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ZARISKI DECOMPOSITIONS ON ARITHMETIC SURFACES

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Abstract. In this paper, we establish the Zariski decompositions of arithmetic $\mathbb{R}$-Cartier divisors of continuous type on arithmetic surfaces and investigate several properties. We also develop the general theory of arithmetic $\mathbb{R}$-Cartier divisors on arithmetic varieties.

Contents

Introduction

Let S be a non-singular projective surface over an algebraically closed field and let $\mathrm{Div}(S)$ be the group of Cartier divisors on S. An element of $\mathrm{Div}(S) \otimes_{\mathbb{Z}} \mathbb{R}$ is called an $\mathbb{R}$-*Cartier divisor* on S. In addition, it is said to be *effective* if it is a linear combination of curves with non-negative real coefficients. The problem of the Zariski decomposition for an effective $\mathbb{R}$-Cartier divisor D is to find a decomposition $D = P + N$ with the following properties:

(1) $P, N \in \mathrm{Div}(S) \otimes_{\mathbb{Z}} \mathbb{R}$.
(2) P is nef, that is, $(P \cdot C) \geq 0$ for all reduced and irreducible curves C on S.
(3) N is effective.
(4) Assuming $N \neq 0$, let $N = c_1 C_1 + \cdots + c_l C_l$ be the decomposition such that $c_1, \ldots, c_l \in \mathbb{R}_{>0}$ and $C_1, \ldots, C_l$ are distinct reduced and irreducible curves on S. Then the following (4.1) and (4.2) hold:
  (4.1) $(P \cdot C_i) = 0$ for all i.
  (4.2) The $l \times l$ matrix given by $\left((C_i \cdot C_j)\right)_{\substack{1 \leq i \leq l \\ 1 \leq j \leq l}}$ is negative definite.

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In 1962, Zariski [24] established the decomposition in the case where $D \in \mathrm{Div}(S)$. By the recent work due to Bauer [1] (see also Section 1), P is characterized by the greatest element in

$$\{M \in \mathrm{Div}(S) \otimes_{\mathbb{Z}} \mathbb{R} \mid D - M \text{ is effective and } M \text{ is nef}\}.$$

In this paper, we would like to consider an arithmetic analogue of the above problem on an arithmetic surface. In order to make the main theorem clear, we need to introduce a lot of concepts and terminology.

• **Green functions for $\mathbb{R}$-Cartier divisors.** Let V be an equidimensional smooth projective variety over $\mathbb{C}$. An element of $\mathrm{Div}(V)_{\mathbb{R}} := \mathrm{Div}(V) \otimes_{\mathbb{Z}} \mathbb{R}$ is called an $\mathbb{R}$-*Cartier divisor* on V. For an $\mathbb{R}$-Cartier divisor D on V, we would like to introduce several types of Green functions for D. We set $D = a_1 D_1 + \cdots + a_l D_l$, where $a_1, \ldots, a_l \in \mathbb{R}$ and D_i's are reduced and irreducible divisors on V. Let $g : V \to \mathbb{R} \cup \{\pm\infty\}$ be a locally integrable function on V. We say g is a *D-Green function of C^∞-type* (resp a *D-Green function of C^0-type*) on V if, for each point $x \in V$, there are a small open neighborhood U_x of x, local equations $f_1, \ldots, f_l$ of $D_1, \ldots, D_l$ over U_x respectively and a C^∞-function (resp. continuous function) u_x over U_x such that

$$g = u_x + \sum_{i=1}^{l} (-a_i) \log |f_i|^2 \quad \text{(a.e.)}$$

holds on U_x. These definitions are counterparts of C^∞-metrics and continuous metrics. Besides them, it is necessary to introduce a degenerated version of semipositive metrics. We say g is a *D-Green function of* $\mathrm{PSH}_{\mathbb{R}}$*-type* on V if the above u_x is taken as a real valued plurisubharmonic function on U_x (i.e., u_x is a plurisubharmonic function on U_x and $u_x(y) \in \mathbb{R}$ for all $y \in U_x$). To say more generally, let $\mathscr{L}^1_{\mathrm{loc}}$ be the sheaf consisting of locally integrable functions, that is,

$$\mathscr{L}^1_{\mathrm{loc}}(U) = \{g : U \to \mathbb{R} \cup \{\pm\infty\} \mid g \text{ is locally integrable}\}$$

for an open set U of V, and let us fix a subsheaf $\mathscr{T}$ of $\mathscr{L}^1_{\mathrm{loc}}$ satisfying the following conditions (in the following (1), (2) and (3), U is an arbitrary open set of V):

(1) If $u, v \in \mathscr{T}(U)$ and $a \in \mathbb{R}_{\geq 0}$, then $u + v \in \mathscr{T}(U)$ and $au \in \mathscr{T}(U)$.
(2) If $u, v \in \mathscr{T}(U)$ and $u \leq v$ almost everywhere, then $u \leq v$.
(3) If $\phi \in \mathscr{O}_V^\times(U)$ (i.e., ϕ is a nowhere vanishing holomorphic function on U), then $\log(|\phi|^2) \in \mathscr{T}(U)$.

This subsheaf $\mathscr{T}$ is called a *type for Green functions on V*. Moreover, $\mathscr{T}$ is said to be *real valued* if $u(x) \in \mathbb{R}$ for any open set U, $u \in \mathscr{T}(U)$ and $x \in U$. Using $\mathscr{T}$, we say g is a *D-Green function of $\mathscr{T}$-type* on V if the above u_x is an element of $\mathscr{T}(U_x)$ for each $x \in V$. The set of all D-Green functions of $\mathscr{T}$-type on V is denoted by $G_{\mathscr{T}}(V; D)$. If $x \notin \mathrm{Supp}(D)$, then, by using (2) and (3) in the properties of $\mathscr{T}$, we can see that the value

$$u_x(x) + \sum_{i=1}^{l} (-a_i) \log |f_i(x)|^2$$

does not depend on the choice of the local expression

$$g = u_x + \sum_{i=1}^{l} (-a_i) \log |f_i|^2 \quad \text{(a.e.)}$$

of g, so that $u_x(x) + \sum_{i=1}^{l}(-a_i)\log|f_i(x)|^2$ is called the *canonical value of g at x* and it is denoted by $g_{\mathrm{can}}(x)$. Note that $g_{\mathrm{can}} \in \mathscr{T}(V \setminus \mathrm{Supp}(D))$ and $g = g_{\mathrm{can}}$ (a.e.) on $V \setminus \mathrm{Supp}(D)$. Further, if $\mathscr{T}$ is real valued, then $g_{\mathrm{can}}(x) \in \mathbb{R}$.

