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On congruence prime criterion for cusp forms on GL_2 over number fields

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1 Introduction

Around 1980, Hida proved an analogue of the class number formula for certain degree 3 L-functions, that is, he found a meaning of the special value of the adjoint L-functions of cusp forms. Let us recall briefly his discovery. We denote by $L(s, \mathrm{Ad}(f))$ the adjoint L-function associated with a normalized eigen cusp form f on GL_2 over the rational number field. For a fixed odd prime p, there are p-optimal periods $\Omega_{f,\pm}$ of f (see Section 3) which are well-defined up to p-adic units. Then Hida proved that $C_{f,p} := \Gamma(1, \mathrm{Ad}(f))L(1, \mathrm{Ad}(f))/\Omega_{f,+}\Omega_{f,-}$ is a p-adic integer. Moreover, we can consider the p-adic valuation of $C_{f,p}$ since the periods $\Omega_{f,\pm}$ are well-defined up to p-adic units. Hence, if we consider an analogue of the class number formula, we expect that the quantity $C_{f,p}$ have some algebraic nature. Hida proved the following theorem.

Theorem([Hi1],[Hi2]) *Let K be a number field which contains all Fourier coefficients of f. Let $\mathfrak{P}$ be a prime of K above p. Assume $p > k-2, p \nmid 6N$. Then $\mathfrak{P}$ divides the value $C_{f,p}$ if and only if $\mathfrak{P}$ is a congruence prime for f (see Definition 4.3 for the definition of congruence prime).*

After this discovery, Hida also proved that the inverse of the p-adic valuation of $C_{f,p}$ equals to the order of the congruence module of f ([Hi3]). Furthurmore, by proving that the order of congruence modules of cusp forms equal to the order of Selmer groups of the adjoint Galois representation associated with cusp forms, it is proved that the inverse of the p-adic valuation of $C_{f,p}$ equals to the order of the Selmer group of the adjoint Galois representation associated with a cusp form f by Taylor-Wiles ([HTU, (CN1)]). This formula is called as non-abelian class number formula.

In this article, we consider an analogue of Hida's theorem for cusp forms on GL_2 over arbitrary number fields. Such analogues of Hida's theorem are obtained by Urban

([Ur]) for the case of cusp forms on GL_2 over imaginary quadratic fields and also by Ghate and Dimitrov for the case of cusp forms on GL_2 over totally real number fields ([D], [Gh]). The main purpose of this article is to give a sufficient condition for a prime ideal $\mathfrak{P}$ to be a congruence prime for f for the case of cusp forms on GL_2 over arbitrary number fields F. The precise statement is as follows.

Let F be a number field. We put $I_F = \{F \hookrightarrow \mathbf{C}\}$. We denote by r_1 (resp. r_2) the number of real (complex) places of F. We put $t = \sum_{\sigma \in I_F} \sigma \in \mathbf{Z}[I_F]$. We denote the strict class number (resp. the discriminant, the ring of integers) of F by h_F (resp. D_F, $\mathcal{O}_F$). We fix an embedding $i_p : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$. We denote by $\mathfrak{p}$ the prime ideal of $\mathcal{O}_F$ above p which is determined by the embedding i_p. We also fix an isomorphism $\overline{\mathbf{Q}}_p \xrightarrow{\sim} \mathbf{C}$. Let f be a normalized newform on GL_2 over F. We denote the weight (resp. the central character, the complex conjugate) of f by $n+2t := \sum_{\sigma \in I_F}(n_\sigma + 2)\sigma$ (resp. χ, f^c). Let $\mathfrak{N}$ be an integral ideal of $\mathcal{O}_F$. We put $U := K_0(p) \cap K_1(\mathfrak{N})$ (for the definitions of $K_0(p)$ and $K_1(\mathfrak{N})$, see Section 2). Let $a_1, \cdots, a_{h_F}$ be a set of elements of $F_{\mathbf{A},f}$ such that $\{a_i\mathcal{O}_F\}_{i=1,\cdots,h_F}$ is a complete set of representatives of the strict class group and $a_i\mathcal{O}_F$ is prime to $\mathfrak{N}p$ for $i = 1, \cdots, h_F$. Assume that $\mathfrak{N}$ is sufficiently large so that $\Gamma_U^i \cap SL_2(F)$ is torsion-free for all $i = 1, \cdots, h_F$, where Γ_U^i is defined to be $GL_2(F) \cap \left(\begin{smallmatrix} a_i & 0 \\ 0 & 1 \end{smallmatrix}\right) GL_2^+(F_\infty)U \left(\begin{smallmatrix} a_i & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1}$. Suppose the f has the level U.

