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On a Stochastic Cash Management Model with Two Sources of Short-term Funds

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Abstract In this paper, we consider a cash management model in which two types of funds are available for a manager to adjust cash level. We assume that the rate of utilizing the two funds for the amount of adjustment is constant. The objective of the paper is to find an optimal policy so as to minimize the expected discounted costs over an infinite horizon. We formulate this cash balance management problem as an impulse control problem and then derive an optimal cash management policy. Moreover, we obtain explicit policy parameters when there is no discount rate, and discuss the properties of the optimal policy.

1. Introduction

The financial manager can increase or decrease the amount of cash by selling or buying short-term securities. A transfer cost is incurred when changing the cash level. When the manager does not make any changes in the cash level, there are costs involved in holding stock or in being understocked. One cash management problem is to find an optimal level of the cash balance in order to minimize the expected total of these costs.

In this paper, we deal with a cash management model in which two types of funds with different transaction costs are available whenever the manager adjusts the cash balance level. The first paper which deals with this type of problem seems to be Daellenbach [4]. He formulated this model by using a dynamic programming formulation in discrete time. Perhaps the paper which is closest to ours in terms of the structure of cost function is Elton and Gruber [5]. However, the existence of an optimal policy remains unproved in their paper. Sato and Sawaki [7] reformulated this problem in continuous time as an extension of Constantinides and Richard [3] and show that there exists an optimal policy for the cash management problem over an infinite-horizon by using impulse control. The policy is described as band policy when the rate of utilizing the two funds for the amount of adjustment is constatnt. However, the effect of the rate of utilizing the two funds to the policy parameters has not been shown in their paper. Therefore, in this paper, we find explicit policy parameters when there is no discount rate, and discuss the properties of the optimal policy.

2. The Analysis of the Model

In this section, we introduce terminologies and notations and then present the problem formulation. Let $(\Omega,\mathscr{F},P)$ be a complete probability space equipped with a filtration $\mathscr{F}_t$ satisfying the usual information structure, and w_t a Brownian motion. Consider a manager who is in charge of the cash management of the company. He/she wishes to control the stochastic cash level X_t. The cash level at time t is given by

$$\begin{cases} dX_t = \mu dt + \sigma dw_t \\ X_0 = x \end{cases} \tag{2.1}$$

where x is the initial cash level. X_t is a Brownian motion with drift μ and a diffusion parameter $\sigma > 0$. The manager can change this cash level by using two sources of funds at any time. Suppose that the sources of funds are short-term borrowing and marketable securities. Let B_t be the amount of short-term debt outstanding at time t.

A policy $v \in \mathscr{V}$ consists of the two sequences $\{(\tau_i, \xi_i), i = 0,1,2,\cdots\}$ of timing for making changes in cash levels $\{\tau_0, \tau_1, \cdots\}$ and the size of control $\{\xi_0, \xi_1, \cdots\}$ such that

$$\begin{cases} 0 \le \tau_i < \tau_{i+1}, i = 0,1,2,\cdots \\ \tau_i \text{ is an } i\text{th stopping time with respect to the filtration } \mathscr{F}_{\tau_i} = \sigma\{X_s^-, s \le \tau_i\}, \\ \xi_i \text{ is } \mathscr{F}_{\tau_i}\text{-measurable.} \end{cases} \tag{2.2}$$

When the cash level changes from x to $x + \xi$, we suppose that the rate of utilizing the two funds for the amount of adjustment ξ is θ $(0 \le \theta \le 1)$, that is, the amount of borrowing is $\theta\xi$, and the amount of securities is $(1-\theta)\xi$.

