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An Application of The Peter-Weyl Theorem to Non-Abelian Lattice Gauge Theory

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ABSTRACT

This paper presents an application of the Peter-Weyl theorem—a key theorem of harmonic analysis on homogeneous space— to get a new formula for the physical states of the Kogut-Susskind Hamiltonian model. A physical state is identified with a matrix element of a suitable representation of the infinite product of the gauge group. The Clebsh-Gordan coefficients for the physical states are obtained. We deduce a handy formula for the plaquette term in the Hamiltonian, so that the matrix elements of the Hamiltonian in the physical space for the (2+1)-dimensional $SU(2)$ model can be calculated without using the Wigner-Eckart theorem for tensor operators.

1. Introduction

In this paper our aim is to show how to apply the Peter-Weyl theory—the structure theory of invariant functions on homogeneous space— to analyze the gauge invariant dynamics of the Kogut-Susskind Hamiltonian model[1][2]. In this model the configuration space is the infinite product of gauge groups assigned to each of lattice links. The state space is defined to be the L^2-functions on the configuration space. The model has a large amount of symmetry— the gauge transformations. This paper presents a new formulation of the gauge invariant functions. In Section 2 we review the Hamiltonian formalism of lattice gauge theory and how to specify the physical states which correspond to the gauge invariant functions. To analyze the gauge invariant functions, one cannot apply directly the Peter-Weyl theory because a gauge transformation acts on the configuration space as neither right- nor left-action, considering the configuration space as a topological group. In Section 3 we will show how to extend the configuration space on which the gauge transformations act as a left-action. The extended configuration space has an extra right-action which relates the extended configuration space to the original one. We will prove a modified version of the Peter-Weyl theorem which tells us that a gauge invariant function is nothing but a matrix element of the suitable representation of the extended configuration space as a topological group. A collection of matrix elements of suitable irreducible representations spans the gauge invariant functions as a linear space. This is the key point of this paper. By this interpretation we get a new presentation of the plaquette term as a matrix element. Then we can study the operation of the plaquette term on the physical states by decomposing a product of two matrix elements into irreducible components. The decomposition is completely described by Clebsh-Gordan coefficients. In Section 4 the formula of the Clebsh-Gordan coefficients for the physical states in (2+1)-dimensional SU(2) case is obtained. Finally the explicit formula for the matrix elements of the Hamiltonian in the physical space is deduced from the Clebsh-Gordan coefficients. In the last section 5 we summarize the results of this paper and discuss the ring structure of the gauge invariant functions which plays an important role in our argument.

2. Hamiltonian formalism of lattice gauge theory

2.1. THE KOGUT-SUSSKIND HAMILTONIAN

We begin with the simplest case. Let G be a compact Lie group, especially $SU(N)$. We fix a basis T^α of its Lie algebra $\mathcal{G}(G)$ such that $\mathrm{Tr}\,\mathrm{T}^\alpha\mathrm{T}^\beta = \frac{1}{2}\delta^{\alpha\beta}$ where the trace is taken in the defining representation. We define a quantum mechanics on G: the configuration space is G, and then the state space is $L^2(G)$ with the invariant measure of G. We should prepare the momentum operator and the position operator.

Define the momentum operators E^α_R, E^α_L as

$$\mathrm{E}^\alpha_R f(x) \equiv \frac{1}{i}\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} f(x\cdot exp\, i\varepsilon\mathrm{T}^\alpha), \quad \mathrm{E}^\alpha_L f(x) \equiv \frac{1}{i}\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} f(exp\,(-i\varepsilon\mathrm{T}^\alpha)\cdot x)$$

where $f(x) \in L^2(G)$. Thus the momentum operator E^α_R (E^α_L) is the infinitesimal form of the right (left) action on $L^2(G)$. Define the position operator $\hat{\mathrm{U}}$,

$$\hat{\mathrm{U}} f(x) \equiv \tau(x) f(x)$$

where τ is the defining representation. This definition means that the $\hat{\mathrm{U}}$ is a $\mathrm{N}\times\mathrm{N}$ matrix of operators and the operation causes to multiply $f(x)$ by a $\mathrm{N}\times\mathrm{N}$ matrix of functions $\tau(x)$.

Then the commutation relations are obtained from the definitions, as

$$[\mathrm{E}^\alpha_R\,,\mathrm{E}^\beta_R] = if^{\alpha\beta\gamma}\mathrm{E}^\gamma_R, \qquad [\mathrm{E}^\alpha_L\,,\mathrm{E}^\beta_L] = if^{\alpha\beta\gamma}\mathrm{E}^\gamma_L, \qquad [\mathrm{E}^\alpha_R\,,\mathrm{E}^\beta_L] = 0$$

$$[\mathrm{E}^\alpha_R\,,\hat{\mathrm{U}}] = \hat{\mathrm{U}}\,\tau(\mathrm{T}^\alpha), \qquad [\mathrm{E}^\alpha_L\,,\hat{\mathrm{U}}] = -\tau(\mathrm{T}^\alpha)\,\hat{\mathrm{U}}$$

where $[\mathrm{T}^\alpha, \mathrm{T}^\beta] = if^{\alpha\beta\gamma}\mathrm{T}^\gamma$. We use also the notation τ for the representation of Lie algebra $\mathcal{G}(G)$ associated with τ.

These relations show that the E^α_R (E^α_L) is similar to the angular momentum. This is more than an analogy, but we will not make further argument here.

Now the Hamiltonian H for the free theory is defined as

$$H \equiv \sum_\alpha (\mathrm{E}^\alpha_R)^2 \;(= \sum_\alpha (\mathrm{E}^\alpha_L)^2).$$

(The last equation is from the Peter-Weyl theorem.)

To know about the dynamics of this simple model, we set a basis of the state space which diagonalizes H. For this purpose we introduce the Peter-Weyl theorem:[3]

Theorem (Peter-Weyl). *Let G be a compact topological group. Then,*

$$L^2(G) = \overline{Lin\{\sqrt{n_\rho}\langle\alpha|\rho|\beta\rangle \text{ ; where } n_\rho = \mathrm{dim}\rho,\ \rho \in \mathcal{D}(G)\}}$$

where $\mathcal{D}(G)$ is a representative set of all irreducible (finite dimensional unitary) representations of G and $\{|\alpha\rangle\}_{1\leq\alpha\leq n_\rho}$ is a orthonormal basis. The symbol Lin means to take liner hull and the Upper line means to take closure. The $\langle\alpha|\rho|\beta\rangle$ stands for the function $\langle\alpha|\rho(g)|\beta\rangle$ on G

We write $\langle\alpha|\rho(g)|\beta\rangle$, in short, as $\rho_{\alpha\beta}$ which is simply a matrix element of the representation ρ. Consequently $\sqrt{n_\rho}\rho_{\alpha\beta}$ spans a reducible (right) module $\overline{Lin\{\sqrt{n_\rho}\rho_{\alpha\beta}\ ;\ 1\leq\beta\leq\ n_\rho\}}$ under the right action on $L^2(G)$ which is equivalent to the n_ρ-times product of ρ. Under left action $\sqrt{n_\rho}\rho_{\alpha\beta}$ spans a reducible (left) module $\overline{Lin\{\sqrt{n_\rho}\rho_{\alpha\beta}\ ;\ 1\leq\alpha\leq\ n_\rho\}}$ which is equivalent to the n_ρ-times product of $\overline{\rho}$, the complex conjugate representation of ρ. This fact and the reality of $\sum_\alpha(\mathrm{E}_R^\alpha)^2$ and $\sum_\alpha(\mathrm{E}_L^\alpha)^2$ prove the equation in the definition of H.

A base $\sqrt{n_\rho}\rho_{\alpha\beta}$ is specified by three natural labels. The first label specifies the representation ρ and the rest two are α and β. In the special case of $SU(2)$ we will use the notation ρ to express a representation and also the half integer value of the total angular momentum of ρ. In this case the eigen value of H with respect to $\sqrt{n_\rho}\rho_{\alpha\beta}$ is $\rho(\rho+1)$. For the labeling one can use three eigen values of H, E_R^3, and E_L^3 which are non-negative half integers. In general case the eigen value is given by the formula which contains the highest weight of ρ and all positive roots of Lie algebra of G. (See APPENDIX A.) We can see that a higher representation corresponds to a higher excitation.