⋆ *$H^0(V,D)$ for an $\mathbb{R}$-Cartier divisor D and its norm arising from a Green function.* Let D be an $\mathbb{R}$-Cartier divisor. If V is connected, then $H^0(V,D)$ is defined by

$$H^0(V,D) := \left\{ \phi \;\middle|\; \begin{array}{l} \phi \text{ is a non-zero rational function} \\ \text{on } V \text{ with } (\phi) + D \geq 0 \end{array} \right\} \cup \{0\}.$$

In general, let $V = V_1 \cup \cdots \cup V_r$ be the decomposition into connected components of V. Then

$$H^0(V,D) := \bigoplus_{i=1}^{r} H^0(V_i, D|_{V_i}).$$

Let g be a D-Green function of C^0-type on V. For $\phi \in H^0(V,D)$, it is easy to see that $|\phi|_g := \exp(-g/2)|\phi|$ coincides with a continuous function almost everywhere on V, so that the supremum norm $\|\phi\|_g$ of ϕ with respect to g is defined by

$$\|\phi\|_g := \operatorname{ess\,sup}\left\{ |\phi|_g(x) \mid x \in V \right\}.$$

• **Arithmetic $\mathbb{R}$-Cartier divisors.** Let X be a d-dimensional generically smooth normal projective arithmetic variety, that is, X is a flat and projective integral scheme over $\mathbb{Z}$ such that X is normal, X is smooth over $\mathbb{Q}$ and the Krull dimension of X is d. Let $\mathrm{Div}(X)$ be the group of Cartier divisors on X. As before, an element of $\mathrm{Div}(X)_{\mathbb{R}} := \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ is called an $\mathbb{R}$-*Cartier divisor* on X. It is said to be *effective* if it is a linear combination of prime divisors with non-negative real coefficients. In addition, for $D, E \in \mathrm{Div}(X)_{\mathbb{R}}$, if $D - E$ is effective, then it is denoted by $D \geq E$ or $E \leq D$.

Let D be an $\mathbb{R}$-Cartier divisor on X and let g be a locally integrable function on $X(\mathbb{C})$. A pair $\overline{D} = (D, g)$ is called an *arithmetic $\mathbb{R}$-Cartier divisor on X* if $F_\infty^*(g) = g$ (a.e.), where F_∞ is the complex conjugation map on $X(\mathbb{C})$. Moreover, $\overline{D}$ is said to be *of C^∞-type* (resp. *of C^0-type*, *of* $\mathrm{PSH}_{\mathbb{R}}$*-type*) if g is a D-Green function of C^∞-type (resp. of C^0-type, of $\mathrm{PSH}_{\mathbb{R}}$-type). More generally, for a fixed type $\mathscr{T}$ for Green functions, $\overline{D}$ is said to be *of $\mathscr{T}$-type* if g is a D-Green function of $\mathscr{T}$-type. For arithmetic $\mathbb{R}$-Cartier divisors $\overline{D}_1 = (D_1, g_1)$ and $\overline{D}_2 = (D_2, g_2)$, we define $\overline{D}_1 = \overline{D}_2$ and $\overline{D}_1 \leq \overline{D}_2$ as follows:

$$\begin{cases} \overline{D}_1 = \overline{D}_2 & \overset{\text{def}}{\iff} \quad D_1 = D_2 \text{ and } g_1 = g_2 \text{ (a.e.)}, \\ \overline{D}_1 \leq \overline{D}_2 & \overset{\text{def}}{\iff} \quad D_1 \leq D_2 \text{ and } g_1 \leq g_2 \text{ (a.e.)}. \end{cases}$$

If $\overline{D} \geq (0,0)$, then $\overline{D}$ is said to be *effective*. Further, the set

$$\{\overline{M} \mid \overline{M} \text{ is an arithmetic } \mathbb{R}\text{-Cartier divisor on } X \text{ and } \overline{M} \leq \overline{D}\}$$

is denoted by $(-\infty, \overline{D}]$.

⋆ *Volume of arithmetic $\mathbb{R}$-Cartier divisors of C^0-type.* Let $\widehat{\mathrm{Div}}_{C^0}(X)_{\mathbb{R}}$ be the group of arithmetic $\mathbb{R}$-Cartier divisors of C^0-type on X. For $\overline{D} \in \widehat{\mathrm{Div}}_{C^0}(X)_{\mathbb{R}}$, we define $H^0(X, D)$, $\hat{H}^0(X, \overline{D})$, $\hat{h}^0(X, \overline{D})$ and $\widehat{\mathrm{vol}}(\overline{D})$ as follows:

$$
\begin{cases}
H^0(X, D) := \left\{ \psi \;\middle|\; \begin{array}{l} \psi \text{ is a non-zero rational function} \\ \text{on } X \text{ with } (\psi) + D \geq 0 \end{array} \right\} \cup \{0\}, \\
\hat{H}^0(X, \overline{D}) := \{ \psi \in H^0(X, D) \mid \|\psi\|_g \leq 1 \}, \\
\hat{h}^0(X, \overline{D}) := \log \#(\hat{H}^0(X, \overline{D})), \\
\widehat{\mathrm{vol}}(\overline{D}) := \limsup_{n \to \infty} \dfrac{\hat{h}^0(X, n\overline{D})}{n^d/d!}.
\end{cases}
$$

Note that

$$
\hat{H}^0(X, \overline{D}) = \left\{ \psi \;\middle|\; \begin{array}{l} \psi \text{ is a non-zero rational function} \\ \text{on } X \text{ with } \widehat{(\psi)} + \overline{D} \geq (0, 0) \end{array} \right\} \cup \{0\}.
$$

