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Author(s)	Nishimoto, Katsuyuki; Wang, Pin-Yu; Lin, Shy-Der
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N -Fractional Calculus of Some Logarithmic Functions and Some Identities

* Katsuyuki Nishimoto , ** Pin-Yu Wang
and
*** Shy-Der Lin

* Institute for Applied Mathematics, Descartes Press Co.
2 -13 -10 Kaguike, Koriyama 963 - 8833, Japan.
** Department of Mechanical Engineering, Nanya Institute of Technology, Chung-Li 32034, Taiwan, ROC.
*** Department of Applied Mathematics, Chung Yuan Christian Uinversity, Chung Li 32023, Taiwan, ROC.

Abstract

In this article, N-fractional calculus of the logarithmic function in title is discussed.

A theorem is presented as follows for example.

Theorem 1. Let $f = f(z) = (\sqrt{z-b} - c)^2 - d \neq 0, 1.$

We have then;

(i)

$$(\log f)_\gamma = -e^{-i\pi\gamma}(z-b)^{-\gamma}\left\{\sum_{k=0}^{\infty}\frac{\Gamma(\frac{1}{2}k+\gamma)}{\Gamma(\frac{1}{2}k+1)}S^k\right.$$

$$\left.+\sum_{k=1}^{\infty}\frac{1}{k}T^k\sum_{m=0}^{\infty}\frac{[2k]_m\Gamma(\frac{1}{2}m+k+\gamma)}{m!\Gamma(\frac{1}{2}m+k)}S^m\right\}$$

$$(|\Gamma(\frac{1}{2}k+\gamma)|, |\Gamma(\frac{1}{2}m+k+\gamma)| < \infty)$$

and

(ii)

$$(\log f)_n = (-1)^{n+1}(z-b)^{-n}\Big\{\sum_{k=0}^{\infty}[\frac{1}{2}k+1]_{n-1}S^k$$

$$+\sum_{k=1}^{\infty}\frac{1}{k}T^k\sum_{m=0}^{\infty}\frac{[2k]_m[\frac{1}{2}m+k]_n}{m!}S^m\Big\}(n\in Z^+)(n\text{-th derivatives })$$

where $S=\dfrac{c}{(z-b)^{1/2}}, T=\dfrac{d}{z-b}, |S|<1, |T|<1,$

and$[\lambda]_k(k\in Z_0^+)$; Notation of Pochhammer.

§0. Introduction (Definition of Fractional Calculus)

(I) Definition. (by K. Nishimoto)([1] Vol.1)

Let $D=\{\underset{-}{\mathrm{D}}, \underset{+}{\mathrm{D}}\}, C=\{\underset{-}{\mathrm{C}}, \underset{+}{\mathrm{C}}\}$

$\underset{-}{\mathrm{C}}$ be a curve along the cut joining two points z and $-\infty+i\ \mathrm{Im}(z)$,

$\underset{+}{\mathrm{C}}$ be a curve along the cut joining two points z and $\infty+i\ \mathrm{Im}(z)$,

$\underset{-}{\mathrm{D}}$ be a domain surrounded by $\underset{-}{\mathrm{C}}$, $\underset{+}{\mathrm{D}}$ be a domain surrounded by $\underset{+}{\mathrm{C}}$,

(Here D contains the points over the curve C)

Moreover, let $f=f(z)$ be a regular function in D $(z\in D)$,

$$f_\nu=(f)_\nu={}_C(f)_\nu=\frac{\Gamma(\nu+1)}{2\pi i}\int_C\frac{f(\xi)}{(\xi-z)^{\nu+1}}d\xi \quad (\nu\notin \mathbf{Z}^-), \quad (0.1)$$

$$(f)_{-m}=\lim_{\nu\to -m}(f)_\nu \quad (m\in \mathbf{Z}^+), \quad (0.2)$$

where $-\pi\le \arg(\xi-z)\le\pi$ for $\underset{-}{\mathrm{C}}$, $0\le\arg(\xi-z)\le 2\pi$ for $\underset{+}{\mathrm{C}}$,

$\xi\ne z,\ z\in \mathbf{C}, \nu\in\mathbf{R}, \quad \Gamma$: Gamma function,

then $(f)_\nu$ is the fractional differintegration of arbitrary order ν (derivatives of order ν for $\nu>0$, and integrals of order $-\nu$ for $\nu<0$), with respect to z, of the function f, if $|(f)_\nu|<\infty$.