Assume that p does not divide $2\mathfrak{N}D_Fh_F\prod_{\sigma:F\hookrightarrow\mathbf{C}} n_\sigma!$. Let K be a number field which contains all Fourier coefficients of f and all conjugates of F over $\mathbf{Q}$. We take the prime $\mathfrak{P}$ of K above $\mathfrak{p}$ which is determined by the fixed embedding $K \hookrightarrow \overline{\mathbf{Q}}_p$. We denote the imprimitive adjoint L-function of f by $L^{\mathrm{i.p.}}(s, \mathrm{Ad}(f))$ whose precise definition will be given at Section 4. For a cusp form f, we define the complex number $w(f)$ which satisfies $W(f) = w(f)f^c$, where W is the Atkin-Lehner involution.

Let Y_U be the real manifold of dimension $2r_1 + 3r_2$ which is introduced in Section 2. We assume that parabolic cohomology groups of a certain local system $\mathscr{L}(n, \chi; \mathbf{C})$ on Y_U are isomorphic to cuspidal cohomology groups. This condition is always satisfied if $n \neq 0$.

The main theorem of this article is as follows.

Main Theorem(Theorem 4.4) *Assume that there exists an element u of $\mathcal{O}_F^\times$ which satisfies $u \equiv 1 \mod \mathfrak{N}$ and*

$$\mathfrak{P} \nmid \prod_{\iota \in \{0,\infty\}^{\Sigma(p)}} \prod_{\mathfrak{p} \in \Sigma(p) \text{ s.t. } \iota_\mathfrak{p} = \infty} \left(\prod_{\sigma \in I(\mathfrak{p})} (\chi^\iota(u) u^{n^\iota} (u^\sigma)^2 - 1) \right),$$

where we denote by $\Sigma(p)$ the set of primes of F above p. (see [Hi4, Section 3] for the definitions of $I(\mathfrak{p})$, χ^ι and u^{n^ι}). Assume that f is ordinary and minimal (see the beginning of Section 4 for the definition of minimal cusp forms). Assume also that $\mathfrak{P}$ divides the value

$$\frac{w(f)N(\mathfrak{N}p)\pi^{2r_2}\Gamma(1, \mathrm{Ad}(f))L^{\mathrm{i.p.}}(1, \mathrm{Ad}(f))}{\Omega^1_{f,\varepsilon}\Omega^2_{f,-\varepsilon}}$$

for some $\varepsilon \in \{\pm\}^{\Sigma(\mathbf{R})}$, where $\Omega^i_{f,\varepsilon}$ is the period of f which is defined in Definition 3.2 for $i = 1, 2$. Then $\mathfrak{P}$ is a congruence prime for f.

This article is organized as follows. In Section 2, we introduce a definition of cusp forms on GL_2 over arbitrary number fields and recall Eichler-Shimura isomorphism. In Section 3, we introduce a definition of periods of cusp forms. We note that almost all the statements and the definitions in Section 2 and 3 are the same as those which appeared in [Hi5]. The sketch of proof of Theorem 4.4 will be given in Section 4.

Notation

For a number field F, we denote the ring of adeles (resp. the idele group) of F by $F_{\mathbf{A}}$ (resp. $F_{\mathbf{A}}^{\times}$). We denote the set of embeddings of F to $\mathbf{C}$ by I_F. Let c be the complex conjugate of $\mathbf{C}$. We denote the number of real (resp. complex) places of F by r_1 (resp. r_2). We denote the ring of integers of F by $\mathcal{O}_F$. We put $\hat{\mathcal{O}}_F = \mathcal{O}_F \otimes_{\mathbf{Z}} \hat{\mathbf{Z}}$. We define F_∞ to be $\prod_{\sigma:\text{place of } F} F_\sigma$. Let D_F be the discriminant of F and h_F the class number of F in the narrow sense. We denote the connected component of the unit of $GL_2(F_\infty)$ by $GL_2^+(F_\infty)$.

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2 Automorphic forms on GL_2

In this section, we introduce the definition of cusp forms on GL_2 over a number field F and Eichler-Shimura isomorphism.

Let $n = \sum_{\sigma \in I_F} n_\sigma \sigma$ be an element of $\mathbf{Z}[I_F]$ which satisfies the following three conditions:

(i) For all $\sigma \in I_F$, n_σ is a non-negative integer.