Given an impulse control $v = \{(\tau_i, \xi_i), i = 0,1,2,\cdots\}$, the state of the system is defined as

$$\begin{cases} dX_t^v = \mu dt + \sigma dw_t, \quad \tau_i < t < \tau_{i+1},\ i \ge 0, \\ X_{\tau_i}^v = X_{\tau_i^-}^v + \xi_i, \quad i \ge 1, \\ dB_t^v = 0, \\ B_{\tau_i}^v = B_{\tau_{i-1}}^v + \theta\xi_i, \quad i \ge 1, \\ X_0^v = x,\ B_0^v = b. \end{cases} \tag{2.3}$$

If $\theta|\xi_i| > B_t^v$ at the time of paying out the debts $(\xi_i < 0)$, then we assume that the amount of difference $\theta|\xi| - B_t^v$ is used to buy securities. When the cash level changes from x to $x + \xi$, the transition costs occurs as follows;

$$F(\xi_i) = \begin{cases} K_B^u + k_B^u \theta \xi_i & \text{if } \xi_i \geq 0 \text{ (Debt finance)} \\ K_B^d + k_B^d \theta |\xi_i| & \text{if } \xi_i < 0 \text{ (Debt extinguishment)} \\ K_S^u + k_S^u (1-\theta) \xi_i & \text{if } \xi_i \geq 0 \text{ (Selling the security)} \\ K_S^d + k_S^d (1-\theta) |\xi_i| & \text{if } \xi_i < 0 \text{ (Buying the security)} \end{cases} \tag{2.4}$$

where K and k are fixed cost and variable cost, respectively, the subscripts B and S indicate borrowings and securities, and the superscripts u (d) represents an increase (decrease) in the cash level. Furthermore, $F(\xi)$ can be rewritten as follows;

$$F(\xi_i) = \begin{cases} K_1 + k_1(\theta)\xi_i, & \text{if } \xi_i \geq 0, \\ K_2 + k_2(\theta)|\xi_i|, & \text{if } \xi_i < 0, \end{cases} \tag{2.5}$$

where $K_1 = K_B^u + K_S^u$, $K_2 = K_B^d + K_S^d$, $k_1(\theta) = k_S^u + (k_B^u - k_S^u)\theta$ and $k_2(\theta) = k_S^d + (k_B^d - k_S^d)\theta$.

We assume that the holding and penalty cost rates are

$$C(B_{\tau_i}^v, X_t^v) = \begin{cases} -pX_t^v, & \text{if } X_t^v \leq 0, \\ h_1 X_t^v, & \text{if } 0 < X_t^v \leq B_{\tau_i}^v, \\ h_1 B_{\tau_i}^v + h_2(X_t^v - B_{\tau_i}), & \text{if } B_{\tau_i}^v < X_t^v, \end{cases} \tag{2.6}$$

where p is the penalty cost, h_1 is the interest rate on short-term debt, and h_2 is the opportunity cost without marketable securities instead of cash. We also assume that the opportunity cost h_2 is less than the interest rate h_1, $h_1 > h_2$, since there are some costs based upon risks of securities.

Here, if the following conditions hold, then an impulse control v is called admissible;

$$E_x \left[\int_0^\infty e^{-\alpha s} C(B_s^v, X_s^v)) ds \right] < \infty, \tag{2.7}$$

$$P(\lim_{i \to \infty} \tau_i \leq T) = 0, \forall T \geq 0, \tag{2.8}$$

$$\lim_{T \to \infty} E_x[e^{-\alpha T} X_T] = 0. \tag{2.9}$$

Assumption 2.1. *We assume that the parameters must satisfy the following inequalities;*

(a) $\max\{k_B^u, k_S^u\} \leq \frac{p}{\alpha} - \frac{h_1 - h_2}{\alpha}$

(b) $\max\{k_B^d, k_S^d\} \leq \frac{h_2}{\alpha}$

where α *is a discount rate,* $0 < \alpha < 1$.

$\frac{p}{\alpha}$ is the present value of the penalty cost of keeping one unit of cash from now to infinity. $\frac{h_1 - h_2}{\alpha}$ is the present cost by borrowing from a bank instead of selling securities. Similarly, $\frac{h_2}{\alpha}$ is the present value of the holding cost of one unit of cash in debt and security from now to infinity.