The continuity of

$$
\widehat{\mathrm{vol}} : \widehat{\mathrm{Pic}}(X)_{\mathbb{Q}} \to \mathbb{R}
$$

is proved in [14], where $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}} := \widehat{\mathrm{Pic}}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. Moreover, in [15], we introduce $\widehat{\mathrm{Pic}}_{C^0}(X)_{\mathbb{R}}$ as a natural extension of $\widehat{\mathrm{Pic}}(X)_{\mathbb{Q}}$ (for details, see [15] or Subsection 5.1) and prove that $\widehat{\mathrm{vol}} : \widehat{\mathrm{Pic}}(X)_{\mathbb{Q}} \to \mathbb{R}$ has the continuous extension

$$
\widehat{\mathrm{vol}} : \widehat{\mathrm{Pic}}_{C^0}(X)_{\mathbb{R}} \to \mathbb{R}.
$$

Theorem 5.2.2 shows that there is a natural surjective homomorphism

$$
\overline{\mathscr{O}}_{\mathbb{R}} : \widehat{\mathrm{Div}}_{C^0}(X)_{\mathbb{R}} \to \widehat{\mathrm{Pic}}_{C^0}(X)_{\mathbb{R}}
$$

such that $\widehat{\mathrm{vol}}(\overline{D}) = \widehat{\mathrm{vol}}(\overline{\mathscr{O}}_{\mathbb{R}}(\overline{D}))$ for all $\overline{D} \in \widehat{\mathrm{Div}}_{C^0}(X)_{\mathbb{R}}$. In particular, by using results in [5], [6], [14], [15], [16] and [22], we have the following properties of $\widehat{\mathrm{vol}} : \widehat{\mathrm{Div}}_{C^0}(X)_{\mathbb{R}} \to \mathbb{R}$ (cf. Theorem 5.2.2 and Theorem 6.6.1):

(1) $\widehat{\mathrm{vol}} : \widehat{\mathrm{Div}}_{C^0}(X)_{\mathbb{R}} \to \mathbb{R}$ is positively homogeneous of degree d, that is, $\widehat{\mathrm{vol}}(a\overline{D}) = a^d \widehat{\mathrm{vol}}(\overline{D})$ for all $a \in \mathbb{R}_{\geq 0}$ and $\overline{D} \in \widehat{\mathrm{Div}}_{C^0}(X)_{\mathbb{R}}$ (cf. [14], [15]).

(2) $\widehat{\mathrm{vol}} : \widehat{\mathrm{Div}}_{C^0}(X)_{\mathbb{R}} \to \mathbb{R}$ is continuous in the following sense: Let $\overline{D}_1, \ldots, \overline{D}_r$, $\overline{A}_1, \ldots, \overline{A}_{r'}$ be arithmetic $\mathbb{R}$-Cartier divisors of C^0-type. For a compact set B in $\mathbb{R}^r$ and a positive number ϵ, there are positive numbers δ and δ' such that, for all $a_1, \ldots, a_r, \delta_1, \ldots, \delta_{r'} \in \mathbb{R}$ and $\phi \in C^0(X)$ with $(a_1, \ldots, a_r) \in B$, $\sum_{j=1}^{r'} |\delta_j| \leq \delta$ and $\|\phi\|_{\sup} \leq \delta'$, we have

$$
\left| \widehat{\mathrm{vol}}\left(\sum_{i=1}^{r} a_i \overline{D}_i + \sum_{j=1}^{r'} \delta_j \overline{A}_j + (0, \phi) \right) - \widehat{\mathrm{vol}}\left(\sum_{i=1}^{r} a_i \overline{D}_i \right) \right| \leq \epsilon.
$$

Moreover, if $\overline{D}_1, \ldots, \overline{D}_r, \overline{A}_1, \ldots, \overline{A}_{r'}$ are C^∞, then there is a positive constant C depending only on X and $\overline{D}_1, \ldots, \overline{D}_r, \overline{A}_1, \ldots, \overline{A}_{r'}$ such that

$$\left| \widehat{\mathrm{vol}}\left(\sum_{i=1}^{r} a_i \overline{D}_i + \sum_{j=1}^{r'} \delta_j \overline{A}_j + (0,\phi) \right) - \widehat{\mathrm{vol}}\left(\sum_{i=1}^{r} a_i \overline{D}_i \right) \right| \leq C \left(\sum_{i=1}^{r} |a_i| + \sum_{j=1}^{r'} |\delta_j| \right)^{d-1} \left(\|\phi\|_{\sup} + \sum_{j=1}^{r'} |\delta_j| \right)$$

for all $a_1, \ldots, a_r, \delta_1, \ldots, \delta_{r'} \in \mathbb{R}$ and $\phi \in C^0(X)$ (cf. [14], [15]).

(3) $\widehat{\mathrm{vol}}(\overline{D})$ is given by "lim", that is,

$$\widehat{\mathrm{vol}}(\overline{D}) = \lim_{t \to \infty} \frac{\hat{h}^0(t\overline{D})}{t^d/d!},$$

where $\overline{D} \in \widehat{\mathrm{Div}}_{C^0}(X)_{\mathbb{R}}$ and $t \in \mathbb{R}_{>0}$ (cf. [5], [15]).

(4) $\widehat{\mathrm{vol}}(-)^{1/d}$ is concave, that is, for arithmetic $\mathbb{R}$-Cartier divisors $\overline{D}_1, \overline{D}_2$ of C^0-type, if $\overline{D}_1$ and $\overline{D}_2$ are pseudo-effective (for the definition of pseudo-effectivity, see SubSection 6.1), then

$$\widehat{\mathrm{vol}}(\overline{D}_1 + \overline{D}_2)^{1/d} \geq \widehat{\mathrm{vol}}(\overline{D}_1)^{1/d} + \widehat{\mathrm{vol}}(\overline{D}_2)^{1/d}$$

(cf. [16], [22]).

(5) (Fujita's approximation theorem for $\mathbb{R}$-Cartier divisors) If $\overline{D}$ is an arithmetic $\mathbb{R}$-Cartier divisor of C^0-type and $\widehat{\mathrm{vol}}(\overline{D}) > 0$, then, for any positive number ϵ, there are a birational morphism $\mu : Y \to X$ of generically smooth and normal projective arithmetic varieties and an ample arithmetic $\mathbb{Q}$-Cartier divisor $\overline{A}$ of C^∞-type on Y (cf. Section 6) such that $\overline{A} \leq \mu^*(\overline{D})$ and $\widehat{\mathrm{vol}}(\overline{A}) \geq \widehat{\mathrm{vol}}(\overline{D}) - \epsilon$ (cf. [6], [22]).

