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ON VORTICITY FORMULATION FOR VISCOUS INCOMPRESSIBLE FLOWS IN THE HALF PLANE

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1. Introduction

In this report we consider the two-dimensional Navier-Stokes equations for viscous incompressible flows under the no-slip boundary conditions:

$$\text{(NS)} \qquad \begin{cases} \partial_t u - \nu\Delta u + u\cdot\nabla u + \nabla p = 0 & t>0, \ \ x\in\mathbb{R}^2_+, \\ \operatorname{div} u = 0 & t\geq 0, \ \ x\in\mathbb{R}^2_+, \\ u = 0 & t\geq 0, \ \ x\in\partial\mathbb{R}^2_+, \\ u|_{t=0} = a & x\in\mathbb{R}^2_+. \end{cases}$$

Here $\mathbb{R}^2_+ = \{x\in\mathbb{R}^2 \mid x_2>0\}$, and $u=u(t,x)=(u_1(t,x),u_2(t,x))$ and $p=p(t,x)$ denote the velocity field and the pressure field, and $\nu>0$ is the viscosity coefficient. We will use the standard notations for derivatives; $\partial_t = \partial/\partial t$, $\partial_j = \partial/\partial x_j$, $\Delta = \sum_{j=1}^2 \partial_j^2$, div $u = \sum_{j=1}^2 \partial_j u_j$, and $u\cdot\nabla u = \sum_{j=1}^2 u_j\partial_j u$.

The system (NS) has been studied quite extensively in various settings. In particular, it is well known that (NS) admits a unique smooth solution, for example, in the energy class; see the books [35, 38]. When $\mathbb{R}^2_+$ is replace by the whole plane $\mathbb{R}^2$ the alternative approach using vorticity fields is also useful and has been well developed by now. Here the vorticity ω of the velocity u is defined by $\omega = \mathrm{Rot}\ u := \partial_1 u_2 - \partial_2 u_1$, and the equation for ω is then formally obtained by acting the Rot operator on the first equation of (NS):

$$(1.1) \qquad \partial_t\omega - \nu\Delta\omega + u\cdot\nabla\omega = 0.$$

The vorticity equation (1.1) in $(0,\infty)\times\mathbb{R}^2$ ensures the uniform bound of vorticity fields by the maximum principle, which is essentially used to show the global existence of smooth solutions to (NS) in the infinite energy class; for example, see [4, 15, 13, 24]. However, in the presence of boundaries, a serious difficulty arises in the study of vorticity fields. Indeed, under the no-slip boundary condition on velocity fields the vorticity fields do not satisfy the boundary conditions such that the classical results in the parabolic PDE theory are directly applied, and this absence of the "normal" boundary conditions has been a crucial obstacle for the detailed mathematical study of (1.1). As is observed in [1], the boundary conditions for vorticity fields are derived from a simple mathematical consideration through the Biot-Savart law. In the case of the half plane these conditions are written as

$$(1.2)\qquad \nu\big(\partial_2 + (-\partial_1^2)^{\frac{1}{2}}\big)\omega = -\partial_2(-\Delta_D)^{-1}(u\cdot\nabla\omega) \qquad t>0,\quad x\in\partial\mathbb{R}^2_+.$$

Here $h = (-\Delta_D)^{-1}f$ denotes the solution to the Poisson equation with the homogeneous Dirichlet condition: $-\Delta h = f$ in $\mathbb{R}^2_+$ and $h = 0$ on $\partial\mathbb{R}^2_+$. Under the compatibility conditions on a such that div $a = 0$ in $\mathbb{R}^2_+$ and $a = 0$ on $\partial\mathbb{R}^2_+$, the equation (1.1) equipped with (1.2) is shown to be equivalent with (NS).

The aim of this report is to analyze the vorticity equations in the half plane by using the vorticity formulation. The details of the proofs for the results are given in the paper [29]. Although (1.2) is not a familiar condition due to the presence of the term $(-\partial_1^2)^{1/2}\omega$, we can derive a solution formula to (1.1)-(1.2) through the Fourier-Laplace transform. This will be stated in Section 2. We note that a solution formula for the (Navier-) Stokes equations is obtained by [36, 40] for $\mathbb{R}^n_+$ with any $n\geq 2$, and it is a basic tool in the study of (NS) in the half space. Our solution formula leads to $L^p - L^q$ estimates of the propagator to the linear vorticity equations, from which the mathematical validity of (1.1)-(1.2), i.e, the (time local) solvability of (1.1)-(1.2) in suitable function spaces, is confirmed; see Theorem 2.2.

