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Shape optimization for partial differential equations/system with mixed boundary conditions

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1 Theory of GJ-integral

Let X and M be real Banach spaces and let X' and M' be their dual spaces, respectively. For $\mathcal{U}_0 \subset X$ ($\mathcal{U}_0 \neq \emptyset$) and an open subset $\mathcal{O}_0 \subset M$ ($\mathcal{O}_0 \neq \emptyset$), we consider a real valued functional $J : \mathcal{U}_0 \times \mathcal{O}_0 \to \mathbb{R}$. In general, for $u \in \mathcal{U}_0$ and $w \in X$, the Gâteaux derivative $\delta_X J(v, \mu)[w] \in \mathbb{R}$ is defined as

$$\delta_X J(u, \mu)[w] = \frac{d}{dt} J(u + tw, \mu)\bigg|_{t=0},$$

when it exists. If $\delta_X J(u, \mu)[w]$ exists, from the linearity of the Gâteaux derivative, $\delta_X J(u, \mu)[\alpha w]$ exists for arbitrary $\alpha \in \mathbb{R}$ and it satisfies and it satisfies

$$\delta_X J(u, \mu)[\alpha w] = \alpha \delta_X J(u, \mu)[w].$$

We use the symbols ∂_X and ∂_M to denote the partial Fréchet derivative operators for $J(u, \mu)$ with respect to $u \in X$ and $\mu \in M$, respectively, and assume the following.

(H1) $[\mu \mapsto J(w, \mu)] \in C^1(\mathcal{O}_0)$ for all $w \in \mathcal{U}_0$, and $\partial_M J : \mathcal{U}_0 \times \mathcal{O}_0 \to M'$ is continuous at $(u(\mu_0), \mu_0)$.

(H2) The Banach space X is reflexive and $\mathcal{U}_0$ is closed and convex in X.

(H3) For the functional $[v \mapsto J(v, \mu_0)]$, u_0 is a unique minimizer over $\mathcal{U}_0$.

(H4) The functional $[v \mapsto J(v, \mu_0)]$ is sequentially lower semicontinuous with respect to the weak topology of X.

(H5) There is a monotone nondecreasing function β_0 defined on $[0, \infty)$ with $\lim_{s\to\infty} \beta_0(s) = \infty$ such that

$$\beta_0 (\|v\|_X) \leq J(v, \mu) \qquad (v \in \mathcal{U}_0,\ \mu \in \mathcal{O}_0).$$

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(H6) For any $\varepsilon > 0$ and $R > 0$, there exists $\delta > 0$ such that

$$|J(v,\mu) - J(v,\mu_0)| \le \varepsilon \qquad (v \in \mathcal{U}_0,\ \|v\|_X \le R,\ \ \mu \in \mathcal{O}_0,\ \|\mu - \mu_0\|_M \le \delta).$$

(H7) For $v \in \mathcal{U}_0$, the function $[t \mapsto J(u_0 + t(v-u_0), \mu_0)]$ belongs to $C^1((0,1])$. Moreover, for a sequence $\{u_n\}_n \subset \mathcal{U}_0$ which weakly converges to u_0 as $n \to \infty$, the condition $\overline{\lim}_{n\to\infty}\delta_X J(u_n,\mu_0)[u_n - u_0] \le 0$ implies that $u_n \to u_0$ strongly in X as $n \to \infty$.

In particular, under the condition (H7),

$$\delta_X J(v,\mu_0)[v-u_0] = \frac{d}{dt}J(u_0 + t(v-u_0),\mu_0)\bigg|_{t=1}$$

exists for all $v \in \mathcal{U}_0$. The condition (H7) is often called the (S_+)-property.

Theorem 1 *Under the conditions (H1)–(H7), $[\mu \mapsto J_*(\mu)]$ is Fréchet differentiable at $\mu = \mu_0$ and the following holds.*

$$D_\mu[J(u(\mu_0),\mu_0)] = \partial_M J(u(\mu_0),\mu_0) \tag{1}$$

where the D_μ denotes the Fréchet differential operator with respect to $\mu \in M$.

See [9] for the proof. Theorem 1 will play an important role in design sensitivity analysis by considering mu to be a design variable. We shall show that Theorem 1 derive an important result in the shape sensitivity analysis of energy.