(6) (The generalized Hodge index theorem for $\mathbb{R}$-Cartier divisors) If $\overline{D}$ is an arithmetic $\mathbb{R}$-Cartier divisor of $(C^0 \cap \mathrm{PSH})$-type and D is nef on every fiber of $X \to \mathrm{Spec}(\mathbb{Z})$, then $\widehat{\mathrm{vol}}(\overline{D}) \geq \widehat{\deg}(\overline{D}^d)$ (see descriptions in "Positivity of arithmetic $\mathbb{R}$-Cartier divisors" below or Proposition 6.4.2 for the definition of $\widehat{\deg}(\overline{D}^d)$) (cf. [14]).

⋆ *Intersection number of an arithmetic* $\mathbb{R}$-*Cartier divisor with a* 1-*dimensional subscheme.* Let $\mathscr{T}$ be a real valued type for Green functions such that $C^0 \subseteq \mathscr{T}$ and $-u \in \mathscr{T}$ whenever $u \in \mathscr{T}$. Let $\overline{D} = (D, g)$ be an arithmetic $\mathbb{R}$-Cartier divisor of $\mathscr{T}$-type. Let C be a 1-dimensional closed integral subscheme of X. Let $D = a_1 D_1 + \cdots + a_l D_l$ be a decomposition such that $a_1, \ldots, a_l \in \mathbb{R}$ and D_i's are Cartier divisors. For simplicity, we assume that D_i's are effective, $C \not\subseteq \mathrm{Supp}(D_i)$ for all i and that C is flat over $\mathbb{Z}$. In this case, $\widehat{\deg}(\overline{D}\big|_C)$ is defined by

$$\widehat{\deg}(\overline{D}\big|_C) := \sum_{i=1}^{l} a_i \log \#(\mathscr{O}_C(D_i)/\mathscr{O}_C) + \frac{1}{2} \sum_{x \in C(\mathbb{C})} g_{\mathrm{can}}(x).$$

In general, see Section 5.3. Let Z be a 1-cycle on X with coefficients in $\mathbb{R}$, that is, there are $a_1, \ldots, a_l \in \mathbb{R}$ and 1-dimensional closed integral subschemes $C_1, \ldots, C_l$ on

X such that $Z = a_1C_1 + \cdots + a_lC_l$. Then $\widehat{\deg}\left(\overline{D} \mid Z\right)$ is defined by

$$\widehat{\deg}\left(\overline{D} \mid Z\right) := \sum_{i=1}^{l} a_i \widehat{\deg}\left(\left.\overline{D}\right|_{C_i}\right).$$

⋆ *Positivity of arithmetic* $\mathbb{R}$*-Cartier divisors.* An arithmetic $\mathbb{R}$-Cartier divisor $\overline{D}$ is said to be *nef* if $\overline{D}$ is of $\mathrm{PSH}_{\mathbb{R}}$-type and $\widehat{\deg}(D|_C) \geq 0$ for all 1-dimensional closed integral subschemes C of X. The cone of all nef arithmetic $\mathbb{R}$-Cartier divisors on X is denoted by $\widehat{\mathrm{Nef}}(X)_{\mathbb{R}}$. Moreover, the cone of all nef arithmetic $\mathbb{R}$-Cartier divisors of C^∞-type (resp. C^0-type) on X is denoted by $\widehat{\mathrm{Nef}}_{C^\infty}(X)_{\mathbb{R}}$ (resp. $\widehat{\mathrm{Nef}}_{C^0}(X)_{\mathbb{R}}$). Further, we say $\overline{D}$ is *big* if $\widehat{\mathrm{vol}}(\overline{D}) > 0$.

Let $\widehat{\mathrm{Div}}_{C^0}^{\mathrm{Nef}}(X)_{\mathbb{R}}$ be the vector subspace of $\widehat{\mathrm{Div}}_{C^0}(X)_{\mathbb{R}}$ generated by $\widehat{\mathrm{Nef}}_{C^0}(X)_{\mathbb{R}}$. Then, by Proposition 6.4.2,

$$\widehat{\mathrm{Div}}_{C^\infty}(X)_{\mathbb{R}} + \widehat{\mathrm{Div}}_{C^0 \cap \mathrm{PSH}}(X)_{\mathbb{R}} \subseteq \widehat{\mathrm{Div}}_{C^0}^{\mathrm{Nef}}(X)_{\mathbb{R}}$$

and the symmetric multi-linear map

$$\widehat{\mathrm{Div}}_{C^\infty}(X)_{\mathbb{R}} \times \cdots \times \widehat{\mathrm{Div}}_{C^\infty}(X)_{\mathbb{R}} \to \mathbb{R}$$

given by $(\overline{D}_1, \ldots, \overline{D}_d) \mapsto \widehat{\deg}(\overline{D}_1 \cdots \overline{D}_d)$ (cf. Proposition-Definition 6.4.1) extends to a unique symmetric multi-linear map

$$\widehat{\mathrm{Div}}_{C^0}^{\mathrm{Nef}}(X)_{\mathbb{R}} \times \cdots \times \widehat{\mathrm{Div}}_{C^0}^{\mathrm{Nef}}(X)_{\mathbb{R}} \to \mathbb{R}$$

such that $(\overline{D}, \ldots, \overline{D}) \mapsto \widehat{\mathrm{vol}}(\overline{D})$ for $\overline{D} \in \widehat{\mathrm{Nef}}_{C^0}(X)_{\mathbb{R}}$.

• **Zariski decompositions on arithmetic surfaces.** Let X be a regular projective arithmetic surface. The main theorem of this paper is the following:

Theorem A (cf. Theorem 9.2.1 and Theorem 9.3.5)**.** *Let* $\overline{D}$ *be an arithmetic* $\mathbb{R}$*-Cartier divisor of* C^0*-type on* X *such that the set*

$$(-\infty, \overline{D}] \cap \widehat{\mathrm{Nef}}(X)_{\mathbb{R}} = \{\overline{M} \mid \overline{M} \textit{ is a nef arithmetic } \mathbb{R}\textit{-Cartier divisor on } X \textit{ and } \overline{M} \leq \overline{D}\}$$

is not empty. Then there is a nef arithmetic $\mathbb{R}$*-Cartier divisor* $\overline{P}$ *of* C^0*-type such that* $\overline{P}$ *gives the greatest element of* $(-\infty, \overline{D}] \cap \widehat{\mathrm{Nef}}(X)_{\mathbb{R}}$*, that is,* $\overline{P} \in (-\infty, \overline{D}] \cap \widehat{\mathrm{Nef}}(X)_{\mathbb{R}}$ *and* $\overline{M} \leq \overline{P}$ *for all* $\overline{M} \in (-\infty, \overline{D}] \cap \widehat{\mathrm{Nef}}(X)_{\mathbb{R}}$*. Moreover, if we set* $\overline{N} = \overline{D} - \overline{P}$*, then the following properties hold:*