Now we proceed to the full (d+1)-dimensional model. Let Γ be a d-dimensional (spatial) cubic lattice: that is $\Gamma\equiv\mathbf{Z}^d$. (Or one may suppose that Γ has cyclic boundary.) We use letters $i\ j\ k\ l$ to refer lattice sites. The lattice Γ inherits natural metric from $\mathbf{R}^d$. Let $i,j\in\Gamma$ be called neighboring sites if their distance of two sites is exactly one. Then each site of lattice has $2d$ neighboring sites. We write L for a set of all links; the set of all ordered pairs of neighboring sites. Call $(i,j)\in L$ positive (negative) if the unit vector $j-i$ has positive (negative) component with respect to a fixed orientation on $\mathbf{R}^d$. Then $L=L^+\cup L^-$ where L^+ (L^-) is a set of positive (negative) pairs.

We assign G to each link and denote as G_{ij} the one assigned to a link (i,j). We call the infinite direct product $\prod_{(i,j)\in L} G_{ij}$ the extended configuration space $\tilde{\mathcal{C}}$. This space will play an important role in the next section.

Define the configuration space $\mathcal{C}$ to be the collection of G_{ij} for all positive links, that is $\mathcal{C} \equiv \prod_{(i,j)\in L^+} G_{ij}$.

We define the state space $\mathcal{H}$ to be $L^2(\mathcal{C})$ with respect to the infinite product measure. This space $L^2(\mathcal{C})$ is naively the infinite tensor product of $L^2(G_{ij})$ assigned to each positive link. A typical element f of $\mathcal{H}$ is a finite sum of terms each of which is a finite product of elements in $L^2(G_{ij})$'s assigned to each positive link.

Similarly, we define the momentum operator E^{α}_{ij} and the position operator $\hat{\mathrm{U}}_{ij}$ for all $(i,j) \in L$.

$$\mathrm{E}^{\alpha}_{ij} f_{kl} \equiv \begin{cases} \mathrm{E}^{\alpha}_{R} f_{kl} & \text{if } (i,j)=(k,l)\ ; \\ \mathrm{E}^{\alpha}_{L} f_{kl} & \text{if } (j,i)=(k,l)\ ; \\ f_{kl} & \text{elsewhere,} \end{cases}$$

$$\hat{\mathrm{U}}_{ij} f_{kl} \equiv \begin{cases} \hat{\mathrm{U}} f_{kl} & \text{if } (i,j)=(k,l)\ ; \\ \hat{\mathrm{U}}^{-1} f_{kl} & \text{if } (j,i)=(k,l)\ ; \\ f_{kl} & \text{elsewhere,} \end{cases}$$

where each of E^{α}_{R}, E^{α}_{L}, $\hat{\mathrm{U}}$ is the same as defined on $L^2(G_{kl})$ for $(k,l) \in L^+$.

Thus these operators obey the commutation relations below:

$$[\mathrm{E}^{\alpha}_{ij}\ , \mathrm{E}^{\beta}_{kl}] = 0 \qquad \text{for } (i,j) \neq (k,l)$$

$$[\mathrm{E}^{\alpha}_{ij}\ , \mathrm{E}^{\beta}_{ij}] = i f^{\alpha\beta\gamma} \mathrm{E}^{\gamma}_{ij}$$

$$\hat{\mathrm{U}}_{ji} = (\hat{\mathrm{U}}_{ij})^{-1} = (\hat{\mathrm{U}}_{ij})^{\dagger}$$

$$[\mathrm{E}^{\alpha}_{ij}\ , \hat{\mathrm{U}}_{kl}] = \begin{cases} \hat{\mathrm{U}}_{kl}\, \tau(\mathrm{T}^{\alpha}) & \text{if } (i,j)=(k,l)\ ; \\ -\tau(\mathrm{T}^{\alpha}) \hat{\mathrm{U}}_{kl} & \text{if } (j,i)=(k,l)\ ; \\ 0 & \text{elsewhere.} \end{cases}$$

Now then, we define

the Kogut-Susskind Hamiltonian

$$H = \sum_{(i,j)\in L^+} (\mathrm{E}_{ij})^2 - K \sum_{\square} \mathrm{Tr}\,\hat{\mathrm{U}}_{\square}$$

where $(\mathrm{E}_{ij})^2 \equiv \sum_{\alpha}(\mathrm{E}_{ij}^{\alpha})^2$, K is a coupling constant, $\hat{\mathrm{U}}_{\square} \equiv \hat{\mathrm{U}}_{ij}\hat{\mathrm{U}}_{jk}\hat{\mathrm{U}}_{kl}\hat{\mathrm{U}}_{li}$ for each plaquette ${}^{l}_{i}\square^{k}_{j}$ which is by definition, a minimum square of connected four links $(i,j),(j,k),(k,l),(l,i)$. The summation of $\square$ is taken for all plaquettes.

A basis of $\mathcal{H}$ which diagonalizes $\sum_{(i,j)\in L^+} (\mathrm{E}_{ij})^2$ is easily obtained from the basis constructed for the simplest case, by multiplying finite number of elements assigned to each positive link.

The Hamiltonian H has a large amount of symmetry due to the remaining (time-independent) gauge symmetries. To see this, define the (infinitesimal) gauge transformation at site i:

$$\mathrm{J}_i^{\alpha} \equiv \sum_{(i,j)\in L} \mathrm{E}_{ij}^{\alpha}$$

where the summation is taken for all j such that $(i,j) \in L$. Then we get $[\mathrm{J}_i^{\alpha}, \mathrm{H}] = 0$ from the commutation relations. We say the state f is physical if $\mathrm{J}_i^{\alpha} f = 0$ for all α and i, and denote the physical subspace $\mathcal{H}_{phy}$.

2.2. Labeling the physical states

We constructed a basis which diagonalizes the kinetic term $\sum_{(i,j)\in L^+} (\mathrm{E}_{ij})^2$. To label a state we should specify a representation and two base vectors for each positive link. Then we get a labeling:

$$\left|\rho_{ij}\ ,\ v_{\alpha}\ ,\ v_{\beta}\right\rangle_{(i,j)\in L^+} \tag{2.1}$$

where v_{α}, v_{β} are bases of the representation space of ρ_{ij}. In $SU(2)$ case one can use eigen values of momentum operators for each label:

$$\left|(\mathrm{E}_{ij})^2\ ,\ \mathrm{E}_{ji}^3\ ,\ \mathrm{E}_{ij}^3\right\rangle_{(i,j)\in L^+} \tag{2.2}$$

where the same symbol is used in place of its eigen value.

Let us now attempt to label the physical states. The construction is closely related to the liner representations of G and their compositions[4][5].

By definition a physical state is singlet under the operation of J_i^α which is a generator of the tensor-product representation of all E_{ij}^α's for fixed i. By fixing orientation we specify a direction by a integer $d = \pm 1, \pm 2, \ldots, \pm n$ where $-d$ is the opposite direction of d. To specify a neighboring site by direction we put a semicolon between site i and direction d. For example,

$$\mathrm{J}_i^\alpha = \sum_{(i,j)\in L} \mathrm{E}_{ij}^\alpha = \sum_{d=1}^{n} (\mathrm{E}_{i;-d}^\alpha + \mathrm{E}_{i;d}^\alpha).$$

Let us refine the labeling (2.1) by composing the representations, connected at site i, one by one. The each step is performed by combining labels for two representations ρ and ρ' using the fact that the tensor product $V_\rho \otimes V_{\rho'}$ of the representation spaces V_ρ, $V_{\rho'}$ is decomposed into irreducible components by the action of the product representation $\rho \otimes \rho'$ of G. That is, from two labels $|\rho, v_i, v_j\rangle$, $|\rho', v'_{i'}, v'_{j'}\rangle$ we construct new label $|\rho, \rho', \rho'', v''_{i''}, v''_{j''}\rangle$ where the ρ'' specifies a irreducible component* of the $\rho \otimes \rho'$ and the vector $v''_{i''}$'s are bases of its representation space.

For simplicity we will explain the case d=2. We combine labels of two positive (negative) links for each site i. Then we have

$$\left|\rho_{i;1}\ ,\ \rho_{i;2}\ ,\ \rho_{i;1}\otimes\rho_{i;2}\ ,\ v_\alpha^+\ ,\ v_\beta^+\ ;\ \rho_{i;-1}\ ,\ \rho_{i;-2}\ ,\ \rho_{i;-1}\otimes\rho_{i;-2}\ ,\ v_{\alpha'}^-\ ,\ v_{\beta'}^-\right\rangle_i$$

where $\rho_{i;1}\otimes\rho_{i;2}$ ($\rho_{i;-1}\otimes\rho_{i;-2}$) stands for a label of irreducible components. Combine $\rho_{i;1}\otimes\rho_{i;2}$ and $\rho_{i;-1}\otimes\rho_{i;-2}$. Then,

$$\begin{aligned}\Big|\rho_{i;1}\ ,\ \rho_{i;2}\ ,\ \rho_{i;-1}\ ,\ \rho_{i;-2}\ ,\ \rho_{i;1}\otimes\rho_{i;2}\ ,\ \rho_{i;-1}\otimes\rho_{i;-2}\ ,\\ \rho_{i;1}\otimes\rho_{i;2}\otimes\rho_{i;-1}\otimes\rho_{i;-2}\ ,\ w_{\alpha''}\ ,\ w_{\beta''}\Big\rangle_i\end{aligned} \tag{2.3}$$

Using the labeling (2.3), we see that a physical state has a label where $\rho_{i;1}\otimes\rho_{i;2}\otimes\rho_{i;-1}\otimes\rho_{i;-2}$ is 0-representation and $w_{\alpha''} = w_{\beta''}$ is its invariant state.

