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Existence and non-existence results of the Fučík type spectrum for the generalized p-Laplace operators

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1 Introduction

In this paper, we consider the existence of $(\alpha,\beta)\in\mathbb{R}^2$ for which the following quasilinear elliptic equation has a non-trivial solution:

$$(F)_{(\alpha,\beta)}\qquad \begin{cases} -\mathrm{div}\, A(x,\nabla u)=\alpha u_+^{p-1}-\beta u_-^{p-1} & \text{in } \Omega,\\ \frac{\partial u}{\partial\nu}=0 & \text{on } \partial\Omega, \end{cases}$$

where ν denotes the outward unit normal vector on $\partial\Omega$, $1<p<\infty$, $\Omega\subset\mathbb{R}^N$ is a bounded domain with C^2 boundary $\partial\Omega$. Here, $A\colon\bar\Omega\times\mathbb{R}^N\to\mathbb{R}^N$ is a map which is strictly monotone in the second variable and satisfies certain regularity conditions (see the following assumption (A)). The equation $(F)_{(\alpha,\beta)}$ contains the corresponding p-Laplacian problem as a special case, and in this case, (α,β) admitting a non-trivial solution to $(F)_{(\alpha,\beta)}$ is said to belong to the *Fučík spectrum* of the p-Laplacian. Although the p-Laplace operator is $(p-1)$-homogeneous, the operator A is not supposed generally to be $(p-1)$-homogeneous in the second variable.

Here, we say that $u\in W^{1,p}(\Omega)$ is a (weak) solution of $(F)_{(\alpha,\beta)}$ if

$$\int_\Omega A(x,\nabla u)\nabla\varphi\,dx=\int_\Omega \alpha u_+^{p-1}\varphi\,dx-\int_\Omega\beta u_-^{p-1}\varphi\,dx$$

for all $\varphi\in W^{1,p}(\Omega)$.

Throughout this paper, we assume that the operator A satisfies the following assumption (A):

(A) $A(x,y)=a(x,|y|)y$, where $a(x,t)>0$ for all $(x,t)\in\bar\Omega\times(0,+\infty)$ and

(i) $A\in C^0(\bar\Omega\times\mathbb{R}^N,\mathbb{R}^N)\cap C^1(\bar\Omega\times(\mathbb{R}^N\setminus\{0\}),\mathbb{R}^N)$;

(ii) there exists a $C_1>0$ such that

$$|D_yA(x,y)|\le C_1|y|^{p-2}\quad\text{for every } x\in\bar\Omega,\ \text{and } y\in\mathbb{R}^N\setminus\{0\};$$

(iii) there exists a $C_0>0$ such that

$$D_yA(x,y)\xi\cdot\xi\ge C_0|y|^{p-2}|\xi|^2\quad\text{for every } x\in\bar\Omega,\ y\in\mathbb{R}^N\setminus\{0\} \text{ and } \xi\in\mathbb{R}^N.$$

(iv) there exists a $C_2 > 0$ such that

$$|D_x A(x,y)| \leq C_2(1+|y|^{p-1}) \quad \text{for every } x \in \bar{\Omega},\ y \in \mathbb{R}^N \setminus \{0\}.$$

Throughout this paper, we assume $C_0 \leq p-1 \leq C_1$ because we can take such desired C_0 and C_1 anew if necessary.

The hypothesis (A) has been considered in the study of the quasilinear elliptic problems (cf. [6], [12], [13]). For example, we can treat the operators like the p-Laplacian with the positive weight and

$$\operatorname{div}\left((|\nabla u|^{p-2}+|\nabla u|^{q-2})(1+|\nabla u|^q)^{\frac{p-q}{q}}\nabla u\right) \quad \text{for } 1 < p \leq q < \infty.$$

