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Periodic solutions of some double-diffusive convection systems based on Brinkman-Forchheimer equations

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1 Introduction

In this paper, we shall consider a double-diffusive convection system based upon Brinkman-Forchheimer Equations in a bounded domain $\Omega \subset \mathbb{R}^N$ with smooth boundary $\partial\Omega$, which is given as follows.

$$
\text{(BF)}\begin{cases}
\partial_t \boldsymbol{u} = \nu\Delta \boldsymbol{u} - a\boldsymbol{u} - \nabla p + \boldsymbol{g}T + \boldsymbol{h}C + \boldsymbol{f}_1 & in\ \Omega\times[0,S],\\
\partial_t T + \boldsymbol{u}\cdot\nabla T = \Delta T + f_2 & in\ \Omega\times[0,S],\\
\partial_t C + \boldsymbol{u}\cdot\nabla C = \Delta C + \rho\Delta T + f_3 & in\ \Omega\times[0,S],\\
\nabla\cdot\boldsymbol{u} = 0 & in\ \Omega\times[0,S],\\
\boldsymbol{u}|_{\partial\Omega} = 0,\ T|_{\partial\Omega} = 0,\ C|_{\partial\Omega} = 0,
\end{cases}
$$

where $\boldsymbol{u}, T, C, p$ are unknown functions and represent the solenoidal velocity, temperature of the fluid, concentration of a solute, pressure of the fluid respectively. Given constant vectors $\boldsymbol{g}, \boldsymbol{h}$ are derived from gravity. The positive constants ρ, a are called the Soret's coefficient and Darcy's coefficient respectively. $\boldsymbol{f}_1, f_2, f_3$ are the given external forces. Throughout this paper, $\partial_t \boldsymbol{u}$ and $\boldsymbol{u}_t$ designate the time derivative of $\boldsymbol{u}$, i.e., $\frac{\partial \boldsymbol{u}}{\partial t}$. In this note, we consider equations (BF) under the time periodic condition with period S.

$$
\boldsymbol{u}(\cdot,0) = \boldsymbol{u}(\cdot,S),\ T(\cdot,0) = T(\cdot,S),\ C(\cdot,0) = C(\cdot,S). \tag{1}
$$

The first equation of (BF) comes from the Brinkman-Forchheimer equation, which describes the behavior of the fluid velocity in some porous medium. Originally, the Brinkman-Forchheimer equation has a convection term and another nonlinear term, and in each term of the equation, there appears another space-dependent function which stands for the rate of the void space in the porous medium (which is called the porosity). However, under some physical conditions we linearize the Brinkman-Forchheimer equation. First we assume that the medium is homogeneous, whence follows that the porosity is constant. Second we presume that the flow is relatively calm and nonlinear terms are very small. This assumption is realized when we are concerned with the porous medium, which disturbs the flow. It is also known that the nonlinear terms in the Brinkman-Forchheimer equation become negligibly small when we deal with the convection of temperature and concentration together. Third we assume that the porosity of the porous medium is sufficiently large. This assumption makes the diffusion term more effective than the nonlinear terms. Under these assumption, we derive the linearized Brinkman-Forchheimer equation given in (BF). Here $\boldsymbol{g}T, \boldsymbol{h}C$ are the effect from gravity.

The second equation and the third equation of (BF) originate from the result of the irreversible thermodynamics. The term $\rho\Delta T$, which is called Soret's effect, describes the certain interaction between the temperature of the fluid and the concentration of a solute. Naturally, the second equation also contains a interaction term $\rho'\Delta C$, which is called Dufore's effect. However, Dufore's effect is generally much smaller than Soret's effect, especially for the case where we deal with liquid fluid. Therefore we here consider only Soret's effect.

There are many studies for (BF), for example, about the dependence of the solutions on the Soret's coefficient ρ and so on. However, to the best of our knowledge, it seems that there are very few studies for the solvability of (BF). In [1], there is a result of the existence of the unique global solution of (BF) under some initial condition.

The system has convection terms $\boldsymbol{u}\cdot\nabla T, \boldsymbol{u}\cdot\nabla C$ as nonlinear terms. In addition, the third equation has the term of $\rho\Delta T$ which may not be small perturbation. In order to solve the periodic problem for (BF), we try to apply an abstract result developed in [2]. However, this abstract result can not be applied directly to (BF) because of the presence of terms $\boldsymbol{u}\cdot\nabla T, \boldsymbol{u}\cdot\nabla C, \boldsymbol{g}T$ and $\boldsymbol{h}C$. In order to cope with this difficulty, we introduce some approximation system involving some dissipation terms and cut-off functions, whose solvability can be assured by the abstract result in [2]. In addition to this, we establish appropriate a priori estimates independent of the approximation parameter and apply standard convergence arguments. In section 2, we prepare some preliminary and our main result is stated. In section 3, we check some conditions required in the abstract theorem to assure the existence of the solution of approximation equations. Making use of the boundedness derived from appropriate a priori estimates in section 4, we discuss the convergence of solutions of approximation equations in section 5.

2 Preliminaries and Main Result

2.1 Notation

In this paper, we use following notations in order to formulate our results.

$$\mathbb{C}_\sigma^\infty(\Omega) = \{\boldsymbol{u} = (u^1, u^2, \cdots, u^N); u^j \in C_0^\infty(\Omega)\ \forall j = 1, 2, \cdots, N,\ \nabla\cdot\boldsymbol{u} = 0\},$$
$$\mathbb{L}^2(\Omega) = (L^2(\Omega))^N, \quad \mathbb{H}^1(\Omega) = (H^1(\Omega))^N = (W^{1,2}(\Omega))^N,$$
$\mathbb{L}_\sigma^2(\Omega)$: The closure of $\mathbb{C}_\sigma^\infty(\Omega)$ under the $\mathbb{L}^2(\Omega)$-norm,

$\mathbb{H}_\sigma^1(\Omega)$: The closure of $\mathbb{C}_\sigma^\infty(\Omega)$ under the $\mathbb{H}^1(\Omega)$-norm,

$H = \mathbb{L}_\sigma^2(\Omega) \times L^2(\Omega) \times L^2(\Omega)$: Hilbert space,

$C_\pi([0, S]; H) = \{U \in C([0, S]; H); U(0) = U(S)\}$,

$\mathcal{P}_\Omega$: The orthogonal projection $\mathbb{L}^2(\Omega)$ onto $\mathbb{L}_\sigma^2(\Omega)$,

$\mathcal{A} = -\mathcal{P}_\Omega\Delta$: The Stokes operator with domain $D(\mathcal{A}) = \mathbb{H}^2(\Omega) \cap \mathbb{H}_\sigma^1(\Omega)$.