We define the total discounted expected cost function for a given policy v as follows;

$$J_{b,x}(v) \equiv E_x^v\left[\int_0^{\infty} e^{-\alpha s}C(B_s^v, X_s^v)ds + \sum_{i=1}^{\infty} e^{-\alpha s}F(\xi_i) \mid X_0^v = x, B_0^v = b\right]. \tag{2.10}$$

Then, the value function is defined as follows;

$$\Phi(b,x) = \inf_{v\in\mathcal{V}} J_{b,x}(v). \tag{2.11}$$

We consider the QVI (Quasi-Variational Inequality) problem to show the existence of an optimal policy that achieves the infimum in equation (2.11) and to obtain a closed-form solution of the value function. In order to derive a QVI, we follow the same approach as Baccarin [1] and Constantinides and Richard [3].

If the manager needs a volume of transaction ξ at time t, then the cash level jumps from x to $x+\xi$, and the amount of short-term debt outstanding jumps from b to $b+\xi$. The total cost caused by this transaction is given as

$$\inf_{\xi}\{F(\xi) + \Phi(b+\theta\xi, x+\xi)\}. \tag{2.12}$$

On the other hand, if the manager does not transact cash in the small interval, then the amount of short-term debt outstanding B_t does not change. Hence, the cost structure is similar to Constantinides and Richard [3]. Here, we define two operators, L and M, as follows;

$$L\phi(b,x) = \alpha\phi(b,x) - \mu\phi^{'}(b,x) - \frac{1}{2}\sigma^2\phi^{''}(b,x) \tag{2.13}$$

$$M\phi(b,x) = \inf_{\xi}\{F(\xi) + \phi(b+\theta\xi, x+\xi)\} \tag{2.14}$$

where $\phi^{'}(b,x) = \frac{\partial\phi(b,x)}{\partial x}$ and $\phi^{''}(b,x) = \frac{\partial^2\phi(b,x)}{\partial x^2}$. Then, the following relations are called QVI for the problem (2.11);

$$L\phi - C \leq 0 \tag{2.15}$$

$$\phi \leq M\phi \tag{2.16}$$

$$(L\phi - C)(\phi - M\phi) = 0 \tag{2.17}$$

The following theorem is given by Korn [6]. It guarantees that the solution of QVI is equal to the value function given by equation (2.11).

Theorem 2.1. *If there exists a solution $\phi \in C^2$ that satisfies the growth conditions*

$$E_x^v\left[\int_0^{\infty}(e^{-\alpha s}\sigma(X_s)\phi^{'}(B_s, X_s))^2 ds\right] < \infty, \tag{2.18}$$

$$\lim_{T\to\infty} E[e^{-rT}\phi(B_T, X_T)] = 0, \tag{2.19}$$

for every process X_t corresponding to an admissible impulse control v, then we have

$$\Phi(b,x) \geq \phi(b,x) \tag{2.20}$$

for every $x \in \mathbb{R}$. Moreover, if the QVI-control corresponding to ϕ, that is, the impulse control v satisfying

(i) $(\tau_0, \xi_0) = (0,0)$

(ii) $\tau_i := \inf\{t \geq \tau_{i-1} : \phi(B_{\tau_{i-1}}, X_{t^-}) = M\phi(B_{\tau_{i-1}}, X_{t^-})\}$

(iii) $\xi_i := \arg\min_\xi E[F(\xi) + \phi(B_{\tau_{i-1}} + \xi, X_{\tau_i^-} + \xi)]$

is admissible, then v attaining $\Phi(b,x)$ is an optimal impulse control, and for every $x \in \mathbb{R}$

$$\Phi(b,x) = \phi(b,x). \tag{2.21}$$

3. Existence of Optimal Cash Management Policy

In this section, we derive a solution for the QVI problem under the assumption that the value function is continuous and twice differentiable. After we guess an optimal policy of the band type, we show that the optimal policy satisfies the hypothesis of Theorem 2.1.

Let $\mathbf{p} := (d_b, D_b, U_b, u_b)$ be the parameters of a control band policy satisfying $d_b < D_b < U_b < u_b$. Then, we suppose that the continuation region has the form of

$$\mathcal{D} \equiv \{(b,x) : d_b < x < u_b\}. \tag{3.1}$$

All of the parameters are expressed as a function of the amount of short-term debt outstanding B_t because the holding and penalty cost C depends on B_t. Recalling the fact that the changing of B_t is exclusive to the transaction time, these parameters are constant in the continuation region.