1.1 Boundary value problems and its Lipschitz perturbation

Let Ω be a bounded domain in $\mathbb{R}^d$ ($d \ge 2$) and $L^p(\Omega,\mathbb{R}^m)$ Lebesgue space of all measurable functions $v : \Omega \to \mathbb{R}^m$ (a real number $1 < p < \infty$ and an integer $m \ge 1$) with $\|v\|_{p,\Omega}$

$$\|v\|_{p,\Omega} = \left(\sum_{i=1}^m \int |v_i|^p\right)^{1/p}, v = (v_1,\cdots,v_m)$$

$\mathrm{P}(f, V(\Omega,\Gamma_D))$: For a given function $f \in L^{p'}(\Omega,\mathbb{R}^m), p' = p/(p-1)$, find u minimizing the following functional

$$\mathcal{E}(v; f,\Omega) = \int_\Omega \left\{\widehat{W}(x,v,\nabla v) - f\cdot v\right\} dx$$

over the space

$$V(\Omega,\Gamma_D) = \left\{v \in W^{1,p}(\Omega,\mathbb{R}^m) : v = 0 \text{ on } \Gamma_D\right\}$$

where Γ_D stands for the part of $\partial\Omega$ and a scalar function $\widehat{W}(\xi,z,\zeta) : \mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{d\times m} \to \mathbb{R}$ is in $C^1\left(\mathbb{R}^d \times \mathbb{R}^m \times \mathbb{R}^{d\times m}\right)$ and $W^{1,p}(\Omega,\mathbb{R}^m)$ denote Sobolev space of functions $v \in L^p(\Omega,\mathbb{R}^m)$ with $\|v\|_{1,p,\Omega}$.

We now give a condition of the existence of minimizers for $\mathrm{P}(f, V(\Omega,\Gamma_D))$.

Theorem 2 *If $\widehat{W}$ satisfies the coercivity condition*

$$\widehat{W}(\xi,z,\zeta) \geq c_1|\zeta|^p + c_2|z|^q + \alpha_1(\xi) \tag{2}$$

for almost every $\xi \in \Omega$ and for every $(z,\zeta) \in \mathbb{R}^m \times \mathbb{R}^{m\times d}$ and for some $\alpha_1 \in L^1(\Omega)$, $c_2 \in \mathbb{R}$, $c_1 > 0$ and $p > q \geq 1$. Assume that $\zeta \to \widehat{W}(\xi,z,\zeta)$ is convex and $\mathcal{E}(0; f,\Omega) < \infty$, then there is a minimizer u

$$\mathcal{E}(u; f,\Omega) = \min_{v\in V(\Omega,\Gamma_D)} \mathcal{E}(v; f,\Omega)$$

attains its minimum.

Furthermore, if $(z,\zeta) \to \widehat{W}(\xi,z,\zeta)$ is strictly convex for almost every $\xi \in \Omega$, then the minimizer is unique.

See [2][Theorem 3.30] for the proof.

We assume the growth conditions of $\widehat{W}$, that is, for almost every $\xi \in \Omega$, for every $(z,\zeta) \in \mathbb{R}^m \times \mathbb{R}^{m\times d}$

$$\begin{aligned} |\widehat{W}(\xi,z,\zeta)| &\leq \alpha_1(\xi) + c(|z|^p + |\zeta|^p) \\ |D_z\widehat{W}(\xi,z,\zeta)| &\leq \alpha_2(\xi) + c(|z|^{p-1} + |\zeta|^{p-1}) \\ |D_z\widehat{W}(\xi,z,\zeta)| &\leq \alpha_3(\xi) + c(|z|^{p-1} + |\zeta|^{p-1}) \end{aligned} \tag{3}$$

where $\alpha_1 \in L^1(\Omega_0)$, $\alpha_2,\alpha_3 \in L^{p(p-1)}(\Omega_0)$ and $c \geq 0$,

$$\begin{aligned} D_\zeta\widehat{W}(u) &= \left.\begin{pmatrix} \frac{\partial}{\partial\zeta_{11}}\widehat{W}(\xi,z,\zeta) & \cdots & \frac{\partial}{\partial\zeta_{m1}}\widehat{W}(\xi,z,\zeta) \\ \vdots & \ddots & \vdots \\ \frac{\partial}{\partial\zeta_{1d}}\widehat{W}(\xi,z,\zeta) & \cdots & \frac{\partial}{\partial\zeta_{md}}\widehat{W}(\xi,z,\zeta) \end{pmatrix}\right|_{(\xi,z,\zeta)=(x,v(x),\nabla v(x))} \\ D_z\widehat{W}(u) &= \left.\left(\frac{\partial}{\partial z_1}\widehat{W}(\xi,z,\zeta),\cdots,\frac{\partial}{\partial z_d}\widehat{W}(\xi,z,\zeta)\right)^T\right|_{(\xi,z,\zeta)=(x,v(x),\nabla v(x))} \end{aligned}$$

Then, we have the following proposition.