(ii) For all $\sigma, \tau \in I_F$, $n_\sigma \equiv n_\tau \bmod 2$.

(iii) For a complex place σ, $n_\sigma = n_{\sigma c}$.

Let $m = \sum_{\sigma \in I_F} m_\sigma \sigma$ be an element of $\mathbf{Z}[I_F]$ which satisfies the following three conditions:

(i) For all $\sigma \in I_F$, m_σ is a non-negative integer.

(ii) For all $\sigma, \tau \in I_F$, $n_\sigma + 2m_\sigma = n_\tau + 2m_\tau$.

(iii) There exists a place σ such that $m_\sigma = 0$.

We denote $\Sigma_{\sigma \in I_F}\sigma \in \mathbf{Z}[I_F]$ by t. We put $\kappa = n + 2m$ and $k = n + 2t \in \mathbf{Z}[I_F]$. We write $[\kappa] = n_\sigma + 2m_\sigma$, where $\sigma \in I_F$. From the definition of m, $[\kappa]$ is independent of the choice of σ. For an ideal $\mathfrak{M}$ of $\mathcal{O}_F$, we define

$$K_0(\mathfrak{M}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\hat{\mathcal{O}}_F) : c \equiv 0 \mod \mathfrak{M} \right\},$$
$$K_1(\mathfrak{M}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\hat{\mathcal{O}}_F) : c, d-1 \equiv 0 \mod \mathfrak{M} \right\}.$$

We define

$$L(n^*; \mathbf{C}) = \bigotimes_{\sigma \in \Sigma(\mathbf{C})} \mathrm{Sym}^{n_\sigma + n_{\sigma c} + 2}(\mathbf{C}),$$

where we take the tensor product over $\mathbf{C}$, $\Sigma(\mathbf{C})$ is the set of complex places of F and $\mathrm{Sym}^{n_\sigma + n_{\sigma c}+2}(\mathbf{C})$ is the symmetric tensor product of the standard representation of $GL_2(F_\sigma)$ for $\sigma \in \Sigma(\mathbf{C})$. We denote a pair of basis of $\mathrm{Sym}^{n_\sigma + n_{\sigma c}+2}(\mathbf{C})$ by $\{\otimes_{\sigma \in I_F} X_\sigma^{n_\sigma - i_\sigma} Y_\sigma^{i_\sigma}; 0 \le i_\sigma \le n_\sigma\}$.

We introduce three elements of $M_2(\mathbf{Q})$:

$$X_+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, X_- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, Z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We consider these elements as elements of $\mathfrak{g} := \mathfrak{gl}_2(F_\sigma) \otimes_{\mathbf{R}} \mathbf{C}$ for an infinite place σ of F, and also consider them as elements of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. The element D in the universal enveloping algebra of $\mathfrak{g}$, which is called the Casimir element, is defined by the following identity:

$$D = X_+X_- + X_-X_+ + \frac{Z^2}{2}.$$

For an infinite place σ of F, $X \in \mathfrak{g}$ and a C^∞-function $f : GL_2(F_{\mathbf{A}}) \to L(n^*; \mathbf{C})$, we define a C^∞-function $X_\sigma f$ by the following properties:

(i) For a place τ which is different from σ, $(X_\sigma f)|_{GL_2(F_\tau)} = f|_{GL_2(F_\tau)}$.

(ii) For $g \in GL_2(F_\sigma)$, the function $(X_\sigma f)|_{GL_2(F_\sigma)}$ satisfies the following identity:

$$(X_\sigma f)|_{GL_2(F_\sigma)}(g) = \left(\frac{d}{dt} f(ge^{tX})\right)|_{t=0}.$$

We define the action of $U(\mathfrak{g})$ by extending the action of $\mathfrak{g}$. For a real place σ, we write $D_\sigma := D \otimes 1 \in U(\mathfrak{gl}_2(F_\sigma \otimes_{\mathbf{R}} \mathbf{C}))$. For a complex place σ, we write $D_\sigma := D \otimes 1$ and $D_{\sigma c} := 1 \otimes D \in U(\mathfrak{gl}_2(F_\sigma \otimes_{\mathbf{R}} \mathbf{C}))$.