In this model, the policy procedure is as follows. First, we determine the values $\mathbf{p}$ based on initial value b of the cycle. If the cash level reaches either d_b or u_b, then we increase the cash level up to D_b or decrease down to U_b. And then, B_t changes from b to $b + \theta(D_b - x)$ or $b - \theta(x - U_b)$.

In $\mathcal{D}$, inequality (2.15) holds as an equality, that is,

$$C(b,x) - \alpha\phi(b,x) + \mu\phi'(b,x) + \frac{1}{2}\sigma^2\phi''(b,x) = 0, \tag{3.2}$$

which has a general solution

$$\phi(b,x) = \begin{cases} c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x} + \frac{h_2}{\alpha}x + \frac{(h_1-h_2)b}{\alpha} + \frac{\mu}{\alpha^2}h_2, & \text{for } b \leq x < u_b, \\ c_3 e^{\lambda_1 x} + c_4 e^{\lambda_2 x} + \frac{h_1}{\alpha}x + \frac{\mu}{\alpha^2}h_1, & \text{for } 0 \leq x \leq \min\{b, u_b\}, \\ c_5 e^{\lambda_1 x} + c_6 e^{\lambda_2 x} - \frac{p}{\alpha}x - \frac{\mu}{\alpha^2}p, & \text{for } d_b < x \leq 0, \end{cases} \tag{3.3}$$

where c_1, c_2, c_3, c_4, c_5 and c_6 are arbitrary constants, and parameter λ_1 and λ_2 are defined as

$$\lambda_1 = -\frac{\mu}{\sigma^2} + \frac{1}{\sigma^2}\sqrt{\mu^2 + 2\alpha\sigma^2}, \quad \lambda_2 = -\frac{\mu}{\sigma^2} - \frac{1}{\sigma^2}\sqrt{\mu^2 + 2\alpha\sigma^2}. \tag{3.4}$$

Here, the matching conditions at the points 0 and b imply that $\phi(b, 0^+) = \phi(b, 0^-)$, $\phi'(b, 0^+) = \phi'(b, 0^-)$, $\phi(b, b^+) = \phi(b, b^-)$ and $\phi'(b, b^+) = \phi'(b, b^-)$ give

$$c_3 = c_1 + \frac{\lambda_2(h_1 - h_2)}{\alpha\lambda_1(\lambda_1 - \lambda_2)}e^{-\lambda_1 b}, \tag{3.5}$$

$$c_4 = c_2 + \frac{\lambda_1(h_2 - h_1)}{\alpha\lambda_2(\lambda_1 - \lambda_2)}e^{-\lambda_2 b}, \tag{3.6}$$

$$c_5 = c_1 + \frac{\lambda_2(h_1 - h_2)}{\alpha\lambda_1(\lambda_1 - \lambda_2)}e^{-\lambda_1 b} - \frac{\lambda_2(h_1 + p)}{\alpha\lambda_1(\lambda_1 - \lambda_2)}, \tag{3.7}$$

$$c_6 = c_2 + \frac{\lambda_1(h_2 - h_1)}{\alpha\lambda_2(\lambda_1 - \lambda_2)}e^{-\lambda_2 b} + \frac{\lambda_1(h_1 + p)}{\alpha\lambda_2(\lambda_1 - \lambda_2)}. \tag{3.8}$$

Thus, the arbitrary constants are reduced to c_1 and c_2.

For $x \le d_b$ and $x \ge u_b$, the cash level is changed, and the inequality (2.16) holds as an equality. Then, the form of cost function ϕ is given by

$$\phi(b, x) = \begin{cases} \phi(b + \theta(D_b - x), D_b) + K_1 + k_1(\theta)(D_b - x), & \text{if } x \le d_b, \\ \phi(b - \theta(x - U_b), U_b) + K_2 + k_2(\theta)(x - U_b), & \text{if } x \ge u_b. \end{cases} \tag{3.9}$$

The following results are the existence of parameters $\mathbf{p}$ and an optimal policy of the problem (2.10) (Sato and Sawaki [7]).

