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Random walk in a finite directed graph subject to a road coloring

Kouji Yano[(1)(2)]

Abstract

A necessary and sufficient condition for a random walk in a finite directed graph subject to a road coloring to be measurable with respect to the driving process is proved to be that the road coloring is synchronizing. The key to the proof is to find out a hidden symmetry in the non-synchronizing case.

Keywords and phrases: Stochastic difference equation, strong solution, random walk on directed graph, road coloring.
AMS 2010 subject classifications: Primary 60J10; secondary 60B15; 05C81; 37H10.

1 Introduction

Let us consider a finite directed graph of constant outdegree. See, for example, Figure 1 below; there are five sites and from each site there are two oneway roads laid. Let us color every road blue or red so that no two roads running from the same site have the same color. See, for example, Figure 2 below; the thick roads are colored red and the thin ones blue.

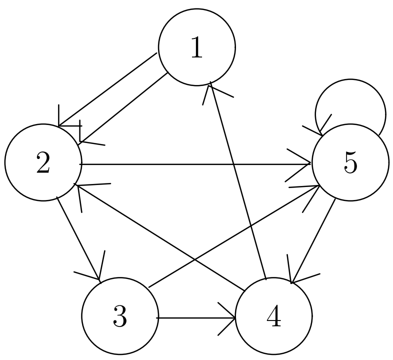

Figure 1.

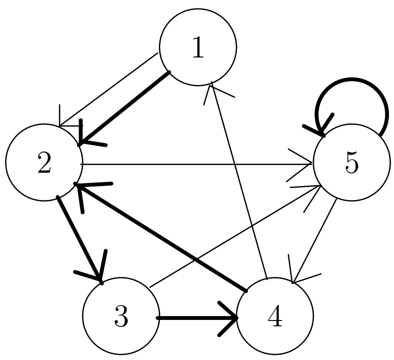

Figure 2.

We call N a *random color* if N is a random variable which takes values in the set of road colors; in this case, red and blue. To put it roughly, we mean by *random walk* a pair of processes $\{X, N\}$ where $N = (N_k)_{k\in\mathbb{Z}}$ is a sequence of colors which are independent and identically distributed (abbreviated as IID) and $X = (X_k)_{k\in\mathbb{Z}}$ is a site-valued process which moves at each step from X_{k-1} to X_k being driven by the random color N_k. We have the following table for instance:

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k	k_0-5	k_0-4	k_0-3	k_0-2	k_0-1	k_0
N_k		blue	red	blue	blue	red
X_k	1	2	3	5	4	2
X_k	2	5	5	4	1	2
X_k	3	5	5	4	1	2
X_k	4	1	2	5	4	2
X_k	5	4	2	5	4	2

In this table, we let k_0 be a certain time and we assume that at the 5 steps to the time $k = k_0$ the random colors are blue, red, blue, blue and red, in this order. Then, no matter how the process X moves before $k_0 - 5$, the value of X_{k_0} falls 2, and therefore the values of X afterward can be known from the values of N. We are interested in the necessary and sufficient condition that we can always know the values of X only from the values of N.

1.1 Random walk subject to a road coloring

Let V be a set of finite symbols. A matrix $[A(y,x)]_{y,x\in V}$ whose entries are non-negative integers is called an *adjacency matrix* and the pair (V,A) a *directed graph*. Note that there may be multiple edges which are not distinguished from each other. We call each element of V a *site*. For each $y,x \in V$, the value of $A(y,x)$ may be regarded as the number of *(oneway) roads* from x to y. (We prefer to write $A(y,x)$ than write $A(x,y)$.)

From each site $x \in V$ there are as many roads as $\deg(x) := \sum_{y\in V} A(y,x)$. The graph (V,A) or the adjacency matrix A is called *d-out* if $\deg(x) \equiv d$; in other words, there are d roads from each site. It is called *of constant outdegree* if it is d-out for some d. Write

$$\Sigma = \text{the set of all mappings from } V \text{ to itself.} \tag{1.1}$$

For $\sigma_1, \sigma_2 \in \Sigma$ and $x \in V$, we write $\sigma_2\sigma_1 x$ simply for $\sigma_2(\sigma_1(x))$. The semigroup Σ acts on V as follows:

$$(\sigma_1\sigma_2)x = \sigma_1(\sigma_2 x), \quad \sigma_1, \sigma_2 \in \Sigma,\ x \in V. \tag{1.2}$$

Each element $\sigma \in \Sigma$ may be identified with the 1-out adjacency matrix $[\sigma(y,x)]_{y,x\in V}$ via equation

$$\sigma(y,x) = 1_{\{y=\sigma x\}}. \tag{1.3}$$

An adjacency matrix A of constant outdegree admits a family $C = \{\sigma^{(1)}, \dots, \sigma^{(d)}\}$ of Σ (possibly with repeated elements) such that

$$A = \sigma^{(1)} + \cdots + \sigma^{(d)}. \tag{1.4}$$

Such a family C will be called a *road coloring*, because C indicates one of the ways of coloring the d roads running from each site so that no two roads from the same site have the same color.

Let μ be a probability law on Σ. We write $\mathrm{Supp}(\mu)$ for the support of μ, i.e.,

$$\mathrm{Supp}(\mu) = \{\sigma \in \Sigma : \mu(\sigma) > 0\}. \tag{1.5}$$

Enumerating $\mathrm{Supp}(\mu)$ as $\{\sigma^{(1)}, \dots, \sigma^{(d)}\}$, we define the adjacency matrix A by (1.4) so that $\mathrm{Supp}(\mu)$ is a road coloring of (V, A). The resulting directed graph (V, A) is called the *directed graph induced by* μ. Now we introduce random walk in a directed graph indexed by $\mathbb{Z} := \{\dots, -1, 0, 1, \dots\}$ as follows.

Definition 1.1. Let μ be a probability law on Σ. A pair of processes $\{X, N\}$ defined on a probability space is called a *μ-random walk* if $X = (X_k)_{k\in\mathbb{Z}}$ and $N = (N_k)_{k\in\mathbb{Z}}$ are processes taking values in V and Σ, respectively, such that the following statements hold:

(i) N_k is independent of $\sigma(X_j, N_j : j \le k-1)$ for each $k \in \mathbb{Z}$;

(ii) $N = (N_k)_{k\in\mathbb{Z}}$ is IID with common law μ;

(iii) it holds that

$$X_k = N_k X_{k-1}, \quad \text{a.s.,} \ \ k \in \mathbb{Z}. \tag{1.6}$$

Let (V, A) denote the directed graph induced by μ. The process $X = (X_k)_{k\in\mathbb{Z}}$ moves at each step from a site to another in (V, A), being driven by the randomly-chosen road colors indicated by $N = (N_k)_{k\in\mathbb{Z}}$ via equation (1.6). This is why we call such a process $\{X, N\}$ a random walk. Note that our definition is different from the one in a lot of literatures; see [20] and references therein.

Following [24], we adopt the following definition.

Definition 1.2. A μ-random walk $\{X, N\}$ is called *strong* if X_k is a.s. measurable with respect to $\sigma(N_j : j \le k)$ for all $k \in \mathbb{Z}$; or equivalently, there exist measurable mappings $f_k : \Sigma^{\mathbb{N}} \to V$ such that

$$X_k = f_k(N_k, N_{k-1}, \dots) \quad \text{a.s. for all } k \in \mathbb{Z}. \tag{1.7}$$

The purpose of this paper is to investigate a necessary and sufficient condition for the μ-random walk to be strong.

1.2 Main theorem

Let $A^1(y,x) = A(y,x)$ and define $A^n(y,x) = \sum_{z\in V} A^{n-1}(y,z)A(z,x)$, $y, x \in V$ recursively for $n \ge 2$. A directed graph (V, A) is called *strongly-connected* if for any $y, x \in V$ there exists $n = n(y,x) \ge 1$ such that $A^n(y,x) \ge 1$; or in other words, one can walk from every site to every other site. The graph (V, A) is called *aperiodic* if the *period* at $x \in V$, i.e., the greatest common divisor among $\{n \ge 1 : A^n(x,x) \ge 1\}$, is one for all $x \in V$. Note that (V, A) is both strongly connected and aperiodic if and only if there exists a positive integer r such that $A^r(y,x) \geq 1$ for all $y, x \in V$. We say that the directed graph (V, A) *satisfies the assumption* **(A)** if it is of constant outdegree, strongly-connected, and aperiodic. We will prove as Theorem 2.8 that under the assumption **(A)** there exists a unique μ-random walk $\{X, N\}$ which is stationary.