Definition 2.1. *Let U be an open compact subgroup $K_0(p) \cap K_1(\mathfrak{N})$ of $GL_2(\hat{\mathcal{O}}_F)$. Let J be a subset of the set of real places of F. A cusp form on $GL_2(F_{\mathbf{A}})$ of weight k, of type J and with respect to U is a C^∞-function $f : GL_2(F_{\mathbf{A}}) \to L(n^*; \mathbf{C})$ which satisfies the following conditions:*

(i) *For any* $\sigma \in I_F$, $D_\sigma f = \left(\frac{n_\sigma^2}{2} + n_\sigma\right) f$.

(ii) *For any* $\gamma \in GL_2(F), z_\infty \in F_\infty^\times$ *and* $g \in GL_2(F_{\mathbf{A}})$, *we have* $f(\gamma z_\infty g) = z_\infty^{-\kappa} f(g)$.

(iii) *There exists a Hecke character* $\chi : F_{\mathbf{A}}^\times \to \mathbf{C}^\times$ *which satisfies the following properties:*

  (a) *For any* $z_\infty \in F_\infty^\times$, $\chi(z_\infty) = z_\infty^{-\kappa}$.

  (b) *For any* $z \in F_{\mathbf{A}}^\times$ *and* $g \in GL_2(F_{\mathbf{A}})$, $f(zg) = \chi(z) f(g)$.

  (c) *The conductor of* χ *divides* $\mathfrak{N}p$.

  The Hecke character χ *is called the central character of* f.

(iv) *For any* $u = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U$ *and* $g \in GL_2(F_{\mathbf{A}})$, $f(gu) = \chi_U(d) f(g)$, *where we define* $\chi_U(d) = \prod_{v \mid \mathfrak{N}p} \chi(d_v)$.

(v) *For any* $u = \left(\left(\begin{pmatrix} \cos\theta_\sigma & -\sin\theta_\sigma \\ \sin\theta_\sigma & \cos\theta_\sigma \end{pmatrix}\right)_{\sigma \in \Sigma(\mathbf{R})}, (u_\sigma)_{\sigma \in \Sigma(\mathbf{C})}\right) \in C_{\infty,+}$,

$$f\left(gu : \otimes_{\sigma \in \Sigma(\mathbf{C})} \begin{pmatrix} S_\sigma \\ T_\sigma \end{pmatrix}\right)$$
$$= e^{\sqrt{-1}(-\sum_{\sigma \in J} \theta_\sigma k_\sigma + \sum_{\sigma \in \Sigma(\mathbf{R}) \setminus J} \theta_\sigma k_\sigma)} f\left(g : \otimes_{\sigma \in \Sigma(\mathbf{C})} u_\sigma \begin{pmatrix} S_\sigma \\ T_\sigma \end{pmatrix}\right).$$

(vi) *For any* $g \in GL_2(F_{\mathbf{A}})$, *we have* $\int_{U(F)\backslash U(F_{\mathbf{A}})} f(vg)du = 0$, *where we define* $U(F) = \{v = \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right); u \in F\}$ *and* $U(F_{\mathbf{A}}) = \{v = \left(\begin{smallmatrix} 1 & u \\ 0 & 1 \end{smallmatrix}\right); u \in F_{\mathbf{A}}\}$.

Let us denote by $S_{k,J}(U)$ *the space of cusp forms on* $GL_2(F_{\mathbf{A}})$. *In particular, we denote by* $S_{k,J}(U,\chi)$ *the subspace of* $S_{k,J}(U)$ *which consists of cusp forms with a central character* χ.

We put $Y_U = GL_2(F)\backslash GL_2(F_{\mathbf{A}})/F_\infty^\times \prod_{\sigma:\text{real}} SO_2(\mathbf{R}) \prod_{w:\text{complex}} SU_2(\mathbf{C})U$. We denote by $L(n,\chi,\mathbf{C})$ the $GL_2(\hat{\mathcal{O}}_F)$-module $L(n;\mathbf{C})$ with twisted action by χ (see [Hi5, Section 3] for the precise definition). We denote by $\mathcal{L}(n,\chi;\mathbf{C})$ the local system on Y_U which is determined by $L(n,\chi;\mathbf{C})$. In this article, we will consider parabolic cohomology groups $H^{\cdot}_{\text{par}}$ and cuspidal cohomology groups $H^{\cdot}_{\text{cusp}}$ of $\mathcal{L}(n,\chi;\mathbf{C})$. For the definitions of parabolic cohomology groups and cuspidal cohomology groups, we refer to [Ha] and [Hi5].