2.2 Subdifferential Operator

Let φ be a proper lower semi-continuous convex function from H to $(-\infty, +\infty]$. Define the effective domain of φ by $D(\varphi) = \{U \in H; \varphi(U) < +\infty\}$ and the subdifferential of φ by

$$\partial\varphi(U) = \{f \in H; \varphi(V) - \varphi(U) \leqslant (f, V - U)_H \text{ for all } V \in H\}$$

with domain $D(\partial\varphi) = \{U \in H; \partial\varphi(U) \neq \varnothing\}$.

In the later arguments, it will be shown that the leading terms can be given as the subdifferential of some lower semi-continuous convex function. Generally, subdifferential operators are multivalued maximal monotone operators. However, since the subdifferential operators used in this note are always single-valued, we restrict ourselves to the single-valued subdifferential operators.

2.3 Reduction to an Abstract Problem

In this section, we shall reduce problem (BF) to an abstract periodic problem in the Hilbert space H. Operating the projection $\mathcal{P}_\Omega$ to the first equation of (BF) to erase the pressure term ∇p, we obtain the following equations:

$$\begin{aligned} &\partial_t \boldsymbol{u} = \nu \mathcal{P}_\Omega \Delta \boldsymbol{u} - a\boldsymbol{u} + \mathcal{P}_\Omega \boldsymbol{g} T + \mathcal{P}_\Omega \boldsymbol{h} C + \mathcal{P}_\Omega \boldsymbol{f}_1, \\ &\partial_t T + \boldsymbol{u}\cdot\nabla T = \Delta T + f_2, \\ &\partial_t C + \boldsymbol{u}\cdot\nabla C = \Delta C + \rho\Delta T + f_3. \end{aligned} \tag{2}$$

Here we introduce the inner product of H as follows:

$$(U_1, U_2)_H = (\boldsymbol{u}_1, \boldsymbol{u}_2)_{\mathbb{L}^2_\sigma} + (T_1, T_2)_{L^2} + \frac{1}{9\rho^2}(C_1, C_2)_{L^2} \quad \text{for } U_i = (\boldsymbol{u}_i, T_i, C_i), (i = 1, 2). \tag{3}$$

The inner product of C has a coefficient which depends on ρ in order to deal with the term $\rho\Delta T$ as a small perturbation.

Define φ by

$$\varphi(U) = \begin{cases} \frac{\nu}{2}\|\nabla \boldsymbol{u}\|^2_{\mathbb{L}^2} + \frac{1}{2}\|\nabla T\|^2_{L^2} + \frac{1}{18\rho^2}\|\nabla C\|^2_{L^2} & \text{if } U \in D(\varphi) = \mathbb{H}^1_\sigma \times H^1_0 \times H^1_0, \\ +\infty & \text{if } U \in H \backslash D(\varphi). \end{cases} \tag{4}$$

Then it is easy to see that φ becomes a lower semi-continuous convex function from H into $(-\infty, +\infty]$. Moreover the subdifferential $\partial\varphi$ is given by

$$\partial\varphi(U) = \begin{pmatrix} -\nu\mathcal{P}_\Omega \Delta \boldsymbol{u} \\ -\Delta T \\ -\Delta C \end{pmatrix} \quad \text{with domain } D(\partial\varphi) = (\mathbb{H}^2 \cap \mathbb{H}^1_\sigma) \times (H^2 \cap H^1_0) \times (H^2 \cap H^1_0). \tag{5}$$

Furthermore, we put

$$\begin{aligned} &U(t) = \begin{pmatrix} \boldsymbol{u}(t) \\ T(t) \\ C(t) \end{pmatrix}, \quad \frac{dU}{dt}(t) = \begin{pmatrix} \partial_t \boldsymbol{u}(t) \\ \partial_t T(t) \\ \partial_t C(t) \end{pmatrix}, \\ &BU(t) = \begin{pmatrix} a\boldsymbol{u}(t) - \mathcal{P}_\Omega \boldsymbol{g} T(t) - \mathcal{P}_\Omega \boldsymbol{h} C(t) \\ \boldsymbol{u}\cdot\nabla T \\ \boldsymbol{u}\cdot\nabla T - \rho\Delta T \end{pmatrix}, \quad F(t) = \begin{pmatrix} \boldsymbol{f}_1(t) \\ f_2(t) \\ f_3(t) \end{pmatrix}. \end{aligned} \tag{6}$$

Then (2) is reduced to the following abstract periodic problem in H:

$$\text{(AP)}\begin{cases} \frac{dU(t)}{dt} + \partial\varphi(U(t)) + B(U(t)) = F(t) \quad t \in [0,S], \\ U(0) = U(S). \end{cases} \tag{7}$$

2.4 Known Abstract Theorem

In order to prove the existence of a periodic solution, we apply the following theorem given in [2] .

Theorem2.1 Let the following assumptions (A.1) – (A.4) be satisfied by (AP).

(A.1) For any $L \in (0, +\infty)$, the set $\{U \in H; \varphi(U) + \|U\|_H^2 \leqslant L\}$ is compact in H.

(A.2) $B(\cdot)$ is φ-demiclosed in the following sense:

$U_n \to U$ strongly in $C([0,S];H), \partial\varphi(U_n) \rightharpoonup \partial\varphi(U)$ weakly in $L^2(0,S;H), B(U_n) \rightharpoonup b$ weakly in $L^2(0,S;H)$, then $b(t) = B(U(t))$ holds for almost every $t \in [0,S]$.

(A.3) There exists a monotone increasing function $\ell(\cdot)$ and a positive constant $k \in [0,1)$ such that $\|B(U)\|_H^2 \leqslant k\|\partial\varphi(U)\|_H^2 + \ell(\|U\|_H)(\varphi(U)+1)^2$, for a.e.$t \in [0,S]$, $\forall U \in D(\partial\varphi)$.

(A.4) There exist positive constants α, K such that $\langle -\partial\varphi(U) - B(U), U\rangle_H + \alpha\varphi(U) \leqslant K$, for a.e.$t \in [0,S], \forall U \in D(\partial\varphi)$.