Let us recall the known results in the special case of $A(x,y)=|y|^{p-2}y$ that is, p-Laplace problem and $C_0 = C_1 = p-1$. The set of all points $(\alpha,\beta) \in \mathbb{R}^2$ for which the equation

$$-\Delta_p u = \alpha u_+^{p-1} - \beta u_-^{p-1} \text{ in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial\Omega \tag{1}$$

has a non-trivial solution is called the *Fučík spectrum* of the p-Laplacian under the Neumann boundary condition. In this paper, we denote the Fučík spectrum of p-Laplacian by Θ_p. It is well known that the first eigenvalue $\mu_1 = 0$ of $-\Delta_p$ is simple and every eigenfunction corresponding to $\mu_1 = 0$ is a constant function. Therefore, Θ_p contains the lines $\{0\} \times \mathbb{R}$ and $\mathbb{R} \times \{0\}$ (we call these lines as "the trivial lines"). Furthermore, by the same argument as in [5], it can be proved that there exists a Lipschitz continuous curve contained in Θ_p which is called "the first nontrivial curve" $\mathscr{C}$ (see Section 2). In the p-Laplacian case, many authors have treated the Fučík spectrum (see [5], [7], [8], [10] under the Dirichlet boundary condition and [2], [3] for Neumann boundary condition).

Let us return to the general case. In [14], D. Motreanu and the present author treated the equation

$$-\operatorname{div} A(x,\nabla u) = f(x,u) \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega \tag{2}$$

with the following nonlinearity:

$$f(x,u) = \begin{cases} \alpha_0 u_+^{p-1} - \beta_0 u_-^{p-1} + o(|u|^{p-1}) & \text{at } 0, \\ \alpha u_+^{p-1} - \beta u_-^{p-1} + o(|u|^{p-1}) & \text{at } \infty \end{cases}$$

for (α_0,β_0), $(\alpha,\beta) \in \mathbb{R}^2$. Roughly speaking, by constructing two curves $\tilde{\mathscr{C}}$ and $\underline{\mathscr{C}}$ related to the map A (see section 3), it was shown that the equation (2) has a sign-changing solution in the case where (α,β) is below the curve $\underline{\mathscr{C}}$ and (α_0,β_0) is above the curve $\tilde{\mathscr{C}}$. In the p-Laplacian case, we see that two curves $\tilde{\mathscr{C}}$ and $\underline{\mathscr{C}}$ coincide with the first nontrivial curve $\mathscr{C}$. Moreover, if the first nontrivial curve lies between (α_0,β_0) and (α,β), then equation $-\Delta_p u = f(x,u)$ in Ω (under the Dirichlet boundary condition) has a non-trivial solution. Therefore, even for the general case of A, it seems reasonable to expect the existence of uncountably many Fučík type spectrum between $\tilde{\mathscr{C}}$ and $\underline{\mathscr{C}}$.

Mainly, this paper consists of results in [14] and [15]. In the final section, we see further results and several questions concerning our problem.

2 The first nontrivial curve contained in Θ_p

Here, we recall the result for the special case of $A(x,y) = |y|^{p-2}y$, that is, p-Laplacian problems (note that we can take $C_0 = C_1 = p-1$ in (A)). The construction of the curve $\mathscr{C}$ contained in the Fučík spectrum is carried out by the same argument as in [5]: For $s \geq 0$, we define

$$J_s(u) := \int_\Omega |\nabla u|^p\,dx - s\int_\Omega u_+^p\,dx \quad \text{for } u \in W^{1,p}(\Omega), \quad \tilde{J}_s := J_s|_S$$
$$S := \left\{ u \in W^{1,p}(\Omega)\,;\ \int_\Omega |u|^p\,dx = 1 \right\},$$
$$\Sigma := \{ \gamma \in C([0,1],S)\,;\ \gamma(0) = \psi_1,\ \gamma(1) = -\psi_1 \},$$

where $\psi_1 = 1/|\Omega|^{1/p}$ (so $\|\psi_1\|_p = 1$). Here, the set $C([0,1],S)$ denotes the set of continuous functions from $[0,1]$ to S with the topology induced by the $W^{1,p}(\Omega)$ norm. Finally, we set

$$c(s) := \inf_{\gamma\in\Sigma} \max_{t\in[0,1]} \tilde{J}_s(\gamma(t)). \tag{3}$$