The following theorem is well-known as Eichler-Shimura isomorphism.

Theorem 2.2. (Eichler-Shimura)([Hi5, Proposition 3.1]) *We put* $\chi_0 = \chi|_{\hat{\mathcal{O}}_F^\times}$. *For* $r = 1$ *or* 2, *there exists a Hecke-equivariant isomorphism:*

$$\delta^r : \oplus_J \oplus_\psi S_{k,J}(U,\psi) \xrightarrow{\sim} H^{r_1+rr_2}_{\mathrm{cusp}}(Y_U, \mathscr{L}(n,\chi;\mathbf{C})),$$

where J *runs over the power set of the set of real places and* ψ *runs over all Hecke character* $F_{\mathbf{A}}^\times \to \mathbf{C}$ *of infinity type* $z^{-\kappa}$ *and its restriction to* $\hat{\mathcal{O}}_F^\times$ *is equal to* χ_0.

We note that, if $n \neq 0$, parabolic cohomology groups are isomorphic to cuspidal cohomology groups by [Ha, Proposition 3.2.4]. For simplicity, in this article we always assume that parabolic cohomology groups are isomorphic to cuspidal cohomology groups.

In the next section, we define p-optimal periods of a normalized Hecke eigen cusp form f by using the above theorem.

3 Definition of period

In this section, we define periods of cusp forms.

There exists the natural action of $\prod_{v:\mathrm{real}} O_2(\mathbf{R})/SO_2(\mathbf{R}) \cong \{\pm 1\}^{\Sigma(\mathbf{R})}$ on $H^{r_1+rr_2}_{\mathrm{par}}$ $(Y_U, \mathscr{L}(n,\chi;\mathbf{C}))$. Let $\varepsilon : \{\pm 1\}^{\Sigma(\mathbf{R})} \to \mathbf{C}^\times$ be a character and f a normalized Hecke eigenform of $S_{k,J}(U,\chi)$. We naturally identify ε as the element of $\{\pm\}^{\Sigma(\mathbf{R})}$. We denote by $H^q_{\mathrm{par}}(Y_U, \mathscr{L}(n,\chi;\mathbf{C}))[f,\varepsilon,\chi]$ the subspace of $H^q_{\mathrm{par}}(Y_U, \mathscr{L}(n,\chi;\mathbf{C}))$ such that the operator $T(\mathfrak{q})$ (resp. $\left[U \begin{pmatrix} \varpi_{\mathfrak{q}'} & 0 \\ 0 & \varpi_{\mathfrak{q}'} \end{pmatrix} U\right]$, $\iota \in \{\pm 1\}^{\Sigma(\mathbf{R})}$) acts as scalar multiplication by the Hecke eigenvalue of f for $T(\mathfrak{q})$ for all prime ideal $\mathfrak{q}$ of F (resp.by $\chi(\varpi_{\mathfrak{q}'})$ for all prime ideal $\mathfrak{q}'$ of F which is prime to $\mathfrak{N}P$, by $\varepsilon(\iota)$).

Then, by Theorem 2.2, we obtain the following proposition under the assumption that parabolic cohomology groups are isomorphic to cuspidal cohomology groups.

Proposition 3.1. *There exists a Hecke-equivariant isomorphism:*

$$\delta^r_{J,\varepsilon} : S_{k,J}(U,\chi) \xrightarrow{\sim} H^{r_1+rr_2}_{\mathrm{par}}(Y_U, \mathscr{L}(n,\chi;\mathbf{C}))[\varepsilon,\chi].$$

For $r = 1$ or 2, we note that $\delta^r_{J,\varepsilon}(f)$ is an element of $H^{r_1+rr_2}_{\mathrm{par}}(Y_U, \mathscr{L}(n,\chi;\mathbf{C}))[f,\varepsilon,\chi]$ and that the $\mathbf{C}$-vector space $H^{r_1+rr_2}_{\mathrm{par}}(Y_U, \mathscr{L}(n,\chi;\mathbf{C}))[f,\varepsilon,\chi]$ is 1-dimensional by the multiplicity one theorem.

Let K be a number field which contains all Fourier coefficients of f and the all conjugates of F over $\mathbf{Q}$. We denote the ring of integers of K by $\mathcal{O}_K$. We denote by $\mathfrak{P}$ the prime ideal of $\mathcal{O}_K$ which is determined by the fixed embedding $\overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}}_p$. We denote the completion of K at $\mathfrak{P}$ by $K_{\mathfrak{P}}$ and denote the ring of integers of $K_{\mathfrak{P}}$ by $\mathcal{O}_{K,\mathfrak{P}}$.

