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Smallest complex nilpotent orbits with real points

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Abstract

In this paper, we show that there uniquely exists a real minimal nilpotent orbit in a non-compact simple Lie algebra $\mathfrak{g}$ if $(\mathfrak{g}, \mathfrak{k})$ is of non-Hermitian type. For the cases where $\mathfrak{g}$ is isomorphic to $\mathfrak{su}^*(2k)$, $\mathfrak{so}(n-1,1)$, $\mathfrak{sp}(p,q)$, $\mathfrak{e}_{6(-26)}$ or $\mathfrak{f}_{4(-20)}$, the complexification $\mathcal{O}^{G_{\mathbb{C}}}_{\min,\mathfrak{g}}$ of such the real minimal nilpotent orbit in $\mathfrak{g}$ is not the complex minimal nilpotent orbit in $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} + \sqrt{-1}\mathfrak{g}$. For such cases, we also determine $\mathcal{O}^{G_{\mathbb{C}}}_{\min,\mathfrak{g}}$ by describing the weighted Dynkin diagram of it.

1 Introduction and main results

Let $\mathfrak{g}_{\mathbb{C}}$ be a complex simple Lie algebra. In this paper, an adjoint nilpotent orbit in $\mathfrak{g}_{\mathbb{C}}$ will be simply called a complex nilpotent orbit in $\mathfrak{g}_{\mathbb{C}}$. It is well-known that there exists a unique non-zero complex nilpotent orbit $\mathcal{O}^{G_{\mathbb{C}}}_{\min}$ in $\mathfrak{g}_{\mathbb{C}}$, which is called a complex minimal nilpotent orbit, with the following property: The closure of $\mathcal{O}^{G_{\mathbb{C}}}_{\min}$ in $\mathfrak{g}_{\mathbb{C}}$ is just $\mathcal{O}^{G_{\mathbb{C}}}_{\min} \sqcup \{0\}$. By the uniqueness of such $\mathcal{O}^{G_{\mathbb{C}}}_{\min}$, for any non-zero complex nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}_{\mathbb{C}}$, the closure of $\mathcal{O}$ contains $\mathcal{O}^{G_{\mathbb{C}}}_{\min}$. In other words, $\mathcal{O}^{G_{\mathbb{C}}}_{\min}$ is minimum in $\mathcal{N}/G_{\mathbb{C}}$ without the zero-orbit, where $\mathcal{N}/G_{\mathbb{C}}$ denotes the set of complex nilpotent orbits in $\mathfrak{g}_{\mathbb{C}}$ with the closure ordering.

Let $\mathfrak{g}$ be a non-compact real form of $\mathfrak{g}_{\mathbb{C}}$. Namely, $\mathfrak{g}$ is a non-compact real simple Lie algebra without complex structures and $\mathfrak{g}_{\mathbb{C}}$ is the complexification of $\mathfrak{g}$. Our concern in this paper is in real minimal nilpotent orbits in $\mathfrak{g}$. Here, we say that a non-zero real nilpotent orbit $\mathcal{O}^G$ in $\mathfrak{g}$ is minimal if the closure

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of $\mathcal{O}^G$ in $\mathfrak{g}$ is just $\mathcal{O}^G \sqcup \{0\}$. In general, real minimal nilpotent orbits are not unique for real simple $\mathfrak{g}$.

If the complex minimal nilpotent orbit $\mathcal{O}^{G_\mathbb{C}}_{\min}$ in $\mathfrak{g}_\mathbb{C}$ meets $\mathfrak{g}$, then the intersection $\mathcal{O}^{G_\mathbb{C}}_{\min} \cap \mathfrak{g}$ is the union of all real minimal nilpotent orbits in $\mathfrak{g}$. It is known that $\mathcal{O}^{G_\mathbb{C}}_{\min}$ meets $\mathfrak{g}$ if and only if $\mathfrak{g}$ is not isomorphic to $\mathfrak{su}^*(2k)$ $(k \geq 2)$, $\mathfrak{so}(n-1,1)$ $(n \geq 5)$, $\mathfrak{sp}(p,q)$ $(p \geq q \geq 1)$, $\mathfrak{f}_{4(-20)}$ nor $\mathfrak{e}_{6(-26)}$ (see Brylinski [3, Theorem 4.1]). In particular, if $(\mathfrak{g}, \mathfrak{k})$ is of Hermitian type, then $\mathcal{O}^{G_\mathbb{C}}_{\min}$ meets $\mathfrak{g}$, where $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$. Furthermore, for the cases where $\mathcal{O}^{G_\mathbb{C}}_{\min}$ meets $\mathfrak{g}$, the number of real minimal nilpotent orbits (i.e. the number of adjoint orbits in $\mathcal{O}^{G_\mathbb{C}}_{\min} \cap \mathfrak{g}$) is two if $(\mathfrak{g}, \mathfrak{k})$ is of Hermitian type; one if $(\mathfrak{g}, \mathfrak{k})$ is of non-Hermitian type.

In this paper, we study real minimal nilpotent orbits in $\mathfrak{g}$ including the cases where $\mathcal{O}^{G_\mathbb{C}}_{\min}$ does not meets $\mathfrak{g}$. For any real non-compact simple Lie algebra $\mathfrak{g}$ without complex structures, we put

$$\mathcal{N}_\mathfrak{g}/G_\mathbb{C} := \{\, \text{Complex nilpotent orbits in } \mathfrak{g}_\mathbb{C} \text{ meeting } \mathfrak{g} \,\}$$

and consider the closure ordering on it. Our first main result is here:

Theorem 1.1. *There uniquely exists a complex nilpotent orbit $\mathcal{O}^{G_\mathbb{C}}_{\min,\mathfrak{g}}$ in $\mathfrak{g}_\mathbb{C}$ which is minimum in $\mathcal{N}_\mathfrak{g}/G_\mathbb{C}$ without the zero-orbit (i.e. for any non-zero complex nilpotent orbit $\mathcal{O}$ in $\mathfrak{g}$, if $\mathcal{O} \cap \mathfrak{g} \neq \emptyset$, then the closure of $\mathcal{O}$ in $\mathfrak{g}_\mathbb{C}$ contains $\mathcal{O}^{G_\mathbb{C}}_{\min,\mathfrak{g}}$). Furthremore, the intersection $\mathcal{O}^{G_\mathbb{C}}_{\min,\mathfrak{g}} \cap \mathfrak{g}$ is the union of all real minimal nilpotent orbits in $\mathfrak{g}$.*

We will construct such $\mathcal{O}^{G_\mathbb{C}}_{\min,\mathfrak{g}}$ as the complex adjoint orbit through a non-zero longest restricted root vector in $\mathfrak{g}$. By the definition of $\mathcal{O}^{G_\mathbb{C}}_{\min,\mathfrak{g}}$, the complex minimal nilpotent orbit $\mathcal{O}^{G_\mathbb{C}}_{\min}$ is not our $\mathcal{O}^{G_\mathbb{C}}_{\min,\mathfrak{g}}$ if and only if $\mathcal{O}^{G_\mathbb{C}}_{\min}$ does not meet $\mathfrak{g}$ (namely, $\mathfrak{g}$ is isomorphic to $\mathfrak{su}^*(2k)$ $(k \geq 2)$, $\mathfrak{so}(n-1,1)$ $(n \geq 5)$, $\mathfrak{sp}(p,q)$ $(p \geq q \geq 1)$, $\mathfrak{f}_{4(-20)}$ or $\mathfrak{e}_{6(-26)}$). This means that for such cases, a non-zero longest restricted root vector in $\mathfrak{g}$ is not a longest root vector in $\mathfrak{g}_\mathbb{C}$.