Then, it can be proved that $c(s)$ is a positive critical value of $\tilde{J}_s$ with $c(0) = \mu_2$, where μ_2 is the second eigenvalue of the p-Laplacian under the Neumann boundary condition. Moreover, we can see that $c(s)$ is continuous, strictly decreasing in $s \geq 0$ and $c(s) + s$ is strictly increasing in $s \geq 0$ (refer to [1, Lemma2.2] and [5, Proposition 4.1]). Then, $\mathscr{C}$ is defined as follows:

$$\mathscr{C} := \{ (c(s)+s, c(s))\,;\ s \geq 0 \} \cup \{ (c(s), c(s)+s)\,;\ s \geq 0 \}.$$

Finally, we remark that in the case of $N \geq p$, it is shown in [3] that $c(s) \to 0$ as $s \to \infty$, whence the asymptotic lines of the first nontrivial curve are the trivial lines $\mathbb{R} \times \{0\}$ and $\{0\} \times \mathbb{R}$. However, if $N < p$, then $c(s) \to \bar{\lambda}$ as $s \to \infty$, where $\bar{\lambda}$ is a positive constant defined by

$$\bar{\lambda} = \inf_B \int_\Omega |\nabla u|^p\,dx, \quad \text{where } B := \{ u \in S\,;\ u(x_0) = 0 \text{ for some } x_0 \in \overline{\Omega} \}.$$

This yields that the trivial lines are not the asymptotic lines of the first nontrivial curve.

3 Existence and non-existence results

To state the results for $(F)_{(\alpha,\beta)}$, we define curves $\underline{\mathscr{C}}$ and $\tilde{\mathscr{C}}$ by

$$\underline{\mathscr{C}} := \frac{C_0}{p-1}\mathscr{C} := \{ (aC_0/(p-1), bC_0/(p-1))\ ;\ (a,b) \in \mathscr{C} \},$$
$$\tilde{\mathscr{C}} := \frac{C_1}{p-1}\mathscr{C} = \{ (aC_1/(p-1), bC_1/(p-1))\ ;\ (a,b) \in \mathscr{C} \},$$

where C_0 and C_1 are positive constants satisfying (A). First, we state the elementary results for the equation $(F)_{(\alpha,\beta)}$ which is shown in [14].

Proposition 1 ([14, Proposition 2]) *The following assertions hold:*

(i) *if* $\alpha\beta < 0$ *or* $\max\{\alpha,\beta\} < 0$ *holds, then* $(F)_{(\alpha,\beta)}$ *has no non-trivial solutions;*

(ii) *if* u *is a non-trivial solution of* $(F)_{(\alpha,\beta)}$ *with* $\min\{\alpha,\beta\} > 0$, *then* u *changes sign;*

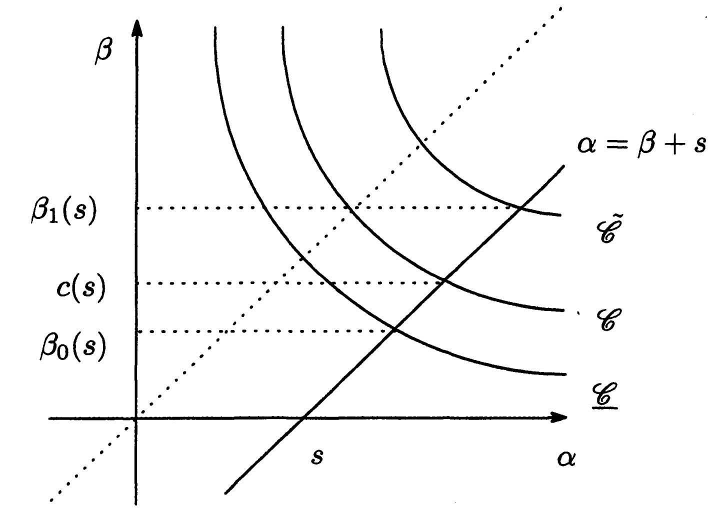
(iii) *if* u *is a non-trivial solution of* $(F)_{(\alpha,\beta)}$ *with* $\alpha\beta = 0$, *then* u *is a constant function;*

(iv) *if* $0 < \alpha < \alpha'$ *and* $0 < \beta < \beta'$ *for some* $(\alpha',\beta') \in \underline{\mathscr{C}}$, *then* $(F)_{(\alpha,\beta)}$ *has no non-trivial solutions.*

Define $\beta_0(s)$ and $\beta_1(s)$ for $s \geq 0$ by

$$\beta_0(s) := \frac{C_0}{p-1}\, c\left(\frac{p-1}{C_0}\, s\right), \quad \beta_1(s) := \frac{C_1}{p-1}\, c\left(\frac{p-1}{C_1}\, s\right),$$

where $c(\cdot)$ is a function defined by (3) (see the following figure):



Now, we state existence results.