For $\sigma \in \Sigma$, we write $\sigma V = \{\sigma x : x \in V\}$. A road coloring C is called *synchronizing* if there exists a sequence $s = (\sigma_p, \ldots, \sigma_1)$ of road colors such that the composition $\langle s \rangle := \sigma_p \cdots \sigma_1$ maps V onto a singleton; or in other words, those who walk in the directed graph being driven by the road colors $\sigma_1, \ldots, \sigma_p$ in this order will arrive at a common site.

Now one of our main results is as follows.

Theorem 1.3. *Let μ be a probability law on Σ and let $\{X, N\}$ be a μ-random walk. Suppose that the directed graph induced by μ satisfies the assumption* **(A)**. *Then the following three assertions are equivalent:*

(i) $\mathrm{Supp}(\mu)$ *is synchronizing.*

(ii) *The limit* $\lim_{l \to -\infty} N_k N_{k-1} \cdots N_{l+1}$ *exists a.s. for all* $k \in \mathbb{Z}$.

(iii) *The μ-random walk $\{X, N\}$ is strong.*

Theorem 1.3 will be proved in Section 4. Note that the most difficult part of Theorem 1.3 is to show that (iii) implies (i). We shall prove the contraposition:

$$\text{If } \mathrm{Supp}(\mu) \text{ is non-synchronizing, then the } \mu\text{-random walk } \{X, N\} \text{ is non-strong.} \quad (1.8)$$

Remark 1.4. Rosenblatt [15] (see also [13]) showed that the subsemigroup generated by the support of a limit law of infinite convolution product of a probability law on a compact semigroup is completely simple. Thus of particular interest is to study the case where the support of μ generates a completely simple subsemigroup of Σ via its Rees decomposition. In this particular case, the remarkable result by Mukherjea–Sun [14, Theorem 3.1] gives a necessary and sufficient condition for the almost sure convergence of product of independent random variables. It must have something to do with our Theorem 1.3, but we do not proceed in this direction in this paper.

1.3 A typical sufficient condition

To prove (1.8), we need to find some extra randomness which is not measurable with respect to $\sigma(N_j : j \leq k)$. The key to the proof is to reveal a certain symmetry, or to put it more precisely, to construct another random walk from the original random walk by stopping it at certain stopping times and the problem is then reduced to the proposition given as follows.

Let $\mathfrak{S}(V)$ denote the permutation group of V, which may be regarded as a subgroup of Σ.

Proposition 1.5. *Let* $\{X, N\}$ *be a* μ*-random walk. Suppose that* $\mathrm{Supp}(\mu)$ *is contained in* $\mathfrak{S}(V)$ *and that*

$$\mu^{*n}\left(\sigma \in \Sigma : \sigma(i) = j\right) \underset{n\to\infty}{\longrightarrow} \frac{1}{\sharp(V)}, \quad i, j \in V \tag{1.9}$$

where $\sharp(V)$ *stands for the number of elements of* V *and* μ^{*n} *for the* n*-times convolution of* μ*. Then the stationary law of* X *is uniform law on* V *and that* $\{X, N\}$ *is non-strong.*

Proof. Let λ denote the stationary law for the process $X = (X_k)_{k\in\mathbb{Z}}$. Since $X_0 = N_0 N_{-1} \cdots N_{-n+1} X_{-n}$, we have

$$\lambda(j) = P(X_0 = j) = \sum_{i\in V} P(N_0 N_{-1} \cdots N_{-n+1}(i) = j) P(X_{-n} = i) \tag{1.10}$$

$$= \sum_{i\in V} \mu^{*n}\left(\sigma \in \Sigma : \sigma(i) = j\right) \lambda(i) \tag{1.11}$$

$$\underset{n\to\infty}{\longrightarrow} \frac{1}{\sharp(V)} \sum_{i\in V} \lambda(i) = \frac{1}{\sharp(V)}. \tag{1.12}$$

Let f be an arbitrary function on V and let l be a negative integer. Since $X_0 = N_0 N_{-1} \cdots N_{l+1} X_l$, we have

$$E[f(X_0)|\sigma(N_j : j \geq l+1)] = E[f(\sigma X_l)]|_{\sigma = N_0 N_{-1}\cdots N_{l+1}} \tag{1.13}$$

$$= \left. \int_V f(\sigma x)\lambda(\mathrm{d}x) \right|_{\sigma = N_0 N_{-1}\cdots N_{l+1}} \tag{1.14}$$

Since $\mathrm{Supp}(\mu) \subset \mathfrak{S}(V)$ and since the uniform law on V is $\mathfrak{S}(V)$-invariant, we see that

$$E[f(X_0)|\sigma(N_j : j \geq l+1)] = \int_V f(x)\lambda(\mathrm{d}x) = E[f(X_0)]. \tag{1.15}$$

Letting $l \to -\infty$, we have

$$E[f(X_0)|\sigma(N_j : j \in \mathbb{Z})] = E[f(X_0)]. \tag{1.16}$$

This shows that X_0 is independent of $\sigma(N_j : j \in \mathbb{Z})$. □

1.4 Backgrounds

Let us give a brief remark on the backgrounds of this study.

a). Road coloring problem. What we call the *road coloring problem* is the following:

Does a directed graph admit at least one synchronizing road coloring? (1.17)

This problem was first posed by Adler–Goodwyn–Weiss [1] (see also [2]) in the context of the isomorphism problem of symbolic dynamics with a common topological entropy. It was solved only recently by Trakhtman(1) [18] after important contributions by Friedman [11] and Culik–Karhumäki–Kari [10], so that the problem is now a theorem:

(1) Trahtman ([18],[19]) = Trakhtman (as he now spells his name).

Theorem 1.6 ([11],[10],[18]). *A directed graph which satisfies the assumption* **(A)** *admits at least one synchronizing road coloring.*

See also [8], [6] and [5] for some other developments before Trakhtman [18].

b). Tsirelson's equation in discrete time. Stochastic equation (1.6) is related to the study [9] by Tsirelson[(1)], who introduced, in order to construct an example of a stochastic differential equation which has a non-strong solution, a stochastic equation indexed by the negative integer

$$X_k = N_k + X_{k-1}, \quad k \in -\mathbb{N}, \tag{1.18}$$

where X and N take values in the one-dimensional torus $\mathbb{R}/\mathbb{Z}$.

Yor [24] obtained a necessary and sufficient condition in terms of the law of the driving process for a strong solution of equation (1.18) to exist. Hirayama–Yano [12] studied Tsirelson's equation in discrete time

$$X_k = N_k X_{k-1}, \quad k \in -\mathbb{N}, \tag{1.19}$$

for processes $(X_k)_{k\in-\mathbb{N}}$ and $(N_k)_{k\in-\mathbb{N}}$ taking values in a compact group and obtained a necessary and sufficient condition in terms of infinite product of the driving process for a strong solution of equation (1.19) to exist.

For other contributions of Tsirelson's equation in discrete time, see Akahori–Uenishi–Yano [3] and Takahashi [17]. Several reviews on this topic can be found in [21] and [23].

c). Finite-state Markov chain. As we shall see later in Theorem 2.8, for a μ-random walk $\{X, N\}$, the process X is a finite-state Markov chain which is stationary, irreducible and aperiodic. Yano–Yasutomi [22] studied its converse and proved the following: Any finite-state Markov chain which is stationary, irreducible and aperiodic can be realized as a μ-random walk subject to a synchronizing road coloring.

1.5 Organization of this paper

The remainder of this paper is organized as follows. In Section 2, we introduce some more notations and discuss existence and uniqueness of μ-random walks. In Section 3, we give several examples which help the reader to understand our main theorems deeply. Section 4 is devoted to the proof of Theorem 1.3. In Section 5, we discuss periodic case.

[(1)] Cirel′son ([9]) = Tsirel′son ([24]) = Tsirelson (as he now spells his name).

2 Notations and preliminary facts

2.1 Directed graphs and their road colorings

If the set V consists of m elements, we may and do write $V = \{1, \dots, m\}$. We shall idenfity $i \in V$ with v_i where $\{v_1, \dots, v_m\}$ is the standard basis of $\mathbb{R}^m$ defined as

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ v_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \dots, \ v_m = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}. \tag{2.1}$$

We remark that the product σx has two meanings: one is the image of the site $x \in V$ by the mapping $\sigma \in \Sigma$, and the other is the usual product among an $m \times m$-matrix $[\sigma(y, x)]_{y,x \in V}$ and a m-vector (or $m \times 1$-matrix) x in $\mathbb{R}^m$.