(II) On the fractional calculus operator N^ν [3]

Theorem A Let fractional calculus operator (Nishimoto's operator) N^ν be

$$N^\nu=\left(\frac{\Gamma(\nu+1)}{2\pi i}\int_C\frac{d\xi}{(\xi-z)^{\nu+1}}\right) \quad (\nu\notin\mathbf{Z}^-), \quad \text{[Refer to (1.1)]} \quad (0.3)$$

with

$$N^{-m} = \lim_{\nu \to -m} N^{\nu} \quad (m \in \mathbf{Z}^{+}), \tag{0.4}$$

and define the binary operation $\circ$ as

$$N^{\beta} \circ N^{\alpha} f = N^{\beta} N^{\alpha} f = N^{\beta}(N^{\alpha} f) \quad (\alpha,\ \beta \in \mathbf{R}), \tag{0.5}$$

then the set

$$\{N^{\nu}\} = \{N^{\mu} | \nu \in \mathbf{R}\} \tag{0.6}$$

is an Abelian product group (having continuous index ν) which has the inverse transform operator $(N^{\nu})^{-1} = N^{-\nu}$ to the fractional calculus operator N^{ν} , for the function f such that $f \in F = \{f; 0 \neq |f_{\nu}| < \infty, \nu \in \mathbf{R}\}$, where $f = f(z)$ and $z \in \mathbf{C}$. (viz. $-\infty < \nu < \infty$).
(For our convenience, we call $N^{\beta} \circ N^{\alpha}$ as product of N^{β} and N^{α})
Theorem B The "F.O.G.$\{N^{\nu}\}$" is an " Action product group which has continuous index ν " for the set of F.(F.O.G.: Fractional calculus operator group)[3]

Theorem C. Let

$$S := \{\pm N^{\nu}\} \cup \{0\} = \{N^{\nu}\} \cup \{-N^{\nu}\} \cup \{0\} (\nu \in R) \tag{0.7}$$

Then the set S is a commutative ring for the function $f \in F$, when the identity

$$N^{\alpha} + N^{\beta} = N^{\gamma} (N^{\alpha}, N^{\beta}, N^{\gamma} \in S) \tag{0.8}$$

holds. [5]

(III) Lemma 1 We have [1]

(i)

$$((z-c)^{\beta})_{\alpha} = e^{-i\pi\alpha} \frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)} (z-c)^{\beta-\alpha} \quad (|\frac{\Gamma(\alpha-\beta)}{\Gamma(-\beta)}| < \infty), \tag{0.9}$$

(ii)

$$(\log(z-c))_\alpha = -e^{-i\pi\alpha}\Gamma(\alpha)(z-c)^{-\alpha} \quad (|\Gamma(\alpha)| < \infty), \quad (0.10)$$

(iii)

$$((z-c)^{-\alpha})_{-\alpha} = -e^{i\pi\alpha}\frac{1}{\Gamma(\alpha)}\log(z-c) \quad (|\Gamma(\alpha)| < \infty), \quad (0.11)$$

(iv)

$$(u\cdot v)_\alpha = \sum_{k=0}^{\infty}\frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha+1-k)}u_{\alpha-k}v_k \quad (u=u(z), v=v(z)) \quad (0.12)$$

where $z \neq c$ in (0.9), and $z-c \neq 0, 1$ in (0.10) and (0.11).