Theorem 2 ([15]) *For every* $s \geq 0$ *and* $R > 0$, *there exists a* $\beta \in [\beta_0(s), \beta_1(s)]$ *such that* $(F)_{(\beta+s,\beta)}$ *and* $(F)_{(\beta,\beta+s)}$ *have at least one sign-changing solution* $u \in C^1(\bar{\Omega})$ *with* $\int_\Omega |u|^p\,dx \leq R^p$.

Theorem 3 ([15]) *Let* $s \geq 0$, $\varepsilon > 0$ *and* $R_2 > R_1 > 0$ *be constants satisfying*

$$R_2 > \max\left\{\frac{\beta_1(s)+s+\varepsilon}{\min\{\beta_0(s),\varepsilon\}},\ \frac{C_1(\beta_1(s)+s+\varepsilon)^2}{C_0(\beta_1(s)+\varepsilon)^2},\ \frac{s(C_1-C_0)}{C_0(\beta_1(s)+\varepsilon)}\right\}^{1/p} R_1.$$

Then, there exists a $\beta \in [\beta_0(s), \beta_1(s)+\varepsilon]$ *such that* $(F)_{(\beta+s,\beta)}$ *and* $(F)_{(\beta,\beta+s)}$ *have at least one sign-changing solution* $u \in C^1(\bar{\Omega})$ *with* $R_1^p \leq \int_\Omega |u|^p\,dx \leq R_2^p$.

3.1 Variational setting and notations

In what follows, we define the norm of $W := W^{1,p}(\Omega)$ by $\|u\|^p := \|\nabla u\|_p^p + \|u\|_p^p$, where $\|u\|_q$ denotes the norm of $L^q(\Omega)$ for $u \in L^q(\Omega)$ $(1 \le q \le \infty)$. Define $G(x,y) := \int_0^{|y|} a(x,t)t\,dt$, then we can easily see that

$$\nabla_y G(x,y) = A(x,y) \quad \text{and} \quad G(x,0) = 0$$

for every $x \in \bar{\Omega}$.

Remark 4 *The following assertions hold:*

(i) *for all $x \in \bar{\Omega}$, $A(x,y)$ is maximal monotone and strictly monotone in y;*

(ii) $|A(x,y)| \le \frac{C_1}{p-1}|y|^{p-1}$ *for every $(x,y) \in \bar{\Omega} \times \mathbb{R}^N$;*

(iii) $A(x,y)y \ge \frac{C_0}{p-1}|y|^p$ *for every $(x,y) \in \bar{\Omega} \times \mathbb{R}^N$;*

(iv) *$G(x,y)$ is convex in y for all x and satisfies the following inequalities:*

$$A(x,y)y \ge G(x,y) \ge \frac{C_0}{p(p-1)}|y|^p \quad \text{and} \quad G(x,y) \le \frac{C_1}{p(p-1)}|y|^p \tag{4}$$

for every $(x,y) \in \bar{\Omega} \times \mathbb{R}^N$,

where C_0 and C_1 are the positive constants described in (A).

For parameters $s \ge 0$ and $\beta \in \mathbb{R}$, we define the C^1 functionals $I_{\beta,s}$ and $I_{\beta,s}^+$ on $W^{1,p}(\Omega)$ by

$$I_{\beta,s}(u) := \int_\Omega G(x,\nabla u)\,dx - \frac{\beta+s}{p}\int_\Omega u_+^p\,dx - \frac{\beta}{p}\int_\Omega u_-^p\,dx$$

with

$$\langle I'_{\beta,s}(u), v\rangle = \int_\Omega A(x,\nabla u)\,\nabla v\,dx - (\beta+s)\int_\Omega u_+^{p-1}\,v\,dx + \beta\int_\Omega u_-^{p-1}\,v\,dx,$$