There exists the natural homomorphism

$$H^{r_1+rr_2}_{\mathrm{par}}(Y_U, \mathscr{L}(n,\chi;\mathcal{O}_{K,\mathfrak{P}}))[f,\varepsilon,\chi] \to H^{r_1+rr_2}_{\mathrm{par}}(Y_U, \mathscr{L}(n,\chi;\mathbf{C}))[f,\varepsilon,\chi].$$

We see that the image of the above map is free of rank one over $\mathcal{O}_{K,\mathfrak{P}}$ by the universal coefficient theorem. We fix a generator $\eta^r_{f,\varepsilon}$ of the $\mathcal{O}_{K,\mathfrak{P}}$-module.

Definition 3.2. *For a normalized Hecke eigenform $f \in S_{k,J}(U,\chi)$, $r = 1,2$ and $\varepsilon \in \{\pm\}^{\Sigma(\mathbf{R})}$, we define a p-optimal period $\Omega^r_{f,\varepsilon} \in \mathbf{C}^\times$ of cusp form f by the following identity:*

$$\delta^r_{J,\varepsilon}(f) = \Omega^r_{f,\varepsilon}\eta^r_{f,\varepsilon}.$$

The definition of the period $\Omega^r_{f,\varepsilon}$ depends on the choice of a generator $\eta^r_{f,\varepsilon}$, hence $\Omega^r_{f,\varepsilon}$ is uniquely determined up to multiplication by p-adic units.

4 Integrality of L-values and congruence prime criterion

In this section, we introduce an integrality of special values of adjoint L-functions of cusp forms (Theorem 4.2) and the main theorem of this article (Theorem 4.4).

An irreducible admissible representation π_v of $GL_2(F_v)$ is called minimal, if conductor of π_v is minimal in the set $\{\mathrm{cond}(\pi_v \otimes \eta); \eta : F_v^\times \to \mathbf{C}^\times\}$. For a newform f, we denote by π_f the admissible representation which is determined by the right translation of f by $GL_2(F)$. We decompose π_f to the restricted tensor product $\otimes'_{v:\mathrm{place}}\pi_{f,v}$. A newform f is called minimal, if $\pi_{f,v}$ is minimal for all finite places v.

Definition 4.1. *For a place v of F, we define:*

$$L^{\mathrm{i.p.}}_v(s,\mathrm{Ad}(f)) = \begin{cases} \frac{1}{(1-\frac{\mu_v(\varpi_v)}{\nu_v(\varpi_v)}q^{-s})(1-q^{-s})(1-\frac{\nu_v(\varpi_v)}{\mu_v(\varpi_v)}q^{-s})} & (v \nmid \mathfrak{N}p, \pi_{f,v} \cong \pi(\mu_v,\nu_v) : \textit{principal series}), \\ \frac{1}{1-q_v^{-s}} & (v \mid \mathfrak{N}P, \pi_{f,v} : \textit{principal series and minimal}), \\ \frac{1}{1-q_v^{-s-1}} & (v \mid \mathfrak{N}P, \pi_{f,v} : \textit{special and minimal}), \\ \Gamma_{\mathbf{R}}(s+1)\Gamma_{\mathbf{C}}(s+k_v-1) & (v : \textit{real}), \\ \Gamma_{\mathbf{C}}(s)^2\Gamma_{\mathbf{C}}(s+k_v-1)^2 & (v : \textit{complex}), \\ 1 & (v : \textit{otherwise}). \end{cases}$$

We also define:

$$L^{\mathrm{i.p.}}(s,\mathrm{Ad}(f)) = \prod_{v:\text{finite place}} L^{\mathrm{i.p.}}_v(s,\mathrm{Ad}(f)),$$
$$\Gamma(s,\mathrm{Ad}(f)) = \prod_{v:\text{infinite place}} L^{\mathrm{i.p.}}_v(s,\mathrm{Ad}(f)).$$

Theorem 4.2. *Assume that p does not divide $2\mathfrak{N}D_Fh_F\prod_{\sigma:F\hookrightarrow\mathbf{C}}n_\sigma!$. Then, for any element ε of $\{\pm\}^{\Sigma(\mathbf{R})}$, the value*

$$\frac{w(f)N(\mathfrak{N}P)\pi^{2r_2}\Gamma(1,\mathrm{Ad}(f))L^{\mathrm{i.p.}}(1,\mathrm{Ad}(f))}{\Omega^1_{f,\varepsilon}\Omega^2_{f,-\varepsilon}}$$

is an element of $\mathcal{O}_{K,\mathfrak{P}}$.