The identity mapping e is identified with the identity matrix. We write

$$o_1 = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \ o_2 = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \dots, \ o_m = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix}. \tag{2.2}$$

The set Σ is a subsemigroup of the semigroup consisting of all $m \times m$ matrices, but it is not a group, because the elements $o_1, \dots, o_m$ do not possess its inverse in Σ; in fact, we have

$$o_i x = o_i, \quad x \in V, \ i = 1, \dots, m. \tag{2.3}$$

Let Σ_0 be a subset of Σ. A sequence $s = (\sigma_p, \dots, \sigma_1)$ of Σ_0 is called a *word in* Σ_0, and then $\langle s \rangle$ denotes the product $\sigma_p \cdots \sigma_1 \in \Sigma$. We note that the road coloring Σ_0 is synchronizing if and only if $\langle s \rangle = o_i$ for some word s in Σ_0 and some $i = 1, \dots, m$. To study the non-synchronizing cases, we adopt the following terminology due to [18] for (ii) and to [10] for (iii).

Definition 2.1. Let Σ_0 be a subset of Σ. Let V_0 be a subset of V.

(i) V_0 is called *synchronizing* if $\langle s \rangle V_0$ is a singleton for some word s in Σ_0.

(ii) V_0 is called a *deadlock* if V_0 has no synchronizing pair.

(iii) V_0 is called *stable* if the subset $\langle s \rangle V_0$ is synchronizing for all word s in Σ_0.

Note that there may exist a pair which is synchronizing but non-stable; an example will be given in Section 3.3. We also adopt the following terminology due to [18].

Definition 2.2. A subset V_0 of V is called an *F-clique* if V_0 is a deadlock and is of the form $V_0 = \langle s \rangle V$ for some word s in Σ_0.

We write $\sharp(\cdot)$ for the cardinality and set

$$\widehat{m} = \min\{\sharp(\langle s \rangle V) : s \text{ is a word in } \Sigma_0\}. \tag{2.4}$$

The following lemma, restated using Trakhtman's terminology in [18], can be found in Friedman [11].

Lemma 2.3 ([11],[18]). *The following assertions hold:*

(i) *If $V_0 = \langle s \rangle V$ for some word s in Σ_0 such that $\widehat{m} = \sharp(\langle s \rangle V)$, then V_0 is an F-clique.*

(ii) *For any F-clique V_0 and for all word s in Σ_0, the subset $\langle s \rangle V_0$ is also an F-clique and satisfies $\sharp(V_0) = \sharp(\langle s \rangle V_0)$.*

(iii) *Every F-clique has $\widehat{m}$ elements.*

Let us give the proof of Lemma 2.3 for completeness of this paper.

Proof. (i) Suppose that $\langle s \rangle V$ had a synchronizing pair. Then there would exist another word s' in Σ_0 such that $\sharp(\langle s's \rangle V) < \sharp(\langle s \rangle V) = \widehat{m}$. This contradicts the minimality of $\widehat{m}$. Thus we obtain (i).

(ii) Let V_0 be an F-clique and s be a word in Σ_0. Suppose $\langle s \rangle V_0$ were not an F-clique. Then it would admit a synchronizing pair, and so would V_0. This is a contradiction.

Suppose that $\sharp(V_0) > \sharp(\langle s \rangle V_0)$. Then V_0 would admit a synchronizing pair, which is again a contradiction.

Hence we obtain (ii).

(iii) Let V_0 be an F-clique and s be a word in Σ_0 such that $\widehat{m} = \sharp(\langle s \rangle V)$. Then, by (ii), we see that

$$\widehat{m} \leq \sharp(V_0) = \sharp(\langle s \rangle V_0) \leq \sharp(\langle s \rangle V) = \widehat{m}, \tag{2.5}$$

which shows that $\widehat{m} = \sharp(V_0)$. □

2.2 Random walks

We deal with a pair of two processes $X = (X_k)_{k\in\mathbb{Z}}$ and $N = (N_k)_{k\in\mathbb{Z}}$ defined on a common probability space. We need the σ-fields generated by these processes up to time $k \in \mathbb{Z}$ given by

$$\mathcal{F}_k^X = \sigma(X_j : j \leq k), \quad \mathcal{F}_k^N = \sigma(N_j : j \leq k), \quad \mathcal{F}_k^{X,N} = \sigma(X_j, N_j : j \leq k). \tag{2.6}$$

We also need the following σ-fields for each $k, l \in \mathbb{Z}$ with $k > l$:

$$\mathcal{F}_{k,l}^X = \sigma(X_j : j = k, k-1, \dots, l+1), \quad \mathcal{F}_{k,l}^N = \sigma(N_j : j = k, k-1, \dots, l+1) \tag{2.7}$$

and

$$\mathcal{F}_{k,l}^{X,N} = \sigma(X_j, N_j : j = k, k-1, \dots, l+1). \tag{2.8}$$

Definition 2.4. Two pairs of processes $\{X, N\}$ and $\{X', N'\}$ are called *identical in law* if

$$((X_k)_{k\in\mathbb{Z}}, (N_k)_{k\in\mathbb{Z}}) \stackrel{\mathrm{d}}{=} ((X'_k)_{k\in\mathbb{Z}}, (N'_k)_{k\in\mathbb{Z}}) . \tag{2.9}$$

In this case, we write $\{X, N\} \stackrel{\mathrm{d}}{=} \{X', N'\}$.

Definition 2.5. A pair of processes $\{X, N\}$ is called *stationary* if, for any $n \in \mathbb{Z}$, $\{X, N\}$ and $\{(X_{k+n})_{k\in\mathbb{Z}}, (N_{k+n})_{k\in\mathbb{Z}}\}$ are identical in law.

Lemma 2.6. *Let $\{X, N\}$ and $\{X', N'\}$ be two pairs of processes. Then the following assertions hold:*

(i) *If $\{X, N\} \stackrel{\mathrm{d}}{=} \{X', N'\}$ and if $\{X, N\}$ is a μ-random walk (resp. stationary), then $\{X', N'\}$ is also a μ-random walk (resp. stationary).*

(ii) *If $\{X, N\}$ and $\{X', N'\}$ are μ-random walks, and if $X_k \stackrel{\mathrm{d}}{=} X'_k$ for all $k \in \mathbb{Z}$, then $\{X, N\} \stackrel{\mathrm{d}}{=} \{X', N'\}$.*

Proof. Claim (i) is obvious. Let us prove Claim (ii). Let $k_0 \in \mathbb{Z}$. Since $X_{k_0-1} \stackrel{\mathrm{d}}{=} X'_{k_0-1}$ and $N \stackrel{\mathrm{d}}{=} N'$ and since $X_k = N_k N_{k-1} \cdots N_{k_0} X_{k_0-1}$ for $k \geq k_0$, we see that $(X_k, N_k)_{k\geq k_0} \stackrel{\mathrm{d}}{=} (X'_k, N'_k)_{k\geq k_0}$. Since $k_0 \in \mathbb{Z}$ is arbitrary, we obtain $(X_k, N_k)_{k\in\mathbb{Z}} \stackrel{\mathrm{d}}{=} (X'_k, N'_k)_{k\in\mathbb{Z}}$. □

The *convolution* of two probability laws μ_1 and μ_2 on Σ will be denoted by $\mu_1 * \mu_2$, which is a probability law on Σ such that

$$(\mu_1 * \mu_2)(\sigma) = \sum_{\sigma_1\in\Sigma} \mu_1(\sigma_1) \sum_{\sigma_2\in\Sigma} \mu_2(\sigma_2) 1_{\{\sigma_1\sigma_2=\sigma\}}, \quad \sigma \in \Sigma. \tag{2.10}$$

For a probability law μ on Σ and λ on V, the convolution of μ and λ will also be denoted by $\mu * \lambda$, which is a probability law on V such that

$$(\mu * \lambda)(y) = \sum_{\sigma\in\Sigma} \mu(\sigma) \sum_{x\in V} \lambda(x) 1_{\{\sigma x=y\}}, \quad y \in V. \tag{2.11}$$

By (1.2), we see that

$$(\mu_1 * \mu_2) * \lambda = \mu_1 * (\mu_2 * \lambda). \tag{2.12}$$

We write $\mu^{*1} = \mu$ and define μ^{*n} for $n \geq 2$ recursively by $\mu^{*n} = \mu^{*(n-1)} * \mu$.

Lemma 2.7. *Let $\{X, N\}$ be a μ-random walk. For $k \in \mathbb{Z}$, let λ_k denote the law of X_k. Then the following convolution equation holds:*

$$\lambda_k = \mu * \lambda_{k-1}, \quad k \in \mathbb{Z}. \tag{2.13}$$

Conversely, if probability laws μ and $\{\lambda_k : k \in \mathbb{Z}\}$ are given and the convolution equation (2.13) is satisfied, then there exists a μ-random walk $\{X, N\}$ and X_k has law λ_k for all $k \in \mathbb{Z}$. The μ-random walk is unique up to identity in law.

Proof. Let $\{X, N\}$ be a μ-random walk. Since $X_k = N_k X_{k-1}$ and since N_k is independent of X_{k-1}, we obtain (2.13).