$$I_{\beta,s}^+(u) := \int_\Omega G(x,\nabla u)\,dx - \frac{\beta+s}{p}\int_\Omega u_+^p\,dx$$

for u, $v \in W^{1,p}(\Omega)$. In this paper, we use the following notations:

$$\begin{aligned}
&B(r) := \{u \in W\,;\ \|u\| \le r\}, && B_p(r) := \{u \in W\,;\ \|u\|_p \le r\},\\
&D(r,r') := \{u \in W\,;\ r \le \|u\| \le r'\}, && D_p(r,r') := \{u \in W\,;\ r \le \|u\|_p \le r'\}\\
&rS := \{u \in W\,;\ \|u\|_p = r\}, && rS_+ := \{u \in W\,;\ \|u_+\|_p = r\}
\end{aligned}$$

for $r' \ge r > 0$. Here, we note that the topology of all subsets above are induced by the $W^{1,p}(\Omega)$ norm. We set

$$K(I_{\beta,s}) := \{u \in W\,;\ I'_{\beta,s}(u) = 0\} \quad \text{and} \quad I_{\beta,s}^c := \{u \in W\,;\ I_{\beta,s}(u) \le c\}$$

for $c \in \mathbb{R}$.

Remark 5 *Let $u \in W^{1,p}(\Omega)$ be a critical point of $I_{\beta,s}$, namely, u satisfies the equality*

$$\int_\Omega A(x,\nabla u)\nabla\varphi\,dx = (\beta+s)\int_\Omega u_+^{p-1}\varphi\,dx - \beta\int_\Omega u_-^{p-2}\varphi\,dx$$

for every $\varphi \in W^{1,p}(\Omega)$. Then, because of $u \in L^\infty(\Omega)$ (see Appendix in [14]), we see $u \in C^{1,\gamma}(\overline{\Omega})$ $(0<\gamma<1)$ by the regularity result (cf. [11]).

By Theorem 3 in [4], u satisfies $(\mathrm{F})_{(\beta+s,\beta)}$ *in the distribution sense and the boundary condition*

$$0 = \frac{\partial u}{\partial \nu_A} := A(\cdot,\nabla u)\nu = a(\cdot,|\nabla u|)\frac{\partial u}{\partial \nu} \quad \text{in } W^{-1/q,q}(\partial\Omega)$$

for every $1<q<\infty$ (see [4] for the definition of $W^{-1/q,q}(\partial\Omega)$). Since $u \in C^{1,\gamma}(\overline{\Omega})$ and $a(x,y)>0$ for every $y \neq 0$, u satisfies the Neumann boundary condition, that is, $\frac{\partial u}{\partial \nu}(x)=0$ for every $x \in \partial\Omega$.

By Proposition 1 and the remark above (note also that $A(x,y)$ is odd in y), it is sufficient to prove the following theorems for the proofs of Theorem 2 and 3.

Theorem 6 ([15]) *For every $s \geq 0$ and $R>0$, there exists a $\beta \in [\beta_0(s),\beta_1(s)]$ such that $K(I_{\beta,s}) \cap B_p(R) \setminus \{0\} \neq \emptyset$.*

Theorem 7 ([15]) *Let $s \geq 0$, $\varepsilon>0$ and $R_2>R_1>0$ be constants satisfying (3) as in Theorem 3. Then, there exists a $\beta \in [\beta_0(s),\beta_1(s)+\varepsilon]$ such that $K(I_{\beta,s}) \cap D_p(R_1,R_2) \neq \emptyset$.*

Roughly speaking, to show the existence of a non-trivial critical point near zero of $I_{\beta,s}$, we see the variation of the critical groups at 0 for $I_{\beta,s}$ when a parameter β changes from $\beta_0(s)$ to $\beta_1(s)$. Moreover, it is necessary to construct a flow for which $B_p(R)$ (or $D_p(R_1,R_2)$) is invariant. Furthermore, we shall produce suitable paths to see that 0-th reduced homology group is trivial. For this purpose, we need to consider the constrained variational problems. The key point of our proof is to introduce a Finsler manifold rS_+.

Finally, we state the result characterizing $c(s)$ by Morse theory.