(1) $\widehat{\mathrm{vol}}(\overline{D}) = \widehat{\mathrm{vol}}(\overline{P}) = \widehat{\deg}(\overline{P}^2)$.
(2) $\widehat{\deg}(\left.\overline{P}\right|_C) = 0$ *for all* 1*-dimensional closed integral subschemes* C *with* $C \subseteq \mathrm{Supp}(N)$.
(3) *If* $\overline{L}$ *is an arithmetic* $\mathbb{R}$*-Cartier divisor of* $\mathrm{PSH}_{\mathbb{R}}$*-type on* X *such that* $0 \leq \overline{L} \leq \overline{N}$ *and* $\deg(\left.\overline{L}\right|_C) \geq 0$ *for all* 1*-dimensional closed integral subschemes* C *with* $C \subseteq \mathrm{Supp}(N)$*, then* $\overline{L} = 0$.

Note that the condition $(-\infty, \overline{D}] \cap \widehat{\mathrm{Nef}}(X)_{\mathbb{R}} \neq \emptyset$ is guaranteed if $\hat{h}^0(X, a\overline{D}) \neq 0$ for some $a \in \mathbb{R}_{>0}$ (cf. Proposition 9.3.2). The above decomposition $\overline{D} = \overline{P} + \overline{N}$ is called the *Zariski decomposition of* $\overline{D}$ and we say $\overline{P}$ (resp. $\overline{N}$) is the *positive part* (resp. the *negative part*) of the decomposition. For example, let $\mathbb{P}^1_{\mathbb{Z}} = \operatorname{Proj}(\mathbb{Z}[x, y])$, $C_0 = \{x = 0\}$, $z = x/y$ and $\alpha, \beta \in \mathbb{R}_{>0}$ with $\alpha > 1$ and $\beta < 1$. Then the positive part of an arithmetic Cartier divisor

$$(C_0, -\log|z|^2 + \log\max\{\alpha^2|z|^2, \beta^2\})$$

of $(C^0 \cap \mathrm{PSH})$-type on $\mathbb{P}^1_{\mathbb{Z}}$ is

$$(\theta C_0, -\theta\log|z|^2 + \log\max\{\alpha^2|z|^{2\theta}, 1\}),$$

where $\theta = \log\alpha/(\log\alpha - \log\beta)$ (cf. Subsection 9.4). This example shows that an $\mathbb{R}$-Cartier divisor is necessary for the arithmetic Zariski decomposition. In addition, an example in Remark 9.4.3 shows that the Arakelov Chow group consisting of admissible metrics due to Arakelov-Faltings is insufficient to get the Zariski decomposition.

We assume that $N \neq 0$. Let $N = c_1C_1 + \cdots + c_lC_l$ be the decomposition of N such that $c_1, \ldots, c_l \in \mathbb{R}_{>0}$ and C_i's are distinct 1-dimensional closed integral subschemes on X. Let $(C_1, g_1), \ldots, (C_l, g_l)$ be effective arithmetic Cartier divisors of $\mathrm{PSH}_{\mathbb{R}}$-type such that

$$c_1(C_1, g_1) + \cdots + c_l(C_l, g_l) \leq \overline{N},$$

which is possible by Proposition 2.4.2 and Lemma 9.1.3. Then, by using Lemma 1.2.3, the above (3) yields an inequality

$$(-1)^l \det\left(\widehat{\deg}\left((C_i, g_i)\big|_{C_j}\right)\right) > 0.$$

This is a counterpart of the property (4.2) of the Zariski decomposition on an algebraic surface. On the other hand, our Zariski decomposition is a refinement of Fujita's approximation theorem due to Chen [6] and Yuan [22] on an arithmetic surface. Actually Fujita's approximation theorem on an arithmetic surface is a consequence of the above theorem (cf. Proposition 9.3.7).

Let $\overline{D}$ be an effective arithmetic $\mathbb{R}$-Cartier divisor of C^0-type. For each $n \geq 1$, we set $F_n(\overline{D})$ and $M_n(\overline{D})$ as follows:

$$\begin{cases} F_n(\overline{D}) = \dfrac{1}{n}\displaystyle\sum_C \min\left\{\mathrm{mult}_C((\phi) + nD) \mid \phi \in \hat{H}^0(X, n\overline{D}) \setminus \{0\}\right\} C, \\ M_n(\overline{D}) = D - F_n(\overline{D}). \end{cases}$$

Let $V(n\overline{D})$ be a complex vector space generated by $\hat{H}^0(X, n\overline{D})$. It is easy to see that

$$g_{M_n(\overline{D})} := g + \frac{1}{n}\log\operatorname{dist}(V(n\overline{D}); ng)$$

is an $M_n(\overline{D})$-Green function of C^∞-type (for the definition of distorsion functions, see Subsection 3.2). Then we have the following:

Theorem B (Asymptotic orthogonality)**.** *If $\overline{D}$ is big, then*

$$\lim_{n\to\infty} \widehat{\deg}\left(\left(M_n(\overline{D}), g_{M_n(\overline{D})}\right) \Big| \ F_n(\overline{D})\right) = 0.$$

• **Technical results for the proof of the arithmetic Zariski decomposition.** In order to get the greatest element of $(-\infty, \overline{D}] \cap \widehat{\mathrm{Nef}}(X)_{\mathbb{R}}$, we need to consider the nefness of the limit of a convergent sequence of nef arithmetic $\mathbb{R}$-Cartier divisors. The following theorem is our solution for this problem:

Theorem C (cf. Theorem 7.1). *Let X be a regular projective arithmetic surface. Let $\{\overline{M}_n = (M_n, h_n)\}_{n=0}^{\infty}$ be a sequence of nef arithmetic $\mathbb{R}$-Cartier divisors on X with the following properties:*