Theorem 1.1 claims that $\mathcal{O}^{G_\mathbb{C}}_{\min,\mathfrak{g}} \cap \mathfrak{g}$ is the union of all real minimal nilpotent orbits in $\mathfrak{g}$. Our second main result is here:

Theorem 1.2. *For the cases where the complex minimal nilpotent orbit $\mathcal{O}^{G_\mathbb{C}}_{\min}$ does not meet $\mathfrak{g}$, there exists a unique real minimal nilpotent orbit in $\mathfrak{g}$. In particular, the complex nilpotent orbit $\mathcal{O}^{G_\mathbb{C}}_{\min,\mathfrak{g}}$ in Theorem 1.1 (which is not $\mathcal{O}^{G_\mathbb{C}}_{\min}$ in these cases) is the complexification of the unique real minimal nilpotent orbit in $\mathfrak{g}$.*

Therefore, we have the following corollary:

Corollary 1.3. *Let $\mathfrak{g}$ be a non-compact real simple Lia algebra without complex structures. If $(\mathfrak{g}, \mathfrak{k})$ is of non-Hermitian type, there uniquely exists a real minimal nilpotent orbit in $\mathfrak{g}$. If $(\mathfrak{g}, \mathfrak{k})$ is of Hermitian type, there are just two real miniamal nilpotent orbits in $\mathfrak{g}$.*

By Theorem 1.2, our $\mathcal{O}^{G_{\mathbb{C}}}_{\min,\mathfrak{g}}$ is just the complexification of the unique real minimal nilpotent orbit in $\mathfrak{g}$ for the cases where $\mathfrak{g}$ is isomorphic to $\mathfrak{su}^*(2k)$ $(k \geq 2)$, $\mathfrak{so}(n-1,1)$ $(n \geq 5)$, $\mathfrak{sp}(p,q)$ $(p \geq q \geq 1)$, $\mathfrak{f}_{4(-20)}$ or $\mathfrak{e}_{6(-26)}$. We will determine our $\mathcal{O}^{G_{\mathbb{C}}}_{\min,\mathfrak{g}}$ by describing the weighted Dynkin diagram of it for such cases (recall that for another cases, $\mathcal{O}^{G_{\mathbb{C}}}_{\min,\mathfrak{g}}$ is just $\mathcal{O}^{G_{\mathbb{C}}}_{\min}$). The result is here (see also Table 2 in §2 for the weighted Dynkin diagrams of $\mathcal{O}^{G_{\mathbb{C}}}_{\min}$):

Theorem 1.4. *For the cases where $\mathcal{O}^{G_{\mathbb{C}}}_{\min,\mathfrak{g}} \neq \mathcal{O}^{G_{\mathbb{C}}}_{\min}$, the weighted Dynkin diagram of $\mathcal{O}^{G_{\mathbb{C}}}_{\min,\mathfrak{g}}$ are the following:*

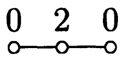
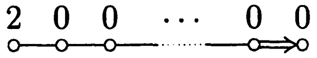
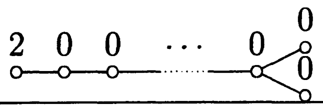
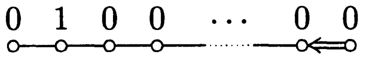
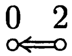
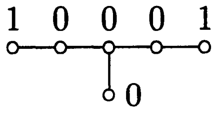
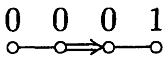
$\mathfrak{g}$	$\dim_{\mathbb{C}} \mathcal{O}^{G_{\mathbb{C}}}_{\min,\mathfrak{g}}$	*Weighted Dynkin diagram of* $\mathcal{O}^{G_{\mathbb{C}}}_{\min,\mathfrak{g}}$
$\mathfrak{su}^*(2k)$	$8k-8$	 $(k \geq 3)$  $(k = 2)$
$\mathfrak{so}(n-1,1)$	$2n-4$	 *(n is odd, $n \geq 5$)*  *(n is even, $n \geq 6$)*
$\mathfrak{sp}(p,q)$	$4(p+q)-2$	 $(p+q \geq 3, p \geq q \geq 1)$  $(p = q = 1)$
$\mathfrak{e}_{6(-26)}$	32	
$\mathfrak{f}_{4(-20)}$	22	

Table 1: List of $\mathcal{O}^{G_{\mathbb{C}}}_{\min,\mathfrak{g}}$ for $\mathfrak{su}^*(2k)$, $\mathfrak{so}(n-1,1)$, $\mathfrak{sp}(p,q)$, $\mathfrak{e}_{6(-26)}$ and $\mathfrak{f}_{4(-20)}$.

This works motivated by recent works [7], by Joachim Hilgert, Toshiyuki Kobayashi and Jan Möllers, on the construction of an L^2-model of irreducible unitary representations of real reductive groups with smallest Gelfand-Kirillov dimension; and [8], by Toshiyuki Kobayashi and Yoshiki Oshima, on the classification of reductive symmetric pairs $(\mathfrak{g}, \mathfrak{h})$ with a $(\mathfrak{g}, K)$-module which is discretely decomposable as an $(\mathfrak{h}, H \cap K)$-module.

2 Preliminary results for weighted Dynkin diagrams of complex minimal nilpotent orbits

In this section, we recall weighted Dynkin diagrams of complex minimal nilpotent orbits in complex simple Lie algebras.

Let $\mathfrak{g}_{\mathbb{C}}$ be a complex semisimple Lie algebra, and denote by $G_{\mathbb{C}}$ the inner automorphism group of $\mathfrak{g}_{\mathbb{C}}$. Fix a Cartan subalgebra $\mathfrak{h}_{\mathbb{C}}$ of $\mathfrak{g}_{\mathbb{C}}$. We denote by $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ the root system of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. Then, the root system $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ can be regarded as a subset of the dual space $\mathfrak{h}^*$ of

$$\mathfrak{h} := \{\, H \in \mathfrak{h}_{\mathbb{C}} \mid \alpha(H) \in \mathbb{R} \ ({}^{\forall}\alpha \in \Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}))\,\}.$$

We write $W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ for the Weyl group of $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ acting on $\mathfrak{h}$. Take a positive system $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ of the root system $\Delta(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. Then, a closed Weyl chamber

$$\mathfrak{h}_+ := \{\, H \in \mathfrak{h} \mid \alpha(H) \geq 0 \ ({}^{\forall}\alpha \in \Delta_+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}}))\,\}$$

is a fundamental domain of $\mathfrak{h}$ under the action of $W(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$.

Let Π be the simple system of $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. Then, for any $H \in \mathfrak{h}$, we can define a map

$$\Psi_H : \Pi \to \mathbb{R}, \ \alpha \mapsto \alpha(H).$$

We call Ψ_H the weighted Dynkin diagram corresponding to $H \in \mathfrak{h}$, and $\alpha(H)$ the weight on a node $\alpha \in \Pi$ of the weighted Dynkin diagram. Since Π is a basis of $\mathfrak{h}^*$, the map

$$\Psi : \mathfrak{h} \to \mathrm{Map}(\Pi, \mathbb{R}), \ H \mapsto \Psi_H$$

is a linear isomorphism (between vector spaces). Furthermore,

$$\mathfrak{h}_+ \to \mathrm{Map}(\Pi, \mathbb{R}_{\geq 0}), \; H \mapsto \Psi_H$$

is also bijective.