Then for every $F \in L^2(0,S;H)$, (AP) has a strong solution $U \in C_\pi([0,S];H)$, such that

$$\begin{cases} dU/dt \in L^2(0,S;H), \\ \partial\varphi(U), B(U) \in L^2(0,S;H), \\ \varphi(U) \text{ is absolutely continuous on } [0,S] \text{ and } \varphi(U(0)) = \varphi(U(S)). \end{cases}$$

Here $U(t) \in C_\pi([0,S];H) = \{U \in C(0,S;H); U(0) = U(S)\}$ is said to be a strong solution of (AP) if $U(t)$ is an H-valued absolutely continuous function on $[0,S]$ and belongs to $D(\partial\varphi)$ (the domain of $\partial\varphi$) for a.e.$t \in [0,S]$ and $U, \partial\varphi(U), B(U)$ satisfy (AP) for a.e.$t \in [0,S]$.

2.5 Approximation Equations

For the case of the initial boundary value problem treated in [1], the existence of a local solution is assured by applying a result for abstract Cauchy problems developed in [3]. When one tries to follow the same strategy as in [1], i.e., to apply Theorem2.1 to (AP), one faces some difficulties. The worst one arises in (A.3). More precisely, according to the estimate given in [1], we have

$$\|B(U)\|_H^2 \leqslant \frac{1}{3}\|\partial\varphi(U)\|_H^2 + \alpha\varphi(U)^3 + \beta\|U\|_H^2,$$

where the growth order for $\varphi(U)$ is cubic which does not satisfy the required growth order in (A.3). Additionally, when the constant vectors $\boldsymbol{g}, \boldsymbol{h}$ are very large, it is difficult to check whether condition (A.4) is satisfied.

From these reasons, we are led to introduce some relaxed approximation problems, for which the conditions (A.3) and (A.4) are satisfied. More precisely, we replace the T, C by their cut-off function, $[T]_\varepsilon, [C]_\varepsilon$, and we add some dissipation terms to second and third equation. Indeed, we consider the following approximation equations.

$$
\begin{cases}
\partial_t \boldsymbol{u} = \nu \mathcal{P}_\Omega \Delta \boldsymbol{u} - a\boldsymbol{u} + \mathcal{P}_\Omega \boldsymbol{g}[T]_\varepsilon + \mathcal{P}_\Omega \boldsymbol{h}[C]_\varepsilon + \mathcal{P}_\Omega \boldsymbol{f}_1, \\
\partial_t T + \boldsymbol{u}\cdot\nabla T = \Delta T - \varepsilon |T|^{p-2} T + f_2, \\
\partial_t C + \boldsymbol{u}\cdot\nabla C = \Delta C + \rho \Delta T - \varepsilon |C|^{p-2} C + f_3,
\end{cases} \tag{8}
$$

where cut-off function $[T]_\varepsilon$ is defined as follows:

$$
[T]_\varepsilon = \begin{cases} T & \text{if } |T| \leqslant 1/\varepsilon, \\ (\text{Sgn } T) 1/\varepsilon & \text{if } |T| \geqslant 1/\varepsilon, \qquad \varepsilon > 0, \end{cases} \tag{9}
$$

and p is a large exponent to be fixed later on.

Then we shall reduce these approximation equations (8) to an abstract problem similar to (AP). For the perturbation term, we replace it by

$$
B_\varepsilon(U) = \begin{pmatrix} a\boldsymbol{u} - \mathcal{P}_\Omega \boldsymbol{g}[T]_\varepsilon - \mathcal{P}_\Omega \boldsymbol{h}[C]_\varepsilon \\ \boldsymbol{u}\cdot\nabla T \\ \boldsymbol{u}\cdot\nabla C - \rho \Delta T \end{pmatrix}. \tag{10}
$$

We need to modify the lower semicontinuous convex function φ by φ_ε as follows;

$$
\varphi_\varepsilon(U) = \varphi(U) + \psi_\varepsilon(U) = \begin{cases} \frac{\nu}{2}\|\nabla \boldsymbol{u}\|^2_{\mathbb{L}^2} + \frac{1}{2}\|\nabla T\|^2_{L^2} + \frac{1}{18\rho^2}\|\nabla C\|^2_{L^2} + \frac{\varepsilon}{p}\|T\|^p_{L^p} + \frac{\varepsilon}{9\rho^2 p}\|C\|^p_{L^p} \\ \qquad \text{if } U \in D(\varphi_\varepsilon) = D(\varphi) \cap (\mathbb{H}^1_\sigma \times L^p \times L^p), \\ +\infty \qquad \text{if } U \in H \backslash D(\varphi_\varepsilon). \end{cases} \tag{11}
$$

Because ψ_ε is lower semicontinuous convex function on H and Fréchet differentiable on $D(\psi_\varepsilon) = \mathbb{L}^2_\sigma(\Omega) \times L^p(\Omega) \times L^p(\Omega)$, the subdifferential of ψ_ε coincides with the dissipation term:

$$
\partial \psi_\varepsilon(U) = (0, \varepsilon |T|^{p-2} T, \varepsilon |C|^{p-2} C)^t. \tag{12}
$$

In general, the sum of two subdifferentials is not always maximal monotone. But for this case, we have the following good property:

$$
\begin{aligned}
(\partial\varphi(U), \partial\psi_\varepsilon(U))_H &= (-\Delta T, \varepsilon|T|^{p-2}T)_{L^2} + (-\Delta C, \varepsilon |C|^{p-2} C)_{L^2} \\
&= \varepsilon(p-1)\int_\Omega |T|^{p-2}|\nabla T|^2 dx + \varepsilon(p-1)\int_\Omega |C|^{p-2}|\nabla C|^2 dx \geqslant 0.
\end{aligned} \tag{13}
$$

By virtue of (13), together with Proposition2.17, Theorem4.4 and Proposition4.6 in Brézis[4], we can deduce that $\partial\varphi + \partial\psi_\varepsilon$ becomes maximal monotone, and hence we get $\partial(\varphi + \psi_\varepsilon) = \partial\varphi + \partial\psi_\varepsilon$ with $D(\partial(\varphi + \psi_\varepsilon)) = D(\partial\varphi) \cap D(\partial\psi_\varepsilon)$.