Let μ and $\{\lambda_k : k \in \mathbb{Z}\}$ be given such that (2.13) holds. Then, by the Kolmogorov extension theorem, we may construct a (possibly time-inhomogeneous) Markov chain $(N_k, X_{k-1})_{k\in\mathbb{Z}}$ with state space $V \times \Sigma$ so that the marginal law for each $k \in \mathbb{Z}$ is given as

$$P(N_k = \sigma, X_{k-1} = x) = \mu(\sigma)\lambda_{k-1}(x), \quad \sigma \in \Sigma,\ x \in V \tag{2.14}$$

and the one-step transition probabilities are given as

$$P(N_k = \varsigma, X_{k-1} = y \mid N_{k-1} = \sigma, X_{k-2} = x) = \mu(\varsigma)1_{\{y=\sigma x\}} \tag{2.15}$$

for all $\sigma, \varsigma \in \Sigma$ and all $x, y \in V$. It is easy to see that the so constructed pair of processes $\{(X_k)_{k\in\mathbb{Z}}, (N_k)_{k\in\mathbb{Z}}\}$ is as desired. The uniqueness is immediate from Lemma 2.6. $\square$

2.3 Aperiodic case

Since the index of our μ-random walk varies in $\mathbb{Z}$, existence and uniqueness of μ-random walks are not obvious. The following theorem assures the existence and uniqueness in the aperiodic case.

Theorem 2.8. *Let μ be a probability law on Σ and suppose that the directed graph induced by μ satisfies the assumption* **(A)**. *Then the following assertion holds:*

(i) *There exists a μ-random walk $\{X, N\}$ in V.*

(ii) *The μ-random walk is unique up to identity in law.*

(iii) *The μ-random walk $\{X, N\}$ is stationary.*

(iv) *The common law λ of $X = (X_k)_{k\in\mathbb{Z}}$ is a unique probability law satisfying*

$$\mu * \lambda = \lambda. \tag{2.16}$$

(v) *The tail σ-field $\mathcal{F}^{X,N}_{-\infty} := \bigcap_k \mathcal{F}^{X,N}_k$ is a.s. trivial.*

This theorem is an immediate consequence of the classical Perron–Frobenius theory on infinite product of stochastic matrices. We call $B = (B(y, x))_{x,y=1,\dots,m}$ a *stochastic matrix* if $B(y, x) \geq 0$ for $x, y \in V$ and

$$\sum_{y\in V} B(y, x) = 1, \quad x \in V. \tag{2.17}$$

We call a column vector $u = {}^\top\left[u(1) \quad \cdots \quad u(m)\right]$ a *stochastic vector* if $u(x) \geq 0$ for all $x = 1, \dots, m$ and $\sum_{x=1}^m u(x) = 1$.

Theorem 2.9 (Perron–Frobenius). *Let $B = (B(y,x))_{x,y=1,\dots,m}$ be a stochastic matrix and suppose that there exists a positive integer r such that every entry of B^r is positive. Then there exists a stochastic vector u such that*

$$B^n(y,x) \underset{n\to\infty}{\longrightarrow} u(y) \quad \text{for all } y = 1, \dots, m, \tag{2.18}$$

where $B^n(y,x)$ is the (y,x)-entry of B^n, i.e., $B^n = (B^n(y,x))_{x,y=1,\dots,m}$. The vector u is the unique stochastic vector such that

$$Bu = u. \tag{2.19}$$

For the proof of Theorem 2.9, see, e.g., [16].

Let us give the proof of Theorem 2.8 for completeness of this paper.

Proof of Theorem 2.8. (i) Define a $(V \times V)$-matrix $B = (B(y,x))_{x,y\in V}$ by

$$B = \sum_{\sigma\in\Sigma} \mu(\sigma)\sigma, \tag{2.20}$$

or in other words,

$$B(y,x) = \mu(\sigma \in \Sigma : \sigma x = y), \quad x, y \in V. \tag{2.21}$$

Note that B is a stochastic matrix and that $B(y,x)$ is positive if $A(y,x) \geq 1$. Since (V,A) satisfies the assumption **(A)**, there exists a positive integer r such that every entry of A^r is greater than or equal to 1, and hence that every entry of B^r is positive. Thus we may apply Theorem 2.9 to see that there exists a probability law λ on V such that

$$\mu^{*n}(\{\sigma \in \Sigma : \sigma x = y\}) \underset{n\to\infty}{\longrightarrow} \lambda(y), \quad x, y \in V, \tag{2.22}$$

and that λ is the unique probablity law such that

$$\mu * \lambda = \lambda. \tag{2.23}$$

By this convolution equation, we may apply Lemma 2.7 to construct $\{X, N\}$ such that N has common law μ and X has common law λ. This is as desired.

(ii) Let $\{X, N\}$ be a μ-random walk. For each $k \in \mathbb{N}$, let λ_k denote the law of X_k. Then we have

$$\lambda_k = \mu^{*(k-l)} * \lambda_l, \quad k, l \in \mathbb{Z},\ k > l. \tag{2.24}$$

Let $k \in \mathbb{Z}$ be fixed. Since V is finite, there exist a subsequence $l(n) \to -\infty$ and a probability law $\widetilde{\lambda}$ such that $\lambda_{l(n)} \xrightarrow{\mathrm{w}} \widetilde{\lambda}$. Then, for any $y \in V$, we have

$$\begin{aligned}
\lambda_k(y) =& \mu^{*(k-l(n))} * \lambda_{l(n)}(y) && (2.25)\\
=& \sum_{x\in V} \mu^{*(k-l(n))}(\{\sigma \in \Sigma : \sigma x = y\}) \lambda_{l(n)}(x) && (2.26)\\
\underset{n\to\infty}{\longrightarrow}& \lambda(y) \sum_{x\in V} \widetilde{\lambda}(x) = \lambda(y). && (2.27)
\end{aligned}$$

This shows that X_k has law λ for each $k \in \mathbb{Z}$. This proves uniqueness by Lemma 2.6.

Claims (iii) and (iv) have already been proved.

(v) Suppose that there were $A \in \mathcal{F}_{-\infty}^{X,N}$ such that $0 < P(A) < 1$. Define $P' = P(\cdot \mid A)$. Then it is easy to see that $\{X, N\}$ under P' is also a μ-random walk. Thus Claim (ii) shows that $P'((X,N) \in \cdot) = P((X,N) \in \cdot)$. This contradicts the fact that $P'(A) = 1 > P(A)$. This proves Claim (v).

The proof is now complete. $\square$

2.4 Proofs of easy parts of Theorem 1.3

Let us prove easy parts of Theorem 1.3.

Proof of [(i) $\Rightarrow$ (ii)] *of Theorem 1.3.* Suppose that (i) holds. Let $s = (\sigma_p, \ldots, \sigma_1)$ be a word in $\mathrm{Supp}(\mu)$ such that $\langle s \rangle V$ is a singleton. Define, for each $k \in \mathbb{Z}$, a random time

$$T(k) = \max\{l = k-1, k-2, \ldots : \ N_{l+p} = \sigma_p, \ldots, N_{l+1} = \sigma_1\}. \tag{2.28}$$

Here we note that the random time $k - T(k)$ is a stopping time with respect to the filtration $\{\mathcal{F}_{k,k-n}^N : n = 1, 2, \ldots\}$. Then we see that $T(k)$ is finite a.s. and that

$$\lim_{l \to -\infty} N_{k,l} = N_{k,T(k)+p} \quad \text{a.s.} \tag{2.29}$$

Thus we obtain (ii). $\square$

Proof of [(ii) $\Rightarrow$ (iii)] *of Theorem 1.3.* Suppose that (ii) holds. Denote

$$Y_k = \lim_{l \to -\infty} N_k N_{k-1} \cdots N_{l+1} \quad \text{a.s. for } k \in \mathbb{Z}. \tag{2.30}$$

Then, for any fixed $x_0 \in V$, a pair of processes

$$\{(Y_k x_0)_{k \in \mathbb{Z}}, (N_k)_{k \in \mathbb{Z}}\} \tag{2.31}$$

is a strong μ-random walk which is identical in law to $\{X, N\}$. This proves (iii). $\square$

3 Illustrative examples

Before proceeding to prove our main theorems, we give illustrative examples.

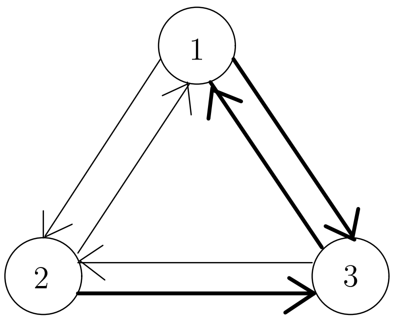

Figure 3.