§1. Preliminary

Theorem D. below for the fractional calculus of a logarithmic function is reported by K. Nishimoto (cf. J. Frac. Calc. Vol. 29, May (2006), pp. 35-44)[12].

Theorem D. We have

(i)

$$(\log((z-b)^\beta - c))_\gamma = -e^{-i\pi\gamma}\beta(z-b)^{-\gamma}\Gamma(\gamma) \times \sum_{k=0}^{\infty}\frac{\Gamma(\beta k+\gamma)}{\Gamma(\gamma)\Gamma(\beta k+1)}\left(\frac{c}{(z-b)^\beta}\right)^k \quad \left(\left|\frac{\Gamma(\beta k+\gamma)}{\Gamma(\beta k+1)}\right| < \infty\right) \quad (1.1)$$

and

(ii)

$$(\log((z-b)^\beta - c))_m = (-1)^{m+1}\beta(z-b)^{-m}\Gamma(m) \times \sum_{k=0}^{\infty}\frac{\Gamma(\beta k+m)}{\Gamma(m)\Gamma(\beta k+1)}\left(\frac{c}{(z-b)^\beta}\right)^k \quad (m\in Z^+)$$

where $(z-b)^\beta - c \neq 0, 1$, and $|c/(z-b)^\beta| < 1$,

and $[\lambda]_k = \lambda(\lambda+1)\cdots(\lambda+k-1) = \Gamma(\lambda+k)/\Gamma(\lambda)$ with $[\lambda]_0 = 1$,

(Notation of Pochhammer). (1.2)

And the Theorem E below is reported by K. Nishimoto already (cf. JFC Vol. 32, Nov. (2007), pp. 17-28)[13].

Theorem E. We have
(i)

$$(\log(((z-b)^\beta - c)^\alpha - d))_\gamma = -e^{-i\pi\gamma}(z-b)^{-\gamma}$$
$$\times\left[\alpha\beta\sum_{k=0}^{\infty}\frac{\Gamma(\beta k+\gamma)}{\Gamma(\beta k+1)}\left(\frac{c}{(z-b)^\beta}\right)^k\right.$$
$$\left.+\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{d}{(z-b)^{\alpha\beta}}\right)^k\sum_{m=0}^{\infty}\frac{[\alpha k]_m\Gamma(\beta m+\alpha\beta k+\gamma)}{m!\Gamma(\beta m+\alpha\beta k)}\left(\frac{c}{(z-b)^\beta}\right)^m\right]$$
$$\left(\left|\frac{\Gamma(\beta k+\gamma)}{\Gamma(\beta k+1)}\right|,\left|\frac{\Gamma(\beta m+\alpha\beta k+\gamma)}{\Gamma(\beta m+\alpha\beta k)}\right|<\infty\right) \tag{1.3}$$

and
(ii)

$$(\log(((z-b)^\beta - c)^\alpha - d))_n$$
$$=(-1)^{n+1}(z-b)^{-n}\left[\alpha\beta\sum_{k=0}^{\infty}[\beta k+1]_{n-1}\left(\frac{c}{(z-b)^\beta}\right)^k\right.$$
$$\left.+\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{d}{(z-b)^{\alpha\beta}}\right)^k\sum_{m=0}^{\infty}\frac{[\alpha k]_m[\beta m+\alpha\beta k]_n}{m!}\left(\frac{c}{(z-b)^\beta}\right)^m\right]$$
$$\text{where}((z-b)^\beta-c)^\alpha-d\neq 0,1.(n\in Z^+)$$
$$\text{and}\left|\frac{d}{((z-b)^\beta-c)^\alpha}\right|,\left|\frac{c}{(z-b)^\beta}\right|,\left|\frac{d}{(z-b)^{\alpha\beta}}\right|<1. \tag{1.4}$$

