL+L^2}{\sqrt{(A+L)^2-|k|^2}+\sqrt{A^2-|k|^2}}\\&\lesssim\frac{A}{\sqrt{A^2-|k|^2}}L\lesssim\sqrt{\frac{N}{L}}L=\sqrt{NL}.\end{aligned}$$

With all of them together, we reach the bound $\lesssim N^{\frac{3}{2}}L^{\frac{1}{2}}$. □

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