(a) *There is an arithmetic Cartier divisor $\overline{D} = (D, g)$ of C^0-type such that $\overline{M}_n \leq \overline{D}$ for all $n \geq 1$.*
(b) *There is a proper closed subset E of X such that* $\mathrm{Supp}(D) \subseteq E$ *and* $\mathrm{Supp}(M_n) \subseteq E$ *for all $n \geq 1$.*
(c) $\lim_{n\to\infty} \mathrm{mult}_C(M_n)$ *exists for all 1-dimensional closed integral subschemes C on X.*
(d) $\limsup_{n\to\infty} (h_n)_{\mathrm{can}}(x)$ *exists in $\mathbb{R}$ for all $x \in X(\mathbb{C}) \setminus E(\mathbb{C})$.*

Then there is a nef arithmetic $\mathbb{R}$-Cartier divisor $\overline{M} = (M, h)$ on X such that $\overline{M} \leq \overline{D}$,

$$M = \sum_C \left(\lim_{n\to\infty} \mathrm{mult}_C(M_n)\right) C$$

and that $h_{\mathrm{can}}|_{X(\mathbb{C})\setminus E(\mathbb{C})}$ is the upper semicontinuous regularization of the function given by $x \mapsto \limsup_{n\to\infty} (h_n)_{\mathrm{can}}(x)$ over $X(\mathbb{C}) \setminus E(\mathbb{C})$.

Moreover, for the first property $\widehat{\mathrm{vol}}(\overline{P}) = \widehat{\mathrm{vol}}(\overline{D})$ of the arithmetic Zariski decomposition, it is necessary to observe the following behavior of distorsion functions (cf. Remark 9.3.9), which is a consequence of Gromov's inequality for an $\mathbb{R}$-Cartier divisor (cf. Proposition 3.1.1).

Theorem D (cf. Theorem 3.2.3). *Let V be an equidimensional smooth projective variety over $\mathbb{C}$ and let D be an $\mathbb{R}$-Cartier divisor on V. Let $R = \bigoplus_{n\geq 0} R_n$ be a graded subring of $\bigoplus_{n\geq 0} H^0(V, nD)$. If g is a D-Green function of C^∞-type, then there is a positive constant C with the following properties:*

(1) $\mathrm{dist}(R_n; ng) \leq C(n+1)^{3\dim V}$ *for all $n \geq 0$.*
(2) $\dfrac{\mathrm{dist}(R_n; ng)}{C(n+1)^{3\dim V}} \cdot \dfrac{\mathrm{dist}(R_m; mg)}{C(m+1)^{3\dim V}} \leq \dfrac{\mathrm{dist}(R_{n+m}; (n+m)g)}{C(n+m+1)^{3\dim V}}$ *for all $n, m \geq 0$.*

The most difficult point for the proof of the arithmetic Zariski decomposition is to check the continuous property of the positive part. For this purpose, the following theorem is needed:

Theorem E (cf. Theorem 4.6). *Let V be an equidimensional smooth projective variety over $\mathbb{C}$. Let A and B be $\mathbb{R}$-Cartier divisors on V with $A \leq B$. If there is an A-Green function h of C^∞-type such that $dd^c([h]) + \delta_A$ is represented by either a positive C^∞-form or the zero form, then, for a B-Green function g_B of C^0-type, there is an A-Green function g of* $(C^0 \cap \mathrm{PSH})$*-type such that g is the greatest element of the set*

$$G_{\mathrm{PSH}}(V; A)_{\leq g_B} := \{u \in G_{\mathrm{PSH}}(V; A) \mid u \leq g_B \text{ (a.e.)}\}$$

modulo null functions, that is, $g \in G_{\mathrm{PSH}}(V; A)_{\leq g_B}$ and $u \leq g$ (a.e.) *for all $u \in G_{\mathrm{PSH}}(V; A)_{\leq g_B}$.*

For the proof, we actually use a recent regularity result due to Berman-Demailly [3]. Even starting from an arithmetic Cartier divisor $\overline{D}$ of C^{∞}-type, it is not expected that the positive part $\overline{P}$ is of C^{∞}-type again (cf [17]). It could be that $\overline{P}$ is of $C^{1,1}$-type.

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1. Zariski decompositions in vector spaces

Logically the contexts of this section are not necessary except Lemma 1.2.3. They however give an elementary case for our considerations and provide a good overview of our paper.

1.1. In the paper [1], Bauer presents a simple proof of the existence of Zariski decompositions on an algebraic surface. Unfortunately, he uses liner series on the algebraic surface to show the negative definiteness of the negative part of the Zariski decomposition. In this section, we would like to give a linear algebraic proof without using any materials of algebraic geometry. The technical main result for our purpose is Lemma 1.2.3. After writing the first draft of this paper, Bauer, Caibăr and Kennedy kindly informed me that, in the paper [2], they had independently obtained several results similar to the contexts of this section. Their paper is written for a general reader.

Let V be a vector space over $\mathbb{R}$. Let $\boldsymbol{e} = \{e_\lambda\}_{\lambda\in\Lambda}$ be a basis of V and let $\boldsymbol{\phi} = \{\phi_\lambda\}_{\lambda\in\Lambda}$ be a family of elements of $\operatorname{Hom}_{\mathbb{R}}(V, \mathbb{R})$ such that $\phi_\lambda(e_\mu) \geq 0$ for $\lambda \neq \mu$. This pair $(\boldsymbol{e}, \boldsymbol{\phi})$ of $\boldsymbol{e}$ and $\boldsymbol{\phi}$ is called a *system of Zariski decompositions* in V.