We shall discuss the (2+1)-dimensional case of $SU(2)$ in detail. For $SU(2)$ any representation and its complex conjugate are the same one.

⋆ It is not sufficient to specify the type of representation of the irreducible component, because, in case of $SU(N)$ ($N > 2$), the product representation may have many irreducible components which are of the same type. This difficulty was mentioned in Ref 4. Also see the reference.[6]

Thus $(\mathrm{E}^{\alpha}_{i;-d})^2 = (\mathrm{E}^{\alpha}_{i-d;d})^2$ for $d = 1, 2$. Then labeling (2.3)becomes:

$$\big|(\mathrm{E}_{i;1})^2\ ,\ (\mathrm{E}_{i;2})^2\ ,\ (\mathrm{E}_{i;1}+\mathrm{E}_{i;2})^2\ ,\ (\mathrm{E}_{i;-1}+\mathrm{E}_{i;-2})^2\ ,\ (\mathrm{J}_i)^2\ ,\ \mathrm{J}^3_i\ ,\ \mathrm{J}^3_i\big\rangle_i$$

For the physical states $(\mathrm{J}_i)^2 = \mathrm{J}^3_i = 0$. This implies $(\mathrm{E}_{i;1}+\mathrm{E}_{i;2})^2 = (\mathrm{E}_{i;-1}+\mathrm{E}_{i;-2})^2$ because the product $\rho_1 \otimes \rho_2$ contains 0-representation if and only if $\rho_1 = \overline{\rho}_2 = \rho_2$. Furthermore this 0-representation has multiplicity 1 in $\rho_1 \otimes \overline{\rho}_2$. Thus we need three indices for labeling the physical states:

$$\big|(\mathrm{E}_{i;1})^2\ ,\ (\mathrm{E}_{i;2})^2\ ,\ (\mathrm{E}_{i;1}+\mathrm{E}_{i;2})^2\big\rangle_i$$

or

$$\big|\rho_{i;1}, \rho_{i;2}, \rho_{i;12}\big\rangle_i$$

where $\rho_{i;d}$'s are the total angular momenta which satisfy the triangle rule:

$$|\rho_{i;1} - \rho_{i;2}| \le \rho_{i;12} \le \rho_{i;1} + \rho_{i;2}.$$

A graphical presentation of this labeling was developed in Ref 5. (Or another presentation using Kagomé lattice[7].)

3. Harmonic analysis on the extended configuration space

3.1. Structure theorems

The Peter-Weyl theorem is, as we have seen, a structure theorem of the function on a compact topological group. This theorem can be extended on a compact homogeneous space G/K for a compact topological group G and its closed subgroup K. We denote $\mathcal{D}(G, K)$ the subset of $\mathcal{D}(G)$ each of which element has at least one nonzero K-invariant vector. We choose a basis $\{|\alpha\rangle\}_{1\le\alpha\le n_\rho}$ so that the first m^K_ρ vectors are K-invariant i.e. $\rho(k)|\alpha\rangle = |\alpha\rangle$ for $1 \le \alpha \le m^K_\rho$ and all $k \in K$. The m^K_ρ is called multiplicity of ρ. The statement is

Theorem. *Let K be a closed subgroup of a compact group G. Then,*

$$L^2(G/K) = \overline{Lin\{\sqrt{n_\rho}\langle\alpha|\rho|\beta\rangle\ ;\ \text{where } 1\le\alpha\le n_\rho,\, 1\le\beta\le m^K_\rho,\ \rho \in \mathcal{D}(G,K)\}}. \tag{3.1}$$

The left hand side $L^2(G/K)$ means the right K-invariant functions on G and the right one derives from the original theorem merely by replacing the ket vector with the K-invariant one. Furthermore We can extend this theorem to the case $L\backslash G/K$.

Corollary. *Let K,L be closed subgroups of a compact group G. Then,*

$$L^2(L\backslash G/K) = \overline{Lin\{\sqrt{n_\rho}\langle\alpha|'\rho|\beta\rangle \text{ ; where } {\scriptstyle 1\leq\alpha\leq m^L_\rho,\, 1\leq\beta\leq m^K_\rho},\ \rho\in\mathcal{D}(G,K,L)\}} \tag{3.2}$$

where $\mathcal{D}(G,K,L) \equiv \mathcal{D}(G,K)\cap\mathcal{D}(G,L)$, and $\{|\alpha\rangle'\}_{1\leq\alpha\leq n_\rho}$ is another basis for L, and $L^2(L\backslash G/K)$ is the closure of a set of right K-invariant, left L-invariant functions i.e. $f(lxk) = f(x)$ for all $x\in G$, $k\in K$, $l\in L$.

Proof. We will show only inclusion ($\subset$). Let $f\in L^2(G/K\backslash L)$. We may assume that f has the form of the right hand side of theorem (3.1):

$$f(x) = \langle\alpha|\rho(x)|K\rangle$$

where $|K\rangle$ is a K-invariant vector. Since f is left L-invariant, for any $l\in L$

$$f(x) = f(lx) = \langle\alpha|\rho(l)\rho(x)|K\rangle = \sum_\gamma \rho(l)_{\alpha\gamma}\langle\gamma|\rho(x)|K\rangle$$

From the linear independency of $\langle\alpha|\rho(x)|K\rangle$, we have

$$\rho(l)_{\alpha\gamma} = \delta_{\alpha\gamma} \quad \text{for all} \quad \gamma$$

This implies

$$\langle\alpha|\rho(l) = \langle\alpha|. \qquad \Box$$

3.2. Extended configuration space

In this subsection we will study the relationship between $\tilde{\mathcal{C}}$ and $\mathcal{C}$, and the action of gauge transformation group on them. Identifying each positive and corresponding negative links we get, as has been suggested, $\mathcal{C}$ from $\tilde{\mathcal{C}}$. Precisely, we define G_{glue} to be a subgroup of $\tilde{\mathcal{C}}$.

$$\tilde{G}_{glue} \equiv \{\mathrm{U}\in\tilde{\mathcal{C}} \text{ ; } \mathrm{U}_{ij} = \mathrm{U}_{ji} \text{ for all possible site indices } i,j\}$$

where we consider the equation $\mathrm{U}_{ij} = \mathrm{U}_{ji}$ under the canonical identification $G_{ij} = G_{ji} = G$. Then, we have the equivalence as homogeneous spaces.

$$\tilde{\mathcal{C}}/\tilde{G}_{glue} \cong \mathcal{C}$$

where the identification map is give by the extension of the map $G_{ij}\times G_{ji}\to G_{ij}$ such that $(g_+,\, g_-)\mapsto g_+\cdot g_-^{-1}$ for each positive link (i,j).

Recall that the gauge transformations, by definition, act on the function space; however, their finite form are realized by the action on $\mathcal{C}$ as

$$\mathrm{J}_i(g)\mathrm{U}_{kl} \equiv \begin{cases} g^{-1}\cdot \mathrm{U}_{kl} & \text{if } k=i\ ; \\ \mathrm{U}_{kl}\cdot g & \text{if } l=i\ ; \\ \mathrm{U}_{kl} & \text{otherwise} \end{cases}$$

where $\{\mathrm{U}_{kl}\}_{(k,l)\in L^+}$ is an element of $\mathcal{C}$. (One can see easily that the definition of $\mathrm{J}_i(g)$ coincides with the usual definition of the gauge transformation in the lattice gauge theory, which is generated by g at the site i.) We denote G_{gauge} the group generated by all $\mathrm{J}_i(g)$'s for all site i and all $g \in G$. It is very important that $\mathrm{J}_i(g)$ can be lifted up onto $\tilde{\mathcal{C}}$ in such a factorized form $\tilde{\mathrm{J}}_i(g)$ as

$$\tilde{\mathrm{J}}_i(g)\mathrm{U}_{kl} \equiv \begin{cases} g^{-1}\cdot \mathrm{U}_{kl} & \text{if k=i}\ ; \\ \mathrm{U}_{kl} & \text{otherwise.} \end{cases}$$

Then we have the following commutative diagram.

$$\begin{array}{ccccc} \tilde{\mathcal{C}} & \xrightarrow{\pi} & \tilde{\mathcal{C}}/\tilde{G}_{glue} & \cong & \mathcal{C} \\ \big\downarrow \tilde{\mathrm{J}}_i(g) & & & & \big\downarrow \mathrm{J}_i(g) \\ \tilde{\mathcal{C}} & \xrightarrow{\pi} & \tilde{\mathcal{C}}/\tilde{G}_{glue} & \cong & \mathcal{C} \end{array}$$

where π is the natural projection.