Proposition 3 *If u is the solution of $P(f, V(\Omega,\Gamma_D))$ and $\widehat{W}$ satisfy Condition (3), then*

$$\int_\Omega \left\{D_\zeta\widehat{W}(x,u,\nabla u) : \nabla v + D_z\widehat{W}(x,u,\nabla u)v - fv\right\} dx = 0$$

for all $v \in W_0^{1,p}(\Omega,\mathbb{R}^m)$, where $A : B = A_{ij}B_{ij}$ for two matrices A and B.

See [2][Theorem 3.37] for the proof.

Putting

$$F(v) = \int_\Omega \widehat{W}(x,v,\nabla v)dx$$

we have for $X = W^{1,p}(\Omega, \mathbb{R}^m)$,

$$\begin{aligned} \langle \delta_X F(v), w \rangle_X &= \lim_{\epsilon \to 0} \epsilon^{-1}[F(v + \epsilon w) - F(v)] \\ &= \int_\Omega \left\{ D_\varsigma \widehat{W}(x, v, \nabla v) : \nabla w + D_z \widehat{W}(x, v, \nabla v) w \right\} dx \end{aligned}$$

The operator $[v \mapsto \delta_X F(v)]$ is called uniformly monotone, if there is a strictly monotone increasing continuous function $a : \mathbb{R}_+ \to \mathbb{R}_+$ with $a(0) = 0$ and $\lim_{t\to\infty} a(t) = +\infty$ such that

$$\langle \delta_X F(v) - \delta_X F(w), v - w \rangle_X \geq a(\|v - w\|_X)\|v - w\|_X$$

If $[v \mapsto \delta_X F(v)]$ is uniformly monotone, then the condition (H7) is satisfied. Indeed, taking a sequence $u_n \to u$ weakly in X, we have

$$\begin{aligned} \langle \delta_X F(u_n) - f, u_n - u \rangle_X &= \langle \delta_X F(u_n) - \delta_X F(u), u_n - u \rangle_X \\ &\geq a(\|v - w\|_X)\|v - w\|_X \end{aligned}$$

Since X is reflexive, the strong convergence $u_n \to u$ follows from

$$a(\|v - w\|_X) \leq \|\delta_X F(u_n) - f\|_{X'}$$

1.1.1 Perturbation

We choose a bounded convex domain Ω_0 with $\overline{\Omega} \subset \Omega_0$, and define $M = W^{1,\infty}(\Omega_0, \mathbb{R}^d)$ and

$$\mathcal{O}_0 = \left\{ \varphi \in M : \ |\varphi - \varphi_0|_{\mathrm{Lip},\Omega_0} < a_0 < 1, \ \overline{\varphi(\Omega)} \subset \Omega_0 \right\}, \tag{4}$$

where $a_0 \in (0,1)$ is a fixed number and we denote by φ_0 the identity map on $\mathbb{R}^d$, i.e., $\varphi_0(x) = x$ ($x \in \mathbb{R}^d$). Then $\varphi \in \mathcal{O}_0$ becomes a bi-Lipschitz transform from Ω onto $\varphi(\Omega)$.

For the domain $\varphi(\Omega)$, $\varphi \in \mathcal{O}_0$, we consider the problem $\mathrm{P}(f, V(\varphi(\Omega), \varphi(\Gamma_D)))$: Find $u(t)$ minimizing the following functional

$$\mathcal{E}(\boldsymbol{v}; f, \varphi(\Omega)) = \int_{\varphi(\Omega)} \left\{ \widehat{W}(x, w, \nabla w) - f \cdot w \right\} dx$$

over the space

$$V(\varphi(\Omega), \varphi(\Gamma_D)) = \left\{ w \in W^{1,p}(\varphi(\Omega), \mathbb{R}^m) : w = 0 \text{ on } \varphi(\Gamma_D) \right\}$$