§2. N-Fractional Calculus of A Logarithmic Function

Theorem 1. Let

$$f=f(z)=(\sqrt{z-b}-c)^2-d\neq 0,1 \tag{2.1}$$

We have then;
(i)

$$(\log f)_\gamma = -e^{-i\pi\gamma}(z-b)^{-\gamma}\left\{\sum_{k=0}^{\infty}\frac{\Gamma(\frac{1}{2}k+\gamma)}{\Gamma(\frac{1}{2}k+1)}S^k\right.$$

$$\left.+\sum_{k=1}^{\infty}\frac{1}{k}T^k\sum_{m=0}^{\infty}\frac{[2k]_m\Gamma(\frac{1}{2}m+k+\gamma)}{m!\Gamma(\frac{1}{2}m+k)}S^m\right\}$$

$$(\ |\Gamma(\frac{1}{2}k+\gamma)|, |\Gamma(\frac{1}{2}m+k+\gamma)|<\infty) \tag{2.2}$$

and
(ii)

$$(\log f)_n = -(-1)^n(z-b)^{-n}\left\{\sum_{k=0}^{\infty}[\frac{1}{2}k+1]_{n-1}S^k\right.$$

$$\left.+\sum_{k=1}^{\infty}\frac{1}{k}T^k\sum_{m=0}^{\infty}\frac{[2k]_m[\frac{1}{2}m+k]_n}{m!}S^m\right\}(n\in Z^+)(n\text{-th derivatives })$$

$$\text{where } S=\frac{c}{(z-b)^{1/2}}, T=\frac{d}{z-b}, |S|<1, |T|<1, \tag{2.3}$$

Proof of (i). Set $\beta = 1/2$ and $\alpha = 2$ in Theorem E (i), We obtain (2.2) under the conditions stated before.

Theorem 2. Let $f = f(z)$ be (2.1). We have then
(i)

$$(\log f)_\gamma = -e^{-i\pi\gamma}\frac{\Gamma(\gamma)}{2}(z-b)^{-\gamma}$$

$$\times\sum_{k=0}^{\infty}\frac{\Gamma(\frac{1}{2}k+\gamma)}{\Gamma(\gamma)\Gamma(\frac{1}{2}k+1)}\left\{\left(\frac{c+\sqrt{d}}{(z-b)^{1/2}}\right)^k+\left(\frac{c-\sqrt{d}}{(z-b)^{1/2}}\right)^k\right\}$$

$$(\ |\Gamma(\frac{1}{2}k+\gamma)|<\infty) \tag{2.4}$$

and

(ii)

$$(\log f)_n = -(-1)^n \frac{\Gamma(n)}{2}(z-b)^{-n}$$
$$\times \sum_{k=0}^{\infty} \frac{[\frac{1}{2}k+1]_{n-1}}{\Gamma(n)} \left\{ \left(\frac{c+\sqrt{d}}{(z-b)^{1/2}} \right)^k + \left(\frac{c-\sqrt{d}}{(z-b)^{1/2}} \right)^k \right\} (n\text{-derivatives})(n \in Z+)$$
$$\text{where} \quad \left| \frac{c+\sqrt{d}}{(z-b)^{1/2}} \right|, \left| \frac{c-\sqrt{d}}{(z-b)^{1/2}} \right| < 1. \tag{2.5}$$

Proof of (i). We have

$$\log f = \log\{\sqrt{z-b} - c - \sqrt{d}\}\{\sqrt{z-b} - c + \sqrt{d}\}$$
$$= \log((z-b)^{1/2} - (c+\sqrt{d})) + \log((z-b)^{1/2} - (c-\sqrt{d})) \tag{2.6}$$

Operate N-fractional calculus operator N^γ to the both side of (2.6), we have then

$$(\log f)_\gamma = (\log((z-b)^{1/2} - (c+\sqrt{d})))_\gamma + (\log((z-b)^{1/2} - (c-\sqrt{d})))_\gamma \tag{2.7}$$