Thus, we have another abstract problem associated with approximation problems:

$$
(\text{AP})_\varepsilon \begin{cases} \frac{dU(t)}{dt} + \partial\varphi_\varepsilon(U(t)) + B_\varepsilon(U(t)) = F(t) \quad t \in [0, S], \\ U(0) = U(S). \end{cases} \tag{14}
$$

2.6 Main Result

Our main result is stated as follows:

Theorem2.2 Let $N \leqslant 3$ and $(f_1, f_2, f_3) \in L^2(0, S; H)$. Then (BF) has a solution $(\boldsymbol{u}, T, C)$ satisfying

$$\begin{cases} \partial_t \boldsymbol{u}, \mathcal{A}\boldsymbol{u} \in \mathbb{L}^2(0, S; \mathbb{L}^2_\sigma(\Omega)), \\ \boldsymbol{u} \in \mathbb{C}([0, S]; \mathbb{H}^1_\sigma(\Omega)), \\ \partial_t T, \partial_t C, \Delta T, \Delta C \in L^2([0, S]; L^2(\Omega)), \\ T, C \in C_\pi([0, S]; H^1_0(\Omega)). \end{cases}$$

3 Solvability of Approximation Equations

In this section, we are going to verify that Th.2.1 can be applied to $(\mathrm{AP})_\varepsilon$, that is to say, we are going to check (A.1)-(A.4).

In what follows, let the space dimension N be 3. For the case where $N = 2$, the proof can be done by the same (much easier) arguments.

3.1 Check of (A.1)

(A.1)< Compactness condition >

For any $L \in (0, +\infty)$, the set $\{U \in H; \varphi(U) + \|U\|_H^2 \leqslant L\}$ is compact in H.

Proof. The level set $\{U \in H; \varphi(U) + \|U\|_H^2 \leqslant L\}$ is bounded in the function space $\mathbb{H}^1_\sigma(\Omega) \times H^1_0(\Omega) \times H^1_0(\Omega)$. Therefore it is clear that the level set is compact in H by virtue of Rellich's compactness theorem. □

3.2 Check of (A.2)

(A.2)< φ-demiclosedness condition >

$B(\cdot)$ is φ-demiclosed

Proof. Assume

$$\begin{cases} \boldsymbol{u}_k \to \boldsymbol{u} \text{ strongly in } C([0, S], \mathbb{L}^2_\sigma(\Omega)), \\ T_k \to T \text{ strongly in } C([0, S], L^2(\Omega)), \\ C_k \to C \text{ strongly in } C([0, S], L^2(\Omega)), \end{cases} \tag{15}$$

$$\begin{cases} -\nu \mathcal{P}_\Omega \Delta \boldsymbol{u}_k \rightharpoonup -\nu \mathcal{P}_\Omega \Delta \boldsymbol{u} \text{ weakly in } L^2(0, S; \mathbb{L}^2_\sigma(\Omega)), \\ -\Delta T_k + \varepsilon |T_k|^{p-2} T_k \rightharpoonup -\Delta T + \varepsilon |T|^{p-2} T \text{ weakly in } L^2(0, S; L^2(\Omega)), \\ -\Delta C_k + \varepsilon |C_k|^{p-2} C_k \rightharpoonup -\Delta C + \varepsilon |C|^{p-2} C \text{ weakly in } L^2(0, S; L^2(\Omega)), \end{cases} \tag{16}$$

and let

$$\begin{cases} a\boldsymbol{u}_k - \mathcal{P}_\Omega \boldsymbol{g}[T_k]_\varepsilon - \mathcal{P}_\Omega \boldsymbol{h}[C_k]_\varepsilon \rightharpoonup h_1 \text{ weakly in } L^2(0, S; \mathbb{L}^2_\sigma(\Omega)), \\ \boldsymbol{u}_k \cdot \nabla T_k \rightharpoonup h_2 \text{ weakly in } L^2(0, S; L^2(\Omega)), \\ \boldsymbol{u}_k \cdot \nabla C_k - \rho \Delta T_k \rightharpoonup h_3 \text{ weakly in } L^2(0, S; L^2(\Omega)). \end{cases} \tag{17}$$

From the strong convergences of (15), we easily get

$$h_1 = a\boldsymbol{u} - \mathcal{P}_\Omega \boldsymbol{g}[T]_\varepsilon - \mathcal{P}_\Omega \boldsymbol{h}[C]_\varepsilon \tag{18}$$

and additionally from (13), we derive the weak convergence $-\Delta T_k \rightharpoonup -\Delta T, -\Delta C_k \rightharpoonup -\Delta C$.

Because $\boldsymbol{u}_k$ is a solenoidal function, applying the integration by parts, we obtain

$$\langle \boldsymbol{u}_k \cdot \nabla T_k, \phi\rangle = -\langle \boldsymbol{u}_k T_k, \nabla\phi\rangle \to -\langle \boldsymbol{u}T, \nabla\phi\rangle = \langle \boldsymbol{u}\cdot\nabla T, \phi\rangle \tag{19}$$

for all $\phi \in C_0^\infty(\Omega \times (0,S))$. Consequently we find $h_2 = \boldsymbol{u}\cdot\nabla T$. Similarly, we also find $h_3 = \boldsymbol{u}\cdot\nabla C - \rho\Delta T$. □

3.3 Check of (A.3)

(A.3)< Boundedness condition >

For all $U \in D(\partial\varphi)$, there exists a monotone increasing function $\ell(\cdot)$ and a constant $k \in [0,1)$ such that $\|B(U)\|_H^2 \leqslant k\|\partial\varphi(U)\|_H^2 + \ell(\|U\|_H)(\varphi(U)+1)^2$.

Proof. By the definition of $B(U)$ and the inner product of H, we get

$$\|B_\varepsilon(U)\|_H^2 \leqslant \beta\|U\|_H^2 + \int_\Omega |\boldsymbol{u}\cdot\nabla T|^2 dx + \frac{2}{9\rho^2}\int_\Omega |\boldsymbol{u}\cdot\nabla C|^2 dx + \frac{2}{9}\|\Delta T\|_{L^2}^2 \tag{20}$$

for some constant β.