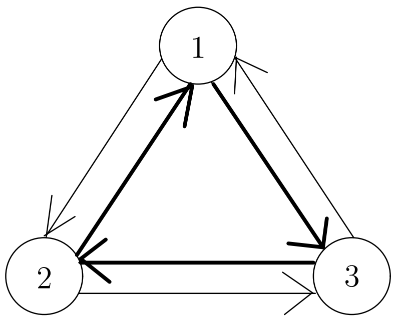

Figure 4.

3.1 Synchronizing case

Let $V = \{1, 2, 3\}$ and consider the following adjacency matrix:

$$A = \begin{bmatrix} A(1,1) & A(1,2) & A(1,3) \\ A(2,1) & A(2,2) & A(2,3) \\ A(3,1) & A(3,2) & A(3,3) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}. \tag{3.1}$$

We can easily verify that the graph (V, A) satisfies the assumption **(A)**. Consider a road coloring $\{\sigma^{(1)}, \sigma^{(2)}\}$ of (V, A) given as

$$\sigma^{(1)} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{bmatrix}, \quad \sigma^{(2)} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}. \tag{3.2}$$

See Figure 3, where the thick roads are colored $\sigma^{(1)}$ and the thin ones $\sigma^{(2)}$. Since

$$\sigma^{(1)}\sigma^{(2)} = o_3, \tag{3.3}$$

we see that the road coloring $\{\sigma^{(1)}, \sigma^{(2)}\}$ is synchronizing.

Let $p, q > 0$ with $p + q = 1$ and let μ be a probability law on Σ such that

$$\mu(\{\sigma^{(1)}\}) = p, \quad \mu(\{\sigma^{(2)}\}) = q. \tag{3.4}$$

Let $\{X, N\}$ be a μ-random walk in (V, A). Solving equation (2.16), we see that the common law λ of X is given as

$$\begin{bmatrix} \lambda(1) \\ \lambda(2) \\ \lambda(3) \end{bmatrix} = \frac{1}{2 + pq} \begin{bmatrix} 1 - pq \\ q + pq \\ p + pq \end{bmatrix} \tag{3.5}$$

where we write $\lambda(i)$ simply for $\lambda(\{i\})$, $i = 1, 2, 3$. For each $k \in \mathbb{Z}$, we have

$$T(k) = \inf\{n = 1, 2, \dots, : \ N_{k-n+1} = \sigma^{(1)}, \ N_{k-n+2} = \sigma^{(2)}\} \tag{3.6}$$

and then we have

$$X_k = N_{k,k-T(k)+2} v_3, \quad k \in \mathbb{Z}, \tag{3.7}$$

which shows that the μ-random walk $\{X, N\}$ is strong.

3.2 Non-synchronizing case: an easy example

Let $V = \{1, 2, 3\}$ and let A as defined in (3.1). Consider a road coloring $\{\sigma^{(1)}, \sigma^{(2)}\}$ of (V, A) given as

$$\sigma^{(1)} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad \sigma^{(2)} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{3.8}$$

See Figure 4, where the thick roads are colored $\sigma^{(1)}$ and the thin ones $\sigma^{(2)}$. Since

$$\sigma^{(1)}\sigma^{(2)} = \sigma^{(2)}\sigma^{(1)} = \mathrm{id}, \tag{3.9}$$

the set $G = \{\sigma^{(1)}, \sigma^{(2)}, \mathrm{id}\}$ is a group, and thus we see that the road coloring $\{\sigma^{(1)}, \sigma^{(2)}\}$ is non-synchronizing.

Let μ be a probability law on Σ such that $\mathrm{Supp}(\mu) = \{\sigma^{(1)}, \sigma^{(2)}\}$. Let $\{X, N\}$ be a μ-random walk in (V, A). Solving equation (2.16), we see that the common law λ of X is uniform on V, i.e.,

$$\begin{bmatrix} \lambda(1) \\ \lambda(2) \\ \lambda(3) \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \tag{3.10}$$

Now, by Proposition 1.5, we conclude that $\{X, N\}$ is non-strong.

3.3 Non-synchronizing case: a difficult example

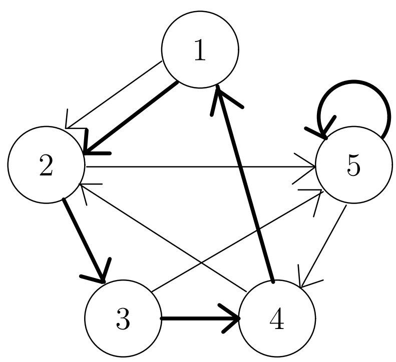

Figure 5.

Let $V = \{1, 2, 3, 4, 5\}$ and consider the following adjacency matrix:

$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}. \tag{3.11}$$

We can easily verify that the graph (V, A) satisfies the assumption **(A)**. Consider a road coloring $\{\sigma^{(1)}, \sigma^{(2)}\}$ of (V, A) given as

$$\sigma^{(1)} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \sigma^{(2)} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}, \tag{3.12}$$

See Figure 5, where the thick roads are colored $\sigma^{(1)}$ and the thin ones $\sigma^{(2)}$. It is easy to see that the F-cliques are

$$\{1, 3, 5\} \quad \text{and} \quad \{2, 4, 5\}. \tag{3.13}$$

In particular, the road coloring is non-synchronizing. The pairs

$$\{1, 2\}, \{1, 4\}, \{2, 3\}, \{3, 4\} \tag{3.14}$$

are all synchronizing. Since

$$\{1, 4\} \xrightarrow{\sigma^{(1)}} \{1, 2\} \xrightarrow{\sigma^{(2)}} \{2, 5\}, \quad \{2, 3\} \xrightarrow{\sigma^{(1)}} \{3, 4\} \xrightarrow{\sigma^{(2)}} \{2, 5\}, \tag{3.15}$$

and since $\{2, 5\}$ is a deadlock, we see that the pairs (3.14) are non-stable.

Let $p, q > 0$ with $p + q = 1$ and let μ be as defined in (3.4). Let $\{X, N\}$ be a μ-random walk in (V, A). Solving equation (2.16), we see that the common law λ of X is given as

$$\begin{bmatrix} \lambda(1) \\ \lambda(2) \\ \lambda(3) \\ \lambda(4) \\ \lambda(5) \end{bmatrix} = \frac{1}{3(1+p)} \begin{bmatrix} p \\ 1 \\ p \\ 1 \\ 1+p \end{bmatrix}. \tag{3.16}$$

Thus we may find the uniformity:

$$\lambda(\{1, 2\}) = \lambda(\{3, 4\}) = \lambda(\{5\}). \tag{3.17}$$

Note that this is a special case of Theorem 4.1 given in the next section. One may expect that some symmetry lies behind this uniformity, but it seems hidden because $\{1, 2\}$ and $\{3, 4\}$ cannot be interchanged. We will reveal a certain hidden symmetry behind this uniformity in the proof of Claim (1.8).

4 Non-strongness of the μ-random walk in the non-synchronizing case

This section is devoted to the proof of Claim (1.8), which will complete the proof of Theorem 1.3.

4.1 Uniformity

Suppose that $\{X, N\}$ and λ be as in Theorem 2.8. Let $\mathrm{Supp}(\mu) = \{\sigma^{(1)}, \dots, \sigma^{(d)}\}$ and suppose that $\mathrm{Supp}(\mu)$ is non-synchronizing. Then, by (i) of Lemma 2.3, there exists a word $s = (\sigma_p, \dots, \sigma_1)$ in $\mathrm{Supp}(\mu)$ such that $\langle s \rangle V$ is an F-clique. We enumerate $\langle s \rangle V$ as

$$\langle s \rangle V = \{\widehat{x}_1, \dots, \widehat{x}_{\widehat{m}}\} \tag{4.1}$$

where

$$\widehat{m} = \min\{\sharp(\langle s \rangle V) : s \text{ is a word in } \mathrm{Supp}(\mu)\}. \tag{4.2}$$

Since $\mathrm{Supp}(\mu)$ is non-synchronizing, we have

$$\widehat{m} \geq 2. \tag{4.3}$$

Set

$$V_i = \{x \in V : \ \langle s \rangle x = \widehat{x}_i\}, \quad i = 1, \dots, \widehat{m}. \tag{4.4}$$

Then the family $\{V_1, \dots, V_{\widehat{m}}\}$ is a partition of the state space V. Note that this partition of V may depend on the choice of the word s in $\mathrm{Supp}(\mu)$ such that $\langle s \rangle V$ is an F-clique. The following theorem is crucial to our proof of Claim (1.8), which does not matter whatever we choose as such a word s in $\mathrm{Supp}(\mu)$.

Theorem 4.1. *It holds that*

$$\lambda(V_1) = \cdots = \lambda(V_{\widehat{m}}) = \frac{1}{\widehat{m}}. \tag{4.5}$$

We shall postpone the proof of Theorem 4.1 until Section 4.3.