A triple (H, X, Y) is said to be an $\mathfrak{sl}_2$-triple in $\mathfrak{g}_\mathbb{C}$ if

$$[H, X] = 2X, \; [H, Y] = -2Y, \; [X, Y] = H \quad (H, X, Y \in \mathfrak{g}_\mathbb{C}).$$

For any $\mathfrak{sl}_2$-triple (H, X, Y) in $\mathfrak{g}_\mathbb{C}$, the elelements X and Y are nilpotent in $\mathfrak{g}_\mathbb{C}$, and H is hyperbolic in $\mathfrak{g}_\mathbb{C}$ (i.e. $\mathrm{ad}_{\mathfrak{g}_\mathbb{C}} H \in \mathrm{End}(\mathfrak{g}_\mathbb{C})$ is diagonalizable with only real eigenvalues).

Combining the Jacobson–Morozov theorem with Kostant [9], for any complex nilpotent orbit $\mathcal{O}^{G_\mathbb{C}}$, there uniquely exists an element $H_\mathcal{O}$ of $\mathfrak{h}_+$ with the following property: There exists $X, Y \in \mathcal{O}^{G_\mathbb{C}}$ such that $(H_\mathcal{O}, X, Y)$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}_\mathbb{C}$. Furthermore, by Malcev [10], the following map is injective:

$$\{\, \text{Complex nilpotent orbits in } \mathfrak{g}_\mathbb{C} \,\} \hookrightarrow \mathfrak{h}_+, \; \mathcal{O}^{G_\mathbb{C}} \mapsto H_\mathcal{O}.$$

The weighted Dynkin diagram corresponding to $H_\mathcal{O}$ is called the weighted Dynkin diagram of $\mathcal{O}^{G_\mathbb{C}}$. Dynkin [6] proved that for any complex nilpotent orbit $\mathcal{O}^{G_\mathbb{C}}$, any weight of the weighted Dynkin diagram of $\mathcal{O}^{G_\mathbb{C}}$ is given by 0, 1 or 2, and classified weighted Dynkin diagrams of complex nilpotent orbits (More precisely, Dynkin [6] classified $\mathfrak{sl}_2$-triples in $\mathfrak{g}_\mathbb{C}$. See Bala–Carter [2] for more details).

In the rest of this subsection, we suppose that $\mathfrak{g}_\mathbb{C}$ is simple. Let ϕ be the highest root of $\Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$. Then, the complex minimal nilpotent orbit in $\mathfrak{g}_\mathbb{C}$ can be written by

$$\mathcal{O}^{G_\mathbb{C}}_{\min} = G_\mathbb{C} \cdot \mathfrak{g}_\phi \setminus \{0\}.$$

We define the element $H_{\phi^\vee}$ of $\mathfrak{h}$ by

$$\alpha(H_{\phi^\vee}) = \frac{2\langle \alpha, \phi \rangle}{\langle \phi, \phi \rangle}$$

for any $\alpha \in \mathfrak{h}^*$ (where $\langle \ , \ \rangle$ is the inner product on $\mathfrak{h}^*$ induced by the Killing form on $\mathfrak{g}_\mathbb{C}$). Namley, $H_{\phi^\vee}$ is the element of $\mathfrak{h}$ corresponding to the coroot $\phi^\vee$ of ϕ. Since ϕ is dominant, $H_{\phi^\vee}$ is in $\mathfrak{h}_+$. Furthermore, $H_{\phi^\vee}$ is the hyperbolic element corresponding to $\mathcal{O}^{G_\mathbb{C}}_{\min}$ since we can find $X_\phi \in \mathfrak{g}_\phi$, $Y_\phi \in \mathfrak{g}_{-\phi}$ such that $(H_{\phi^\vee}, X_\phi, Y_\phi)$ is an $\mathfrak{sl}_2$-triple. The list of weighted Dynkin diagrams of $\mathcal{O}^{G_\mathbb{C}}_{\min}$ for all simple $\mathfrak{g}_\mathbb{C}$ can be found in Collingwood–McGovern [4, Ch.5.4 and 8.4].

Recall that our concern in this paper is in real simple Lie algebras $\mathfrak{su}^*(2k)$, $\mathfrak{so}(n-1,1)$, $\mathfrak{sp}(p,q)$, $\mathfrak{e}_{6(-26)}$ and $\mathfrak{f}_{4(-20)}$. The complexifications of such algebras are $\mathfrak{sl}(2k,\mathbb{C})$, $\mathfrak{so}(n,\mathbb{C})$, $\mathfrak{sp}(p+q,\mathbb{C})$, $\mathfrak{e}_{6,\mathbb{C}}$ and $\mathfrak{f}_{4,\mathbb{C}}$, respectively. For the convenience of the reader, we give a list of weighted Dynkin diagrams of complex minimal nilpotent orbits in such complex simple Lie algebras.

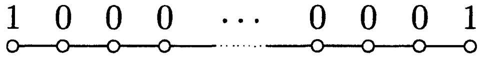
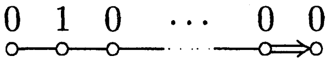
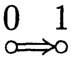
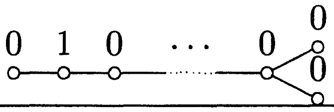
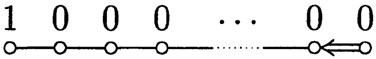
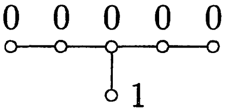
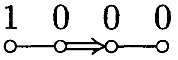
$\mathfrak{g}_{\mathbb{C}}$	$\dim_{\mathbb{C}} \mathcal{O}^{G_{\mathbb{C}}}_{\min}$	Weighted Dynkin diagram of $\mathcal{O}^{G_{\mathbb{C}}}_{\min,\mathfrak{g}}$
$\mathfrak{sl}(n,\mathbb{C})$	$2n$	 $(n \geq 2)$
$\mathfrak{so}(n,\mathbb{C})$	$2n-6$	 (n is odd, $n \geq 7$)
		 ($n = 5$)
		 (n is even, $n \geq 6$)
$\mathfrak{sp}(n,\mathbb{C})$	$2n$	 $(n \geq 2)$
$\mathfrak{e}_{6,\mathbb{C}}$	22	
$\mathfrak{f}_{4,\mathbb{C}}$	16	

Table 2: List of weighted Dynkin diagrams of $\mathcal{O}^{G_{\mathbb{C}}}_{\min}$ for $\mathfrak{sl}(n,\mathbb{C})$, $\mathfrak{so}(n,\mathbb{C})$, $\mathfrak{sp}(n,\mathbb{C})$, $\mathfrak{e}_{6,\mathbb{C}}$ and $\mathfrak{f}_{4,\mathbb{C}}$.

3 Outline of a proof of Theorem 1.1

Let $\mathfrak{g}_{\mathbb{C}}$ be a complex simple Lie algebra and $\mathfrak{g}$ a non-compact real form of $\mathfrak{g}$ with a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. In this section, we describe an idea of the proof of Theorem 1.1.