Let us fix several notations which work only in this section. For $\lambda \in \Lambda$, the coefficient of x at e_λ in the linear combination of x with respect to the basis $\boldsymbol{e}$ is denoted by $x(\lambda;\boldsymbol{e})$, that is, $x = \sum_\lambda x(\lambda;\boldsymbol{e})e_\lambda$. Let $\leq_{\boldsymbol{e}}$ be an order relation of V given by

$$x \leq_{\boldsymbol{e}} y \quad \overset{\text{def}}{\Longleftrightarrow} \quad x(\lambda;\boldsymbol{e}) \leq y(\lambda;\boldsymbol{e}) \text{ for all } \lambda \in \Lambda.$$

We often use $y \geq_{\boldsymbol{e}} x$ instead of $x \leq_{\boldsymbol{e}} y$. $\operatorname{Supp}(x;\boldsymbol{e})$, $[x, y]_{\boldsymbol{e}}$, $(-\infty, x]_{\boldsymbol{e}}$, $[x, \infty)_{\boldsymbol{e}}$, $\operatorname{Nef}(\boldsymbol{\phi})$ and $\operatorname{Num}(\boldsymbol{\phi})$ are defined as follows:

$$\begin{cases} \operatorname{Supp}(x;\boldsymbol{e}) := \{\lambda \in \Lambda \mid x(\lambda;\boldsymbol{e}) \neq 0\}, \\ [x, y]_{\boldsymbol{e}} := \{v \in V \mid x \leq_{\boldsymbol{e}} v \leq_{\boldsymbol{e}} y\}, \\ (-\infty, x]_{\boldsymbol{e}} := \{v \in V \mid v \leq_{\boldsymbol{e}} x\}, \\ [x, \infty)_{\boldsymbol{e}} := \{v \in V \mid v \geq_{\boldsymbol{e}} x\}, \\ \operatorname{Nef}(\boldsymbol{\phi}) := \left\{v \in V \mid \phi_\lambda(v) \geq 0 \text{ for all } \lambda \in \Lambda\right\}, \\ \operatorname{Num}(\boldsymbol{\phi}) := \{v \in V \mid \phi_\lambda(v) = 0 \text{ for all } \lambda \in \Lambda\}. \end{cases}$$

For an element x of V, a decomposition $x = y + z$ is called a *Zariski decomposition* of x with respect to $(\boldsymbol{e}, \boldsymbol{\phi})$ if the following conditions are satisfied:

(1) $y \in \operatorname{Nef}(\boldsymbol{\phi})$ and $z \geq_{\boldsymbol{e}} 0$.
(2) $\phi_\lambda(y) = 0$ for all $\lambda \in \operatorname{Supp}(z;\boldsymbol{e})$.
(3) $\left\{x \in \sum_{\lambda\in\operatorname{Supp}(z;\boldsymbol{e})} \mathbb{R}_{\geq 0} e_\lambda \mid \phi_\lambda(x) \geq 0 \text{ for all } \lambda \in \operatorname{Supp}(z;\boldsymbol{e})\right\} = \{0\}$.

We call y (resp. z) the *positive part* of x (resp. *negative part* of x).

The purpose of this section is to give the proof of the following proposition.

Proposition 1.1.1. *For an element x of V, we have the following:*

(1) *The following are equivalent:*
  (1.1) *A Zariski decomposition of x with respect to $(\boldsymbol{e},\boldsymbol{\phi})$ exists.*
  (1.2) $(-\infty,x]_{\boldsymbol{e}} \cap \operatorname{Nef}(\boldsymbol{\phi}) \neq \emptyset$.

(2) *If a Zariski decomposition exists, then it is uniquely determined.*

(3) *If a Zariski decomposition of x with respect to $(\boldsymbol{e},\boldsymbol{\phi})$ exists and the negative part z of x is non-zero, then z has the following properties:*
  (3.1) *Let Q be the matrix given by $(\phi_\lambda(e_\mu))_{\lambda,\mu\in\operatorname{Supp}(z;\boldsymbol{e})}$. Then*

$$(-1)^{\#(\operatorname{Supp}(z;\boldsymbol{e}))}\det Q > 0.$$

  Moreover, if Q is symmetric, then Q is negative definite.
  (3.2) $\{e_\lambda\}_{\lambda\in\operatorname{Supp}(z;\boldsymbol{e})}$ *is linearly independent on $V/\operatorname{Num}(\boldsymbol{\phi})$.*

1.2. Proofs. Here let us give the proof of Proposition 1.1.1.

For $x_1,\ldots,x_r \in V$, $\max_{\boldsymbol{e}}\{x_1,\ldots,x_r\} \in V$ is given by

$$\max_{\boldsymbol{e}}\{x_1,\ldots,x_r\} := \sum_{\lambda\in\Lambda} \max\{x_1(\lambda;\boldsymbol{e}),\ldots,x_r(\lambda;\boldsymbol{e})\}e_\lambda.$$

Let us begin with the following lemma.

Lemma 1.2.1. *If $x_1,\ldots,x_r \in \operatorname{Nef}(\boldsymbol{\phi})$, then $\max\{x_1,\ldots,x_r\} \in \operatorname{Nef}(\boldsymbol{\phi})$.*

Proof. It is sufficient to see that if $\phi_\lambda(x_i) \geq 0$ for all i, then $\phi_\lambda(\max_{\boldsymbol{e}}\{x_1,\ldots,x_r\}) \geq 0$. We set $z = \max_{\boldsymbol{e}}\{x_1,\ldots,x_r\}$. Note that $\operatorname{Supp}(z-x_1;\boldsymbol{e}) \cap \cdots \cap \operatorname{Supp}(z-x_r;\boldsymbol{e}) = \emptyset$. Thus there is i with $\lambda \notin \operatorname{Supp}(z-x_i;\boldsymbol{e})$. Then $\phi_\lambda(z-x_i) \geq 0$, and hence

$$\phi_\lambda(z) = \phi_\lambda(z-x_i) + \phi_\lambda(x_i) \geq 0.$$

□

Lemma 1.2.2. *Let x be an element of V such that $(-\infty,x]_{\boldsymbol{e}} \cap \operatorname{Nef}(\boldsymbol{\phi}) \neq \emptyset$. Then there is the greatest element y in $(-\infty,x]_{\boldsymbol{e}} \cap \operatorname{Nef}(\boldsymbol{\phi})$, that is, $y \in \operatorname{Nef}(\boldsymbol{\phi}) \cap (-\infty,x]_{\boldsymbol{e}}$ and $y \geq_{\boldsymbol{e}} v$ for all $v \in \operatorname{Nef}(\boldsymbol{\phi}) \cap (-\infty,x]_{\boldsymbol{e}}$. This greatest element y is denoted by*

$$\max(\operatorname{Nef}(\boldsymbol{\phi}) \cap (-\infty,x]_{\boldsymbol{e}}).$$

Further, y and $z := x - y$ satisfy the following properties:

(a) $y \in \operatorname{Nef}(\boldsymbol{\phi})$, $z \geq_{\boldsymbol{e}} 0$ *and* $x = y + z$.
(b) $\phi_\lambda(y) = 0$ *for all* $\lambda \in \operatorname{Supp}(z;\boldsymbol{e})$.
(c) $\left\{v \in \sum_{\lambda\in\operatorname{Supp}(z;\boldsymbol{e})} \mathbb{R}_{\geq 0}e_\lambda \mid \phi_\lambda(v) \geq 0 \text{ for all } \lambda \in \operatorname{Supp}(z;\boldsymbol{e})\right\} = \{0\}$.