We begin with the estimate for the convection terms. Since $div\boldsymbol{u} = 0$, the integration by parts give

$$\int_\Omega |\boldsymbol{u}\cdot\nabla T|^2 dx = \int_\Omega \nabla T\cdot \boldsymbol{u}(\boldsymbol{u}\cdot\nabla T)dx = -\int_\Omega T\boldsymbol{u}\nabla(\boldsymbol{u}\cdot\nabla T)dx \leqslant \int_\Omega |T||\boldsymbol{u}||\nabla(\boldsymbol{u}\cdot\nabla T)|dx. \tag{21}$$

Hence by the elliptic estimate and Hölder's inequality, we have

$$\begin{aligned}\int_\Omega |\boldsymbol{u}\cdot\nabla T|^2 dx &\leqslant \beta\Big(\int_\Omega |T||\boldsymbol{u}||\boldsymbol{u}||\Delta T|dx + \int_\Omega |T||\boldsymbol{u}||\nabla\boldsymbol{u}||\nabla T|dx\Big)\\ &\leqslant \beta(\|T\|_{L^{12}}\|\boldsymbol{u}\|_{\mathbf{L}^6}\|\boldsymbol{u}\|_{\mathbf{L}^4}\|\Delta T\|_{L^2} + \|T\|_{L^{12}}\|\boldsymbol{u}\|_{\mathbf{L}^6}\|\nabla\boldsymbol{u}\|_{\mathbf{L}^4}\|\nabla T\|_{L^2})\end{aligned} \tag{22}$$

Then, by virtue of the fact $\|\boldsymbol{u}\|_{\mathbf{L}^4}^4 \leqslant \|\boldsymbol{u}\|_{\mathbf{L}^2}\|\boldsymbol{u}\|_{\mathbf{L}^6}^3$, Sobolev's inequality and Young's inequality, we get

$$\begin{aligned}\int_\Omega |\boldsymbol{u}\cdot\nabla T|^2 dx &\leqslant \frac{1}{9}\|\Delta T\|_{L^2}^2 + \beta(\|\nabla\boldsymbol{u}\|_{\mathbf{L}^2}^4 + \|T\|_{L^{12}}^{16}\|\boldsymbol{u}\|_{\mathbf{L}^2_\sigma}^4)\\ &\quad + \frac{1}{6}\|\mathcal{A}\boldsymbol{u}\|_{\mathbf{L}^2_\sigma}^2 + \beta(\|\nabla\boldsymbol{u}\|_{\mathbf{L}^2}^4 + \|\nabla T\|_{L^2}^4 + \|T\|_{L^{12}}^{16}),\end{aligned} \tag{23}$$

for some constant β. The convection term for C can be estimated by the same way.

Hence we obtain

$$\|B(U)\|_H^2 \leqslant \frac{1}{3}\|\partial\varphi_\varepsilon\|_H^2 + \ell(\|U\|_H)(\varphi_\varepsilon(U)+1)^2. \tag{24}$$

Thus the assumption (A.3) is assured with $k = 1/3$, provided that $p \geqslant 12$. □

In the above arguments, in order to estimate $\|B(U)\|_H^2$ from above by $\|\nabla\boldsymbol{u}\|_{\mathbf{L}^2}^4, \|\nabla T\|_{L^2}^4$ and $\|\nabla C\|_{L^2}^4$, we need the additional terms $\|\nabla T\|_{L^{12}}^{16}$ and $\|\nabla C\|_{L^{12}}^{16}$, which can be covered by the presence of the dissipation terms in approximation equations.

3.4 Check of (A.4)

(A.4)< Angular condition >

For all $U \in D(\partial\varphi)$, there exist positive constants α and K such that

$$\langle -\partial\varphi(U) - B(U), U\rangle_H + \alpha\varphi(U) \leqslant K.$$

Proof. Calculating the $H-$inner product between U and $B_\varepsilon(U)$, we have

$$\begin{aligned}(\partial\varphi_\varepsilon(U), U)_H &= \nu\|\nabla \boldsymbol{u}\|^2_{\mathbf{L}^2_\sigma} + \|\nabla T\|^2_{L^2} + \frac{1}{9\rho^2}\|\nabla C\|^2_{L^2} + \varepsilon\|T\|^p_{L^p} + \frac{\varepsilon}{9\rho^2}\|C\|^p_{L^p} \\ &\geqslant 2\varphi_\varepsilon(U)\end{aligned} \tag{25}$$

Moreover, noting that $(\boldsymbol{u}\cdot\nabla T, T)_{L^2} = (\boldsymbol{u}\cdot\nabla C, C)_{L^2} = 0$ and the cut-off function is bounded by $1/\varepsilon$, we get

$$\begin{aligned}(B_\varepsilon(U), U)_H &\geqslant a\|\boldsymbol{u}\|^2_{\mathbf{L}^2_\sigma} - |\boldsymbol{g}|\|\boldsymbol{u}\|_{\mathbf{L}^2_\sigma}\|[T]_\varepsilon\|_{L^2} - |\boldsymbol{h}|\|\boldsymbol{u}\|_{\mathbf{L}^2_\sigma}\|[C]_\varepsilon\|_{L^2} - \frac{1}{9\rho}\|\nabla T\|_{L^2}\|\nabla C\|_{L^2} \\ &\geqslant a\|\boldsymbol{u}\|^2_{\mathbf{L}^2_\sigma} - 2\left(\frac{a}{2}\|\boldsymbol{u}\|^2_{\mathbf{L}^2_\sigma} + \frac{\beta}{\varepsilon^2}\right) - \frac{1}{2}\|\nabla T\|^2_{L^2} - \frac{1}{18\rho^2}\|\nabla C\|^2_{L^2} \\ &\geqslant -\frac{2\beta}{\varepsilon^2} - \varphi_\varepsilon(U).\end{aligned} \tag{26}$$

Hence we get

$$(-\partial\varphi_\varepsilon(U) - B_\varepsilon(U), U)_H + \varphi_\varepsilon(U) \leqslant \frac{2\beta}{\varepsilon^2}, \tag{27}$$

whence follows (A.4) with $K = \frac{2\beta}{\varepsilon^2}$ and $\alpha = 1$. □

4 A Priori Estimates

In this section, we are going to establish some a priori estimates independent of the approximation parameter ε. In what follows, we donate by $(\boldsymbol{u}_\varepsilon, T_\varepsilon, C_\varepsilon)$ the periodic solutions of approximation equations (8).

Throughout this section we set $Q = [0, S] \times \Omega$ and donates by γ the general constant depending on $\|\boldsymbol{f}_1\|_{\mathbf{L}^2(0,S;\mathbf{L}^2_\sigma(\Omega))}$, $\|f_2\|_{L^2(0,S;L^2(\Omega))}$, $\|f_3\|_{L^2(0,S;L^2(\Omega))}$, $|\boldsymbol{g}|$ and $|\boldsymbol{h}|$ but not on ε.