We denote $\tilde{G}_{gauge}$ the group generated by all $\tilde{\mathrm{J}}_i(g)$'s for all site i and all $g \in G$, which is realized as a subgroup of $\tilde{\mathcal{C}}$:

$$\tilde{G}_{gauge} = \{\mathrm{U} \in \tilde{\mathcal{C}}\ ;\ \mathrm{U}_{ij} = \mathrm{U}_{ik} \text{ for all possible site indices } i,j,k\}.$$

This means that the left $\tilde{G}_{gauge}$ action (as a subgroup) and the action of $\tilde{\mathrm{J}}_i(g)$'s on $\tilde{\mathcal{C}}$ are the same. Then we have

$$\tilde{G}_{gauge}\backslash\ \tilde{\mathcal{C}}\ /\tilde{G}_{glue} \cong \mathcal{C}/G_{gauge}$$

From this relation our main theorem follows.

Theorem. *There is 1-1 correspondence between gauge invariant functions on $\mathcal{C}$ and both left $\tilde{G}_{gauge}$- and right $\tilde{G}_{glue}$-invariant functions on $\tilde{\mathcal{C}}$.*

Combining with the corollary (3.2) we get the structure theorem for gauge invariant functions.

Structure theorem for gauge invariant functions.

$$\{\sqrt{n_{\tilde\rho}}\langle O^{\tilde\rho\,(\alpha)}_{gauge}|\tilde\rho|O^{\tilde\rho}_{glue}\rangle\ ;\ \tilde\rho\in\mathcal{D}(\tilde{\mathcal{C}},\tilde G_{glue},\tilde G_{gauge})\}$$

is a complete orthonormal basis of gauge invariant functions.

The states $|O^{\tilde\rho\,(\alpha)}_{gauge}\rangle$, $|O^{\tilde\rho}_{glue}\rangle$ are invariant bases for each of $\tilde G_{glue}$ and $\tilde G_{gauge}$ and the index α distinguishes the $\tilde G_{gauge}$-multiplicity of $\tilde\rho$. On the other hand $\tilde G_{glue}$-multiplicity is one. A representation $\tilde\rho\in\mathcal{D}(\tilde{\mathcal{C}},\tilde G_{glue},\tilde G_{gauge})$ is an infinite product of irreducible representations of G assigned to each links—but only finite number of them are non trivial, and has invariant bases for each of $\tilde G_{glue}$ and $\tilde G_{gauge}$.

Proof. Any gauge invariant function f on $\mathcal{C}$ has the corresponding both left $\tilde G_{gauge}$- and right $\tilde G_{glue}$-invariant function $\tilde f\equiv f\circ\pi$ on $\tilde{\mathcal{C}}$. The function $\tilde f$ belongs to $L^2(\tilde{\mathcal{C}})$ if and only if f belongs to $L^2(\mathcal{C})$. Then we have

$$\tilde f\in L^2(\tilde G_{gauge}\backslash\tilde{\mathcal{C}}/\tilde G_{glue}).$$

Apply the corollary (3.2) to the triple of groups $\tilde{\mathcal{C}}$, $\tilde G_{glue}$, and $\tilde G_{gauge}$, then we get the component expression of f as desired form. The uniqueness of $|O^{\tilde\rho}_{glue}\rangle$ is from the fact that tensor product of two irreducible representations of G contains, if it exists, unique 0-component and $|O_{glue}\rangle$ is the tensor product of all stable vectors each of which is associated to the 0-component of the tensor product of the two representations of each positive link and corresponding negative link. $\square$

We should notice that $\tilde{\mathcal{C}}$, $\tilde G_{glue}$, and $\tilde G_{gauge}$ are compact groups because the infinite product of compact groups is also a compact topological group under the product topology. Furthermore $\tilde{\mathcal{C}}$ has the product measure[8] which is the invariant Haar measure. Thus the Peter-Weyl theorem is applicable to $\tilde{\mathcal{C}}$.

3.3. Identification of plaquette term

Through the correspondence between states and functions, the operator $\hat{\mathrm{U}}_{ij}$ is nothing but the multiplicative operator $\tau(\mathrm{U}_{ij})$ on the function space. Furthermore, the plaquette term $\mathrm{Tr}\,\hat{\mathrm{U}}_\square\equiv\mathrm{Tr}(\hat{\mathrm{U}}_{ij}\hat{\mathrm{U}}_{jk}\hat{\mathrm{U}}_{kl}\hat{\mathrm{U}}_{li})$,assigned to the plaquette ${}^{l}_{i}\square^{k}_{j}$, corresponds to $\mathrm{Tr}\,\tau(\hat{\mathrm{U}}_\square)$ that is a gauge invariant function. Based on the structure theorem for gauge invariant functions one can find the expression of this function as a matrix element. Define a representation $\rho_{\Box}$ of $\tilde{\mathcal{C}}$ as the infinite product of representations of G so that we assign the defining representation to links (i,j),(j,k),(k,l),(l,i) and assign the complex conjugate of the defining representation to links (j,i),(k,j),(l,k),(i,l) and assign 0-representation to other links. We write $|O_{ab,cd}\rangle$ for the stable base of 0-representation component of the tensor product of the defining representation of the link (a,b) and the complex conjugate of the defining representation of (c,d).

Then $|O_{glue}\rangle$ is a tensor product $|O_{ij,ji}\rangle|O_{jk,kj}\rangle|O_{kl,lk}\rangle|O_{li,il}\rangle$ and $|O_{gauge}\rangle$ is a tensor product $|O_{ij,il}\rangle|O_{jk,ji}\rangle|O_{kl,kj}\rangle|O_{li,lk}\rangle$ where we omit the infinite product of stable bases of 0-representations assigned to the other links.

In this case $\tilde{G}_{gauge}$-multiplicity is also one. The following proposition gives the matrix element expression of the plaquette term.

Proposition. *In the case of $SU(N)$*

$$\mathrm{Tr}\,\tau(\hat{\mathrm{U}}_{\Box}) = N^4\langle O_{gauge}|\rho_{\Box}|O_{glue}\rangle$$

Proof. We need more information about $|O_{ab,cd}\rangle$. We write $|\alpha\rangle$ for the α-th natural base of the defining representation of $SU(N)$. In the case of $SU(2)$ define $|1\rangle^* \equiv |2\rangle$ and $|2\rangle^* \equiv -|1\rangle$, otherwise define $|n\rangle^* \equiv |n\rangle$. Then we have

$$|O_{ab,cd}\rangle = \frac{1}{\sqrt{N}}\sum_{\alpha=1}^{N}|\alpha\rangle_{ab}|\alpha\rangle^*_{cd}$$

It is easy to see that the right hand side is eliminated by the action of all generators of tensor-product representation. Furthermore $|O_{ab,cd}\rangle$ has the contraction property:

$$\langle\alpha|_{ab}\langle\beta|^*_{cd}\rho_{ab}\overline{\rho}_{cd}|O_{ab,cd}\rangle = \frac{1}{\sqrt{N}}\sum_{\gamma=1}^{N}(\rho_{ab})_{\alpha\gamma}(\rho_{cd})^*_{\beta\gamma}$$

where ρ is the defining representation $\tau(\mathrm{U})$. We should notice that for the case of $SU(2)$ we have $\rho = \overline{\rho}$, then the complex conjugate term $(\rho_{cd})^*_{\beta\gamma}$ of the right hand side comes from the twisted definition of $|1\rangle^*$ and $|2\rangle^*$.