We fix a maximal abelian subspace $\mathfrak{a}$ of $\mathfrak{p}$ (such $\mathfrak{a}$ is called a maximally split abelian subspace of $\mathfrak{g}$) and write $\Sigma(\mathfrak{g},\mathfrak{a})$ for the restricted root system for $(\mathfrak{g}, \mathfrak{a})$. For any restricted root ξ of $\Sigma(\mathfrak{g}, \mathfrak{a})$, we define $A_{\xi^\vee} \in \mathfrak{a}$ by

$$\eta(A_{\xi^\vee}) = \frac{2(\xi, \eta)}{(\xi, \xi)} \quad ({}^\forall \eta \in \mathfrak{a}^*)$$

(where (,) is the inner product on $\mathfrak{a}^*$ induced by the Killing form on $\mathfrak{g}$). Namley, $A_{\xi^\vee}$ is the element of $\mathfrak{a}$ corresponding to the coroot $\xi^\vee$ of ξ. Then, the fact below holds:

Fact 3.1. *For any restricted root ξ of $\Sigma(\mathfrak{g}, \mathfrak{a})$ and any non-zero root vector X_ξ in $\mathfrak{g}_\xi$, there exists $Y_\xi \in \mathfrak{g}_{-\xi}$ such that $(A_{\xi^\vee}, X_\xi, Y_\xi)$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$.*

We fix an ordering on $\mathfrak{a}$ and write $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ for the positive system of $\Sigma(\mathfrak{g}, \mathfrak{a})$ corresponding to the ordering on $\mathfrak{a}$. We denote by λ the highest root of $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ with respect to the ordering on $\mathfrak{a}$. Next two lemmas give charactorizations of the highest root λ of $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ (we omit proofs of the two lemmas in this paper):

Lemma 3.2. *The highest root λ of $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ is a unique dominant longest root of $\Sigma(\mathfrak{g}, \mathfrak{a})$.*

Lemma 3.3. *Let ξ be a root of $\Sigma(\mathfrak{g}, \mathfrak{a})$. If ξ is not the highest root λ, then for any non-zero root vector X_ξ in $\mathfrak{g}_\xi$, there exists a positive root η in $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ and a root vector $X_\eta \in \mathfrak{g}_\eta$ such that $[X_\xi, X_\eta] \neq 0$. In particular, $\xi = \lambda$ if and only if $\xi + \eta \in \mathfrak{a}^*$ is not a root of $\Sigma(\mathfrak{g}, \mathfrak{a})$ for any $\eta \in \Sigma^+(\mathfrak{g}, \mathfrak{a})$.*

We write $G_{\mathbb{C}}$ for the inner automorphism group of $\mathfrak{g}_{\mathbb{C}}$. Then, the following two propositions hold:

Proposition 3.4. *For any non-zero real nilpotent orbit $\mathcal{O}'_0$ in $\mathfrak{g}$. Then, there exists a non-zero highest root vector X_λ in $\mathfrak{g}_\lambda$ such that X_λ is in the closure of $\mathcal{O}'_0$ in $\mathfrak{g}$.*

Proposition 3.5. *For any two highest root vectors X_λ, X'_λ in $\mathfrak{g}_\lambda$, there exists $g_{\mathbb{C}} \in G_{\mathbb{C}}$ such that $g_{\mathbb{C}} X_\lambda = X'_\lambda$.*

Proof of Proposition 3.4. There is no loss of generality in assuming that the ordering on $\mathfrak{a}$ is lexicographic. Let us put $\mathfrak{m} = Z_{\mathfrak{k}}(\mathfrak{a})$. Then, $\mathfrak{g}$ can be decomposed as

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\xi \in \Sigma(\mathfrak{g}, \mathfrak{a})} \mathfrak{g}_\xi.$$

For any $X' \in \mathfrak{g}$, we denote by

$$X' = X'_{\mathfrak{m}} + X'_{\mathfrak{a}} + \sum_{\xi \in \Sigma(\mathfrak{g},\mathfrak{a})} X'_{\xi} \quad (X'_{\mathfrak{m}} \in \mathfrak{m},\ X'_{\mathfrak{a}} \in \mathfrak{a},\ X'_{\xi} \in \mathfrak{g}_{\xi}).$$

We put $\overline{\mathcal{O}'_0}$ to the closure of $\mathcal{O}'_0$ in $\mathfrak{g}$ and fix an element X' in $\overline{\mathcal{O}'_0}$. Let us denote by λ' the highest one of

$$\Sigma_{X'} := \{\, \xi \in \Sigma(\mathfrak{g},\mathfrak{a}) \mid X'_{\xi} \neq 0 \,\}$$

with respect to the ordering on $\mathfrak{a}$ (if $X' \neq 0$, then $\Sigma_{X'}$ is not empty since X' is nilpotent element in $\mathfrak{g}$). As a first step of the proof, we shall prove that the root vector $X'_{\lambda'}$ is also in $\overline{\mathcal{O}'_0}$. We take $A' \in \mathfrak{a}$ satisfying that

$$\xi(A') < \lambda'(A') \quad ({}^{\forall}\xi \in \Sigma_{X'} \setminus \{\lambda'\}).$$

(such A' exists since λ' is highest in $\Sigma_{X'}$ with respect to the lexicographic ordering on $\mathfrak{a}$). Let us put

$$X'_k := \frac{1}{e^{k\lambda'(A')}} \exp(\mathrm{ad}_{\mathfrak{g}}\, kA')X' \quad (\text{for } k \in \mathbb{N})$$

Then, X'_k is in $\overline{\mathcal{O}'_0}$ for any k since $\overline{\mathcal{O}'_0}$ is stable by positive scalars. Furthermore,

$$\lim_{k\to\infty} X'_k = \lim_{k\to\infty} \sum_{\xi \in \Sigma_{X'}} e^{k(\xi(A')-\lambda'(A'))} X'_{\xi} = X'_{\lambda'}.$$

This means that $X'_{\lambda'}$ is in $\overline{\mathcal{O}'_0}$. To complete the proof, we only need to show that there exists $X' \in \overline{\mathcal{O}'_0}$ such that $\lambda' = \lambda$ (where λ' is the highest one of $\Sigma_{X'}$). Let λ_0 be the highest one of

$$\Sigma_{\overline{\mathcal{O}'_0}} := \{\, \xi \in \Sigma(\mathfrak{g},\mathfrak{a}) \mid {}^{\exists}X' \in \overline{\mathcal{O}'_0} \text{ such that } X'_{\xi} \neq 0 \,\}$$

(namely, $\Sigma_{\overline{\mathcal{O}'_0}} = \bigcup_{X' \in \overline{\mathcal{O}'_0}} \Sigma_{X'}$) with respect to the ordering on $\mathfrak{a}$. Then, we can find a root vector X'_{λ_0} in $\mathfrak{g}_{\lambda_0} \cap \overline{\mathcal{O}'_0}$ by the argument avobe. We assume that $\lambda_0 \neq \lambda$. Then, by Lemma 3.3, there exists $\eta \in \Sigma^+(\mathfrak{g},\mathfrak{a})$ and $X_\eta \in \mathfrak{g}_\eta$ such that $[X_\eta, X'_{\lambda_0}] \neq 0$. In particular, for the element $X'' := \exp(\mathrm{ad}_{\mathfrak{g}}(X_\eta))X'_{\lambda_0}$ in $\overline{\mathcal{O}'_0}$, we obtain that

$$\lambda_0 + \eta \in \Sigma_{X''} \subset \Sigma_{\overline{\mathcal{O}'_0}}.$$