4.1 First Energy Estimate for T_ε

Multiplying the second equation of (8) by T_ε and integrating over Ω, we have

$$\frac{1}{2}\frac{d}{dt}\|T_\varepsilon\|^2_{L^2(\Omega)} + \frac{1}{2}\|\nabla T_\varepsilon\|^2_{L^2(\Omega)} + \varepsilon\|T_\varepsilon\|^p_{L^p(\Omega)} = \int_\Omega f_2 T_\varepsilon dx. \tag{28}$$

Since the periodic condition gives

$$\int_0^S \frac{d}{d\tau}\|T_\varepsilon(\tau)\|^2_{L^2(\Omega)} d\tau = \|T_\varepsilon(S)\|^2_{L^2(\Omega)} - \|T_\varepsilon(0)\|^2_{L^2(\Omega)} = 0, \tag{29}$$

integrating (28) over $[0,S]$, we obtain

$$\|\nabla T_\varepsilon\|^2_{L^2(0,S;L^2(\Omega))} \leqslant \|f_2\|_{L^2(0,S;L^2(\Omega))}\|T_\varepsilon\|_{L^2(0,S;L^2(\Omega))}. \tag{30}$$

By $T_\varepsilon \in C([0,S];L^2(\Omega))$, there exists $t_0 \in [0,S]$ where $\|T_\varepsilon(t)\|_{L^2(\Omega)}$ attains its minimum, i.e.,

$$\|T_\varepsilon(t_0)\|_{L^2(\Omega)} = \min_{0\leqslant t\leqslant S}\|T_\varepsilon(t)\|_{L^2(\Omega)}. \tag{31}$$

Hence applying Poincaré's inequality and Cauchy's inequality, we have

$$\|T_\varepsilon(t_0)\|_{L^2(\Omega)} \leqslant \frac{M}{S}\|f_2\|_{L^2(0,S;L^2(\Omega))} \leqslant \gamma, \tag{32}$$

where M is a constant depending on the Poincaré constant. Then, integrating (28) over $[t_0,t]$ $(t_0 \leqslant t \leqslant t_0+S)$ and over $[t_0,t_0+S]$, we deduce

$$\|T_\varepsilon\|_{C([0,S];L^2(\Omega))}, \|\nabla T_\varepsilon\|_{L^2(0,S;L^2(\Omega))}, \varepsilon\|T_\varepsilon\|^p_{L^p(0,S;L^p(\Omega))} \leqslant \gamma. \tag{33}$$

4.2 First Energy Estimate for C_ε

Multiplying the third equation of (8) by C_ε and integrating over Ω, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|C_\varepsilon\|^2_{L^2} + \|\nabla C_\varepsilon\|^2_{L^2} + \varepsilon\|C_\varepsilon\|^p_{L^p} &= \rho\int_\Omega C_\varepsilon\Delta T_\varepsilon dx + \int_\Omega f_2 C_\varepsilon dx. \\ &\leqslant \frac{1}{2}\|\nabla C_\varepsilon\|^2_{L^2} + \frac{\rho^2}{2}\|\nabla T_\varepsilon\|^2_{L^2} + \|f_2\|^2_{L^2}\|C_\varepsilon\|^2_{L^2}\end{aligned} \tag{34}$$

Since we already know the boundedness of $\|\nabla T_\varepsilon\|^2_{L^2(0,S;L^2(\Omega))}$, repeating the same arguments as above, we obtain

$$\|C_\varepsilon\|_{C([0,S];L^2(\Omega))}, \|\nabla C_\varepsilon\|_{L^2(0,S;L^2(\Omega))}, \varepsilon\|C_\varepsilon\|^p_{L^p(0,S;L^p(\Omega))} \leqslant \gamma. \tag{35}$$

4.3 First Energy Estimate for $\boldsymbol{u}_\varepsilon$

Multiplying the first equation of (8) by $\boldsymbol{u}_\varepsilon$ and integrating over Ω, we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\boldsymbol{u}_\varepsilon\|^2_{\mathbf{L}^2_\sigma} + \nu\|\nabla\boldsymbol{u}_\varepsilon\|^2_{\mathbf{L}^2} + a\|\boldsymbol{u}_\varepsilon\|^2_{\mathbf{L}^2_\sigma} &= \int_\Omega [T_\varepsilon]_\varepsilon\, \boldsymbol{g}\cdot\boldsymbol{u}_\varepsilon dx + \int_\Omega [C_\varepsilon]_\varepsilon\, \boldsymbol{h}\cdot\boldsymbol{u}_\varepsilon dx + \int_\Omega \boldsymbol{f}_1\cdot\boldsymbol{u}_\varepsilon dx \\ &\leqslant |\boldsymbol{g}|\|T_\varepsilon\|_{L^2}\|\boldsymbol{u}_\varepsilon\|_{\mathbf{L}^2_\sigma} + |\boldsymbol{h}|\|C_\varepsilon\|_{L^2}\|\boldsymbol{u}_\varepsilon\|_{\mathbf{L}^2_\sigma} + \|\boldsymbol{f}_1\|_{\mathbf{L}^2}\|\boldsymbol{u}_\varepsilon\|_{\mathbf{L}^2_\sigma} \\ &\leqslant \gamma\|\boldsymbol{u}_\varepsilon\|_{\mathbf{L}^2_\sigma}\end{aligned} \tag{36}$$

by (33) and (35). Then, as above, we get

$$\|\boldsymbol{u}_\varepsilon\|_{C([0,S];L^2(\Omega))}, \|\nabla\boldsymbol{u}_\varepsilon\|_{L^2(0,S;L^2(\Omega))} \leqslant \gamma. \tag{37}$$

4.4 Second Energy Estimate for $\boldsymbol{u}_\varepsilon$

Multiplying the first equation of (8) by $\partial_t \boldsymbol{u}_\varepsilon$ and integrating over Ω, we have

$$\begin{aligned} \|\partial_t \boldsymbol{u}_\varepsilon\|^2_{\mathbf{L}^2_\sigma} + \frac{\nu}{2}\frac{d}{dt}\|\nabla \boldsymbol{u}_\varepsilon\|^2_{\mathbf{L}^2} + \frac{a}{2}\frac{d}{dt}\|\boldsymbol{u}_\varepsilon\|^2_{\mathbf{L}^2_\sigma} &= \int_\Omega ([T_\varepsilon]_\varepsilon \boldsymbol{g} + [C_\varepsilon]_\varepsilon \boldsymbol{h} + \boldsymbol{f}_1)\cdot \partial_t \boldsymbol{u}_\varepsilon dx \\ &\leqslant (|\boldsymbol{g}|\|T_\varepsilon\|_{L^2} + |\boldsymbol{h}|\|C_\varepsilon\|_{L^2} + \|\boldsymbol{f}_1\|_{\mathbf{L}^2})\|\partial_t \boldsymbol{u}_\varepsilon\|_{\mathbf{L}^2_\sigma} \\ &\leqslant \gamma \|\partial_t \boldsymbol{u}_\varepsilon\|_{\mathbf{L}^2_\sigma} \end{aligned} \tag{38}$$