Now we have

$$(\log((z-b)^{1/2} - (c+\sqrt{d})))_\gamma = -e^{-i\pi\gamma} \frac{1}{2}(z-b)^{-\gamma}\Gamma(\gamma)$$
$$\times \sum_{k=0}^{\infty} \frac{\Gamma(\frac{1}{2}k+\gamma)}{\Gamma(\gamma)\Gamma(\frac{1}{2}k+1)} \left(\frac{c+\sqrt{d}}{(z-b)^{1/2}} \right)^k$$
$$\left(\left|\Gamma(\frac{1}{2}k+\gamma)\right| < \infty \right) \tag{2.8}$$

and

$$(\log((z-b)^{1/2} - (c-\sqrt{d})))_\gamma = -e^{-i\pi\gamma} \frac{1}{2}(z-b)^{-\gamma}\Gamma(\gamma)$$
$$\times \sum_{k=0}^{\infty} \frac{\Gamma(\frac{1}{2}k+\gamma)}{\Gamma(\gamma)\Gamma(\frac{1}{2}k+1)} \left(\frac{c-\sqrt{d}}{(z-b)^{1/2}} \right)^k$$
$$\left(\left|\Gamma(\frac{1}{2}k+\gamma)\right| < \infty \right) \tag{2.9}$$

Therefore, we obtain (2.4) from (2.7), (2.8), and (2.9) clearly.
Proof of(ii). Set $\gamma = n$ in (2.4).

§3. Some Identities

Theorem 3. We have the identities below;

(i)
$$\sum_{k=1}^{\infty}\sum_{m=0}^{\infty}\frac{[2k]_m\Gamma(\frac{1}{2}m+k+\gamma)}{k\cdot m!\,\Gamma(\frac{1}{2}m+k)}S^m T^k$$
$$=\frac{1}{2}\sum_{k=0}^{\infty}\frac{\Gamma(\frac{1}{2}k+\gamma)}{\Gamma(\frac{1}{2}k+1)}\left\{\left(\frac{c+\sqrt{d}}{(z-b)^{1/2}}\right)^k+\left(\frac{c-\sqrt{d}}{(z-b)^{1/2}}\right)^k-2S^k\right\}, \qquad (3.1)$$
$$\left(\left|\Gamma(\tfrac{1}{2}k+\gamma)\right|,\quad \left|\Gamma(\tfrac{1}{2}m+k+\gamma)\right|<\infty\right)$$

and

(ii)
$$\sum_{k=1}^{\infty}\sum_{m=0}^{\infty}\frac{[2k]_m[\frac{1}{2}m+k]_n}{k\cdot m!}S^m T^k$$
$$=2\sum_{k=0}^{\infty}[\tfrac{1}{2}k+1]_{n-1}\left\{\left(\frac{c+\sqrt{d}}{(z-b)^{1/2}}\right)^k+\left(\frac{c-\sqrt{d}}{(z-b)^{1/2}}\right)^k-2S^k\right\}\quad (n\in Z^+) \qquad (3.2)$$

where

$$S=c/(z-b)^{1/2}\,,\quad T=d/(z-b)\,,\quad |S|<1,\quad |T|<1,\quad |(c\pm\sqrt{d})/(z-b)^{1/2}|<1\,.$$

Proof of (i). It is clear from Theorem 1 (i) and Theorem 2 (i).
Proof of (ii). Set $\gamma=n$ in (i).

Corollary 1. We have the following identities;
(i)

$$(z-b-d)^{-\gamma}=\frac{1}{2}(z-b)^{-\gamma}\sum_{k=0}^{\infty}\frac{\Gamma(\frac{1}{2}k+\gamma)}{\Gamma(\gamma)\Gamma(\frac{1}{2}k+1)}$$
$$\times\left\{\left(\frac{\sqrt{d}}{(z-b)^{1/2}}\right)^k+\left(\frac{-\sqrt{d}}{(z-b)^{1/2}}\right)^k\right\}$$
$$\left(|\Gamma(\frac{1}{2}k+\gamma)|<\infty\right) \qquad (3.3)$$