We define a pushforward operator φ_* which transforms a function v on Ω to a function $\varphi_* v = v \circ \varphi^{-1}$ on $\varphi(\Omega)$. For $q \in [1, \infty]$, φ_* is a linear topological isomorphism from $L^q(\Omega)$ onto $L^q(\varphi(\Omega))$, and a linear topological isomorphism from $W^{1,q}(\Omega)$ onto $W^{1,q}(\varphi(\Omega))$. For $v \in V(\Omega, \Gamma_D)$, we get the equivalence,

$$\mathcal{E}(\varphi_* v, f, \varphi(\Omega)) = \int_\Omega \left\{ \widehat{W}\left(\varphi(x), v(x), [A(\varphi)(x)]\nabla v(x)\right) - f \circ \varphi(x) v(x) \right\} \kappa(\varphi)(x) dx \tag{5}$$

where

$$A(\varphi) = \left(\nabla \varphi^T\right)^{-1} \in L^\infty(\Omega_0, \mathbb{R}^{d \times d}), \quad \kappa(\varphi) = \det \nabla \varphi^T \in L^\infty(\Omega_0, \mathbb{R}).$$

We denote the right-hand side of (5) by $J(v, \varphi)$, and apply Theorem 1 to $J(v, \varphi)$.

If $\widehat{W}$ satisfy the growth condition (3) and

$$|D_\xi \widehat{W}(\xi, z, \zeta)| \le \alpha_1(\xi) + c(|z|^p + |\zeta|^p)$$

then we have the following proposition.

Proposition 4 *Suppose that $f \in W^{1,p/(p-1)}(\Omega_0)$. Then $J \in C^1(X \times \mathcal{O}_0)$ and*

$$\begin{aligned} \partial_M J(u, \varphi_0)[\mu] &= \frac{d}{dt} \int_\Omega \widehat{W}(x + t\mu(x), u(x), [A(\varphi_0 + t\mu)(x)]\nabla u(x))\, \kappa(\varphi_0 + t\mu)(x) dx \Big|_{t=0} \\ &\quad - \int_\Omega f \circ (\varphi_0 + t\mu)(x) v(x) \kappa(\varphi_0 + t\mu)(x) dx \Big|_{t=0} \qquad (6) \\ &= \int_\Omega \left(D_\xi W(u) \cdot \mu - (D_\zeta W(u))^T (\nabla \mu^T) \nabla u + W(u) \mathrm{div}\mu - \mathrm{div}(f \cdot \mu) v \right) dx. \end{aligned}$$

where $(D_\zeta W(u))^T (\nabla \mu^T) \nabla u = \sum_{i,j,k} \partial_{\zeta_{ij}} \widehat{W}(x, u, \nabla u)(\partial_j \mu_k)(\partial_k u_i)$.

1.1.2 Definition of GJ-integral

For an open subset ω in $\mathbb{R}^d$ and $\rho \in W^{1,\infty}(\mathbb{R}^d, \mathbb{R}^d)$, GJ-integral [5, 6, 7, 8]

$$\mathcal{J}_\omega(u, \rho) = P_\omega(u, \rho) + R_\omega(u, \rho)$$

is defined by

$$P_\omega(u, \rho) = \int_{\partial(\omega \cap \Omega)} \left\{ \widehat{W}(u)(\rho \cdot n) - \widehat{T}(u) \cdot (\nabla u \cdot \rho) \right\} ds$$

$$R_\omega(u, \rho) = -\int_{\omega \cap \Omega} \left\{ \nabla_\xi \widehat{W}(u) \cdot \rho + f \cdot (\nabla u \cdot \rho) - \left(\nabla_\zeta \widehat{W}(u) \right)^T (\nabla \rho^T) \nabla u + \widehat{W}(u) \,\mathrm{div}\, \rho \right\} dx$$

where $n = (n_1, \cdots, n_d)^T$ is the outward unit normal of $\partial(\omega \cap \Omega)$, i-th component of $\widehat{T}(u)$ is $n_j \nabla_{\zeta_{ij}} \widehat{W}(x, u, \nabla u)$ and ds the surface(line) element of $\partial(\omega \cap \Omega)$.

Proposition 5 *If $u|_{\omega\cap\Omega} \in W^{2,p}(\omega \cap \Omega, \mathbb{R}^m)$ and the divergence formula*

$$\int_{\omega\cap\Omega} \nabla \widehat{W}(u) \cdot \rho dx = \int_{\partial(\omega\cap\Omega)} \widehat{W}(u)(\rho \cdot n) ds - \int_{\omega\cap\Omega} \widehat{W}(u) \,\mathrm{div}\, \rho dx \qquad (7)$$

holds, then

$$\mathcal{J}_\omega(u, \rho) = 0 \ \textit{for all}\ \rho \in W^{1,\infty}(\Omega_0, \mathbb{R}^d) \qquad (8)$$

Consider the perturbation $\Omega(t), 0 \le t < \epsilon$ of Ω and the problems $\mathrm{P}(f, V(\Omega(t), \Gamma_D(t)))$ that are given by φ_t such as $\Omega(t) = \varphi_t(\Omega)$ and $\Gamma_D(t) = \varphi_t(\Gamma_D)$.