On the other hand, in view of (33), (35) and (37), we find that $\|\varphi_\varepsilon(U(t))\|_{L^1(0,S)} \leqslant \gamma$. Hence, since $\varphi_\varepsilon(U(t))$ is absolutely continuous on $[0,S]$, there exists $t_1 \in [0,S]$, where $\varphi_\varepsilon(U(t))$ attains its minimum at $t = t_1$, i.e.,

$$\varphi_\varepsilon(U_\varepsilon(t_1)) = \min_{0\leqslant t \leqslant S} \varphi_\varepsilon(U_\varepsilon(t)) \leqslant \frac{1}{S}\int_0^S \varphi_\varepsilon(U_\varepsilon(\tau))d\tau \leqslant \frac{\gamma}{S} \tag{39}$$

whence follows

$$\|\nabla \boldsymbol{u}_\varepsilon(t_1)\|^2_{\mathbf{L}^2}, \|\nabla T_\varepsilon(t_1)\|^2_{L^2}, \|\nabla C_\varepsilon(t_1)\|^2_{L^2}, \varepsilon\|T_\varepsilon(t_1)\|^p_{L^p}, \varepsilon\|C_\varepsilon(t_1)\|^p_{L^p} \leqslant \gamma. \tag{40}$$

Then integrating (38) over $[t_1, t]$ and $[t_1, t_1+S]$, we derive

$$\|\nabla \boldsymbol{u}_\varepsilon\|_{C(0,S;L^2(\Omega))}, \|\partial_t \boldsymbol{u}_\varepsilon\|_{L^2(0,S;L^2_\sigma(\Omega))} \leqslant \gamma. \tag{41}$$

Furthermore, by using the first equation of (8), we also obtain

$$\|\mathcal{A}\boldsymbol{u}_\varepsilon\|_{L^2(0,S;L^2_\sigma(\Omega))} \leqslant \gamma. \tag{42}$$

4.5 Second Energy Estimate for T_ε

Multiplying the second equation of (8) by $-\Delta T_\varepsilon$ and integrating over Ω, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\nabla T_\varepsilon\|^2_{L^2} + \|\Delta T_\varepsilon\|^2_{L^2} &= -\varepsilon(p-1)\int_\Omega |T_\varepsilon|^{p-2}|\nabla T_\varepsilon|^2 dx - \int_\Omega \Delta T_\varepsilon \boldsymbol{u}_\varepsilon \cdot \nabla T_\varepsilon dx - \int_\Omega f_2 \Delta T_\varepsilon dx. \\ &\leqslant \int_\Omega |\Delta T_\varepsilon||\boldsymbol{u}_\varepsilon||\nabla T_\varepsilon| dx + \|f_2\|_{L^2}\|\Delta T_\varepsilon\|_{L^2}. \end{aligned} \tag{43}$$

Here, using again $\|T_\varepsilon\|^2_{L^3} \leqslant \|T_\varepsilon\|_{L^2}\|T_\varepsilon\|_{L^6}$, we get

$$\begin{aligned} \int_\Omega |\boldsymbol{u}_\varepsilon||\nabla T_\varepsilon||\Delta T_\varepsilon| dx &\leqslant \|\boldsymbol{u}_\varepsilon\|_{\mathbf{L}^6}\|\nabla T_\varepsilon\|_{L^3}\|\Delta T_\varepsilon\|_{L^2} \\ &\leqslant \kappa \|\nabla \boldsymbol{u}_\varepsilon\|_{\mathbf{L}^2}\|\nabla T_\varepsilon\|^{1/2}_{L^2}\|\Delta T_\varepsilon\|^{3/2}_{L^2} \\ &\leqslant \frac{1}{4}\|\Delta T_\varepsilon\|^2_{L^2} + \kappa^4\|\nabla \boldsymbol{u}_\varepsilon\|^4_{\mathbf{L}^2}\|\nabla T_\varepsilon\|^2_{L^2} \end{aligned} \tag{44}$$

where κ is the constant which depends on Sobolev's embedding constant. Therefore, using previous estimates, we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla T_\varepsilon\|^2_{L^2(\Omega)} + \frac{1}{4}\|\Delta T_\varepsilon\|^2_{L^2(\Omega)} \leqslant \gamma\|\nabla T_\varepsilon\|^2_{L^2(\Omega)} + \frac{1}{2}\|f_2\|^2_{L^2(\Omega)}. \tag{45}$$

Then, by Gronwall's inequality, we obtain

$$\|\nabla T_\varepsilon\|_{C(0,S;L^2(\Omega))}, \|\Delta T_\varepsilon\|_{L^2(Q)} \leqslant \gamma. \tag{46}$$

Next multiplying the second equation of (8) by $\partial_t T_\varepsilon$ and integrating over Ω, we get

$$\|\partial_t T_\varepsilon\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{d}{dt}\|\nabla T_\varepsilon\|^2_{L^2(\Omega)} + \frac{\varepsilon}{p}\frac{d}{dt}\|T_\varepsilon\|^p_{L^p(\Omega)} = \int_\Omega \partial_t T_\varepsilon \boldsymbol{u}_\varepsilon \cdot \nabla T_\varepsilon dx + \int_\Omega f_2 \partial_t T_\varepsilon dx. \tag{47}$$

The above argument with ΔT_ε replaced by $\partial_t T_\varepsilon$ gives

$$\frac{1}{4}\|\partial_t T_\varepsilon\|^2_{L^2(\Omega)} + \frac{1}{2}\frac{d}{dt}\|\nabla T_\varepsilon\|^2_{L^2(\Omega)} + \frac{\varepsilon}{p}\frac{d}{dt}\|T_\varepsilon\|^p_{L^p(\Omega)} \leqslant \gamma\|\nabla T_\varepsilon\|^2_{L^2(\Omega)} + \frac{1}{2}\|f_2\|^2_{L^2(\Omega)}, \tag{48}$$

whence follows

$$\|\partial_t T_\varepsilon\|_{L^2(Q)}, \sup_{0\leqslant t\leqslant S} \varepsilon\|T_\varepsilon\|^p_{L^p(\Omega)} \leqslant \gamma. \tag{49}$$