4.2 Constructing a permutation process

Denote $\widehat{V} = \{1, \dots, \widehat{m}\}$ and its permutation group by $\mathfrak{S}(\widehat{V})$. We decompose V into the disjoint union $\bigcup_{i=1}^{\widehat{m}} V_i$ where

$$V_i = \{x \in V : \langle s \rangle x = \widehat{x}_i\}, \quad i \in \widehat{V}. \tag{4.6}$$

By (ii) and (iii) of Lemma 2.3, we see that

$$\sharp(\langle s \rangle \sigma \langle s \rangle V) = \widehat{m} \quad \text{for any } \sigma \in \Sigma, \tag{4.7}$$

and hence

$$(\langle s \rangle \sigma)\{\widehat{x}_1, \dots, \widehat{x}_{\widehat{m}}\} = \{\widehat{x}_1, \dots, \widehat{x}_{\widehat{m}}\}. \tag{4.8}$$

This yields that there exists a mapping $M : \Sigma \ni \sigma \mapsto M[\sigma] \in \mathfrak{S}(\widehat{V})$ such that

$$M[\sigma](i) = j \quad \text{if and only if} \quad \sigma\widehat{x}_i \in V_j, \tag{4.9}$$

where $i, j \in \widehat{V}$.

Let $\{X, N\}$ be a μ-random walk in (V, A). Set $T_1 = 0$ and define $T_0, T_{-1}, \dots$ recursively by

$$T_\kappa = \max\{l \leq T_{\kappa+1} - p : N_{l+p} = \sigma_p, \dots, N_{l+2} = \sigma_2, N_{l+1} = \sigma_1\}. \tag{4.10}$$

By the second Borel–Cantelli lemma, we see that the decreasing sequence $(T_\kappa)_{\kappa \in -\mathbb{N}}$ is well-defined a.s. Note that, for any $\kappa \in -\mathbb{N}$ and any $l \in -\mathbb{N}$, we have

$$\{T_\kappa = l\} \in \mathcal{F}^N_{0,l}. \tag{4.11}$$

Now we define an $\mathfrak{S}(\widehat{V})$-valued process $(\widehat{N}_\kappa)_{\kappa \in -\mathbb{N}}$ as

$$\widehat{N}_\kappa = M\left[N_{T_\kappa} \cdots N_{T_{\kappa-1}+p+2} N_{T_{\kappa-1}+p+1}\right] \quad \text{if } T_{\kappa-1} < T_\kappa - p \tag{4.12}$$

and $\widehat{N}_\kappa =$ identity if $T_{\kappa-1} = T_\kappa - p$. We may write $\widehat{\mu}$ for the law of $\widehat{N}_0$ on $\mathfrak{S}(\widehat{V})$. Then it is obvious that $(\widehat{N}_\kappa)_{\kappa \in -\mathbb{N}}$ has common law $\widehat{\mu}$ since

$$N_{T_\kappa} \cdots N_{T_{\kappa-1}+p+2} N_{T_{\kappa-1}+p+1} \overset{\mathrm{d}}{=} N_0 \cdots N_{T_{-1}+p+2} N_{T_{-1}+p+1}. \tag{4.13}$$

By the aperiodicity assumption, there exists a constant r such that from any $x \in V$ to any $y \in V$ there exists a path of length r. For a technical reason, we introduce the following assumption:

$$\begin{aligned} p > r \quad & \text{and, for any } q = p-1, p-2, \dots, p-r, \\ & (\sigma_p, \sigma_{p-1}, \dots, \sigma_{p-q+2}, \sigma_{p-q+1}) \neq (\sigma_q, \sigma_{q-1}, \dots, \sigma_2, \sigma_1). \end{aligned} \tag{4.14}$$

To prove Theorem 4.1, we may assume (4.14) without loss of generality. For this, it suffices to replace s by another word $\widetilde{s}$ in $\mathrm{Supp}(\mu)$ defined as follows. Set $\widetilde{p} = p + 2r$ and

$$\widetilde{\sigma}_i = \begin{cases} \sigma_i & \text{if } i = 1, 2, \dots, p, \\ \sigma^{(1)} & \text{if } i = p+1, p+2, \dots, p+r, \\ \sigma^{(2)} & \text{if } i = p+r+1, p+r+2, \dots, p+2r, \end{cases} \tag{4.15}$$

and then define

$$\widetilde{s} = (\widetilde{\sigma}_{\widetilde{p}}, \dots \widetilde{\sigma}_2, \widetilde{\sigma}_1). \tag{4.16}$$

Then it is obvious that the sequence $\widetilde{s}$ satisfies (4.14). By (ii) and (iii) of Lemma 2.3, we see that $\langle \widetilde{s} \rangle V$ is also an F-clique, and that $\langle \widetilde{s} \rangle V = \{\widetilde{x}_1, \dots, \widetilde{x}_{\widehat{m}}\}$. Set

$$\widetilde{V}_i = \{x \in V : \langle \widetilde{s} \rangle x = \widetilde{x}_i\}, \quad i = 1, \dots, \widehat{m}. \tag{4.17}$$

We then note that $(\widetilde{V}_1, \dots, \widetilde{V}_{\widehat{m}})$ is a permutation of $(V_1, \dots, V_{\widehat{m}})$, which shows that the replacement of s by $\widetilde{s}$ does not matter in the proof of Theorem 4.1.

Lemma 4.2. *Suppose that* (4.14) *holds. Then, for any* $i, j \in \widehat{V}$, *it holds that*

$$\widehat{\mu}^{*n}\left(\left\{\pi \in \mathfrak{S}(\widehat{V}) : \pi(i) = j\right\}\right) \underset{n\to\infty}{\longrightarrow} \frac{1}{\widehat{m}}. \tag{4.18}$$

Proof. By the strong-connectedness property of (V, A) we see that, for any $i, j \in \widehat{V}$, there exists a word $s' = (\sigma'_r, \ldots, \sigma'_1)$ in $\mathrm{Supp}(\mu)$ such that

$$\langle s' \rangle \, \widehat{x}_i \in V_j. \tag{4.19}$$

We define an event B by

$$\begin{aligned} B = \{N_0 = \sigma'_r, \ldots,\ N_{-r+2} = \sigma'_2,\ N_{-r+1} = \sigma'_1, \\ N_{-r} = \sigma_p, \ldots,\ N_{-r-p+2} = \sigma_2,\ N_{-r-p+1} = \sigma_1\}. \end{aligned} \tag{4.20}$$

Since we assume that (4.14) holds, we see that

$$T_{-1} = -r - p \text{ and } \widehat{N}_0(i) = j \text{ on the event } B. \tag{4.21}$$

Therefore, for any $i, j \in \widehat{V}$, we have

$$\widehat{\mu}(\{\pi \in \mathfrak{S}(\widehat{V}) : \pi(i) = j\}) \geq P(B) > 0. \tag{4.22}$$

Now we may apply Theorem 2.9 to see that there exists a probability law $(\rho(j) : j \in \widehat{V})$ such that

$$\widehat{\mu}^{*n}(\{\pi \in \mathfrak{S}(\widehat{V}) : \pi(i) = j\}) \underset{n\to\infty}{\longrightarrow} \rho(j), \quad i, j \in \widehat{V}. \tag{4.23}$$

Since, for any n and $j \in \widehat{V}$, we have

$$\sum_{i \in \widehat{V}} \widehat{\mu}^{*n}(\{\pi \in \mathfrak{S}(\widehat{V}) : \pi(i) = j\}) = \sum_{i \in \widehat{V}} \widehat{\mu}^{*n}(\{\pi \in \mathfrak{S}(\widehat{V}) : \pi(j) = i\}) = 1, \tag{4.24}$$

we see that $\rho(j) = 1/\widehat{m}$ for all $j \in \widehat{V}$. The proof is now complete. $\square$

4.3 Constructing a new random walk

Define $\widehat{X}_\kappa \in \widehat{V}$ as follows:

$$\widehat{X}_\kappa = i \quad \text{if} \quad X_{T_\kappa} \in V_i. \tag{4.25}$$

We will prove gradually that the process $\{\widehat{X}, \widehat{N}\} = \{(\widehat{X}_\kappa)_{\kappa \in -\mathbb{N}}, (\widehat{N}_\kappa)_{\kappa \in -\mathbb{N}}\}$ is a $\widehat{\mu}$-random walk indexed by $-\mathbb{N}$. The first step is the following.