By definition we have $\rho_{\Box} = \rho_{ij}\,\overline{\rho}_{ji}\,\rho_{jk}\,\overline{\rho}_{kj}\,\rho_{kl}\,\overline{\rho}_{lk}\,\rho_{li}\,\overline{\rho}_{il}$. Then, using contraction property, we get

$$
\begin{aligned}
&N^4\langle O_{gauge}|\rho_{\Box}|O_{glue}\rangle \\
&= N^4\langle O_{ij,il}|\langle O_{jk,ji}|\langle O_{kl,kj}|\langle O_{li,lk}| \\
&\qquad \times \rho_{ij}\,\overline{\rho}_{ji}\,\rho_{jk}\,\overline{\rho}_{kj}\,\rho_{kl}\,\overline{\rho}_{lk}\,\rho_{li}\,\overline{\rho}_{il}|O_{ij,ji}\rangle|O_{jk,kj}\rangle|O_{kl,lk}\rangle|O_{li,il}\rangle \\
&= \sum_{\text{Greek indices}} (\rho_{ij})_{\alpha_1\beta_1}(\rho_{ji})^*_{\gamma_1\beta_1}(\rho_{jk})_{\gamma_1\beta_2}(\rho_{kj})^*_{\gamma_2\beta_2}(\rho_{kl})_{\gamma_2\beta_3}(\rho_{lk})^*_{\gamma_3\beta_3}(\rho_{li})_{\gamma_3\beta_4}(\rho_{il})^*_{\alpha_1\beta_4} \\
&= \sum_{\text{Greek indices}} (\rho_{ij})_{\alpha_1\beta_1}(\rho_{ji})^{-1}_{\beta_1\gamma_1}(\rho_{jk})_{\gamma_1\beta_2}(\rho_{kj})^{-1}_{\beta_2\gamma_2}(\rho_{kl})_{\gamma_2\beta_3}(\rho_{lk})^{-1}_{\beta_3\gamma_3}(\rho_{li})_{\gamma_3\beta_4}(\rho_{il})^{-1}_{\beta_4\alpha_1} \\
&= \mathrm{Tr}\,(\tau(\mathrm{U}_{ij})\tau(\mathrm{U}_{ji})^{-1}\tau(\mathrm{U}_{jk})\tau(\mathrm{U}_{kj})^{-1}\tau(\mathrm{U}_{kl})\tau(\mathrm{U}_{lk})^{-1}\tau(\mathrm{U}_{li})\tau(\mathrm{U}_{il})^{-1}) \\
&= \mathrm{Tr}\,\tau(\mathrm{U}_{\Box})
\end{aligned}
$$

To get the last equation, we use the correspondence between $L^2(\tilde{\mathcal{C}})$ and $L^2(\mathcal{C})$. □

4. Formula of the matrix elements of H

4.1. Clebsh-Gordan coefficients

It is trivial that the product of two gauge invariant functions is also gauge invariant. But this implies a non-trivial consequence that the tensor product of two bases of the physical states which we chose in the previous section would be decomposed into the linear combination of the same basis. A base of the physical states is a matrix element of a suitable irreducible representation $\tilde{\rho}$ of $\tilde{\mathcal{C}}$. Thus the decomposition coefficients are the special case of the Clebsh-Gordan coefficients of $\tilde{\mathcal{C}}$.

We write simply f for the function which corresponds to a base of the physical states. Then the irreducible representation ρ^f of $\tilde{\mathcal{C}}$ corresponds to f. (We will omit the tilde for representations.) We write simply $|O^f_{glue}\rangle$ for $|O^{\rho_f}_{glue}\rangle$ and $|O^f_{gauge}\rangle$ for $|O^{\rho_f\,(\alpha)}_{gauge}\rangle$.

Then we have the matrix element expression:

$$
f = \sqrt{n_{\rho^f}}\langle O^f_{gauge}|\rho_f|O^f_{glue}\rangle
$$

where $n_{\rho^f} \equiv \prod_i \prod_{d=1}^{2}(2\rho_{i;d}+1)(2\rho_{i;-d}+1) = \prod_i [(2\rho_{i;1}+1)(2\rho_{i;2}+1)]^2$. (We write simply $\rho_{i;d}$ for $\rho^f_{i;d}$.)

We choose another base f'. The product $f' \cdot f$ can be rewritten as

$$\begin{aligned} f' \cdot f &= \sqrt{n_{\rho^{f'}}} \langle O^{f'}_{gauge} | \rho^{f'} | O^{f'}_{glue} \rangle \cdot \sqrt{n_{\rho_f}} \langle O^{f}_{gauge} | \rho^{f} | O^{f}_{glue} \rangle \\ &= \sqrt{n_{\rho^{f'}}} \sqrt{n_{\rho^f}} \langle O^{f'}_{gauge} | \langle O^{f}_{gauge} | \rho^{f'} \otimes \rho^{f} | O^{f'}_{glue} \rangle | O^{f}_{glue} \rangle \end{aligned}.$$

The state $|O^{f'}_{gauge}\rangle|O^{f}_{gauge}\rangle$ is $\tilde{G}_{gauge}$-invariant state of the representation $\rho^{f'} \otimes \rho^{f}$. Thus $|O^{f'}_{gauge}\rangle|O^{f}_{gauge}\rangle$ can be decomposed into irreducible $\tilde{G}_{gauge}$-invariant states with respect to the decomposition of $\rho^{f'} \otimes \rho^{f}$ into irreducible representations. For $|O^{f'}_{glue}\rangle|O^{f}_{glue}\rangle$ the situation is the same.

Here we restrict ourselves to the (2+1)-dimensional $SU(2)$ case. On $SU(2)$ an irreducible representation is specified by a non-negative half integer ρ, and its basis is labeled by a half integer m ($|m| \leq \rho$). We write $|\rho, m\rangle$ for this base—the state which has total momentum ρ and has z-component m.

An irreducible component of the tensor product of two irreducible representations, is completely determined by the Clebsh-Gordan coefficients $C(\rho_1\rho_2\rho_3; m_1m_2m_3)$. The state $|\rho_{12}, m\rangle$ of the irreducible component ρ_{12} of the tensor product $\rho_1 \otimes \rho_2$ is given as

$$|(\rho_1 + \rho_2)\rho_{12}, m\rangle = \sum_{m_1, m_2} C(\rho_1\rho_2\rho_{12}; m_1m_2m)|\rho_1, m_1\rangle|\rho_2, m_2\rangle$$

and its inverse relation

$$|\rho_1, m_1\rangle|\rho_2, m_2\rangle = \sum_{\rho_{12}, m} C(\rho_1\rho_2\rho_{12}; m_1m_2m)|(\rho_1 + \rho_2)\rho_{12}, m\rangle$$

where we use the notation $(\rho_1 + \rho_2)\rho_{12}$ which means ρ_{12}-component of the product representation $\rho_1 \otimes \rho_2$. At this time we may not need extra-index for $SU(2)$ to specify which component of the same class of the representation we choose. The component is unique. (See APPENDIX B for detailed formula.)

The Clebsh-Gordan coefficients describe completely the composition of two representations. But, when one compose more than two representations the order of composition becomes a problem. For later application we shall consider the composition order problem for four representations. For the total representation space of the composite representation ρ of ρ_1, ρ_2, ρ_3, and ρ_4, we will choose two bases

$$\{\, |((\rho_1 + \rho_2)\rho_{12} + (\rho_3 + \rho_4)\rho_{34})\rho, \mathrm{m}\rangle \,\}_{\rho_{12}, \rho_{34}, \mathrm{m}} \quad \text{and} \quad \{\, |((\rho_1 + \rho_3)\rho_{13} + (\rho_2 + \rho_4)\rho_{24})\rho, \mathrm{m}\rangle \,\}_{\rho_{13}, \rho_{24}, \mathrm{m}}.$$

Then we introduce the Wigner's $9j$-symbol (the curly brackets) which combines these two bases:

$$
\begin{aligned}
&\left|((\rho_1+\rho_2)\rho_{12}+(\rho_3+\rho_4)\rho_{34})\rho,m\right\rangle \\
&\quad=\sum_{\rho_{13},\rho_{24}}\left|((\rho_1+\rho_3)\rho_{13}+(\rho_2+\rho_4)\rho_{24})\rho,m\right\rangle \\
&\qquad\qquad\times\sqrt{(2\rho_{12}+1)(2\rho_{34}+1)(2\rho_{13}+1)(2\rho_{24}+1)}\begin{Bmatrix}\rho_1 & \rho_2 & \rho_{12}\\ \rho_3 & \rho_4 & \rho_{34}\\ \rho_{13} & \rho_{24} & \rho\end{Bmatrix}.
\end{aligned}
$$

(See also APPENDIX B.)