[M1] For each $t \in [0, \epsilon)$, φ_t is 1-1 mapping and has the inverse φ_t^{-1}.

[M2] $[t \mapsto \varphi_t] \in C^1\left([0,\epsilon), W^{1,\infty}\left(\Omega_0, \mathbb{R}^d\right)\right)$.

Theorem 6 *If $\widetilde{E}(v,\varphi) = \mathcal{E}(\varphi_* v, f, \varphi(\Omega))$ satisfies the conditions [H1] - [H7], then it follows for all φ_t satisfying [M1] and [M2] that*

$$\frac{d}{dt}\mathcal{E}\left(u\left(t\right); f, \Omega\left(t\right)\right)\bigg|_{t=0} = -R_{\Omega}\left(u, \mu_{\varphi}\right) - \int_{\partial\Omega} f \cdot u\left(\mu_{\varphi} \cdot n\right) ds \tag{9}$$

where $\mu_{\varphi} = d\varphi_t/dt|_{t=0}$.

2 Application to shape optimization (Energy)

In Theorem 6, we get the shape sensitivity analysis of the potential energy $\Omega \mapsto \mathcal{E}(u; f, \Omega)$. We introduce Azegami's method[1, 4] to find optimum shape Ω^o assuming that the cost function is the energy, that is, find u^o and Ω^o such that

$$\mathcal{E}(f; u^o, \Omega^o) \leq \mathcal{E}(f; u(\widetilde{\Omega}), \widetilde{\Omega}) \quad \text{for all domain } \widetilde{\Omega}$$

under some restrictions, where $u(\widetilde{\Omega})$ is the solution of $\mathrm{P}(f, V(\widetilde{\Omega}, \widetilde{\Gamma_D}))$.

2.1 Azegami's method

Let $\mathcal{V}(\Omega)$ be the subspace of $W^{1,2}(\Omega, \mathbb{R}^d)$ and let $b_\Omega(V,\mu)$ be a bilinear defined on $\mathcal{V}(\Omega) \times \mathcal{V}(\Omega)$ stisfying the following conditions.

[A1] $b_\Omega(V,\mu) \leq \alpha_8 \|V\|_{1,2,\Omega} \|\mu\|_{1,2,\Omega}$ for all $V, \mu \in \mathcal{V}(\Omega)$ with a constant $\alpha_8 > 0$.

[A2] $b_\Omega(V,V) \geq \alpha_9 \|V\|_{1,2,\Omega}^2$ for all $V \in \mathcal{V}(\Omega)$ with a constant $\alpha_9 > 0$.

Consider the variational problem $\Pi(u, f, \Omega)$: Under the condition of Theorem 6, find $V^o \in \mathcal{V}(\Omega)$ such that

$$b_\Omega(V^o, \mu) = R_\Omega(u, \mu) + \int_{\partial\Omega} f \cdot u(\mu \cdot n) ds \quad \text{for all } \mu \in \mathcal{V}(\Omega) \tag{10}$$

The mapping $\varphi_t(x) = x + tV^o(x)$ from Ω to $\mathbb{R}^d$ is 1-1 if t is near 0. Unfortunately, $[\mu \mapsto R_\Omega(u,\mu)]$ is linear functional on $W^{1,\infty}(\Omega, \mathbb{R}^d)$, and is not on $W^{1,2}(\Omega, \mathbb{R}^d)$. To extend it on $W^{1,2}(\Omega, \mathbb{R}^d)$, we need slightly smoothness of u.