Corollary 8 ([15]) *Let $C_0=C_1=p-1$ (that is, the case of p-Laplace operator). Then, for every $s \geq 0$*

$$c(s) = \min\left\{\beta>0\,;\,\widetilde{H}_0\left(I^0_{\beta,s}\setminus\{0\}\right)=0\right\}$$

holds, where $c(s)$ is a function defined by (3) and $\widetilde{H}_*$ denotes the reduced homology groups.

This corollary means that the mountain pass value $c(s)$ is attained by some continuous path $\gamma_s \in \Sigma$ for each $s \geq 0$.

4 The constrained variational problems

Throughout this section, we fix any $s \geq 0$. Thus, set $I_{\beta,s}(\cdot) = I_\beta(\cdot)$ for $\beta \in \mathbb{R}$ to simplify the notation. First, we define C^1 functionals Φ and Φ_+ on W by $\Phi(u) := \frac{1}{p}\|u\|_p^p$ and $\Phi_+(u) := \frac{1}{p}\|u_+\|_p^p$ for $u \in W$. Because r^p/p is a regular value of Φ and Φ_+ for each $r > 0$, it is well known that the norm of the derivative at $u \in (rS)$ or $u \in (rS_+)$ of the restriction of I_β or I_β^+ to rS or rS_+ is defined as follows:

$$\begin{aligned} \|\tilde{I_\beta}'(u)\|_* &:= \min\left\{ \|I_\beta'(u) - t\Phi'(u)\|_{W^*}\,;\, t \in \mathbb{R} \right\} \\ &= \sup\left\{ \langle I_\beta'(u), v\rangle\,;\, v \in T_u(rS),\ \|v\| = 1 \right\}, \\ \|(\tilde{I_\beta^+})'(u)\|_* &:= \min\left\{ \|(I_\beta^+)'(u) - t\Phi_+'(u)\|_{W^*}\,;\, t \in \mathbb{R} \right\}, \end{aligned} \tag{5}$$

where $T_u(rS)$ denotes the tangent space of rS at u, that is, $T_u(rS) = \{v \in W\,;\, \int_\Omega |u|^{p-2}uv\,dx = 0\}$ (cf. section 5.3 in [17] for (5)). It is known that rS and rS_+ are C^1 Finsler manifolds (cf. section 27.4 and 27.5 in [9]). Hence, rS and rS_+ are locally path connected. Concerning rS_+, the following result is proved.

Corollary 9 ([15]) *rS_+ is path connected for each $r > 0$.*

To state our results for constrained variational problems, we set the following open subsets of rS or rS_+ as follows:

$$\mathcal{O}(I_\beta, r, b) := \{u \in rS\,;\, I_\beta(u) < b\}, \quad \mathcal{O}^+(I_\beta^+, r, b) := \left\{u \in rS_+\,;\, I_\beta^+(u) < b\right\}$$

for $r > 0$ and $\beta, b \in \mathbb{R}$. Then, we have the following existence result.

Lemma 10 ([15]) *Let $\beta \in \mathbb{R}$, $r > 0$ and $b \in \mathbb{R}$. Then, any nonempty maximal open connected subset of $\mathcal{O}(I_\beta, r, b)$ or $\mathcal{O}^+(I_\beta^+, r, b)$ contains at least one critical point of $I_\beta|_{rS}$ or $I_\beta^+|_{rS_+}$, respectively.*

The above lemma plays an important role for the proof of constructing a suitable path. It is the developed result from one as in [5] for the manifold S.

5 Further results and remaining questions

Finally, the present author would like to take up two questions. First one is "Is the set Θ_A closed?" where Θ_A denotes the set of all (α, β) such that $(F)_{(\alpha,\beta)}$ has a non-trivial solution. Of course, in the case where A is $(p-1)$-homogeneous in the second variable, we know that the above question is true. Second is "When dose Θ_A contain a similar curve to the first nontrivial curve $\mathscr{C}$?" We state the following result related to the first question.