Lemma 4.3. *Suppose that* (4.14) *holds. Then, for fixed* $\kappa \in -\mathbb{N}$*, the following assertions hold:*

(i) $\widehat{X}_\kappa = \widehat{N}_\kappa \widehat{X}_{\kappa-1}$ *holds a.s.;*

(ii) $\widehat{X}_{\kappa-1}$ *is indepenent of* $\widehat{N}_\kappa$*.*

(iii) $P(\widehat{X}_{\kappa-1} = i) = \lambda(V_i)$ *for all* $i \in \widehat{V}$*.*

Proof. Claim (i) is obvious by definition. Let us prove (ii) and (iii) at the same time. Let $l \in -\mathbb{N}$, $\pi \in \mathfrak{S}(\widehat{V})$ and $i \in \widehat{V}$. Then we have

$$\begin{aligned}
&P\left(\widehat{N}_\kappa = \pi,\ \widehat{X}_{\kappa-1} = i,\ T_{\kappa-1} = l\right) && (4.26)\\
=&P\left(\widehat{N}_\kappa = \pi,\ X_l \in V_i,\ T_{\kappa-1} = l\right) && (4.27)\\
=&P\left(\widehat{N}_\kappa = \pi,\ T_{\kappa-1} = l\right) P(X_l = i) \qquad \text{(by independence)} && (4.28)\\
=&P\left(\widehat{N}_\kappa = \pi,\ T_{\kappa-1} = l\right) \lambda(V_i) \qquad \text{(by stationarity).} && (4.29)
\end{aligned}$$

Summing up by $l \in -\mathbb{N}$, we obtain

$$P\left(\widehat{N}_\kappa = \pi,\ \widehat{X}_{\kappa-1} = i\right) = P\left(\widehat{N}_\kappa = \pi\right) \lambda(V_i), \tag{4.30}$$

which proves Claims (ii) and (iii). ∎

The second step is to prove Theorem 4.1.

Proof of Theorem 4.1. Define a probability law $\widehat{\lambda}$ on $\widehat{V}$ by

$$\widehat{\lambda}(i) = \lambda(V_i) \quad \text{for all } i \in \widehat{V}. \tag{4.31}$$

By Lemma 4.3, we have

$$\widehat{\lambda} = \widehat{\mu} * \widehat{\lambda}. \tag{4.32}$$

Iterating this convolution equation, we obtain, for any fixed $i \in \widehat{V}$,

$$\begin{aligned}
\widehat{\lambda}(\{i\}) =&(\widehat{\mu}^{*n} * \widehat{\lambda})(\{i\}) && (4.33)\\
=&\sum_{j \in \widehat{V}} \widehat{\mu}^{*n}(\{\pi \in \mathfrak{S}(\widehat{V}) : \pi(j) = i\}) \widehat{\lambda}(\{j\}) && (4.34)\\
\underset{n\to\infty}{\longrightarrow}& \frac{1}{\widehat{m}} \sum_{j \in \widehat{V}} \widehat{\lambda}(\{j\}) = \frac{1}{\widehat{m}} && (4.35)
\end{aligned}$$

where we have used Lemma 4.2. This completes the proof. ∎

The third step is the following, which reveals a symmetry hidden behind the uniformity (4.5) in Theorem 4.1.

Theorem 4.4. *Suppose that* (4.14) *holds. Then* $\{(\widehat{X}_\kappa)_{\kappa\in-\mathbb{N}}, (\widehat{N}_\kappa)_{\kappa\in-\mathbb{N}}\}$ *is a* $\widehat{\mu}$*-random walk indexed by* $-\mathbb{N}$*, i.e.,*

(i) $\widehat{N}_\kappa$ *is independent of* $\mathcal{F}^{\widehat{X},\widehat{N}}_{\kappa-1}$ *for all* $\kappa \in -\mathbb{N}$*;*

(ii) $(\widehat{N}_\kappa)_{\kappa\in-\mathbb{N}}$ *is IID with common law* $\widehat{\mu}$*;*

(iii) $\widehat{X}_\kappa = \widehat{N}_\kappa \widehat{X}_{\kappa-1}$ *holds a.s. for all* $\kappa \in -\mathbb{N}$*.*

Moreover, it holds that

(iv) $\widehat{X}_\kappa$ *has uniform law on* $\widehat{V}$ *for all* $\kappa \in -\mathbb{N}$*.*

Proof. We have already shown (ii), (iii) and (iv). Let us prove (i). For this, it suffices to prove that, for any fixed $\kappa \in -\mathbb{N}$,

$$\widehat{N}_0, \widehat{N}_{-1}, \dots, \widehat{N}_\kappa \text{ and } \widehat{X}_{\kappa-1} \text{ are independent.} \tag{4.36}$$

Let $B_\kappa \in \sigma(\widehat{N}_0, \widehat{N}_{-1}, \dots, \widehat{N}_\kappa)$ and $i \in \widehat{V}$. Let $l \in -\mathbb{N}$. Since $B_\kappa \cap \{T_{\kappa-1} = l\} \in \sigma(N_0, \dots, N_{l+1})$, we have

$$\begin{aligned}
&P\left(B_\kappa,\ \widehat{X}_{\kappa-1} = i,\ T_{\kappa-1} = l\right) && && (4.37)\\
=&P\left(B_\kappa,\ X_l \in V_i,\ T_{\kappa-1} = l\right) && && (4.38)\\
=&P\left(B_\kappa,\ T_{\kappa-1} = l\right) P\left(X_l \in V_i\right) && \text{(by independence)} && (4.39)\\
=&\frac{1}{\widehat{m}} P\left(B_\kappa,\ T_{\kappa-1} = l\right) && \text{(by Theorem 4.1).} && (4.40)
\end{aligned}$$

This proves that $\widehat{X}_{\kappa-1}$ is independent of $\sigma(\widehat{N}_0, \widehat{N}_{-1}, \dots, \widehat{N}_\kappa)$.

Let $B_{\kappa+1} \in \sigma(\widehat{N}_0, \widehat{N}_{-1}, \dots, \widehat{N}_{\kappa+1})$ and $\pi \in \mathfrak{S}(\widehat{V})$. Let $l, l' \in -\mathbb{N}$ with $l > l'$. Since $B_{\kappa+1} \cap \{T_\kappa = l\} \in \sigma(N_0, \dots, N_{l+1})$, we obtain

$$\begin{aligned}
&P\left(B_{\kappa+1},\ \widehat{N}_\kappa = \pi,\ T_\kappa = l,\ T_{\kappa-1} = l'\right) && (4.41)\\
=&P\left(B_{\kappa+1},\ T_\kappa = l,\ M\left[N_l N_{l-1} \cdots N_{l'+1}\right] = \pi,\ T_{-1} \circ \theta_l = l'\right) && (4.42)\\
=&P\left(B_{\kappa+1},\ T_\kappa = l\right) P\left(M\left[N_l N_{l-1} \cdots N_{l'+1}\right] = \pi,\ T_{-1} \circ \theta_l = l'\right) && (4.43)\\
=&P\left(B_{\kappa+1},\ T_\kappa = l\right) P\left(M\left[N_0 N_{-1} \cdots N_{l'-l+1}\right] = \pi,\ T_{-1} = l' - l\right) && (4.44)\\
=&P\left(B_{\kappa+1},\ T_\kappa = l\right) P\left(\widehat{N}_0 = \pi,\ T_{-1} = l' - l\right), && (4.45)
\end{aligned}$$

where we write

$$T_{-1} \circ \theta_l = \max\{k \le -p : N_{l+k+p} = \sigma_p, \dots, N_{l+k+1} = \sigma_1\}. \tag{4.46}$$

Summing up (4.41)-(4.45) by $l' \in -\mathbb{N}$, we have

$$P\left(B_{\kappa+1},\ \widehat{N}_\kappa = \pi,\ T_\kappa = l\right) = P\left(B_{\kappa+1},\ T_\kappa = l\right) P\left(\widehat{N}_0 = \pi\right). \tag{4.47}$$

This shows that $\widehat{N}_\kappa$ is independent of $\sigma(\widehat{N}_0, \dots, \widehat{N}_{\kappa+1})$. Therefore, we conclude that $\widehat{N}_0, \widehat{N}_{-1}, \dots, \widehat{N}_{\kappa+1}$ and $\widehat{N}_\kappa$ are independent, which completes the proof. $\square$

For $k \in -\mathbb{N}$, we write

$$K(k) = \max\{\kappa \in -\mathbb{N} : k - p \geq T_\kappa\}. \tag{4.48}$$

Note that

$$L(k) := T_{K(k)} = \max\{l \leq k - p : N_{l+p} = \sigma_p, \dots, N_{l+2} = \sigma_2, N_{l+1} = \sigma_1\} \tag{4.49}$$

and that

$$\{L(k) = l\} \in \sigma(N_k, N_{k-1}, \dots, N_{l+1}), \quad l \leq k - p. \tag{4.50}$$

The following theorem proves Claim (1.8).