4.6 Second Energy Estimate for C_ε

We multiply the third equation of (8) by $-\Delta C_\varepsilon$ or $\partial_t C_\varepsilon$ and integrating over Ω. Since we already obtain the a priori bounds for T_ε, by the same arguments as above, we get

$$\frac{1}{2}\frac{d}{dt}\|\nabla C_\varepsilon\|^2_{L^2} + \frac{1}{8}\|\Delta C_\varepsilon\|^2_{L^2} \leqslant \gamma\|\nabla \boldsymbol{u}_\varepsilon\|^4_{L^2}\|\nabla C_\varepsilon\|^2_{L^2} + \rho^2\|\Delta T_\varepsilon\|^2_{L^2} + \frac{1}{2}\|f_3\|^2_{L^2}, \tag{50}$$

$$\frac{1}{8}\|\partial_t C_\varepsilon\|^2_{L^2} + \frac{1}{2}\frac{d}{dt}\|\nabla C_\varepsilon\|^2_{L^2} + \frac{\varepsilon}{p}\frac{d}{dt}\|C_\varepsilon\|^p_{L^p} \leqslant \gamma\|\nabla \boldsymbol{u}_\varepsilon\|^4_{L^2}\|\nabla C_\varepsilon\|^2_{L^2} + \rho^2\|\Delta T_\varepsilon\|^2_{L^2} + \frac{1}{2}\|f_2\|^2_{L^2}. \tag{51}$$

Hence we obtain

$$\|\nabla C_\varepsilon\|_{C(0,S;L^2(\Omega))}, \|\Delta C_\varepsilon\|_{L^2(Q)}, \|\partial_t C_\varepsilon\|_{L^2(Q)}, \sup_{0\leqslant t\leqslant S} \varepsilon\|C_\varepsilon\|^p_{L^p(\Omega)} \leqslant \gamma. \tag{52}$$

5 Convergence

In this section, making use of a priori estimates given in the previous section, we shall discuss the convergence of solutions of the approximation equations.

We first recall

$$\sup_{0\leqslant t\leqslant S} \varphi_\varepsilon(U_\varepsilon(t)) \leqslant \gamma. \tag{53}$$

Therefore by virtue of Rellich's compactness theorem, the sequence of the solution $\{U_\varepsilon(t)\}_{\varepsilon>0}$ is precompact in H for all $t \in [0, S]$. Moreover, noting

$$\begin{aligned}\|U_\varepsilon(t) - U_\varepsilon(s)\|_H &= \|\int_s^t \partial_\tau U_\varepsilon(\tau)d\tau\|_H \leqslant \int_s^t \|\partial_\tau U_\varepsilon(\tau)\|_H d\tau \\ &\leqslant (\int_s^t \|\partial_\tau U_\varepsilon(\tau)\|^2_H d\tau)^{1/2}(\int_s^t 1^2 d\tau)^{1/2} \leqslant \gamma|t-s|^{1/2},\end{aligned} \tag{54}$$

we see that $\{U_\varepsilon(t)\}_{\varepsilon>0}$ forms an equi-continuous subset in $C_\pi([0, S]; H)$. Hence, applying Ascoli's theorem, there exists a sequence $U_n = U_{\varepsilon_n}$ with $\varepsilon_n \to 0$ as $n \to \infty$ such that

$$U_n \to U \quad \text{strongly in } C_\pi([0, S]; H) \text{ as } n \to \infty. \tag{55}$$

Furthermore, we have

$$\begin{aligned} &\frac{dU_n}{dt} \rightharpoonup \frac{dU}{dt} = (\partial_t \boldsymbol{u}, \partial_t T, \partial_t C)^t \text{ weakly in } L^2(0,S;H) \text{ as } n \to \infty, \\ &\nabla U_n \rightharpoonup \nabla U = (\nabla \boldsymbol{u}, \nabla T, \nabla C)^t \text{ weakly in } L^\infty(0,S;H) \text{ as } n \to \infty, \\ &\partial\varphi(U_n) \rightharpoonup \partial\varphi(U) = (\mathcal{A}\boldsymbol{u}, -\Delta T, -\Delta C)^t \text{ weakly in } L^2(0,S;H) \text{ as } n \to \infty. \end{aligned} \tag{56}$$

Since the fact that U_t and $\partial\varphi(U)$ belong to $L^2(0,S;H)$ implies the absolute continuity of ∇U, we easily find

$$\nabla U = (\nabla \boldsymbol{u}, \nabla T, \nabla C)^t \in C_\pi([0,S];H). \tag{57}$$

Now it remains to show that the limit function $(\boldsymbol{u}, T, C)$ gives a solution of (2). Since the terms in the second and third equations of (8) except the dissipation terms are all bounded in $L^2(Q)$, we find that the dissipation terms are also bounded in $L^2(Q)$. Therefore, there exists a sequence $\{T_{n_k}\}_{k\in\mathbb{N}}$ such that

$$\varepsilon_{n_k}|T_{n_k}|^{p-2}T_{n_k} \rightharpoonup \exists\chi \quad \text{weakly in } L^2(0,S;L^2(\Omega)) \text{ as } k \to \infty. \tag{58}$$

On the other hand, by (49), we get

$$\|\varepsilon|T_\varepsilon|^{p-2}T_\varepsilon\|_{L^{p'}}^{p'} = \varepsilon^{p'}\|T_\varepsilon\|_{L^p}^p = \varepsilon^{p'-1}(\varepsilon\|T\|_{L^p}^p) \leqslant \varepsilon^{p'-1}\gamma, \tag{59}$$

which implies that $\chi = 0$. Similarly we find that $\varepsilon_{n_k}|T_{n_k}|^{p-2}T_{n_k} \rightharpoonup 0$ weakly in $L^2(Q)$. Thus we obtain

$$\partial\varphi_{\varepsilon_n}(U_{\varepsilon_n}) \rightharpoonup \partial\varphi(U) \text{ weakly in } L^2(0,S;H).$$

From the strong convergences of U_{ε_n}, cut-off functions $[T_{\varepsilon_n}]_{\varepsilon_n}$ and $[C_{\varepsilon_n}]_{\varepsilon_n}$ weakly converge to original function T, C in $L^2(0,S;L^2(\Omega))$. Hence, we get

$$B_{\varepsilon_n}(U_{\varepsilon_n}) \rightharpoonup B(U) \text{ weakly in } L^2(0,S;H).$$

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