Definition 4.3. *For a newform f in $S_{k,J}(U,\chi)$, if there exists a newform h in $S_{k,J}(U,\chi)$ which is distinct from f and a prime ideal $\mathfrak{P}$ of a number field K which contains all Fourier coefficients of f and h such that $a(\mathfrak{a},f) \equiv a(\mathfrak{a},h) \mod \mathfrak{P}$ for any ideal $\mathfrak{a}$ of $\mathcal{O}_F$, then we call $\mathfrak{P}$ as a congruence prime for f.*

The main theorem of this paper is as follows.

Theorem 4.4. *Let f be an element of $S_{k,J}(U,\chi)$. We assume that f is a newform and that the automorphic representation $\pi_{f,v}$ of $GL_2(F_v)$ which is associated with f is minimal for any finite place v. We put $n = k - 2t$. Assume that there exists an element u of $\mathcal{O}_F^\times$ which satisfies the following two conditions*

(i) $u \equiv 1 \mod \mathfrak{N}$

(ii) $\mathfrak{P}$ does not divide the value $\displaystyle\prod_{\iota\in\{0,\infty\}^{\Sigma(p)}} \prod_{\mathfrak{p}\in\Sigma(p) \text{s.t.} \iota_{\mathfrak{p}}=\infty} \left(\prod_{\sigma\in I(\mathfrak{p})} (\chi^{\iota}(u)u^{n^{\iota}}(u^{\sigma})^2 - 1) \right)$.

We also assume that parabolic cohomology groups $H^q_{\text{par}}(Y_U, \mathscr{L}(n,\chi;\mathbf{C}))$ are isomorphic to cuspidal cohomology groups $H^q_{\text{cusp}}(Y_U, \mathscr{L}(n,\chi;\mathbf{C}))$ for $0 \le q \le 2r_1 + 3r_2$. If f is ordinary and $\mathfrak{P}$ divides the value

$$\frac{w(f)\mathrm{N}(\mathfrak{N}P)\pi^{2r_2}\Gamma(1,\mathrm{Ad}(f))L^{\mathrm{i.p.}}(1,\mathrm{Ad}(f))}{\Omega^1_{f,\varepsilon}\Omega^2_{f,-\varepsilon}},$$

for some character $\varepsilon : \{\pm 1\}^{\Sigma(\mathbf{R})} \to \{\pm 1\}$, then $\mathfrak{P}$ is a congruence prime for f.

(Sketch of proof of Theorem 4.2)

There exists a homomorphism of local system on Y_U

$$\mathscr{L}(n,\chi;\mathcal{O}_{K,\mathfrak{P}}) \otimes \mathscr{L}(n,\overline{\chi};\mathcal{O}_{K,\mathfrak{P}}) \to \mathcal{O}_{K,\mathfrak{P}}.$$

If the class number in narrow sense h_F is 1, this homomorphism is induced by the following pairing of $\mathcal{O}_{K,\mathfrak{P}}$-module

$$L(n,\chi;\mathcal{O}_{K,\mathfrak{P}}) \times L(n,\overline{\chi};\mathcal{O}_{K,\mathfrak{P}}) \to \mathcal{O}_{K,\mathfrak{P}}$$

$$(\otimes_{\sigma\in I_F} \sum_{i=0}^{n_\sigma} u_{\sigma,i}X_\sigma^{n_\sigma-i}Y_\sigma^i, \otimes_{\sigma\in I_F} \sum_{i=0}^{n_\sigma} v_{\sigma,i}X_\sigma^{n_\sigma-i}Y_\sigma^i) \mapsto \prod_{\sigma\in I_F}\sum_{j=0}^{n_\sigma} \frac{(-1)^j u_{\sigma,j}v_{\sigma,n-j}}{\binom{n_\sigma}{j}}.$$