This contradicts the definition of λ_0. Thus, $\lambda_0 = \lambda$. □

Proof of Proposition 3.5. Let $A_{\lambda^\vee}$ be the element in $\mathfrak{a}$ corresponding to the coroot $\lambda^\vee$ of the highest root λ. We put

$$(\mathfrak{g}_{\mathbb{C}})_2 = \{ X \in \mathfrak{g}_{\mathbb{C}} \mid [A_{\lambda^\vee}, X] = 2X \}.$$

Then, $\mathfrak{g}_\lambda$ is included in $(\mathfrak{g}_{\mathbb{C}})_2$. We note that there exists $X, Y \in \mathfrak{g}_{\mathbb{C}}$ such that $(A_{\lambda^\vee}, X, Y)$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}_{\mathbb{C}}$ (in fact, we can find such X, Y in $\mathfrak{g}_\lambda$ by Fact 3.1). Therefore, we can use Malcev's theorem. Namely, for any two non-zero vectors X and X' in $(\mathfrak{g}_{\mathbb{C}})_2$, there exists $g_{\mathbb{C}} \in G_{\mathbb{C}}$ such that $g_{\mathbb{C}} X = X'$. Since $\mathfrak{g}_\lambda \subset (\mathfrak{g}_{\mathbb{C}})_2$, the proof is completed. $\square$

By using Proposition 3.4 and Proposition 3.5, Theorem 1.1 follows by taking $\mathcal{O}^{G_{\mathbb{C}}}_{\mathrm{min},\mathfrak{g}}$ as

$$\mathcal{O}^{G_{\mathbb{C}}}_{\mathrm{min},\mathfrak{g}} := G_{\mathbb{C}} \cdot \mathfrak{g}_\lambda \setminus \{0\}.$$

4 Outline of a proof of Theorem 1.2

Let us consider the same setting in §3. Recall that $\mathcal{O}^{G_{\mathbb{C}}}_{\mathrm{min},\mathfrak{g}}$ is not the complex minimal nilpotent orbit $\mathcal{O}^{G_{\mathbb{C}}}_{\mathrm{min}}$ if and only if $\mathcal{O}^{G_{\mathbb{C}}}_{\mathrm{min}}$ does not meet $\mathfrak{g}$. The proposition below give a characterization of $\mathfrak{g}$ for which $\mathcal{O}^{G_{\mathbb{C}}}_{\mathrm{min}}$ is not $\mathcal{O}^{G_{\mathbb{C}}}_{\mathrm{min},\mathfrak{g}}$ (see Proposition 5.6 for another characterizations of it).

Proposition 4.1. *The following conditions on* $\mathfrak{g}$ *are equivalent:*

1. $\dim_{\mathbb{R}} \mathfrak{g}_\lambda \geq 2$.
2. $\mathcal{O}^{G_{\mathbb{C}}}_{\mathrm{min}} \cap \mathfrak{g} = \emptyset$.

We can prove the proposition without any classification, but we omit it in this paper.

Here, we put $\mathfrak{m} := Z_{\mathfrak{k}}(\mathfrak{a})$ and denote by M_0, A to the analytic subgroups of G corresponding to $\mathfrak{m}$, $\mathfrak{a}$, respectively. Then, the connected Lie group $M_0 A$ (which is the analytic subgroup of G corresponding to $\mathfrak{m} \oplus \mathfrak{a}$) acts on $\mathfrak{a}$. Furthermore, the following proposition holds:

Proposition 4.2. *If* $\dim_{\mathbb{R}} \mathfrak{g}_\lambda \geq 2$, *then* $\mathfrak{g}_\lambda \setminus \{0\}$ *is a single* $M_0 A$*-orbit.*

Combining Proposition 3.4, Proposition 4.1 with Proposition 4.2, we obtain Theorem 1.2.

We will use the next lemma to prove Proposition 4.2.

Lemma 4.3. *Suppose that* $\mathfrak{g}$ *has real rank one (i.e.* $\dim_{\mathbb{R}} \mathfrak{a} = 1$*) and* $\dim_{\mathbb{R}} \mathfrak{g}_\lambda \geq 2$*. Then,* $\mathfrak{g}_\lambda \setminus \{0\}$ *is a single* M_0A*-orbit.*

Proof of Lemma 4.3. Let $A_{\lambda^\vee}$ be the element of $\mathfrak{a}$ corresponding to the coroot $\lambda^\vee$ of the highest root λ in $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ (see §3). Since $\mathfrak{g}$ has real rank one, we have $\mathfrak{a} = \mathbb{R}A_{\lambda^\vee}$, and $\mathfrak{g}$ can be written by

$$\mathfrak{g} = \mathfrak{g}_{-\lambda} \oplus \mathfrak{g}_{-\frac{\lambda}{2}} \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{g}_{\frac{\lambda}{2}} \oplus \mathfrak{g}_\lambda$$

($\mathfrak{g}_{\pm\frac{\lambda}{2}}$ can be zero). Let us denote by $\mathfrak{g}_{\mathbb{C}}$, $\mathfrak{m}_{\mathbb{C}}$, $\mathfrak{a}_{\mathbb{C}}$, $(\mathfrak{g}_{\pm\lambda})_{\mathbb{C}}$, $(\mathfrak{g}_{\pm\frac{\lambda}{2}})_{\mathbb{C}}$ the complexification of $\mathfrak{g}$, $\mathfrak{m}$, $\mathfrak{a}$, $\mathfrak{g}_{\pm\lambda}$, $\mathfrak{g}_{\pm\frac{\lambda}{2}}$, respectively. We set

$$(\mathfrak{g}_{\mathbb{C}})_i = \{\, X \in \mathfrak{g}_{\mathbb{C}} \mid [A_{\lambda^\vee}, X] = iX \,\} \quad (\text{for } i \in \mathbb{Z}).$$

Then,

$$(\mathfrak{g}_{\mathbb{C}})_0 = \mathfrak{m}_{\mathbb{C}} \oplus \mathfrak{a}_{\mathbb{C}}, \ (\mathfrak{g}_{\mathbb{C}})_{\pm 1} = (\mathfrak{g}_{\pm\frac{\lambda}{2}})_{\mathbb{C}}, \ (\mathfrak{g}_{\mathbb{C}})_{\pm 2} = (\mathfrak{g}_{\pm\lambda})_{\mathbb{C}}.$$

By Fact 3.1, for any non-zero highest root vector X_λ in $\mathfrak{g}_\lambda$, there exists $Y_\lambda \in \mathfrak{g}_{-\lambda}$ such that $(A_{\lambda^\vee}, X_\lambda, Y_\lambda)$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}_{\mathbb{C}}$. By the theory of representations of $\mathfrak{sl}(2, \mathbb{C})$, we obtain that $[(\mathfrak{g}_{\mathbb{C}})_0, X_\lambda] = (\mathfrak{g}_{\mathbb{C}})_2$. In particular, we have

$$[\mathfrak{m} \oplus \mathfrak{a}, X_\lambda] = \mathfrak{g}_\lambda.$$

Therefore, for the M_0A-orbit $\mathcal{O}^{M_0A}(X_\lambda)$ in $\mathfrak{g}_\lambda$ through X_λ, we obtain that

$$\dim_{\mathbb{R}} \mathcal{O}^{M_0A}(X_\lambda) = \dim_{\mathbb{R}} \mathfrak{g}_\lambda.$$

This means that the M_0A-orbit $\mathcal{O}^{M_0A}(X_\lambda)$ is open in $\mathfrak{g}_\lambda$ for any non-zero root vector X_λ in $\mathfrak{g}_\lambda$. Recall that we are assuming that $\dim_{\mathbb{R}} \mathfrak{g}_\lambda \geq 2$. Hence, $\mathfrak{g}_\lambda \setminus \{0\}$ is connected. Therefore, $\mathfrak{g}_\lambda \setminus \{0\}$ is a single M_0A-orbit. □

We are ready to prove Proposition 4.2.