Proposition 11 *For $R_2 \geq R_1 > 0$, we set*

$$\begin{aligned} \Theta_A(R_1, R_2) &:= \left\{(\alpha, \beta) \in \mathbb{R}^2\,;\, (F)_{(\alpha,\beta)} \text{ has a solution in } D(R_1, R_2)\right\}, \\ \Theta_A(R_1, R_2)_p &:= \left\{(\alpha, \beta) \in \mathbb{R}^2\,;\, (F)_{(\alpha,\beta)} \text{ has a solution in } D_p(R_1, R_2)\right\}. \end{aligned}$$

Then, $\Theta_A(R_1, R_2)$ and $\Theta_A(R_1, R_2)_p$ are closed for any $R_2 \geq R_1 > 0$.

Proof. Let $\{(\alpha_n, \beta_n)\} \subset \Theta_A(R_1, R_2)_p$ (resp. $\Theta_A(R_1, R_2)$) be a sequence satisfying $\alpha_n \to \alpha_0$ and $\beta_n \to \beta_0$ as $n \to \infty$. Because of $(\alpha_n, \beta_n) \in \Theta_A(R_1, R_2)_p$ (resp. $\Theta_A(R_1, R_2)$), there exists a $u_n \in D_p(R_1, R_2)$ (resp. $D(R_1, R_2)$) being a solution of $(F)_{(\alpha_n,\beta_n)}$, that is, $-\mathrm{div}\, A(x, \nabla u_n) = \alpha_n u_{n+}^{p-1} - \beta_n u_{n-}^{p-1}$ in Ω, $\partial u_n/\partial \nu = 0$ on $\partial\Omega$. Then, we can see that $\{u_n\}$ is bounded in $L^\infty(\Omega)$. Indeed, by taking u_n as test function, we have

$$\frac{C_0}{p-1}\|\nabla u_n\|_p^p \le \int_\Omega A(x, \nabla u_n)\nabla u_n \, dx \le \max\{|\alpha_n|, |\beta_n|\}\|u_n\|_p^p \le \max\{|\alpha_n|, |\beta_n|\}R_2^p$$

by Remark 4 (iii). This implies the boundedness of $\|u_n\|$. Moreover, it is known that there exists a positive constant C independ of n such that $\|u_n\|_\infty \le C\|u_n\|$ because u_n is a solution of $(F)_{(\alpha_n,\beta_n)}$ and

$$|\alpha_n t_+^{p-1} - \beta_n t_-^{p-1}| \le \max\{|\alpha_0| + 1, |\beta_0| + 1\}|t|^{p-1} \tag{6}$$

for every $t \in \mathbb{R}$ and sufficiently large n (see Appendix in [14]). Thus, our claim is shown.

Because of the boundedness of $\|u_n\|_\infty$ and (6), the regularity result in [11] guarantees that there exist $\gamma \in (0, 1)$ and $M > 0$ independ of n such that $u_n \in C^{1,\gamma}(\overline{\Omega})$ and $\|u_n\|_{C^{1,\gamma}(\overline{\Omega})} \le M$. Since the inclusion of $C^{1,\gamma}(\overline{\Omega})$ to $C^1(\overline{\Omega})$ is compact, we may assume that u_n converges some u_0 in $C^1(\overline{\Omega})$ by choosing a subsequence. As a result, u_0 is a solution of $(F)_{(\alpha_0,\beta_0)}$ and $u_0 \in D_p(R_1, R_2)$ (resp. $D(R_1, R_2)$). Thus, $(\alpha_0, \beta_0) \in \Theta_A(R_1, R_2)_p$ (resp. $\Theta_A(R_1, R_2)$) holds, whence our conclusion is shown. ∎

For any $s \ge 0$ and $R_2 \ge R_1 > 0$ such that $K(I_{\beta,s}) \cap D_p(R_1, R_2) \ne 0$ for some $\beta > 0$, we can define $c_A(s, R_1, R_2)$ by

$$c_A(s, R_1, R_2) := \inf\{\beta \ge \beta_0(s)\,;\, K(I_{\beta,s}) \cap D_p(R_1, R_2) \ne \emptyset\}.$$

It follows from Proposition 11 that the above infimum is attained, that is,

$$c_A(s, R_1, R_2) = \min\{\beta \ge \beta_0(s)\,;\, K(I_{\beta,s}) \cap D_p(R_1, R_2) \ne \emptyset\}.$$

Then, the present author would like to consider the problem "What properties does $c_A(s, R_1, R_2)$ have?" to answer to the second question.