Theorem 3.1. *Assume that Assumption 2.1 holds. If* $c_1 < 0$, $c_2 > 0$ *and* $b < \frac{1}{\lambda_2}\log\left(\frac{h_1 - h_2}{h_1 + p}\right)$, *then there exist parameters* d_b, D_b, U_b, u_b, $d_b \le D_b \le U_b \le u_b$, *which satisfy conditions (V1)-(S4).*

Theorem 3.2. *Suppose that Assumption 2.1 holds and there exist parameters* $\mathbf{p}$, $c_1 < 0$, $c_2 > 0$, $b < \frac{1}{\lambda_2}\log\left(\frac{h_1 - h_2}{h_1 + p}\right)$ *and a continuous function* ϕ *which satisfy equations (3.3) and (3.9). If the cash level is always greater than or equal to* $\underline{x} < 0$ *and*

$$1 \le \theta(1 - \theta)\frac{(\lambda_1 - \theta\lambda_2)e^{-\lambda_2(b - D_b + \theta(D_b - d_b))} - (\lambda_2 - \theta\lambda_1)e^{-\lambda_1(b - D_b + \theta(D_b - d_b))}}{\lambda_1 - \lambda_2} \tag{3.10}$$

for $D_b - \theta(D_b - d_b) \le b$, *then there exists an optimal policy for the cash management problem (2.11).*

4. Limit Case of a Zero Discount Rate

In this section, we deal with the undiscounted case, $\alpha = 0$, to find the policy parameters explicitly. We consider the value function as the long-term average costs in order to ensure that the value function is always finite. The value function is represented by

$$\Phi(b, x) = \inf_{v \in \mathscr{V}} \lim_{T \to \infty} T^{-1} E_x^v \left[\int_0^T C(B_s^v, X_s^v)ds + \sum_{i=1}^{T} F(\xi_i) \mid X_0^v = x, B_0^v = b \right]. \tag{4.1}$$

As discussed in Constantinides [2], the function Φ satisfies the following differential equation in $x \in (d, u)$:

$$C(b,x) - \gamma_v + \mu\phi'(b,x) + \frac{1}{2}\sigma^2\phi''(b,x) = 0 \tag{4.2}$$

where γ_v is the average cost rate and is given by

$$\gamma_v = \lim_{T\to\infty} T^{-1}E_x^v\left[\int_0^T C(B_s^v, X_s^v)ds + \sum_{i=1}^{N} F(\xi_i) \mid X_0^v = x, B_0^v = b\right]. \tag{4.3}$$

In above equation, N is the index of the last stopping time in interval [0,T]. We assume that there are no drift in the demand for cash, $\mu = 0$, and fixed costs, $K_B^u = K_B^d = K_S^u = K_S^d = 0$. We also assume that $D_b < 0$ and $U_b > 0$. Then, the solution to equation (4.2) is given by

$$\phi(b,x) = \begin{cases} \frac{1}{\sigma^2}\left(\gamma_v x^2 + \frac{p}{3}x^3\right) + c_1 x + c_2 & \text{if } d_b \le x \le 0, \\ \frac{1}{\sigma^2}\left(\gamma_v x^2 - \frac{h_1}{3}x^3\right) + c_1 x + c_2 & \text{if } 0 \le x \le \min\{b, u_b\}, \\ \frac{1}{\sigma^2}\left\{(\gamma_v - (h_1 - h_2)b)x^2 - \frac{1}{3}h_2 x^3\right\} + c_1 x + c_2 & \text{if } 0 \le b \le x \le u_b. \end{cases} \tag{4.4}$$

Let $G(\xi) = F(\xi) + \phi(b + \theta\xi, \xi + d_b)$ in equation (2.12), then $G(\xi)$ is minimized for $\xi = D_b - d_b$. Differentiating $G(\xi)$ with respect to ξ, we obtain

$$\phi'(b, D_b) = \begin{cases} -k_1(\theta) + \frac{1}{\sigma^2}\theta D_b(h_1 - h_2)(3D_b - 2d_b), & \text{if } b \le D_b - \theta(D_b - d_b), \\ -k_1(\theta) - \frac{1}{\sigma^2}D_b(h_1 - h_2)(2b - D_b), & \text{if } D_b - \theta(D_b - d_b) \le b \le D_b, \\ -k_1(\theta), & \text{if } D_b \le b \end{cases} \tag{4.5}$$