Proposition 7 *If $d = 2$ or 3, and the solution u of $P(f, V(\Omega, \Gamma_D))$ is in $W^{1,2p}(\Omega, \mathbb{R}^m)$, then there is a constant $\alpha_{10} > 0$ such that*

$$R_\Omega(u,\mu) + \int_{\partial\Omega} f \cdot u(\mu \cdot n) ds \leq \alpha_{10} \|\mu\|_{1,2,\Omega} \quad \textit{for all } \mu \in W^{1,\infty}(\Omega, \mathbb{R}^d)$$

If $u \in W^{1,2p}(\Omega, \mathbb{R}^m)$, then there is a unique solution $V^o \in W^{1,2}(\Omega, \mathbb{R}^d)$ of $\Pi(u, f, \Omega)$. If $V^o \in W^{1,\infty}(\Omega, \mathbb{R}^d)$, then $[t \mapsto \varphi_t(x) = x + tV^o(x)] \in C^1\left([0,\epsilon), W^{1,\infty}(\Omega, \mathbb{R}^d)\right)$. So we can apply Theorem 6

$$\begin{aligned}\mathcal{E}(u(t); f, \Omega(t)) - \mathcal{E}(u; f, \Omega) &= -t\left\{R_\Omega(u, V^o) + \int_\Omega f \cdot u\,(V^o \cdot u)\,ds\right\} + o(t) \\ &= -tb_\Omega(V^o, V^o) + o(t) \\ &\leq -t\alpha_9 \|V^o\|^2_{1,2,\Omega} + o(t)\end{aligned}$$

2.2 Numerical examples

We now check the method for the following simple two cases: We start from the initial shapes, $\Omega^0 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$,
$\Gamma_D^0 = \{(\cos\theta, \sin\theta) : 0 < \theta < \pi\}$, $\Gamma_N^0 = \{(\cos\theta, \sin\theta) : \pi < \theta < 2\pi\}$.

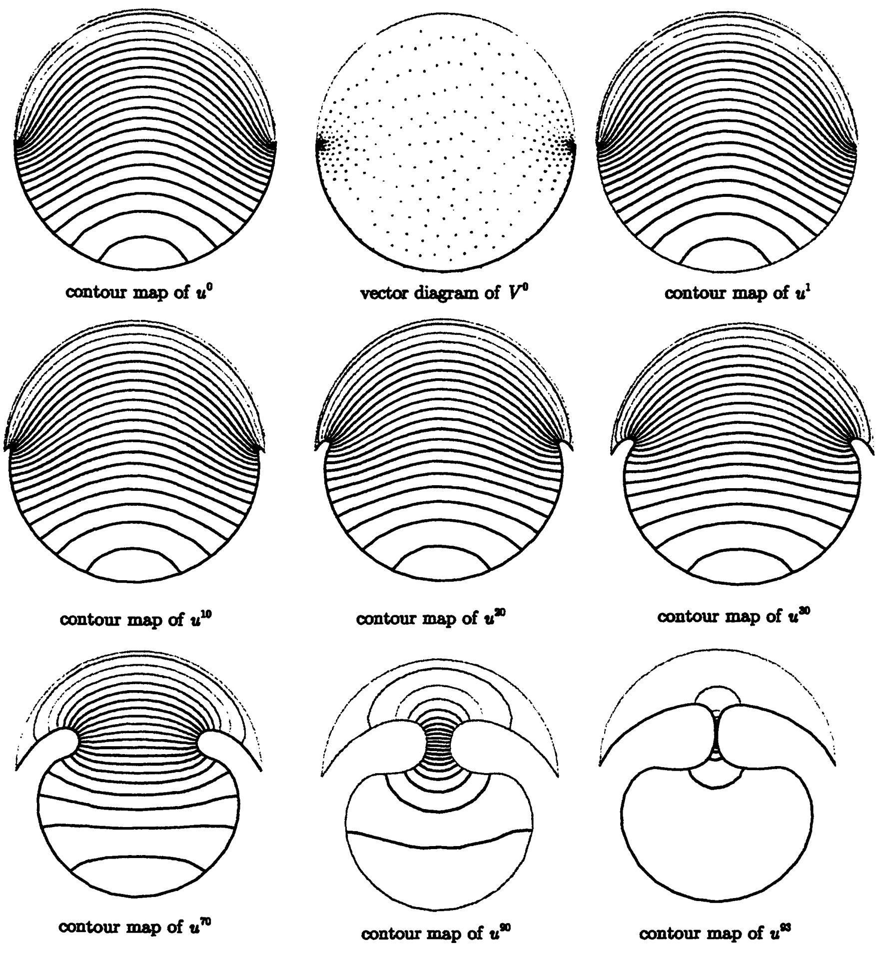

Figure 1: Optimization process under the condisions $|\Omega^i| = \pi$ and $\Gamma_D^i = \Gamma_D$