The solution formula helps to carry out a detailed analysis of vorticity fields even in the region near the boundary. Making use of this advantage, we investigate in Section 3 the behavior of vorticity at the zero viscosity limit $\nu\to 0$. The precise statement of the result will be given in Theorem 3.1. Roughly speaking, we will see that the following asymptotic expansion holds at $\nu\to 0$ near the initial time:

$$(1.3)\qquad \omega(t)\ \sim\ \omega_E(t)\ +\ \omega_{BL}(t), \qquad 0<t\leq c_0\nu^{\frac{1}{3}}.$$

Here ω_E is the vorticity field for the solution to the Euler equations with the initial velocity a, ω_{BL} is the function describing the boundary layer, and c_0 is a constant independent of $0<\nu\ll 1$. The function ω_{BL} is written rather explicitly in terms of the initial data (see (3.3)), and it is a nontrivial function if and only if

$$(1.4)\qquad \partial_2(-\Delta_D)^{-1}(a\cdot\nabla\mathrm{Rot}\ a)\ \not\equiv 0 \qquad \text{on } \partial\mathbb{R}^2_+.$$

When (1.4) holds ω_{BL} will be shown to satisfy

(1.5)
$$c\nu^{-\frac{1}{2}(1-\frac{1}{p})}t^{\frac{1}{2}(1+\frac{1}{p})} \leq \|\omega_{BL}(t)\|_{L^p(\Omega_{\nu t})} \leq \|\omega_{BL}(t)\|_{L^p} \leq C\nu^{-\frac{1}{2}(1-\frac{1}{p})}t^{\frac{1}{2}(1+\frac{1}{p})}$$

for all ν, $t>0$ and $1\leq p\leq\infty$, where $\Omega_{\nu t} = \{x\in\mathbb{R}^2_+ \mid 0\leq x_2\leq(\nu t)^{1/2}\}$ is the region of the boundary layer. In particular, (1.3) and (1.5) imply the

high creation of vorticity near the boundary in L^p for $2 < p \le \infty$ as follows:
(1.6)
$$\|\omega(c_0\nu^{\frac{1}{3}})\|_{L^p(\{0\le x_2\le c_0^{\frac{1}{2}}\nu^{\frac{2}{3}}\})} \ge c'\nu^{-\frac{1}{3}(1-\frac{2}{p})} \to \infty \quad (\nu\to 0) \qquad \text{if} \quad 2<p\le\infty,$$

see Corollary 3.1 for details. We note that, when (1.4) holds, the vorticity creation itself may be proved by contradiction arguments if we do not ask for the concrete estimates such as (1.3) or (1.6). But it will be difficult to gain further insight from such contradiction arguments.

The high creation of vorticity at the zero viscosity limit, which arises due to the nonlinearity of (1.1)-(1.2), is naturally expected from the boundary layer theory. Nevertheless, to the best of the author's knowledge, this phenomenon with explicit estimates has been mathematically observed only under some restricted situations. In [16] the nonlinear instability of the Prandtl boundary layer is proved around linearly-unstable stationary solutions to the Euler equations. As a product of the calculations based on the spectral analysis and the energy argument for velocity fields, it is also shown that there exist a sequence of solutions $\{u^{(\nu)}\}$ to (NS) and $\{T_\nu\}$ such that $\|\mathrm{Rot}\ u^{(\nu)}(T_\nu)\|_{L^\infty} \to \infty$ and $T_\nu \to 0$ as $\nu\to 0$. So in [16] the high vorticity creation in L^∞ is observed around certain class of stationary solutions to the Euler equations. On the other hand, in [32, 33] the asymptotic expansion for solutions to (NS) of the form $u(t,x) = u_E(t,x) + u_P(t,x_1,x_2/\sqrt{\nu}) + O(\sqrt{\nu})$ at $\nu\to 0$ is established for analytic initial data. Here u_E is the solution to the Euler equations and u_P is the solution to the modified Prandtl equations. Hence, the results of [32, 33] imply the high vorticity creation in L^p_{loc} for any $p > 1$, but under the regularity condition of analyticity on initial data.