Recall the argument of the labeling for the physical state in Subsection 2.2. A base f of the physical states corresponds to some $\left|\rho_{i;1},\rho_{i;2},\rho_{i;12}\right\rangle_i$. Then we have explicit formulae:

$$
\left|O^f_{glue}\right\rangle=\prod_i\prod_{d=1}^{2}\left|(\rho_{i;d}+\rho_{i+d;-d})0,0\right\rangle
$$

$$
\begin{aligned}
\left|O^f_{gauge}\right\rangle&=\prod_i\left|(\rho_{i;12}+\rho_{i;-12})0,0\right\rangle\\
&=\prod_i\left|((\rho_{i;1}+\rho_{i;2})\rho_{i;12}+(\rho_{i;-1}+\rho_{i;-2})\rho_{i;-12})0,0\right\rangle
\end{aligned}
$$

Then using the $9j$-symbol, we have

$$
\begin{aligned}
&\left|O^{f'}_{glue}\right\rangle\left|O^f_{glue}\right\rangle\\
&=\prod_i\prod_{d=1}^{2}\left|(\rho'_{i;d}+\rho'_{i+d;-d})0,0\right\rangle\otimes\left|(\rho_{i;d}+\rho_{i+d;-d})0,0\right\rangle\\
&=\prod_i\prod_{d=1}^{2}\left|((\rho'_{i;d}+\rho'_{i+d;-d})0+(\rho_{i;d}+\rho_{i+d;-d}))0,0\right\rangle\\
&=\prod_i\prod_{d=1}^{2}\sum_{\rho,\rho'}\left|((\rho'_{i;d}+\rho_{i;d})\rho+(\rho'_{i+d;-d}+\rho_{i+d;-d})\rho')0,0\right\rangle\sqrt{(2\rho+1)(2\rho'+1)}\begin{Bmatrix}\rho'_{i;d} & \rho'_{i+d;-d} & 0\\ \rho_{i;d} & \rho_{i+d;-d} & 0\\ \rho & \rho' & 0\end{Bmatrix}\\
&=\prod_i\prod_{d=1}^{2}\sum_{\rho}\sqrt{\frac{2\rho+1}{(2\rho'_{i;d}+1)(2\rho_{i;d}+1)}}\left|((\rho'_{i;d}+\rho_{i;d})\rho+(\rho'_{i+d;-d}+\rho_{i+d;-d})\rho)0,0\right\rangle\\
&=\sum_{f''}\prod_i\prod_{d=1}^{2}\sqrt{\frac{2\rho''_{i;d}+1}{(2\rho'_{i;d}+1)(2\rho_{i;d}+1)}}\left|O^{f''}_{glue}\right\rangle
\end{aligned}
$$

where the sum of f'' is taken for all f'' of which $\rho''_{i;d}$ satisfies triangle rule with $\rho_{i;d}$ and $\rho'_{i;d}$.

Similarly,

$$
\begin{aligned}
&\big|O^{f'}_{gauge}\big\rangle\big|O^{f}_{gauge}\big\rangle\\
&=\prod_i\big|(\rho'_{i;12}+\rho'_{i;-12})0,0\big\rangle\otimes\big|(\rho_{i;12}+\rho_{i;-12})0,0\big\rangle\\
&=\prod_i\sum_{\rho''_{i;12}}\sqrt{\tfrac{2\rho''_{i;12}+1}{(2\rho'_{i;12}+1)(2\rho_{i;12}+1)}}\big|((\rho'_{i;12}+\rho_{i;12})\rho''_{i;12}+(\rho'_{i;-12}+\rho_{i;-12})\rho''_{i;12})0,0\big\rangle.
\end{aligned}
$$

Furthermore,

$$(\rho'_{i;12}+\rho_{i;12})\rho''_{i;12}=((\rho'_{i;1}+\rho'_{i;2})\rho'_{i;12}+(\rho_{i;1}+\rho_{i;2})\rho_{i;12})\rho''_{i;12}$$

is a composite of four representations. So is $(\rho'_{i;-12}+\rho_{i;-12})\rho''_{i;12}$. We can change even the composite order of even inner compositions because the coefficients of changing bases have no dependence on m. Thus, we have

$$
\begin{aligned}
&\big|O^{f'}_{gauge}\big\rangle\big|O^{f}_{gauge}\big\rangle\\
&=\sum_{\rho''_{i;12}}\prod_i\sqrt{\tfrac{2\rho''_{i;12}+1}{(2\rho'_{i;12}+1)(2\rho_{i;12}+1)}}\\
&\times\sum_{\rho''_{i;1},\rho''_{i;2},\rho''_{i;-1},\rho''_{i;-2}}\big|(((\rho'_{i;1}+\rho_{i;1})\rho''_{i;1}+(\rho'_{i;2}+\rho_{i;2})\rho''_{i;2})\rho''_{i;12}+((\rho'_{i;-1}+\rho_{i;-1})\rho''_{i;-1}+(\rho'_{i;-2}+\rho_{i;-2})\rho''_{i;-2})\rho''_{i;12})0,0\big\rangle\\
&\times\sqrt{(2\rho'_{i;12}+1)(2\rho_{i;12}+1)(2\rho''_{i;1}+1)(2\rho''_{i;2}+1)}\sqrt{(2\rho'_{i;-12}+1)(2\rho_{i;-12}+1)(2\rho''_{i;-1}+1)(2\rho''_{i;-2}+1)}\\
&\times\begin{Bmatrix}\rho'_{i;1}&\rho'_{i;2}&\rho'_{i;12}\\ \rho_{i;1}&\rho_{i;2}&\rho_{i;12}\\ \rho''_{i;1}&\rho''_{i;2}&\rho''_{i;12}\end{Bmatrix}\begin{Bmatrix}\rho'_{i;-1}&\rho'_{i;-2}&\rho'_{i;-12}\\ \rho_{i;-1}&\rho_{i;-2}&\rho_{i;-12}\\ \rho''_{i;-1}&\rho''_{i;-2}&\rho''_{i;12}\end{Bmatrix}\\
&=\sum_{f''}\prod_i\sqrt{(2\rho''_{i;12}+1)(2\rho'_{i;12}+1)(2\rho_{i;12}+1)}\sqrt{(2\rho''_{i;1}+1)(2\rho''_{i;2}+1)(2\rho''_{i;-1}+1)(2\rho''_{i;-2}+1)}\\
&\times\begin{Bmatrix}\rho'_{i;1}&\rho'_{i;2}&\rho'_{i;12}\\ \rho_{i;1}&\rho_{i;2}&\rho_{i;12}\\ \rho''_{i;1}&\rho''_{i;2}&\rho''_{i;12}\end{Bmatrix}\begin{Bmatrix}\rho'_{i;-1}&\rho'_{i;-2}&\rho'_{i;-12}\\ \rho_{i;-1}&\rho_{i;-2}&\rho_{i;-12}\\ \rho''_{i;-1}&\rho''_{i;-2}&\rho''_{i;12}\end{Bmatrix}\big|O^{f''}_{gauge}\big\rangle.
\end{aligned}
$$

Then, by combining two formulae and normalizing factors, we get

Theorem. (*Clebsh-Gordan coefficients for the tensor product of physical states*)

$$f' \otimes f = \sum_{f''} C(f'f; f'')f''$$

where $C(f'f;f'') \equiv \prod_i \sqrt{(2\rho'_{i;1}+1)(2\rho'_{i;2}+1)(2\rho'_{i;12}+1)}\sqrt{(2\rho_{i;1}+1)(2\rho_{i;2}+1)(2\rho_{i;12}+1)}$

$$\times \sqrt{(2\rho''_{i;-1}+1)(2\rho''_{i;-2}+1)(2\rho''_{i;12}+1)} \begin{Bmatrix} \rho'_{i;1} & \rho'_{i;2} & \rho'_{i;12} \\ \rho_{i;1} & \rho_{i;2} & \rho_{i;12} \\ \rho''_{i;1} & \rho''_{i;2} & \rho''_{i;12} \end{Bmatrix} \begin{Bmatrix} \rho'_{i;-1} & \rho'_{i;-2} & \rho'_{i;12} \\ \rho_{i;-1} & \rho_{i;-2} & \rho_{i;12} \\ \rho''_{i;-1} & \rho''_{i;-2} & \rho''_{i;12} \end{Bmatrix},$$

and the sum of f'' *is taken for all physical* f'' *of which* $\rho''_{i;d}$ *satisfies triangle rule with* $\rho_{i;d}$ *and* $\rho'_{i;d}$ *for* $d = 1, 2, 12$.

4.2. Formula of the matrix elements of H

Now, we will calculate $\operatorname{Tr}\tau(\hat{\mathrm{U}}_\Box) f$ from the last theorem. We knew that $\operatorname{Tr}\tau(\hat{\mathrm{U}}_\Box) = N^4 \langle O_{gauge} | \rho_\Box | O_{glue} \rangle$; the right hand side is normalized properly. Then set f' as

$$\rho'_{i;1} = \rho'_{i;2} = \rho'_{j;-1} = \rho'_{j;2} = \rho'_{j;12} = \rho'_{j;-12} = \rho'_{k;-1} = \rho'_{k;-2} = \rho'_{l;1} = \rho'_{l;-2} = \rho'_{l;12} = \rho'_{l;-12} = \tfrac{1}{2}$$

and otherwise $\rho'_{m;d} = 0$. The non-trivial factors are for i, j, k, l.