When Ω^i was already obtained, find a shape Ω^{i+1} such that

$$\begin{aligned} \mathcal{E}(u^{i+1}; f, \Omega^{i+1}) &< \mathcal{E}(u^i; f, \Omega^i) \\ \mathcal{E}(v; f, \Omega^i) &= \int_{\Omega^i} \left\{ \frac{1}{2}|\nabla v|^2 - fv \right\} dx \\ \mathcal{E}(u^i; f, \Omega^i) &= \min_{v \in V(\Omega^i, \Gamma_D^i)} \mathcal{E}(v; f, \Omega^i) \end{aligned}$$

under the condition: $|\Omega^i| = \pi, i = 0, 1, \cdots$, where $|\Omega^i|$ denotes the area of Ω^i. We find Ω^{i+1} by Azegami's method using the vector field V^i calculated by (10) and for some $t_0^i > 0$

$$\Omega^{i+1} = \{x + t_0^i V^i(x) : x \in \Omega^i\}$$

In the first example, $f = 0.5$, Γ_D is fixed and Γ_N is changeable, so we use for $\mathcal{V}(\Omega)$ in (10), the following

$$\mathcal{V}(\Omega) = \{A \in W^{1,2}(\Omega, \mathbb{R}^d) : A = 0 \qquad \text{on } \Gamma_D\}$$

We get the shapes in Fig.1 with finite element programming language FreeFem++ [3].

At the initial stage of the optimization, the stress concentrations at points $\gamma_1 = (1,0), \gamma_2 = (-1,0)$ are weaken by making a small circular hole near γ_1 and γ_2. The optimization is going to be divided to two parts Ω_D^o and Ω_N^o in which Ω_D^o ruled by Γ_D and Ω_N^o ruled by Neumann boundary condition.

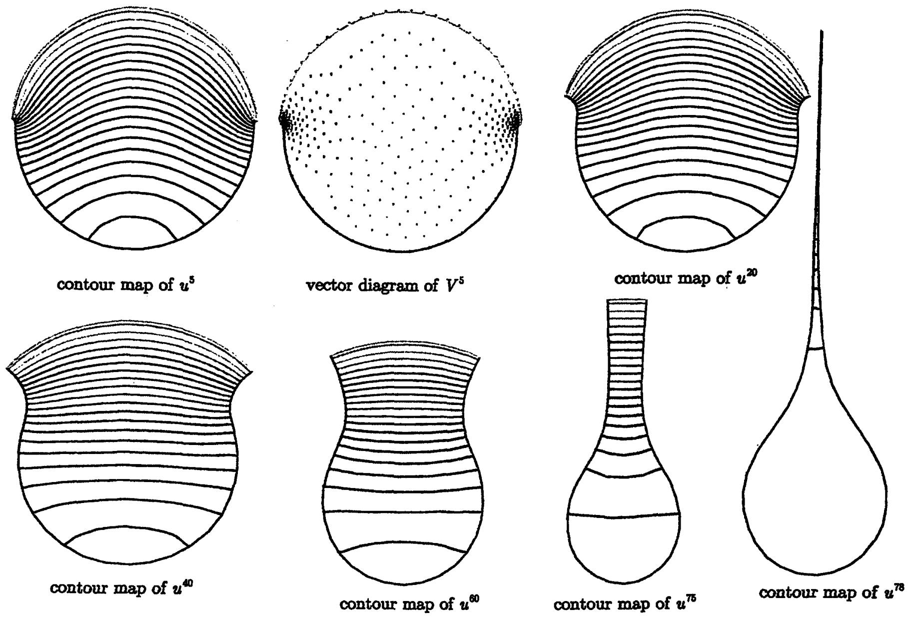

Figure 2: Optimization process under the condisions $|\Omega^i| = \pi$, in the condition to permit a change of Γ_D^i

In the second example, find the optimum shape under the conditions $f = 0.5$, and that Γ_D is changeable. The results are in Fig. 2, and we see that $|\Gamma_D^{i+1}| < |\Gamma_D^i|$ where

$|\Gamma_D^i|$ is the length of the curve Γ_D^i. It's natural because

$$\min_{v\in V(\Omega,\Gamma_D^1)} \mathcal{E}(v;f,\Omega) \le \min_{w\in V(\Omega,\Gamma_D^2)} \mathcal{E}(w;f,\Omega) \quad \text{if } \Gamma_D^1 \subset \Gamma_D^2$$

3 Application to shape optimization (general form)

We now consider the cost functional as follows: For the solution u of $\mathrm{P}(f, V(\Omega, \Gamma_D))$,

$$\mathcal{J}^o(u,\Omega) = \int_\Omega g(u)\,dx$$

3.1 Shape derivative of the solution

In this section, we limit $\mathrm{P}(f, V(\Omega, \Gamma_D))$ to linear case, because the adjoint problem will be used. The following is the main theorem in the section.