In Theorem 3.1 the expansion (1.3) is proved just under the assumptions of some Sobolev regularity on initial data, so the class of initial data we handle with is rather general. Furthermore, ω_{BL} has a simple representation and at least up to the time $c_0\nu^{1/3}$ we do not need the approximation using the Prandtl-type equations in the boundary layer. This observation of the order $\nu^{1/3}$ is newly obtained, though it is still not clear if the power $1/3$ is optimal or not for (1.3) to hold with ω_{BL} in Theorem 3.1; note that this expansion should hold at most only up to the time ν^β for some $\beta > 0$ in general, because the function ω_{BL} in Theorem 3.1 does not take into account the nonlinear interaction in the boundary layer region. It seems to be meaningful to improve the power $1/3$ in (1.3) for initial data in a Sobolev class even if the resulting power would be strictly positive.

The condition (1.4) is necessary and sufficient for the vorticity to exhibit an unbounded growth at $T_\nu = c_0\nu^{1/3}$ as $\nu\to 0$. The meaning of (1.4) is nothing but $\partial_t u_{E,1}|_{\,t=0} \not\equiv 0$ on $\partial\mathbb{R}^2_+$, where $u_E = (u_{E,1}, u_{E,2})$ denotes the solution to the Euler equations with the initial data a. Hence (1.4) represents the nondegenerate condition for u_E to be a nonzero velocity field on the boundary right after the initial time. In such situations it is natural that the boundary layer immediately appears and thus the high vorticity creation occurs near the initial time.

In Theorem 3.1 it is also proved that the L^∞ norm of u is uniformly bounded in $0 < \nu \ll 1$ for the time period $(0, c_0\nu^{1/3})$. In fact, if one does not use the vorticity equations it might be difficult to obtain this uniform bound rigorously for such a "long" time for general initial data. This is one of the advantages of the approach to (NS) from the vorticity formulation.

Before stating the results, let us introduce some function spaces. $C_0^\infty(\mathbb{R}^2_+)$ is the set of smooth functions with compact support in $\mathbb{R}^2_+$; $W_0^{l,p}(\mathbb{R}^2_+)$, $l \in \mathbb{N}$, $1 \le p \le \infty$, is the closure of $C_0^\infty(\mathbb{R}^2_+)$ with respect to the norm of the Sobolev space $W^{l,p}(\mathbb{R}^2_+)$; $C_{0,\sigma}^\infty(\mathbb{R}^2_+)$ denotes the set of all C^∞-vector functions $u = (u_1, u_2)$ with compact support in $\mathbb{R}^2_+$ such that div $u = 0$; $L^p_\sigma(\mathbb{R}^2_+)$ is the closure of $C_{0,\sigma}^\infty(\mathbb{R}^2_+)$ with respect to the norm in $(L^p(\mathbb{R}^2_+))^2$.

2. Solution formula for the half plane case

For (1.1) in $(0,T) \times \mathbb{R}^2_+$ with (1.2) we can apply the Fourier-Laplace transform to derive a solution formula. This formula is considered as a vorticity counterpart of the well-known formula for solutions to (NS) by [36, 40]. For the moment let us consider the linear problem

$$\text{(LV)} \qquad \begin{cases} \partial_t\omega - \nu\Delta\omega = f & t > 0, \quad x \in \mathbb{R}^2_+, \\ \omega|_{t=0} = b & x \in \mathbb{R}^2_+, \end{cases}$$

together with the boundary condition

$$\text{(LBC)} \qquad \nu\big(\partial_2 + (-\partial_1^2)^{\frac{1}{2}}\big)\omega = g \qquad t > 0, \quad x \in \partial\mathbb{R}^2_+.$$

Here f, g, b are assumed to be smooth and decay fast enough at spatial infinity. The integral equation for the vorticity equations will be obtained by taking

$$f = -u\cdot\nabla\omega, \qquad g = -\partial_2(-\Delta_D)^{-1}(u\cdot\nabla\omega)\big|_{x_2=0}, \tag{2.1}$$

and $u = \nabla^\perp(-\Delta_D)^{-1}\omega$ with $\nabla^\perp = (\partial_2, -\partial_1)$. We set

$$\Xi = 2\big(\partial_1^2 + (-\partial_1^2)^{\frac{1}{2}}\partial_2\big), \tag{2.2}$$

$$G(t,x) = \frac{1}{4\pi t}\exp\Big(-\frac{|x|^2}{4t}\Big), \qquad E(x) = -\frac{1}{2\pi}\log|x|, \tag{2.3}$$

$$\Gamma(t,x) = \big(\Xi E * G(t)\big)(x), \tag{2.4}$$

$$(h_1 \star h_2)(x) = \int_{\mathbb{R}^2_+} h_1(x - y^*)h_2(y)\,dy, \qquad y^* = (y_1, -y_2). \tag{2.5}$$