Sketch of a proof of Proposition 4.2. Let $\mathfrak{h}' := [\mathfrak{g}_\lambda, \mathfrak{g}_{-\lambda}] \subset \mathfrak{m} \oplus \mathfrak{a}$. Then $\mathfrak{g}' := \mathfrak{g}_{-\lambda} \oplus \mathfrak{h}' \oplus \mathfrak{g}_\lambda$ becomes a subalgebra of $\mathfrak{g}$ (since $\pm 2\lambda$ is not a root). Furthremore, one can prove that $\mathfrak{g}'$ is a real rank one simeple Lie algebra with a maximally split abelian subspace $\mathfrak{a}' := \mathbb{R}A_{\lambda^\vee}$, where $A_{\lambda^\vee}$ is the element of $\mathfrak{a}$ corresponding to the coroot $\lambda^\vee$ of the highest root λ in $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ (see §3). We put $\mathfrak{m}' \oplus \mathfrak{a}' := Z_{\mathfrak{g}'}(\mathfrak{a}')$ and denote by $M_0'A'$ the analytic subgroup of G corresponding to $\mathfrak{m}' \oplus \mathfrak{a}'$. Then, by Lemma 4.3, we obtain that $\mathfrak{g}_\lambda \setminus \{0\}$ is a single $M_0'A'$-orbit. Since $M_0'A'$ is a subgroup of M_0A, the proof is completed. □

5 Determination of $\mathcal{O}^{G_\mathbb{C}}_{\min,\mathfrak{g}}$

In this section, we determine $\mathcal{O}^{G_\mathbb{C}}_{\min,\mathfrak{g}}$ by describing the weighted Dynkin diagram of $\mathcal{O}^{G_\mathbb{C}}_{\min,\mathfrak{g}}$. Recall that Proposition 4.1 claims that $\mathcal{O}^{G_\mathbb{C}}_{\min} = \mathcal{O}^{G_\mathbb{C}}_{\min,\mathfrak{g}}$ if and only if $\dim_\mathbb{R} \mathfrak{g}_\lambda = 1$. Thus, our concern is in the cases where $\dim_\mathbb{R} \mathfrak{g}_\lambda \geq 2$ (i.e. $\mathfrak{g}$ is isomorphic to $\mathfrak{su}^*(2k)$, $\mathfrak{so}(n-1,1)$, $\mathfrak{sp}(p,q)$, $\mathfrak{e}_{6(-26)}$ or $\mathfrak{f}_{4(-20)}$).

5.1 Satake diagrams and weighted Dynkin diagrams

In order to determine the weighted Dynkin diagram of our $\mathcal{O}^{G_\mathbb{C}}_{\min,\mathfrak{g}}$, we describe some lemmas of relationship between weighted Dynkin diagrams of $\mathfrak{g}_\mathbb{C}$ and Satake diagrams of $\mathfrak{g}$ in this subsection.

Let $\mathfrak{g}_\mathbb{C}$ be a semisimple Lie algebra and $\mathfrak{g}$ a real form of it through this subsection. First, we recall briefly the definition of Satake diagram of a real form $\mathfrak{g}$ of a complex semisimple Lie algebra $\mathfrak{g}_\mathbb{C}$ (see also [1] for more details). Fix a Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of $\mathfrak{g}$. We take a maximal abelian subspace $\mathfrak{a}$ in $\mathfrak{p}$, and extend it to a maximal abelian subspace $\mathfrak{h} = \sqrt{-1}\mathfrak{t} \oplus \mathfrak{a}$ in $\sqrt{-1}\mathfrak{k} \oplus \mathfrak{p}$. Then, the complexification, denoted by $\mathfrak{h}_\mathbb{C}$, of $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}_\mathbb{C}$, and $\mathfrak{h}$ coincide with the real form

$$\{X \in \mathfrak{h}_\mathbb{C} \mid \alpha(X) \in \mathbb{R} \ ({}^\forall \alpha \in \Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C}))\}$$

of $\mathfrak{h}_\mathbb{C}$, where $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ is the root system of $(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$. Let us denote by

$$\Sigma(\mathfrak{g}, \mathfrak{a}) := \{\alpha|_\mathfrak{a} \mid \alpha \in \Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})\} \setminus \{0\} \subset \mathfrak{a}^*$$

the restricted root system of $(\mathfrak{g}, \mathfrak{a})$. We will denote by $W(\mathfrak{g}, \mathfrak{a})$, $W(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ the Weyl group of $\Sigma(\mathfrak{g}, \mathfrak{a})$, $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$, respectively. Fix an ordering on $\mathfrak{a}$ and extend it to an ordering on $\mathfrak{h}$. We write $\Sigma^+(\mathfrak{g}, \mathfrak{a})$, $\Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ for the positive system of $\Sigma(\mathfrak{g}, \mathfrak{a})$, $\Delta(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ corresponding to the ordering on $\mathfrak{a}$, $\mathfrak{h}$, respectively. Then, $\Sigma^+(\mathfrak{g}, \mathfrak{a})$ can be written by

$$\Sigma^+(\mathfrak{g}, \mathfrak{a}) = \{\alpha|_\mathfrak{a} \mid \alpha \in \Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})\} \setminus \{0\}.$$

We denote by Π the fundamental system of $\Delta^+(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$. Then,

$$\overline{\Pi} := \{\, \alpha|_\mathfrak{a} \mid \alpha \in \Pi \,\} \setminus \{0\}$$

is the simple system of $\Sigma^+(\mathfrak{g}, \mathfrak{a})$. Let Π_0 be the set of all simple roots in Π whose restriction to $\mathfrak{a}$ is zero. The Satake diagram $S_\mathfrak{g}$ of $\mathfrak{g}$ consists of the following data: The Dynkin diagram of $\mathfrak{g}_{\mathbb{C}}$ with nodes Π; black nodes Π_0 in $S_{\mathfrak{g}}$; and arrows joining $\alpha \in \Pi \setminus \Pi_0$ and $\beta \in \Pi \setminus \Pi_0$ in $S_{\mathfrak{g}}$ whose restrictions to $\mathfrak{a}$ are the same.