If $x = d_b$ in equation (3.9), then we have

$$\phi'(b, d_b) = \begin{cases} -k_1(\theta) + \frac{1}{\sigma^2}D_b^2\theta(h_1 - h_2), & \text{if } b \le D_b - \theta(D_b - d_b), \\ -k_1(\theta), & \text{if } D_b - \theta(D_b - d_b) \le b. \end{cases} \tag{4.6}$$

By similar arguments we obtain

$$\phi'(b, U_b) = \begin{cases} k_2(\theta) + \frac{1}{\sigma^2}\theta U_b(h_1 - h_2)(3U_b - 2u_b), & \text{if } b \le U_b, \\ k_2(\theta) + \frac{1}{\sigma^2}U_b(h_1 - h_2)\{2(b - \theta(u_b - U_b)) - (1-\theta)U_b\}, & \\ & \text{if } U_b \le b \le U_b + \theta(u_b - U_b), \\ k_2(\theta), & \text{if } U_b + \theta(u_b - U_b) \le b. \end{cases} \tag{4.7}$$

and

$$\phi'(b, u_b) = \begin{cases} k_2(\theta) + \frac{1}{\sigma^2}U_b^2\theta(h_1 - h_2), & \text{if } b \le U_b + \theta(u_b - U_b), \\ k_2(\theta), & \text{if } U_b + \theta(u_b - U_b) \le b. \end{cases} \tag{4.8}$$

When the cash level is changed from $x = d_b$ to $x = D_b$, the cost function is given by equation (3.9) as $x = d_b$. Substituting equation (4.4) into equation (3.9), we have

$$\phi(b, d_b) - \phi(b, D_b) - K_1 - k_1(\theta)(D_b - d_b)$$

$$= \begin{cases} -\frac{1}{\sigma^2}\theta D_b^2(h_1 - h_2)(D_b - d_b), & \text{if } b \le D_b - \theta(D_b - d_b), \\ \frac{1}{\sigma^2} D_b^2(h_1 - h_2)\left(b - \frac{1}{3}D_b\right), & \text{if } D_b - \theta(D_b - d_b) \le b \le D_b, \\ 0, & \text{if } D_b \le b, \end{cases} \quad (4.9)$$

and by similar arguments, we obtain

$$\phi(b, u_b) - \phi(b, U_b) - K_2 - k_2(\theta)(u_b - U_b)$$

$$= \begin{cases} \frac{1}{\sigma^2}\theta U_b^2(h_1 - h_2)(u_b - U_b), & \text{if } b \le U_b, \\ \frac{1}{\sigma^2} U_b^2(h_1 - h_2)\left\{\frac{1}{3}U_b - (b - \theta(u_b - U_b))\right\}, & \text{if } U_b \le b \le U_b + \theta(u_b - U_b), \\ 0, & \text{if } U_b + \theta(u_b - U_b) \le b. \end{cases} \quad (4.10)$$

Then, the parameters of optimal policy (d_b, D_b, U_b, u_b) and arbitrary constants c_1 and c_2 are derived from equations (4.5) -(4.10). The optimal policy is classified into three classes according to the amount of short-term debt outstanding.

Case (i): $0 \le b \le U_b + \theta(u_b - U_b)$

In this case, for $b < \frac{\sigma\sqrt{p(k_1+k_2)}}{h_1 - h_2}$, there exists a unique policy and the parameters is given by

$$u_b = -\frac{I_2}{I_1} + \frac{1}{2I_1}\sqrt{I_2^2 - 4I_1 I_3} > 0, \quad (4.11)$$

$$U_b = \frac{h_2}{h_2 + 6\theta(h_1 - h_2)} u_b > 0, \quad (4.12)$$

$$d_b = D_b = -\frac{h_2 + 8\theta(h_1 - h_2)}{2p} U_b - \frac{1}{p} b(h_1 - h_2) < 0, \quad (4.13)$$