5.1 Asymptotically $(p-1)$ homogeneous case

In this subsection, we deal with the special case where the map $A(x, y)$ is asymptotically $(p-1)$ homogeneous in the following sense:
(AH) there exist a positive function $a_\infty \in C^1(\overline{\Omega}, \mathbb{R})$ and a function $\tilde{a}(x, t)$ on $\overline{\Omega} \times \mathbb{R}$ such that

$$A(x, y) = a_\infty(x)|y|^{p-2}y + \tilde{a}(x, |y|)y \quad \text{for every } x \in \Omega,\ y \in \mathbb{R}^N,$$
$$\text{and} \lim_{t\to+\infty} \frac{\tilde{a}(x, t)}{t^{p-2}} = 0 \quad \text{uniformly in } x \in \overline{\Omega}.$$

For this weight a_∞, we can define the following mountain pass value $c_{a_\infty}(s)$ by the same argument as in $c(s)$, namely

$$c_{a_\infty}(s) := \inf_{\gamma\in\Sigma} \max_{t\in[0,1]} \tilde{J}_{a_\infty,s}(\gamma(t)), \tag{7}$$

$$J_{a_\infty,s}(u) := \int_\Omega a_\infty(x)|\nabla u|^p\,dx - s\int_\Omega u_+^p\,dx, \quad \tilde{J}_{a_\infty,s} := J_{a_\infty,s}\big|_S.$$

It can be proved that the interval $(0, c_{a_\infty}(s))$ has no critical values of $\tilde{J}_{a_\infty,s}$.
Under the hypothesis (AH), we have the following result.

Proposition 12 *Assume (AH). Let $s \geq 0$, $\beta > 0$ and $\{u_n\}$ be a sequence of a solution for $(F)_{(s+\beta,\beta)}$. If $\|u_n\|_p \to \infty$ as $n \to \infty$, then $\beta \geq c_{a_\infty}(s)$ holds, where $c_{a_\infty}(s)$ is the constant defined by (7).*

Proof. Here, we give the sketch of the proof. Set $v_n := u_n/\|u_n\|_p$. Then, by the same argument as in [16, Proposition 36], we can prove that $\{v_n\}$ has a subsequence strongly convergent to a solution v of

$$-\mathrm{div}\,(a_\infty(x)|\nabla u|^{p-2}\nabla u) = (s+\beta)u_+^{p-1} - \beta u_-^{p-1} \quad \text{in } \Omega, \quad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega,$$

where a_∞ is the positive function as in (AH). This means that v is a critical point of $\tilde{J}_{a_\infty,s}$ with $\beta = \tilde{J}_{a_\infty,s}(v)$. Because $\beta > 0$ and $(0, c_{a_\infty}(s))$ contains no critical values of $\tilde{J}_{a_\infty,s}$, we obtain $\beta \geq c_{a_\infty}(s)$. ∎

Corollary 13 *Assume (AH) and $s \geq 0$. Then, we have*

$$\liminf_{R\to\infty} c_A(s, R, \infty) \geq c_{a_\infty}(s),$$

where $c_A(s, R, \infty) := \inf\{\beta \geq \beta_0(s)\,;\, K(I_{\beta,s}) \cap D_p(R,\infty) \neq \emptyset\}$.

Proof. By way of contradiction, we prove our assertion. So, we assume that there exists $s \geq 0$ such that $(0 < \beta_0(s) \leq)\beta := \liminf_{R\to\infty} c_A(s, R, \infty) < c_{a_\infty}(s)$. Then, by choosing a subsequence, we can take a sequence $\{u_n\}$ of a solution for $(F)_{(\beta_n+s,\beta_n)}$ with $\|u_n\|_p \to \infty$ and $\beta_n \to \beta$. By the same argument as in [16, Proposition 36], we can show that β is a critical value of $\tilde{J}_{a_\infty,s}$. Therefore, we have a contradiction because of $0 < \beta < c_{a_\infty}(s)$. ∎

The present author expect that in Theorem 3, we can choose β close to $c_{a_\infty}(s)$ under the additional hypothesis (AH).

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