Second, we define that a weighted Dynkin diagram $\Psi_H \in \mathrm{Map}(\Pi, \mathbb{R})$ "matches" the Satake diagram $S_{\mathfrak{g}}$ of $\mathfrak{g}$ as follows:

Definition 5.1. *Let $\Psi_H \in \mathrm{Map}(\Pi, \mathbb{R})$ be a weighted Dynkin diagram (see §2) and $S_{\mathfrak{g}}$ the Satake diagram of $\mathfrak{g}$ with nodes Π. We say that Ψ_H matches $S_{\mathfrak{g}}$ if all the weights on black nodes are zero and any pair of nodes joined by an arrow has the same weights.*

Remark 5.2. *The concept of "match" defined above is same as "weighted Satake diagrams" in Djocovic [5] and the condition described in Sekiguchi [11, Proposition 1.16].*

Recall that Ψ is a linear isomorphism from $\mathfrak{h}$ to $\mathrm{Map}(\Pi, \mathbb{R})$ (see §2). Then, the next two lemmas hold (we omit proofs of the two lemmas in this paper):

Lemma 5.3. $\Psi : \mathfrak{h} \to \mathrm{Map}(\Pi, \mathbb{R})$ *induces a linear isomorphism below:*

$$\mathfrak{a} \to \{\, \Psi_H \in \mathrm{Map}(\Pi, \mathbb{R}) \mid \Psi_H \text{ matches } S_{\mathfrak{g}} \,\}.$$

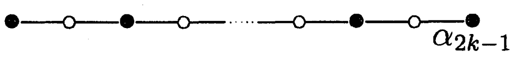
Lemma 5.4. *For each simple root α of Π, we denote by $H_{\alpha^\vee}$ the element in $\mathfrak{h}$ corresponding to the coroot $\alpha^\vee$ of the simple root α. Then, the set*

$$\{\, H_{\alpha^\vee} \mid \alpha \text{ is black in } S_{\mathfrak{g}} \,\} \cup \{\, H_{\alpha^\vee} - H_{\beta^\vee} \mid \alpha \text{ and } \beta \text{ are joined by an arrow in } S_{\mathfrak{g}} \,\}$$

is a basis of $\sqrt{-1}\mathfrak{t}$.

Lemma 5.3 and Lemma 5.4 will be used to compute the weighted Dynkin diagrams of $\mathcal{O}^{G_{\mathbb{C}}}_{\min,\mathfrak{g}}$ for the cases where $\mathcal{O}^{G_{\mathbb{C}}}_{\min,\mathfrak{g}}$ is not the complex minimal nilpotent orbit $\mathcal{O}^{G_{\mathbb{C}}}_{\min}$.

Recall that our concern in this paper is in real simple Lie algebras $\mathfrak{su}^*(2k)$, $\mathfrak{so}(n-1,1)$, $\mathfrak{sp}(p,q)$, $\mathfrak{e}_{6(-26)}$ and $\mathfrak{f}_{4(-20)}$. For the convenience of the reader, we give a list of Satake diagrams of such simple Lie algebras.

$\mathfrak{g}$	Satake diagrams of $\mathfrak{g}$
$\mathfrak{su}^*(2k)$	

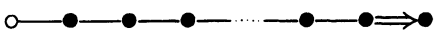
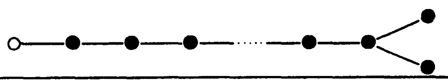
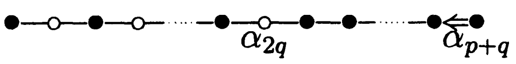
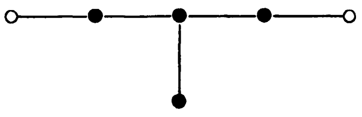
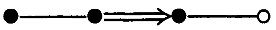
$\mathfrak{so}(n-1,1)$		(n is odd, $n \geq 5$)
		(n is even, $n \geq 6$)
$\mathfrak{sp}(p,q)$		($p \geq q \geq 1$)
$\mathfrak{e}_{6(-26)}$		
$\mathfrak{f}_{4(-20)}$		

Table 3: List of Satake diagrams of $\mathfrak{su}^*(2k)$, $\mathfrak{so}(n-1,1)$, $\mathfrak{sp}(p,q)$, $\mathfrak{e}_{6(-26)}$ and $\mathfrak{f}_{4(-20)}$.

5.2 Computation of weighted Dynkin diagrams of $\mathcal{O}^{G_\mathbb{C}}_{\mathrm{min},\mathfrak{g}}$

We consider the same setting on §5.1 and suppose that $\mathfrak{g}_\mathbb{C}$ is simple and $\mathfrak{g}$ is non-compact. Let us denote by

$$\mathfrak{a}_+ := \{\, A \in \mathfrak{a} \mid \xi(A) \geq 0 \ ({}^\forall \xi \in \Sigma^+(\mathfrak{g},\mathfrak{a}))\,\}.$$

Then $\mathfrak{a}_+$ is a fundamental domain of $\mathfrak{a}$ under the action of $W(\mathfrak{g},\mathfrak{a})$. Since

$$\Sigma^+(\mathfrak{g},\mathfrak{a}) = \{\, \alpha|_\mathfrak{a} \mid \alpha \in \Delta(\mathfrak{g}_\mathbb{C},\mathfrak{h}_\mathbb{C})\,\} \setminus \{0\},$$

the domain $\mathfrak{a}_+$ coincide with $\mathfrak{h}_+ \cap \mathfrak{a}$. Recall that λ is dominant (by Lemma 3.2) and $\mathcal{O}^{G_\mathbb{C}}_{\mathrm{min},\mathfrak{g}}$ contains $\mathfrak{g}_\lambda \setminus \{0\}$ (by the proof of Theorem 1.1). Thus, $A_{\lambda^\vee}$ is the hyperbolic element in $\mathfrak{a}_+$ corresponding to $\mathcal{O}^{G_\mathbb{C}}_{\mathrm{min},\mathfrak{g}}$ (see §2) since we can find $X_\lambda \in \mathfrak{g}_\lambda$, $Y_\lambda \in \mathfrak{g}_{-\lambda}$ such that the triple $(A_{\lambda^\vee}, X_\lambda, Y_\lambda)$ is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}_\mathbb{C}$ by Lemma 3.1 (then, $X_\lambda, Y_\lambda \in \mathcal{O}^{G_\mathbb{C}}_{\mathrm{min},\mathfrak{g}}$). Therefore, to determine the weighted Dynkin diagram of $\mathcal{O}^{G_\mathbb{C}}_{\mathrm{min},\mathfrak{g}}$, we shall compute the weighted Dynkin diagram corresponding to $A_{\lambda^\vee}$.

Let ϕ be the highest root of $\Delta^+(\mathfrak{g}_\mathbb{C},\mathfrak{h}_\mathbb{C})$. Recall that the complex minimal nilpotent orbit $\mathcal{O}^{G_\mathbb{C}}_{\mathrm{min}}$ contains the root space $(\mathfrak{g}_\mathbb{C})_\phi$ without zero, and the weighted Dynkin diagram of $\mathcal{O}^{G_\mathbb{C}}_{\mathrm{min}}$ is the weighted Dynkin diagram corresponding to $H_{\phi^\vee}$ (see §2). The next lemma gives a formula for $A_{\lambda^\vee}$ by $H_{\phi^\vee}$ (we omit a proof of the lemma):

Lemma 5.5. *We denote by τ the anti $\mathbb{C}$-linear involution corresponding to $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \oplus \sqrt{-1}\mathfrak{g}$ (i.e. τ is the complex conjugation of $\mathfrak{g}_{\mathbb{C}}$ with respect to the real form $\mathfrak{g}$). Then, $H_{\phi^\vee}$ is in $\mathfrak{a}$ if and only if $\dim_{\mathbb{R}} \mathfrak{g}_\lambda \geq 2$ and*

$$A_{\lambda^\vee} = \begin{cases} H_{\phi^\vee} & (\textit{if } \dim_{\mathbb{R}} \mathfrak{g}_\lambda = 1), \\ H_{\phi^\vee} + \tau H_{\phi^\vee} & (\textit{if } \dim_{\mathbb{R}} \mathfrak{g}_\lambda \geq 2). \end{cases}$$

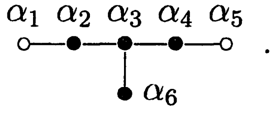
In particular, we have another characterizations of $\mathfrak{g}$ for which $\mathcal{O}^{G_{\mathbb{C}}}_{\min,\mathfrak{g}}$ is not $\mathcal{O}^{G_{\mathbb{C}}}_{\min}$ from Proposition 4.1.