Theorem 2.1. *The integral equation for (LV)-(LBC) is given by*

$$\begin{aligned}\omega(t) \;&=\; e^{\nu t\Delta_N}b \;+\; \Gamma(\nu t)\star b \;-\; \Gamma(0)\star b\\ &+\int_0^t e^{\nu(t-s)\Delta_N}\big(f(s)-g(s)\mathcal{H}^1_{\{x_2=0\}}\big)\,\mathrm{d}s+\int_0^t \Gamma(\nu(t-s))\star\big(f(s)-g(s)\mathcal{H}^1_{\{x_2=0\}}\big)\,\mathrm{d}s\\ &-\int_0^t \Gamma(0)\star\big(f(s)-g(s)\mathcal{H}^1_{\{x_2=0\}}\big)\,\mathrm{d}s.\end{aligned}$$

Here $e^{t\Delta_N}$ is the semigroup for the heat equation in $\mathbb{R}^2_+$ with the homogeneous Neumann boundary condition, $\Gamma(0)\star := \lim_{t\downarrow 0}\Gamma(t)\star$, and $g\mathcal{H}^1_{\{x_2=0\}}$ is the one-dimensional Hausdorff measure with density g defined by

$$\langle h, g\mathcal{H}^1_{\{x_2=0\}}\rangle = \int_{\mathbb{R}} h(x_1,0)g(x_1)\,\mathrm{d}x_1, \qquad h\in C_0(\mathbb{R}^2_+).$$

We note that $\Gamma(0)\star h = \Xi E\star h$ in $\mathbb{R}^2_+$. In the above formula the terms $\Gamma(0)\star$ seem to cause trouble when solving the vorticity equations, for apparently they could give rise to a derivative loss near the boundary. In fact, these terms do not appear in the vorticity equations, due to the following cancellation property.

Proposition 2.1. *If $g = \partial_2(-\Delta_D)^{-1}f\,|_{x_2=0}$ then*

$$\Xi E\star\big(f-g\mathcal{H}^1_{\{x_2=0\}}\big)=0 \qquad \text{in } \mathbb{R}^2_+. \tag{2.6}$$

In particular, we have

$$\Xi E\star b = 0 \quad \text{in } \mathbb{R}^2_+ \qquad \text{if} \quad \partial_2(-\Delta_D)^{-1}b=0 \quad \text{on } \partial\mathbb{R}^2_+. \tag{2.7}$$

The condition in (2.7) is nothing but the no-slip boundary condition for the initial velocity field. Thus, reminding also (2.1), we do not have the problematic terms $\Gamma(0)\star$ in the solution formula for the vorticity equations. It will be useful to rewrite the result of Theorem 2.1 under the conditions in Proposition 2.1.

Corollary 2.1. *Assume that $\partial_2(-\Delta_D)^{-1}b\,|_{x_2=0}=0$ and $g=\partial_2(-\Delta_D)^{-1}f\,|_{x_2=0}$. Then the integral equation for (LV)-(LBC) is given by*

$$\omega(t)=e^{\nu tB}b+\int_0^t e^{\nu(t-s)B}\big(f(s)-g(s)\mathcal{H}^1_{\{x_2=0\}}\big)\,\mathrm{d}s, \tag{2.8}$$

where

$$e^{tB}h = e^{t\Delta_N}h+\Gamma(t)\star h. \tag{2.9}$$

Corollary 2.1 shows that the integral equation for the vorticity equation is written as

(2.10)

$$\omega(t) = e^{\nu tB}b - \int_0^t e^{\nu(t-s)B}\Big(u\cdot\nabla\omega(s) - \partial_2(-\Delta_D)^{-1}\big(u\cdot\nabla\omega\big)(s)\mathcal{H}^1_{\{x_2=0\}}\Big)\,ds,$$

with $u = \nabla^\perp(-\Delta_D)^{-1}\omega$. It is possible to show the local-in-time solvability of (2.10) if $b \in L^p(\mathbb{R}^2_+)$ for some $p \in [1,2)$ by the contraction mapping theorem. There the following $L^p - L^q$ estimates for e^{tB} are essential.