where

$$I_1 = 2h_2\theta(h_1 - h_2)(4h_2 + 3p)(3\theta(h_1 - h_2) + h_2) + h_2^3(h_2 + p) > 0,$$

$$I_2 = 2bh_2(h_1 - h_2)\frac{h_2 + 4\theta(h_1 - h_2)}{h_2 + 6\theta(h_1 - h_2)} > 0,$$

$$I_3 = (h_1 - h_2)^2 b^2 - p\sigma^2(k_1 + k_2).$$

Increasing the variation of demand σ, both u_b and U_b increase but $d_b = D_b$ decrease. From equation (4.12), as the rate of utilizing the two funds θ increases, the amount of the transaction $u_b - U_b$ increases. Since the increase of θ leads to the increase of the amount of paying out the debts and the interest rate is larger than the opportunity cost, $h_1 > h_2$, the cash level is dropped to a lower level by decreasing the amount of debt outstanding. Moreover, by putting equation (4.12) into equation (4.13), we see that $u_b - D_b$ increases in θ. And, it follows from equation (4.13) that the value of $D_b = d_b$ increass as penalty cost p increases.

Case (ii): $U_b + \theta(u_b - U_b) \leq b \leq u_b$

Although we can not obtain the explicit values of the parameters in this case, we have the relationship between the parameters as follows;

$$d_b = D_b = U_b - p^{-\frac{1}{2}}\sqrt{(p+h_1)U_b + (k_1+k_2)\sigma^2}. \tag{4.14}$$

The values U_b and u_b are obtained by solving the following equations;

$$\begin{cases} (h_1U_b - h_2u_b)(u_b - U_b)^2 = u_bU_b(h_1 - h_2)(u_b + U_b - 6b), \\ 4p(u_b - U_b)^2\{(p+h_1)U_b^2 + (k_1+k_2)\sigma^2\} \\ \qquad\qquad = [(h_1 + 2p)U_b^2 - h_2u_b^2 - 2\{(h_1 - h_2)b + pU_b\}u_b]^2. \end{cases} \tag{4.15}$$

Transaction cost k_1 and k_2 defined in equation (2.5) are functions of θ, and the relations between the value of $k_1 + k_2$ and θ are

$$\begin{cases} k_1 + k_2 \text{ is increasing in } \theta, & \text{if } k_B^u + k_B^d > k_S^u + k_S^d, \\ k_1 + k_2 = k_S^u + k_S^d, & \text{if } k_B^u + k_B^d = k_S^u + k_S^d, \\ k_1 + k_2 \text{ is decreasing in } \theta, & \text{if } k_B^u + k_B^d < k_S^u + k_S^d. \end{cases} \tag{4.16}$$

Thus, from equations (4.14) and (4.16), $D_b = d_b$ decreases in θ for $k_B^u + k_B^d > k_S^u + k_S^d$ and increases in θ for $k_B^u + k_B^d < k_S^u + k_S^d$. For $k_B^u + k_B^d = k_S^u + k_S^d$, the parameters d_b, D_b, U_b and u_b are not associated with θ.

Case (iii): $u_b \leq b$

In this case, the policy have the form of reflecting boundaries which are given by

$$\begin{cases} U_b = u_b = \frac{1}{h_1}\sqrt{\frac{h_1p\sigma^2(k_1(\theta)+k_2(\theta))}{h_1+p}}, \\ D_b = d_b = -\frac{1}{p}\sqrt{\frac{h_1p\sigma^2(k_1(\theta)+k_2(\theta))}{h_1+p}}. \end{cases} \tag{4.17}$$

When the amount of short-term debt outstanding is large enough, the policy is the same form as the one for the model of single source of short-term fund (Constantinides and Richard [2]).

5. Conclusion

In this paper, we have formulated a cash management model in which two types of funds are available for the manager to adjust cash level. We showed an explicit solution of the cash management policy under the special case that there are no discount rate, drift of the demand

and fixed costs. Then, we have provided some analytical properties of optimal policy. In future research, we would like to find an optimal fund rate θ at the beginning of each cycle. Moreover, we also would like to modify the cash process to include the jump diffusion.

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