Proposition 5.6. *The following conditions on $\mathfrak{g}$ are equivalent:*

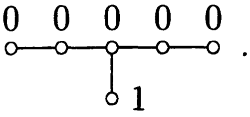
1. $\mathcal{O}^{G_{\mathbb{C}}}_{\min,\mathfrak{g}} \neq \mathcal{O}^{G_{\mathbb{C}}}_{\min}$.
2. $\mathcal{O}^{G_{\mathbb{C}}}_{\min} \cap \mathfrak{g} = \emptyset$.
3. $\dim_{\mathbb{R}} \mathfrak{g}_\lambda \geq 2$.
4. *The highest root ϕ in $\Delta^+(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ is not a real root.*
5. *The weighted Dynkin diagram of $\mathcal{O}^{G_{\mathbb{C}}}_{\min}$ matches the Satake diagram $S_{\mathfrak{g}}$ of $\mathfrak{g}$ (see Definition §5.1).*
6. *$\mathfrak{g}$ is isomorphic to $\mathfrak{su}^*(2k)$, $\mathfrak{so}(n-1,1)$, $\mathfrak{sp}(p,q)$, $\mathfrak{e}_{6(-26)}$ or $\mathfrak{f}_{4(-20)}$, where $k \geq 2$, $n \geq 5$ and $p \geq q \geq 1$.*

We now determine the weighted Dynkin diagram of $\mathcal{O}^{G_{\mathbb{C}}}_{\min,\mathfrak{g}}$ for the cases where $\mathfrak{g}$ is isomorphic to $\mathfrak{su}^*(2k)$, $\mathfrak{so}(n-1,1)$, $\mathfrak{sp}(p,q)$, $\mathfrak{e}_{6(-26)}$ or $\mathfrak{f}_{4(-20)}$. By Lemma 5.5, our purpose is to compute the weighted Dynkin diagram corresponding to $A_{\lambda^\vee} = H_{\phi^\vee} + \tau H_{\phi^\vee}$. We only give the computation for the case $\mathfrak{g} = \mathfrak{e}_{6(-26)}$ below. For the other $\mathfrak{g}$ with $\dim_{\mathbb{R}} \mathfrak{g}_\lambda \geq 2$, we can compute the weighted Dynkin diagram corresponding to $A_{\lambda^\vee}$ by the same way.

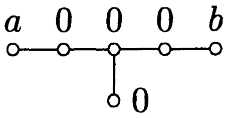
Example 5.7. *Let $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{g}) = (\mathfrak{e}_{6,\mathbb{C}}, \mathfrak{e}_{6(-26)})$. We denote the Satake diagram of $\mathfrak{e}_{6(-26)}$ by*



By Table 2, the weighted Dynkin diagram corresponding to $H_{\phi^\vee}$ *is*



We now compute the weighted Dynkin diagram corresponding to $A_{\lambda^\vee} = H_{\phi^\vee} + \tau H_{\phi^\vee}$. *By Lemma* 5.3, *the weighted Dynkin diagram corresponding to* $A_{\lambda^\vee}$ *matches the Satake diagram of* $\mathfrak{e}_{6(-26)}$. *Thus, we can put the weighted Dynkin diagram corresponding to* $A_{\lambda^\vee}$ *as*

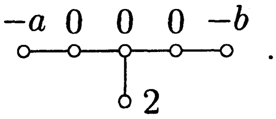


$(a, b \in \mathbb{R})$.

To determine $a, b \in \mathbb{R}$, *we also put*

$$H^{im}_{\phi^\vee} = H_{\phi^\vee} - \tau H_{\phi^\vee} \in \sqrt{-1}\mathfrak{t}.$$

Since $A_{\lambda^\vee} + H^{im}_{\phi^\vee} = 2H_{\phi^\vee}$, *the weighted Dynkin diagram corresponding to* $H^{im}_{\phi^\vee}$ *can be written by*



Namely, we have

$$\begin{aligned} \alpha_1(H^{im}_{\phi^\vee}) &= -a, \\ \alpha_2(H^{im}_{\phi^\vee}) &= \alpha_3(H^{im}_{\phi}) = \alpha_4(H^{im}_{\phi}) = 0, \\ \alpha_5(H^{im}_{\phi^\vee}) &= -b, \\ \alpha_6(H^{im}_{\phi^\vee}) &= 2. \end{aligned}$$

By Lemma 5.4, *the set* $\{ H_{\alpha_2^\vee}, H_{\alpha_3^\vee}, H_{\alpha_4^\vee}, H_{\alpha_6^\vee} \}$ *is a basis of* $\sqrt{-1}\mathfrak{t}$. *Thus,* $H^{im}_{\phi^\vee} \in \sqrt{-1}\mathfrak{t}$ *can be written by*

$$H^{im}_{\phi^\vee} = c_2 H_{\alpha_2^\vee} + c_3 H_{\alpha_3^\vee} + c_4 H_{\alpha_4^\vee} + c_6 H_{\alpha_6^\vee} \quad (c_2, c_3, c_4, c_6 \in \mathbb{R}).$$

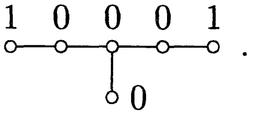
By the Dynkin diagram of $\mathfrak{e}_{6,\mathbb{C}}$, *we can compute*

$$\alpha_i(H_{\alpha_j^\vee}) = \frac{2\langle \alpha_i, \alpha_j \rangle}{\langle \alpha_j, \alpha_j \rangle}$$

for each i, j. Thus, we also have

$$\begin{aligned}
\alpha_1(H^{im}_{\phi^\vee}) &= -c_2, \\
\alpha_2(H^{im}_{\phi^\vee}) &= 2c_2 - c_3, \\
\alpha_3(H^{im}_{\phi^\vee}) &= -c_2 + 2c_3 - c_4 - c_6, \\
\alpha_4(H^{im}_{\phi^\vee}) &= -c_3 + 2c_4, \\
\alpha_5(H^{im}_{\phi^\vee}) &= -c_4, \\
\alpha_6(H^{im}_{\phi^\vee}) &= -c_3 + 2c_6.
\end{aligned}$$

Then, we obtatin that $a = b = 1$. Therefore, the weighted Dynkin diagram of $\mathcal{O}^{G_\mathbb{C}}_{\min,\mathfrak{g}}$ for $\mathfrak{g} = \mathfrak{e}_{6(-26)}$ is



The result of our computation for all $\mathfrak{g}$ with $\dim_\mathbb{R} \mathfrak{g}_\lambda \geq 2$ is Table 1 in §1.

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