Lemma 2.1. (i) *Let $1 \le q < p \le \infty$ or $1 < q \le p < \infty$. Then we have*

$$\|e^{tB}f\|_{L^p} \le Ct^{-\frac{1}{q}+\frac{1}{p}}\|f\|_{L^q} \qquad t > 0.$$

(ii) *Let $1 \le q \le p \le \infty$ and $p > 1$. Then we have*

$$\|e^{tB}(g\mathcal{H}^1_{\{x_2=0\}})\|_{L^p} \le Ct^{-\frac{1}{2}(1+\frac{1}{q}-\frac{2}{p})}\|g\|_{L^q_{x_1}} \qquad t > 0.$$

(iii) *Let $1 \le q \le p \le \infty$ and $k \in \mathbb{N}$. Then we have*

$$\|\nabla^k e^{tB}f\|_{L^p} \le Ct^{-\frac{1}{q}+\frac{1}{p}-\frac{k}{2}}\|f\|_{L^q} \qquad t > 0.$$

(iv) *Let $1 \le q \le p \le \infty$. Assume that $g = \partial_2(-\Delta)^{-1}f\ |_{x_2=0}$. Then we have*

$$\|e^{tB}(f - g\mathcal{H}^1_{\{x_2=0\}})\|_{L^p} \le Ct^{-\frac{1}{q}+\frac{1}{p}-\frac{1}{2}}\|\nabla^\perp(-\Delta_D)^{-1}f\|_{L^q} \qquad t > 0.$$

We conclude this section by stating the local solvability of (2.10).

Theorem 2.2. *Assume that $b \in L^p(\mathbb{R}^2_+)$ for some $p \in (1,2)$. Then there is $T > 0$ such that (2.10) has a unique solution $\omega \in C([0,T); L^p(\mathbb{R}^2_+))$ satisfying $\sup_{0<t<T} t^{1/p-1/4}\|\omega(t)\|_{L^4} < \infty$. If b satisfies the compatibility condition $\partial_2(-\Delta_D)^{-1}b = 0$ on $\partial\mathbb{R}^2_+$ in addition, then the solution $\omega(t)$ converges to b as $t \to 0$ in $L^p(\mathbb{R}^2_+)$.*

Remark 2.1. Even for $b \in L^1(\mathbb{R}^2_+)$ we can construct a local unique solution ω to (2.10) such that $t^{1-1/r}\omega(t) \in L^\infty(0,T; L^r(\mathbb{R}^2_+))$ and $\lim_{t\to 0} t^{1-1/r}\|\omega(t)\|_{L^r} = 0$ with $r = 4/3, 4$. Furthermore, under the smallness assumption of $\|b\|_{L^1}$ it is also possible to show that the solution exists globally in time.

Remark 2.2. By the bootstrap argument using Lemma 2.1 the solution ω in Theorem 2.2 is shown to be smooth in positive time. We note that, in order to ensure that $u = \nabla^\perp(-\Delta_D)^{-1}\omega$ solves (NS), we need the compatibility condition $\partial_2(-\Delta_D)^{-1}b = 0$ on $\partial\mathbb{R}^2_+$ for the initial data.

Remark 2.3. When $b \in L^p(\mathbb{R}^2_+)$ for some $p \in (1,2)$ the related velocity a belongs to $L^q_\sigma(\mathbb{R}^2_+)$ with $1/q = 1/p - 1/2$ by the Hardy-Littlewood-Sobolev inequality. Since $q > 2$ we already know the solvability of (NS) in this case from the L^q theory of the Stokes or the Navier-Stokes equations in the half space; for example, see [36, 11, 37, 42, 12, 28, 14, 40, 34, 10] and references therein. On the other hand, if $b \in L^1(\mathbb{R}^2_+)$ then a belongs to the weak L^2 space. The reader is referred to [25] for the analysis of the Navier-Stokes equations in the weak L^p spaces.