For the i-factor, for instance, we have

$$\sqrt{4(2\rho_{i;1}+1)(2\rho_{i;2}+1)(2\rho_{i;12}+1)(2\rho''_{i;-1}+1)(2\rho''_{i;-2}+1)(2\rho''_{i;12}+1)} \begin{Bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \rho_{i;1} & \rho_{i;2} & \rho_{i;12} \\ \rho''_{i;1} & \rho''_{i;2} & \rho''_{i;12} \end{Bmatrix} \begin{Bmatrix} 0 & 0 & 0 \\ \rho_{i;-1} & \rho_{i;-2} & \rho_{i;12} \\ \rho''_{i;-1} & \rho''_{i;-2} & \rho''_{i;12} \end{Bmatrix}$$

$$= \sqrt{4(2\rho_{i;1}+1)(2\rho_{i;2}+1)(2\rho_{i;12}+1)(2\rho''_{i;-1}+1)(2\rho''_{i;-2}+1)(2\rho''_{i;12}+1)}$$

$$\times\, \delta_{\rho''_{i;12}\rho_{i;12}} \frac{(-)^{\rho_{i;2}+\rho''_{i;1}+\rho_{i;12}+\frac{1}{2}}}{\sqrt{2(2\rho_{i;12}+1)}} \begin{Bmatrix} \rho_{i;12} & \rho_{i;1} & \rho_{i;2} \\ \frac{1}{2} & \rho''_{i;2} & \rho''_{i;1} \end{Bmatrix}$$

$$\times\, \delta_{\rho''_{i;12}\rho_{i;12}} \delta_{\rho''_{i;-1}\rho_{i;-1}} \delta_{\rho''_{i;-2}\rho_{i;-2}} \frac{1}{\sqrt{(2\rho_{i;12}+1)(2\rho_{i;-1}+1)(2\rho_{i;-2}+1)}}$$

$$= \delta_{\rho''_{i;12}\rho_{i;12}} \delta_{\rho''_{i;-1}\rho_{i;-1}} \delta_{\rho''_{i;-2}\rho_{i;-2}} (-)^{\rho_{i;2}+\rho''_{i;1}+\rho_{i;12}+\frac{1}{2}} \sqrt{2(2\rho_{i;1}+1)(2\rho_{i;2}+1)} \begin{Bmatrix} \rho_{i;12} & \rho_{i;1} & \rho_{i;2} \\ \frac{1}{2} & \rho''_{i;2} & \rho''_{i;1} \end{Bmatrix}.$$

Similarly one can calculate the rest j, k, l-factors. Then finally we reproduce the result of Ref 5 in correction of the sign factor.

Matrix elements for the Hamiltonian in the physical space.

$$\langle f'|H|f\rangle = \sum_i \{\rho_{i;1}(\rho_{i;1}+1)+\rho_{i;2}(\rho_{i;2}+1)\}\langle f'|f\rangle - K\sum_{\Box}\langle M^{\Box}_{f'f}\rangle$$

$$where\ \langle M^{\Box}_{f'f}\rangle \equiv \delta_{\rho'_{i;-1}\rho_{i;-1}}\delta_{\rho'_{i;-2}\rho_{i;-2}}\delta_{\rho'_{i;12}\rho_{i;12}}\delta_{\rho'_{j;1}\rho_{j;1}}\delta_{\rho'_{j;-2}\rho_{j;-2}}\delta_{\rho'_{k;1}\rho_{k;1}}\delta_{\rho'_{k;2}\rho_{k;2}}\delta_{\rho'_{k;12}\rho_{k;12}}\delta_{\rho'_{l;2}\rho_{l;2}}\delta_{\rho'_{l;-1}\rho_{l;-1}}$$

$$\times\sqrt{2\rho'_{i;1}+1}\sqrt{2\rho'_{i;2}+1}\sqrt{2\rho'_{j;2}+1}\sqrt{2\rho'_{j;12}+1}\sqrt{2\rho'_{l;1}+1}\sqrt{2\rho'_{l;12}+1}(-)^{\rho'_{i;1}+\rho'_{i;2}+\rho'_{j;2}+\rho'_{j;12}+\rho'_{l;1}+\rho'_{l;12}}$$

$$\times\sqrt{2\rho_{i;1}+1}\sqrt{2\rho_{i;2}+1}\sqrt{2\rho_{j;2}+1}\sqrt{2\rho_{j;12}+1}\sqrt{2\rho_{l;1}+1}\sqrt{2\rho_{l;12}+1}(-)^{\rho_{i;1}+\rho_{i;2}+\rho_{j;2}+\rho_{j;12}+\rho_{l;1}+\rho_{l;12}}$$

$$\times\begin{Bmatrix}\rho_{i;12} & \rho_{i;1} & \rho_{i;2}\\ \frac{1}{2} & \rho'_{i;2} & \rho'_{i;1}\end{Bmatrix}\begin{Bmatrix}\rho_{j;1} & \rho_{j;2} & \rho_{j;12}\\ \frac{1}{2} & \rho'_{j;12} & \rho'_{j;2}\end{Bmatrix}\begin{Bmatrix}\rho_{j;-2} & \rho_{j;12} & \rho_{i;1}\\ \frac{1}{2} & \rho'_{i;1} & \rho'_{j;12}\end{Bmatrix}\begin{Bmatrix}\rho_{k;12} & \rho_{j;1} & \rho_{j;2}\\ \frac{1}{2} & \rho'_{j;2} & \rho'_{l;1}\end{Bmatrix}$$

$$\times\begin{Bmatrix}\rho_{l;2} & \rho_{l;12} & \rho_{l;1}\\ \frac{1}{2} & \rho'_{l;1} & \rho'_{l;12}\end{Bmatrix}\begin{Bmatrix}\rho_{l;-1} & \rho_{i;2} & \rho_{l;12}\\ \frac{1}{2} & \rho'_{l;12} & \rho'_{i;2}\end{Bmatrix}(-)^{\rho_{i;12}+\rho_{j;-2}+\rho_{j;1}+\rho_{k;12}+\rho_{l;2}+\rho_{l;-1}}.$$

5. Discussion

We have developed a new method for studying the structure of the physical space by means of using the Peter-Weyl theorem on the extended configuration space $\tilde{\mathcal{C}}$. The Clebsh-Gordan coefficients for the decomposition of the tensor product of two physical states in (2+1)-dimensional $SU(2)$ model is obtained. From the Clebsh-Gordan coefficients we have deduced the matrix elements of the Hamiltonian without tensor operator calculus. The essential point of our calculus is that each of the interaction plaquette terms can be identified with some physical state.

The formulation in Ref 2, 4 is based on $\mathcal{C}$ and the state space $\mathcal{H}$ is defined to be the infinite tensor product of $|\rho\ ,\ v_\alpha\ ,\ v_\beta\rangle$'s—triplet of a irreducible representation of G and its two base vectors—assigned to each positive links as we have seen in Section 2. (In Ref 4 the author called this space 'the extended Hilbert space'. But in the present paper the word 'extended' has another meaning.) This definition of the state space has an inconvenient property that the state space is not closed under the operation of the tensor product because the tensor product of such two states contains generally a component in which different representations appear at the same link. That is, some component of the product $|\rho'\ ,\ v'_{\alpha'}\ ,\ v'_{\beta'}\rangle|\rho\ ,\ v_\alpha\ ,\ v_\beta\rangle$ at the link (i,j) includes a state on which the eigen values $(\mathrm{E}_{ij})^2$ and $(\mathrm{E}_{ji})^2$ do not coincide. On the other hand the function space $L^2(\mathcal{C})$ has a natural ring structure. In that formulation this ring structure will be lost.

Our new formulation stands on $\tilde{\mathcal{C}}$. The corresponding definition of the extended state space $\tilde{\mathcal{H}}$ is the infinite tensor product of $|\rho\ ,\ v_\alpha\rangle$'s—pair of a irreducible representation of G and one base vector—assigned to each of positive and negative links. In other words $\tilde{\mathcal{C}}$ is the direct sum of the representation spaces of the irreducible representations of $\tilde{\mathcal{C}}$. There different representations can be assigned to the positive link and the corresponding negative link. It is obvious that the $\tilde{\mathcal{H}}$ is closed under the operation of the tensor product. We write $|\rho, v\rangle$ for an element of $\tilde{\mathcal{H}}$ which consists the representation ρ of $\tilde{\mathcal{C}}$ and the base vector v of the representation space of ρ. We define a projection p from the $\tilde{\mathcal{H}}$ to the $L^2(\mathcal{C})$ as

$$p\ :\ |\rho, v\rangle \mapsto \sqrt{n_\rho}\langle\rho, v|\rho|O^\rho_{glue}\rangle$$

where $|O^\rho_{glue}\rangle$ is defined in Section 3 for $\rho \in \mathcal{D}(\tilde{\mathcal{C}}, \tilde{G}_{glue})$, and defined to be 0 elsewhere. It is easy to see that the projection p is a ring homomorphism by using the equation

$$\sqrt{n_\rho}\sqrt{n_{\rho'}}|O^\rho_{glue}\rangle|O^{\rho'}_{glue}\rangle = \sum_{\rho''}\sqrt{n_{\rho''}}|O^{\rho''}_{glue}\rangle$$

where ρ'' moves all irreducible components contained in $\rho\otimes\rho'$. (One can check this equation at least for SU(2) case using the decomposition formula in the present paper. For general case it is still a conjecture.)