and
(ii)

$$(z-b-d)^{-n} = \frac{1}{2}(z-b)^{-n} \sum_{k=0}^{\infty} \frac{[\frac{1}{2}k+1]_{n-1}}{\Gamma(n)} \left\{ \left(\frac{\sqrt{d}}{(z-b)^{1/2}} \right)^k + \left(\frac{-\sqrt{d}}{(z-b)^{1/2}} \right)^k \right\}$$

$$\text{where} \left| \frac{\pm\sqrt{d}}{(z-b)^{1/2}} \right| < 1, (n \in Z^+). \tag{3.4}$$

Proof of (i). Set $c = 0$ in (1).
Indeed we have

$$(\log(z-b-d))_\gamma = -e^{-i\pi\gamma}\Gamma(\gamma)(z-b-d)^{-\gamma}(|\Gamma(\gamma)| < \infty) \tag{3.5}$$

from (2.2), setting $c = 0$.
Next we have

$$(\log(z-b-d))_\gamma = -e^{-i\pi\gamma}\frac{\Gamma(\gamma)}{2}(z-b)^{-\gamma}$$
$$\times \sum_{k=0}^{\infty} \frac{\Gamma(\frac{1}{2}k+\gamma)}{\Gamma(\gamma)\Gamma(\frac{1}{2}k+1)} \left\{ \left(\frac{\sqrt{d}}{(z-b)^{1/2}} \right)^k + \left(\frac{-\sqrt{d}}{(z-b)^{1/2}} \right)^k \right\}$$
$$(|\Gamma(\frac{1}{2}k+\gamma)| < \infty) \tag{3.6}$$

from (2.4), setting $c = 0$.
Therefore we have

$$(z-b-d)^{-\gamma} = \frac{1}{2}(z-b)^{-\gamma} \sum_{k=0}^{\infty} \frac{\Gamma(\frac{1}{2}k+\gamma)}{\Gamma(\gamma)\Gamma(\frac{1}{2}k+1)}$$
$$\times \left\{ \left(\frac{\sqrt{d}}{(z-b)^{1/2}} \right)^k + \left(\frac{-\sqrt{d}}{(z-b)^{1/2}} \right)^k \right\} \tag{3.7}$$

from (3.5) and (3.6).

Proof of (ii). Set $\gamma = n$ in (3.3).

§4. Semi Derivatives

I We have

$$(\log((\sqrt{z-b}-c)^2-d))_{1/2}=i(z-b)^{-1/2}$$
$$\times\left[\sum_{k=0}^{\infty}\frac{\Gamma(\frac{1}{2}k+\frac{1}{2})}{\Gamma(\frac{1}{2}k+1)}S^k+\sum_{k=1}^{\infty}\frac{1}{k}T^k\sum_{m=0}^{\infty}\frac{[2k]_m\Gamma(\frac{1}{2}m+k+\frac{1}{2})}{m!\Gamma(\frac{1}{2}m+k)}S^m\right] \tag{4.1}$$

(S is the one shown in Theorem 1) (semi derivatives) from Theorem 1.(i), setting $\gamma=1/2$.

II We have

$$(\log((\sqrt{z-b}-c)^2-d))_{1/2}=i\frac{\Gamma(\frac{1}{2})}{2}(z-b)^{-1/2}$$
$$\times\sum_{k=0}^{\infty}\frac{\Gamma(\frac{1}{2}k+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}k+1)}\left\{\left(\frac{c+\sqrt{d}}{(z-b)^{1/2}}\right)^k+\left(\frac{c-\sqrt{d}}{(z-b)^{1/2}}\right)^k\right\} \tag{4.2}$$

from Theorem 2. (i).