3. Application to analysis of vorticity at zero viscosity limit

The inviscid limit behavior of solutions to the Navier-Stokes equations is a classical theme in fluid mechanics. However, if the no-slip boundary conditions are imposed on velocity fields, only partial results are known even in the two-dimensional case; so far we need either the analyticity of initial data or the radial symmetry of the domain and the solutions. More precisely, if the initial data is analytic it is proved in [32, 33] that the inviscid limit is described by the Euler equations and the Prandtl equations; we also refer to [2]. When Ω is a disk and the solution possesses a radial symmetry, the inviscid limit is already well studied in various functional settings [30, 5, 26, 27, 22, 31]. On the other hand, [18] gave necessary and sufficient conditions for the convergence of weak solutions of (NS) to that of the Euler equations in the energy class. The analysis in this direction has been developed by several authors [39, 41, 9, 20, 21, 22].

Making use of (2.8), in this section we study the behavior of vorticity fields at the zero viscosity limit and establish the asymptotic expansion near the initial time. The main result is stated as follows.

Theorem 3.1. *Assume that $b = \mathrm{Rot}\ a$ with $a \in L^q_\sigma(\mathbb{R}^2_+) \cap (W^{1,q}_0(\mathbb{R}^2_+))^2$ for some $1 < q < \infty$ and $b \in W^{l,4/3}(\mathbb{R}^2_+)$ for $l \gg 1$. Let ω be the solution to (1.1)-(1.2) in $(0,T) \times \mathbb{R}^2_+$ with the initial data b. Then there are $c_0, C > 0$ such that the following estimates hold for sufficiently small $\nu > 0$:*

$$\|u(t)\|_{L^\infty} \le C \qquad 0 < t \le c_0 \nu^{\frac{1}{3}}, \tag{3.1}$$

$$\|\omega(t) - \omega_E(t) - \omega_{BL}(t)\|_{L^p} \le C\nu^{-\frac{1}{2}(\frac{1}{3}-\frac{1}{p})} t^{\frac{1}{2}(1+\frac{1}{p})} \qquad 0 < t \le c_0 \nu^{\frac{1}{3}}. \tag{3.2}$$

Here $4/3 \le p \le \infty$ and c_0 is independent of ν, and C is independent of ν and $t \in [0, c_0\nu^{1/3}]$. The function ω_E is the vorticity field of the solution to the Euler equation with the initial velocity a. The function ω_{BL} is defined by

$$\omega_{BL}(t,x) = 2\int_0^t (4\pi\nu s)^{-\frac{1}{2}} \exp(-\frac{x_2^2}{4\nu s})\, \mathrm{d}s \cdot \partial_2(-\Delta_D)^{-1}(a\cdot\nabla b)(x_1, 0), \tag{3.3}$$

and in particular, it is nontrivial if and only if

$$\partial_2(-\Delta_D)^{-1}(a\cdot\nabla b) \not\equiv 0 \qquad \text{on } \partial\mathbb{R}^2_+. \tag{3.4}$$

When (3.4) holds ω_{BL} satisfies

$$\|\omega_{BL}(t)\|_{L^p} \le C'\nu^{-\frac{1}{2}(1-\frac{1}{p})}t^{\frac{1}{2}(1+\frac{1}{p})} \qquad t>0, \quad 1\le p\le\infty, \tag{3.5}$$

$$\|\omega_{BL}(t)\|_{L^p(\{0\le x_2\le(\nu t)^{\frac{1}{2}}\})} \ge c_1\nu^{-\frac{1}{2}(1-\frac{1}{p})}t^{\frac{1}{2}(1+\frac{1}{p})} \qquad t>0, \quad 1\le p\le\infty. \tag{3.6}$$

Here the positive constants c_1 and C' are independent of ν and t.

Corollary 3.1. *Under the assumptions of Theorem 3.1, if (3.4) holds in addition, then the high creation of vorticity near the boundary in L^p occurs in the following sense.*

$$\|\omega(c_0\nu^{\frac{1}{3}})\|_{L^p(\{0\le x_2\le c_0^{\frac{1}{2}}\nu^{\frac{2}{3}}\})} \ge c_2\nu^{-\frac{1}{3}(1-\frac{2}{p})} \to \infty \quad (\nu\to 0) \qquad \text{if} \quad 2<p\le\infty.$$

Here $c_2>0$ is independent of ν and $t\in[0,c_0\nu^{1/3}]$.