We note that we have assumed $\prod_{\sigma\in I_F} n_\sigma!$ is invertible in $\mathcal{O}_{K,\mathfrak{P}}$. By using above homomorphism, we have the following homomorphism:

$$[\ ,\ ]_n : H^{r_1+r_2}_{\text{par}}(Y_U, \mathscr{L}(n,\chi;\mathcal{O}_{K,\mathfrak{P}}))' \times H^{r_1+2r_2}_{\text{par}}(Y_U, \mathscr{L}(n,\overline{\chi};\mathcal{O}_{K,\mathfrak{P}}))' \to \mathcal{O}_{K,\mathfrak{P}},$$

where $H^{r_1+rr_2}_{\text{par}}(Y_U,*)'$ is the maximal torsion-free quotient of $H^{r_1+rr_2}_{\text{par}}(Y_U,*)$. There exists a homomorphism

$$[\tau] : H^{r_1+2r_2}_{\text{par}}(Y_U, \mathscr{L}(n,\chi;\mathcal{O}_{K,\mathfrak{P}}))' \to H^{r_1+2r_2}_{\text{par}}(Y_U, \mathscr{L}(n,\overline{\chi};\mathcal{O}_{K,\mathfrak{P}}))'$$

such that $[\tau] \circ \delta^{r}_{J,\varepsilon}(f) = \delta^{r}_{J,\varepsilon}(W(f))$ for $f \in S_{k,J}(U)$, where W is the Atkin-Lehner involution. We put $(x_1, x_2)_n = [x_1, [\tau](x_2)]_n$ for $x_r \in H^{r_1+rr_2}_{\mathrm{par}}(Y_U, \mathscr{L}(n, \chi; \mathcal{O}_{K,\mathfrak{p}}))$ and $r = 1, 2$. Then, by the definition of p-optimal periods, we see that $(\eta^1_{f,\varepsilon}, \eta^2_{f,-\varepsilon})_n = \dfrac{(\delta^1_{J,\varepsilon}, \delta^2_{J,-\varepsilon})_n}{\Omega^1_{f,\varepsilon}\Omega^2_{f,-\varepsilon}}$ is an element of $\mathcal{O}_{K,\mathfrak{p}}$.

On the other hand, by using Rankin-Selberg method, we see that $(\delta^1_{J,\varepsilon}, \delta^2_{J,-\varepsilon})_n$ is the value of imprimitive adjoint L-function at $s = 1$.

Hence, we obtain the theorem.

(Sketch of proof of Theorem 4.4)

The basic strategy of our proof is similar to the proof of Hida's theorem for cusp forms on GL_2 over the rational number field. However, in the case of GL_2 over arbitrary number fields, there are technical problems for the proof of congruence prime criterion. Most important point in the proof is to prove that $(\ ,\ )_n$ is a perfect pairing.

To prove that $(\ ,\ )_n$ is a perfect pairing, it is enough to prove that a cohomology group $H^q(\partial Y^*_U, \mathscr{L}(n, \chi; K/\mathcal{O}_{K,\mathfrak{p}}))$ is a divisible $\mathcal{O}_{K,\mathfrak{p}}$-module for all q, where ∂Y^*_U is the boundary of Borel-Serre compactification Y^*_U of Y_U. To prove this, we use the following lemma.

Lemma 4.5. ([Gh, Lemma 2]) *Let G be a finitely generated group and M a G-module. Assume that there exists an element g of the center of G such that $g - 1 : M \longrightarrow M$ is an automorphism. Then $H^q(G, M) = 0$ for all $q \geq 0$.*

By using the above lemma, Ghate proved that $(\ ,\ )_n$ is a perfect pairing under the assumption that $h_F = 1$, $\mathfrak{N} = \mathcal{O}_F$, $n = 0$ and F is a totally real number field in [Gh]. However, we can prove that $(\ ,\ )_n$ is a perfect pairing for arbitrary $h_F, \mathfrak{N}, n$ and F by using the above lemma and the assumption that an existence of $u \in \mathcal{O}^\times_F$ which satisfies (i) and (ii) in the theorem.

To study congruences between cusp forms, Hida introduced the congruence module $T_{f,p}$ for a cusp form f. Hida proved that, for a prime ideal $\mathfrak{P}$, $\mathfrak{P}$ is a congruence prime for a cusp form f if and only if $\mathfrak{P}$ is an element of the support of $T_{f,p}$. By using the perfectness of $(\ ,\ )_n$, we easily see that, if $\mathfrak{P}$ divides the algebraic part of the special value of imprimitive adjoint L-function at $s = 1$, then $\mathfrak{P}$ is an element of support of $T_{f,p}$. Hence, we see that $\mathfrak{P}$ is a congruence prime for f.

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