Theorem 8 *For any $\vartheta \in C_0^\infty(\Omega;\mathbb{R}^m)$, let u_ϑ be the solution of $P(\vartheta, V(\Omega,\Gamma_D))$. Then we have*

$$\frac{d}{dt}\int_{\Omega(t)} u(t)\cdot\vartheta dx\bigg|_{t=0} = \delta R_\Omega(u,u_\vartheta;\mu_\varphi) + \int_{\partial\Omega} f\cdot u_\vartheta(\mu_\varphi\cdot n)\,ds$$

where $\delta R_\Omega(u,u_\vartheta;\mu_\varphi) = \lim_{\epsilon\to 0}\epsilon^{-1}\{R_\Omega(u+\epsilon u_\vartheta;\mu_\varphi) - R_\Omega(u;\mu_\varphi)\}$. By the estimation

$$\left|\delta R_\Omega(u,u_\vartheta;\mu_\varphi) + \int_{\partial\Omega} f\cdot u(\mu_\varphi\cdot n)\right| \le C_3\|f\|_{1,2,\Omega}\|\vartheta\|_{0,2,\Omega}\|\mu_\varphi\|_{1,\infty,\Omega}$$

and the result that $C_0^\infty(\Omega;\mathbb{R}^m)$ is dense in $W^{0,2}(\Omega;\mathbb{R}^m) = L^2(\Omega;\mathbb{R}^m)$, so $t^{-1}(u(t)\circ\varphi_t - u)$ converges weakly in $L^2(\Omega;\mathbb{R}^m)$ and

$$\frac{d}{dt}\int_{\Omega(t)} u(t)\cdot\vartheta dx\bigg|_{t=0} = \int_\Omega\left(\dot{u} - \mu_\varphi\cdot\nabla u\right)\vartheta dx,\ \dot{u} = \lim_{t\to 0} t^{-1}(u(t)\circ\varphi_t - u)$$

See [8] for the proof.

If $[z\mapsto g(z)] \in W^{1,2}(\mathbb{R}^m;\mathbb{R})$, then $[u(t)\to g(u(t))] \in W^{1,2}(\Omega(t);\mathbb{R})$ and

$$\begin{aligned}\int_{\Omega(t)} g(u(t))\,dx - \int_\Omega g(u)\,dx &= \int_\Omega\{g(u(t)\circ\varphi_t)\kappa(\varphi_t) - g(u)\}\,dx\\ &= \int_\Omega\{(g(u(t)\circ\varphi_t) - g(u))\kappa(\varphi_t) + g(u)(\kappa(\varphi_t)-1)\}\,dx\end{aligned}$$

from which it follows that

$$\begin{aligned}&\frac{d}{dt}\int_{\Omega(t)} g(u(t))\,dx\bigg|_{t=0} = \int_\Omega\{\nabla_z g(u)\,\dot{u} + g(u)\operatorname{div}\mu_\varphi\}\,dx\\ &= \int_\Omega\{\nabla_z g(u)\,\dot{u} - \nabla_z g(u)\,(\mu_\varphi\cdot\nabla u)\}\,dx + \int_{\partial\Omega} g(u)(\mu_\varphi\cdot n)ds\\ &= \int_\Omega \nabla_z g(u)\,u' dx + \int_{\partial\Omega} g(u)(\mu_\varphi\cdot n)ds\end{aligned}$$

Proposition 9 *Let u_g be the solution of $P(\nabla_z g(u), V(\Omega, \Gamma_D))$, then*

$$\int_\Omega \nabla_z g(u)\, u' dx = \delta R_\Omega (u, u_g; \mu_\varphi) + \int_{\partial\Omega} f \cdot u_g (\mu_\varphi \cdot n)\, ds$$

which implies

$$\frac{d}{dt} \int_{\Omega(t)} g(u(t))\, dx \bigg|_{t=0} = \delta R_\Omega (u, u_g; \mu_\varphi) + \int_{\partial\Omega} (f \cdot u_g + g(u)) (\mu_\varphi \cdot n)\, ds \qquad (11)$$

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