III

$$(\log(\sqrt{z-b}-c)^2)_{1/2}=i(z-b)^{-1/2}\sum_{k=0}^{\infty}\frac{\Gamma(\frac{1}{2}k+\frac{1}{2})}{\Gamma(\frac{1}{2}k+1)}S^k\cdot \tag{4.3}$$

(semi derivatives)

References

[1] K. Nishimoto; Fractional Calculus, Vol. 1(1984), Vol.2(1987), Vol. 3(1989), Vol. 4(1991), Vol. 5(1996), Descartes Press, Koriyama, Japan.

[2] K. Nishimoto; An Essence of Nishimoto's Fractional Calculus (Calculus of the 21st Century); Integrals and differentiations of Arbitrary Order (1991), Descartes Press, Koriyama, Japan.

[3] K. Nishimoto; On Nishimoto's fractional calculus operator N^{ν} (On an action group), J. Frac. Calc. Vol.4, Nov. (1993), 1-11.

[4] K. Nishimoto; Unification of the integrals and derivatives (A serendipity in fractional calculus), J. Frac. Calc. Vol. 6, Nov. (1994), 1-14.

[5] K. Nishimoto; Ring and Field produced from The Set of N-Fractional Calculus Operator, J. Frac. Calc. Vol.24, Nov. (2003), 29-36.

[6] K. Nishimoto; On the fractional calculus of functions $(a-z)^{\beta}$ and $\log(a-z)$, J. FRac. Calc. Vol. 3, May(1993),19-27.

[7] K. Nishimoto and S.-T. Tu; Fractional calculus of Psi functions (Generalized Polygamma functions), J. Frac. Calc. Vol. 5, May (1994), 27-34.

[8] S.-T. Tu and K. Nishimoto; On the fractional calculus of functions $(cz-a)^{\beta}$ and $\log(cz-a)$, J. Frac. Calc. Vol. 5, May (1994), 35-43.

[9] K. Nishimoto; N-Fractional Calculus of the Power and Logarithmic Functions, and Some Identities, J. Frac. Calc. Vol. 21, May (2002), 1-6.

[10] K. Nishimoto; Some Theorems for N-Fractional Calculus of Logarithmic Functions I, J. Frac. Calc. Vol. 21, May. (2002), 7-12.

[11] K. Nishimoto; N-Fractional Calclus of Products of Some Power Functions, J. Frac. Calc. Vol. 27, May. (2005), 83-88.

[12] K. Nishimoto; N-Fractional Calculus of Some Composite Functions, J. Frac. Calc. Vol. 29, May. (2006), 35-44.

[13] K. Nishimoto; N-Fractional Calculus of Some Logarithmic Functions, J. Frac. Calc. Vol. 32, Nov (2007), 17-28.

[14] K. Nishimoto; N-Fractional Calculus of Some Functions which Have A Root Signs, J. Frac. Calc. Vol. 33, May (2008), 1-12.

[15] K. Nishimoto; N-Fractional Calculus of Some Functions which Have Multiple Root Signs, J. Frac. Calc. Vol. 33, May (2008), 35-46.

[16] David Dummit and Richard M. Foote; Abstract Algebra, Prentice Hall (1991).

[17] K. B. Oldhanm and J. Spanier; The Fractional Calculus, Academic Press (1974).

[18] S. G. Samko, A.A. Kilbas and O.I. Marichev; Fractional Integrals and Derivatives, and Some Their Applications (1987), Nauka, USSR.

[19] K. S. Miller and B. Ross; An Introduction to The Fractional Calculus and Fractional Differential Equations, John Wiley & Sons, (1993).

[20] V. Kiryakova; Generalized fractional calclus and applications, Pitman Research Notes, No. 301, (1994), Longman.

[21] Igor Podlubny; Fractional Differential Equations (1999), Academic Press.

[22] R. Hilfer (Ed.); Applications of Fractional Calculus in Physics, (2000), World Scientific, Singapore, New Jersey, London, Hong Kong.

[23] S. Moriguchi, K. Udagawa and S. Hitotsumatsu; Mathematical Formulae, Vol. 1, Iwanami Zensho, (1957), Iwannmi, Japan.