We write $L^2(\mathcal{C})^{G_{gauge}}$ for the gauge invariant functions on $\mathcal{C}$ and $\tilde{\mathcal{H}}^{\tilde{G}_{gauge}}$ for $\tilde{G}_{gauge}$-invariant vectors in $\tilde{\mathcal{H}}$. Then the projection p gives the ring isomorphism

$$\tilde{\mathcal{H}}^{\tilde{G}_{gauge}}/Ker(p) \cong L^2(\mathcal{C})^{G_{gauge}}$$

which gives the background of our calculations.

The map $f \mapsto |O^f_{gauge}\rangle$ constructed in Section 4 gives a inverse map of p from $L^2(\mathcal{C})^{G_{gauge}}$ to $\tilde{\mathcal{H}}^{\tilde{G}_{gauge}}$. (But it is not a ring homomorphism.) Under above ring isomorphism, we can study the product $f'\cdot f$ by decomposing $|O^{f'}_{gauge}\rangle \otimes |O^f_{gauge}\rangle$ in $\mathcal{H}^{\tilde{G}_{gauge}}$.

If once the labeling problem mentioned in the footnote in Section 2.2 is solved, the extension of our method to $SU(N)$ $N>2$ case is straightforward.

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APPENDIX A

The operator $\sum_\alpha (\mathrm{E}_R^\alpha)^2$ is equal to *the Casimir element* C in the theory of Lie algebra. The eigen value of C in the irreducible representation ρ of $SU(N)$ is given by

$$C = (\lambda, \lambda + \sum_{\alpha \succ 0} \alpha)$$

where λ is the highest weight of ρ and α moves all positive roots of $SU(N)$.

We can calculate C easily for lower N.

$SU(2)$ case: $SU(2)$ has only one positive root $\alpha = 1$ and representation ρ is specified by half integer j of which $\lambda = j$. Thus $C = j(j+1)$.

$SU(3)$ case: $SU(3)$ has three positive roots $(\frac{1}{2}, \frac{\sqrt{3}}{2})$, $(1, 0)$, and $(\frac{1}{2}, -\frac{\sqrt{3}}{2})$. The representation ρ is specified by its highest weight (j_1, j_2). Thus $C = j_1(j_1+2) + (j_2)^2$.

And generally, if $\lambda' \succ \lambda$ then $C_{\lambda'} > C_\lambda$.

APPENDIX B

We use several properties of the Clebsh-Gordan coefficients, the $6j$-symbols, and the $9j$-symbols.[9][10]

§ Clebsh-Gordan coefficients

The explicit formula of the Clebsh-Gordan coefficients is

$$C(\rho_1\rho_2\rho_3; m_1 m_2 m_3) = \delta_{m_1+m_2, m_3}$$

$$\times \sqrt{(2\rho_3+1)\frac{(\rho_3+\rho_1-\rho_2)!(\rho_3-\rho_1+\rho_2)!(\rho_1+\rho_2-\rho_3)!(\rho_3+m_3)!(\rho_3-m_3)!}{(\rho_1+\rho_2+\rho_3+1)!(\rho_1-m_1)!(\rho_1+m_1)!(\rho_2-m_2)!(\rho_2+m_2)!}}$$

$$\times \sum_\nu \frac{(-)^{\nu+\rho_2+m_2}}{\nu!} \frac{(\rho_2+\rho_3+m_1-\nu)!(\rho_1-m_1+\nu)!}{(\rho_3-\rho_1+\rho_2-\nu)!(\rho_3+m_3-\nu)!(\nu+\rho_1-\rho_2-m_3)!}$$

where the sum over ν is taken for all integer. (We use the convention for negative integer n $\frac{1}{n!} = 0$.)

Especially,

$$C(\rho_1\rho_2 0; m_1 m_2 0) = \delta_{m_1,-m_2}\sqrt{\frac{1}{2\rho_1+1}}(-)^{\rho_1-m_1}.$$

§ 6j-symbol

The definition of the 6j-symbol is

$$\langle (j_1+(j_2+j_3)j)J,m|((j_1+j_2)j'+j_3)J,m\rangle$$
$$\equiv \sqrt{(2j+1)(2j'+1)}(-)^{j_1+j_2+j_3+J}\begin{Bmatrix} j_1 & j_2 & j' \\ j_3 & J & j \end{Bmatrix}.$$

The 6j-symbol has properties below.

i) Invariance under exchanging two columns

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ J_1 & J_2 & J_3 \end{Bmatrix} = \begin{Bmatrix} j_2 & j_1 & j_3 \\ J_2 & J_1 & J_3 \end{Bmatrix} = \begin{Bmatrix} j_1 & j_3 & j_2 \\ J_1 & J_3 & J_2 \end{Bmatrix}.$$

ii) Invariance under exchanging two elements of 1st row and corresponding two elements of 2nd row.

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ J_1 & J_2 & J_3 \end{Bmatrix} = \begin{Bmatrix} J_1 & J_2 & j_3 \\ j_1 & j_2 & J_3 \end{Bmatrix}.$$

iii) For special cases

$$\begin{Bmatrix} j_1 & j_2 & 0 \\ J_1 & J_2 & J_3 \end{Bmatrix} = (-)^{j_1+J_1+J_3}\frac{\delta_{j_1,j_2}\delta_{J_1,J_2}}{\sqrt{(2j_1+1)(2J_1+1)}}$$

$$\begin{Bmatrix} j & j+\frac{1}{2} & \frac{1}{2} \\ J & J+\frac{1}{2} & g+\frac{1}{2} \end{Bmatrix} = (-)^{1+g+j+J}\sqrt{\frac{(1+g+j-J)(1+g-j+J)}{(2j+1)(2j+2)(2J+1)(2J+2)}}$$

$$\begin{Bmatrix} j & j+\frac{1}{2} & \frac{1}{2} \\ J+\frac{1}{2} & J & g \end{Bmatrix} = (-)^{1+g+j+J}\sqrt{\frac{(1-g+j+J)(2+g+j+J)}{(2j+1)(2j+2)(2J+1)(2J+2)}}.$$

§ 9j-symbol

The definition of the $9j$-symbol is

$$\langle((j_1+j_2)j_{12}+(j_3+j_4)j_{34})J,m|((j_1+j_3)j_{13}+(j_2+j_4)j_{24})J,m\rangle$$

$$\equiv\sqrt{(2j_{12}+1)(2j_{34}+1)(2j_{13}+1)(2j_{24}+1)}\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{34} \\ j_{13} & j_{24} & J \end{Bmatrix}.$$

The $9j$-symbol has properties below.

i) Invariance except the sign factor $(-)^{\sum_{i=1}^{9} j_i}$ under exchanging two columns or two rows

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \\ j_7 & j_8 & j_9 \end{Bmatrix}=(-)^{\sum_{i=1}^{9} j_i}\begin{Bmatrix} j_4 & j_5 & j_6 \\ j_1 & j_2 & j_3 \\ j_7 & j_8 & j_9 \end{Bmatrix}=(-)^{\sum_{i=1}^{9} j_i}\begin{Bmatrix} j_2 & j_1 & j_3 \\ j_5 & j_4 & j_6 \\ j_8 & j_7 & j_9 \end{Bmatrix}.$$

ii) For special cases

$$\begin{Bmatrix} j_1 & j_2 & 0 \\ j_4 & j_5 & j_6 \\ j_7 & j_8 & j_9 \end{Bmatrix}=\delta_{j_1,j_2}\delta_{j_6,j_9}\frac{(-)^{j_1+j_5+j_7+j_9}}{\sqrt{(2j_1+1)(2j_9+1)}}\begin{Bmatrix} j_4 & j_5 & j_6 \\ j_8 & j_7 & j_1 \end{Bmatrix}$$

$$\begin{Bmatrix} 0 & 0 & 0 \\ j_4 & j_5 & j_6 \\ j_7 & j_8 & j_9 \end{Bmatrix}=\frac{\delta_{j_4,j_7}\delta_{j_5,j_8}\delta_{j_6,j_9}}{\sqrt{(2j_4+1)(2j_5+1)(2j_6+1)}}.$$

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