The details of the proof of Theorem 3.1 are given in [29]. Let $J(f)$ be the velocity field recovered from f via the Biot-Savart law, i.e.,

$$J(f) = (J_1(f), J_2(f)) = \nabla^\perp(-\Delta_D)^{-1}f, \qquad \nabla^\perp = (\partial_2,-\partial_1). \tag{3.7}$$

Then $J(f)$ satisfies $\nabla\cdot J(f)=0$ in $\mathbb{R}^2_+$ and $J_2(f)=0$ on $\partial\mathbb{R}^2_+$. The function ω_E satisfies the equation

$$\begin{cases} \partial_t\omega_E + u_E\cdot\nabla\omega_E = 0 & t>0, \quad x\in\mathbb{R}^2_+, \\ u_E = J(\omega_E) & t>0, \quad x\in\mathbb{R}^2_+, \\ \omega_E|_{t=0} = b & x\in\mathbb{R}^2_+. \end{cases} \tag{3.8}$$

Eq.(3.8) is equivalent to the Euler equations with the boundary condition $u_{E,2}=0$ on $\partial\mathbb{R}^2_+$. Hence, under the assumption $b\in W^{l,4/3}(\mathbb{R}^2_+)$ with $l\gg 1$ the existence and the uniqueness of solutions to (3.8) follow from the methods developed in the literature [43, 44, 17, 3, 23, 6, 8, 7]. In particular, we can show that $\omega_E\in C^1([0,T);W^{l',4/3}(\mathbb{R}^2_+))$ with $l'\gg 1$ for all $T>0$. Next we consider the second and third expansions of ω which are directly related with ω_E:

$$\begin{cases} \partial_t w_{E,1} - \nu\Delta w_{E,1} = 0 & t>0, \quad x\in\mathbb{R}^2_+, \\ \nu(\partial_2 w_{E,1} + (-\partial_1^2)^{\frac{1}{2}}w_{E,1}) = -J_1(u_E\cdot\nabla\omega_E) & t>0, \quad x\in\partial\mathbb{R}^2_+, \\ w_{E,1}|_{t=0} = 0 & x\in\mathbb{R}^2_+, \end{cases} \tag{3.9}$$

$$\begin{cases} \partial_t w_{E,2} - \nu\Delta w_{E,2} = \nu\Delta\omega_E & t>0, \quad x\in\mathbb{R}^2_+, \\ \nu(\partial_2 w_{E,2} + (-\partial_1^2)^{\frac{1}{2}}w_{E,2}) = -\nu J_1(\Delta\omega_E) & t>0, \quad x\in\partial\mathbb{R}^2_+, \\ w_{E,2}|_{t=0} = 0 & x\in\mathbb{R}^2_+. \end{cases} \tag{3.10}$$

The function $w_{E,1}$ is responsible for the creation of vorticity near the boundary. Set

$$(3.11) \qquad w_E = w_{E,1} + w_{E,2}, \qquad F = J(\omega_E + w_E) \cdot \nabla w_E + J(w_E) \cdot \nabla \omega_E.$$

Then $w = \omega - \omega_E - w_E$ satisfies $w|_{t=0} = 0$ and
(3.12)

$$\begin{cases} \partial_t w - \nu \Delta w = -L(\omega_E + w_E)w - N(w,w) - F & t > 0, \quad x \in \mathbb{R}^2_+, \\ \nu(\partial_2 w + (-\partial_1^2)^{\frac{1}{2}} w) = -J_1(L(\omega_E + w_E)w + N(w,w) + F) & t > 0, \quad x \in \partial\mathbb{R}^2_+. \end{cases}$$

Here

$$(3.13) \qquad L(f)w = J(f) \cdot \nabla w + J(w) \cdot \nabla f, \qquad N(f,g) = J(f) \cdot \nabla g.$$

By the above definitions we can check that each of $J(\omega_E + w_{E,1})$, $J(w_{E,2})$, and $J(w)$, satisfies the no-slip boundary condition (see [29]), and this property will be essentially used in the proof of Theorem 3.1. We note that the above decomposition of ω should be effective only near the initial time $0 < t \le \nu^\beta$ for some $\beta > 0$. For a longer time period we need to take into account the vorticity counterpart of the Prandtl equations, where the verification of such expansion is widely open except for the analytic initial data.

The basic strategy for the proof of Theorem 3.1 is as follows: we will use the integral equations (2.8) for the estimates of $w_{E,1}$ and $w_{E,2}$, and also of w near the boundary. The estimates of w away from the boundary will be obtained by the energy argument. Theorem 3.1 then follows by combining these a priori estimates.

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