Theorem 4.5. *Suppose that* (4.14) *holds. Then, for any $k \in -\mathbb{N}$, it holds that*

$$X_k \in \mathcal{F}_k^N \vee \sigma(\widehat{X}_{K(k)}) \quad a.s. \tag{4.51}$$

and that

$$\widehat{X}_{K(k)} \textit{ is independent of } \mathcal{F}_0^N \textit{ and has uniform law on } \widehat{V}. \tag{4.52}$$

Consequently, if the road coloring is non-synchronizing, i.e., $\widehat{m} \geq 2$, the μ-random walk $\{X, N\}$ is non-strong.

Proof. By definitions of $L(k)$ and $K(k)$, we have

$$\begin{aligned} X_k =& N_k N_{k-1} \cdots N_{L(k)+p+1} N_{L(k)+p} \cdots N_{L(k)+1} X_{L(k)} && (4.53)\\ =& N_k N_{k-1} \cdots N_{L(k)+p+1} \sigma^{(n(1))} \cdots \sigma^{(n(p))} X_{L(k)} && (4.54)\\ =& N_k N_{k-1} \cdots N_{L(k)+p+1} x_i && (4.55) \end{aligned}$$

with $i = \widehat{X}_{K(k)}$. Since $N_k N_{k-1} \cdots N_{L(k)+p+1} \in \mathcal{F}_k^N$, we obtain (4.51).

Let $k' < k$ and let $B \in \sigma(N_j : j \geq k' + 1)$. Let $l \leq k - p$ and $l' \leq \min\{k' - p, l\}$. Note that we have

$$\begin{aligned} & P\left(X_{L(k)} \in V_i,\ L(k) = l,\ L(k') = l',\ B\right) && (4.56)\\ =& P\left(X_l \in V_i,\ L(k) = l,\ L(k') = l',\ B\right) && (4.57)\\ =& P\left(N_l \cdots N_{l'+p+1} \sigma_+ X_{l'} \in V_i,\ L(k) = l,\ L(k') = l',\ B\right) && (4.58)\\ =& \sum_{j=1}^{\widehat{m}} P\left(X_{l'} \in V_j,\ M[N_l \cdots N_{l'+p+1}](j) = i,\ L(k) = l,\ L(k') = l',\ B\right). && (4.59) \end{aligned}$$

Since $X_{l'}$ is independent of $\mathcal{F}^N_{0,l'} := \sigma(N_0, \dots, N_{l'+1})$, we see that

$$P\left(X_{l'} \in V_j \middle| \mathcal{F}^N_{0,l'}\right) = P\left(X_{l'} \in V_j\right) = \frac{1}{\widehat{m}}. \tag{4.60}$$

Since $\{L(k) = l,\ L(k') = l'\} \in \mathcal{F}^N_{0,l'}$, we obtain

$$(4.59) = \frac{1}{\widehat{m}} \sum_{j=1}^{\widehat{m}} P\left(M[N_l \cdots N_{l'+p+1}](j) = i,\ L(k) = l,\ L(k') = l',\ B\right) \tag{4.61}$$

$$= \frac{1}{\widehat{m}} P\left(L(k) = l,\ L(k') = l',\ B\right). \tag{4.62}$$

Summing up in $l \leq k - p$ and $l' \leq k' - p$, we have

$$P\left(\widehat{X}_{K(k)} = i,\ B\right) = P\left(X_{L(k)} \in V_i,\ B\right) = \frac{1}{\widehat{m}} P(B). \tag{4.63}$$

This proves (4.52). □

5 Periodic case

If a directed graph (V, A) is strongly-connected, then it is easy to see that the greatest common divisor among $\{n \geq 1 : A^n(x, x) \geq 1\}$ does not depend on $x \in V$, so that it is called the *period* of (V, A). If the period of (V, A) is greater than one, then (V, A) is called *periodic*.

In this section, we give a brief discussion about when a μ-random walk is strong in the periodic case, although it is not so deep as to involve recent studies of road coloring problems in the periodic case, such as Béal–Berlinkov–Perrin [4], Budzban–Feinsilver [7] and Trakhtman [19].

We shall utilize the following theorem.

Theorem 5.1 (Perron–Frobenius). *Let μ be a probability law on Σ. Suppose that the directed graph induced by μ is strongly-connected and has period $d \geq 2$. Then there exist a partition $\{V^{(1)}, \dots, V^{(d)}\}$ of V and a family $\{\lambda^{(1)}, \dots, \lambda^{(d)}\}$ of probability laws on V such that the following assertions hold:*

(i) *for each $x \in V^{(i)}$, it holds that*

$$\mu(\sigma \in \Sigma : y = \sigma x) \begin{cases} > 0 & \textit{if } y \in V^{(i+1)}, \\ = 0 & \textit{otherwise}, \end{cases} \tag{5.1}$$

where $V^{(d+1)} = V^{(1)}$.

(ii) *for each $i = 1, \dots, d$, the support of $\lambda^{(i)}$ is $V^{(i)}$;*

(iii) *for each* $i = 1, \dots, d$ *and each* $x, y \in V^{(i)}$, *it holds that*

$$\mu^{*nd}(\sigma \in \Sigma : y = \sigma x) \underset{n\to\infty}{\longrightarrow} \lambda^{(i)}(y). \tag{5.2}$$

(iv) *for each* $i = 1, \dots, d$, *it holds that* $\mu^{*d} * \lambda^{(i)} = \lambda^{(i)}$.

For the proof of Theorem 5.1, see, e.g., [16]. Each $V^{(i)}$ will be called a *cyclic part.*

In the sequel, let μ, $\{V^{(1)}, \dots, V^{(d)}\}$ and $\{\lambda^{(1)}, \dots, \lambda^{(d)}\}$ as in Theorem 5.1. The following theorem chracterizes the class of all μ-random walks.

Theorem 5.2. *The following assertions hold:*

(i) *For each* $i = 1, \dots, d$, *there exists a* μ*-random walk* $\{X^{(i)}, N^{(i)}\}$ *in* V *such that* $X_0^{(i)} \in V^{(i)}$ *a.s. Such a* μ*-random walk is unique up to identity in law and the tail* σ*-field* $\mathcal{F}_{-\infty}^{X^{(i)},N^{(i)}}$ *is trivial.*

(ii) *Let* $\{X, N\}$ *be an arbitrary* μ*-random walk in* V. *Then, for each* $i = 1, \dots, d$, *the following hold:*

(ii-1) *the event* $A^{(i)} = \{X_0 \in V^{(i)}\}$ *belongs to the tail* σ*-field* $\mathcal{F}_{-\infty}^{X,N}$;

(ii-2) *if* $P(A^{(i)}) > 0$, *then* $\{X, N\}$ *under* $P(\cdot \mid A^{(i)})$ *is identical in law to* $\{X^{(i)}, N^{(i)}\}$; *in other words, it holds that*

$$P((X, N) \in \cdot) = \sum_{i=1}^{d} P\left(\left(X^{(i)}, N^{(i)}\right) \in \cdot\right) P(A^{(i)}). \tag{5.3}$$

We omit the proof of Theorem 5.2 as it is an easy consequence of Theorem 5.1.

For $i = 1, \dots, d$, we write $\Sigma^{(i)}$ for the set of all mappings from $V^{(i)}$ to itself, and define

$$\mu^{(i)} = \frac{1}{\mu^{*d}(\Sigma^{(i)})} \mu^{*d}|_{\Sigma^{(i)}}. \tag{5.4}$$

Then, by (i) of Theorem 5.1, we see that $\mu^{(i)}$ is a probability law on $\Sigma^{(i)}$. The following theorem gives a necessary and sufficient condition for strongness of μ-random walks.

Theorem 5.3. *Let* $\{X, N\}$ *be a* μ*-random walk in* V.

(A) *If the tail* σ*-field* $\mathcal{F}_{-\infty}^{X,N}$ *is non-trivial, then the* μ*-random walk* $\{X, N\}$ *is non-strong.*

(B) *Suppose that the tail* σ*-field* $\mathcal{F}_{-\infty}^{X,N}$ *is trivial. Take* $i = 1, \dots, d$ *such that* $\{X, N\} \overset{\mathrm{d}}{=} \{X^{(i)}, N^{(i)}\}$, *which is possible by Theorem 5.2. Then the following three assertions are equivalent:*

(i) $\mathrm{Supp}(\mu^{(i)})$ *is synchronizing.*

(ii) *The limit* $\lim_{l\to-\infty} N_k N_{k-1} \cdots N_{l+1}$ *exists a.s. for all* $k \in \mathbb{Z}$.

(iii) *The* μ*-random walk* $\{X, N\}$ *is strong.*

The proof of Theorem 5.3 is an immediate consequence of Theorem 1.3, so that we omit it.

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