Proof. We choose $x' \in (-\infty,x]_{\boldsymbol{e}} \cap \operatorname{Nef}(\boldsymbol{\phi})$. Let us see the following claim.

Claim 1.2.2.1. *There is the greatest element y of $\operatorname{Nef}(\boldsymbol{\phi}) \cap [x',x]_{\boldsymbol{e}}$.*

Proof. Note that $[x',x]_{\boldsymbol{e}} = x' + [0,x-x']_{\boldsymbol{e}}$. Moreover, it is easy to see that

$$\begin{aligned}\operatorname{Nef}(\boldsymbol{\phi}) &\cap [x',x]_{\boldsymbol{e}} \\ &= x' + \left\{v \in [0,x-x']_{\boldsymbol{e}} \mid \phi_\lambda(v) \geq -\phi_\lambda(x') \text{ for all } \lambda \in \operatorname{Supp}(x-x';\boldsymbol{e})\right\}.\end{aligned}$$

Therefore, $\operatorname{Nef}(\boldsymbol{\phi}) \cap [x',x]_{\boldsymbol{e}}$ is a translation of a bounded convex polyhedral set in a finite dimensional vector space $\bigoplus_{\lambda\in\operatorname{Supp}(x-x';\boldsymbol{e})} \mathbb{R}e_\lambda$. Hence $\operatorname{Nef}(\boldsymbol{\phi}) \cap [x',x]_{\boldsymbol{e}}$ is a convex polytope, that is, there are $\gamma_1,\ldots,\gamma_l \in \operatorname{Nef}(\boldsymbol{\phi}) \cap [x',x]_{\boldsymbol{e}}$ such that $\operatorname{Nef}(\boldsymbol{\phi}) \cap [x',x]_{\boldsymbol{e}} = \operatorname{Conv}\{\gamma_1,\ldots,\gamma_l\}$ (cf. [23, Theorem 3.2.5 or Finite basis theorem]). If we set $y = \max\{\gamma_1,\ldots,\gamma_l\}$, then, by Lemma 1.2.1, $y \in \operatorname{Nef}(\boldsymbol{\phi}) \cap [x',x]_{\boldsymbol{e}}$. Moreover, for $v = a_1\gamma_1 + \cdots + a_l\gamma_l \in \mathrm{Nef}(\boldsymbol{\phi}) \cap [x', x]_{\boldsymbol{e}}$ ($a_1, \ldots, a_l \in \mathbb{R}_{\geq 0}$ and $a_1 + \cdots + a_l = 1$), we have

$$y = a_1 y + \cdots + a_l y \geq_{\boldsymbol{e}} a_1\gamma_1 + \cdots + a_l\gamma_l = v.$$

□

This y is actually the greatest element of $(-\infty, x]_{\boldsymbol{e}} \cap \mathrm{Nef}(\boldsymbol{\phi})$. Indeed, if $v \in (-\infty, x]_{\boldsymbol{e}} \cap \mathrm{Nef}(\boldsymbol{\phi})$, then $\max\{v, y\} \in [x', x]_{\boldsymbol{e}} \cap \mathrm{Nef}(\boldsymbol{\phi})$ by lemma 1.2.1, and hence

$$v \leq \max\{v, y\} \leq y.$$

Let us check the properties (a), (b) and (c). First of all, (a) is obvious. In order to see (b) and (c), we may assume that $z \neq 0$.

(b) We assume that $\phi_\lambda(y) > 0$ for $\lambda \in \mathrm{Supp}(z; \boldsymbol{e})$. Let ϵ be a sufficiently small positive number. Then $y + \epsilon e_\lambda \leq_{\boldsymbol{e}} x$ and

$$\phi_\mu(y + \epsilon e_\lambda) = \phi_\mu(y) + \epsilon\phi_\mu(e_\lambda) \geq 0$$

for all $\mu \in \Lambda$ because $0 < \epsilon \ll 1$. Thus $y + \epsilon e_\lambda \in \mathrm{Nef}(\boldsymbol{\phi})$, which contradicts to the maximality of y. Therefore, $\phi_\lambda(y) = 0$ for $\lambda \in \mathrm{Supp}(z; \boldsymbol{e})$.

(c) Next we assume that there is $v \in \left(\sum_{\lambda \in \mathrm{Supp}(z;\boldsymbol{e})} \mathbb{R}_{\geq 0} e_\lambda\right) \setminus \{0\}$ such that $\phi_\lambda(v) \geq 0$ for all $\lambda \in \mathrm{Supp}(z; \boldsymbol{e})$. Then there is a sufficiently small positive number ϵ' such that $y + \epsilon' v \leq_{\boldsymbol{e}} x$. Note that $\phi_\mu(y + \epsilon' v) \geq 0$ for all μ, which yields a contradiction, as before. □

Lemma 1.2.3. *Let W be a vector space over $\mathbb{R}$. Let $e_1, \ldots, e_n \in W$ and $\phi_1, \ldots, \phi_n \in \mathrm{Hom}_{\mathbb{R}}(W, \mathbb{R})$ with the following properties:*

(a) $\{(a_1, \ldots, a_n) \in \mathbb{R}^n_{\geq 0} \mid a_1 e_1 + \cdots + a_n e_n = 0\} = \{(0, \ldots, 0)\}$.
(b) $\phi_i(e_j) \geq 0$ *for all* $i \neq j$.
(c) $\{x \in \mathbb{R}_{\geq 0} e_1 + \cdots + \mathbb{R}_{\geq 0} e_n \mid \phi_i(x) \geq 0$ *for all* $i\} = \{0\}$.

Then we have the following:

(1) *Let Q be the $(n \times n)$-matrix given by $(\phi_i(e_j))$. Then there are $(n \times n)$-matrices A and B with the following properties:*
  (1.1) *A (resp. B) is a lower (resp. upper) triangle matrix consisting of non-negative numbers.*
  (1.2) $\det A > 0$, $\det B > 0$ *and*

$$AQB = \begin{pmatrix} -1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -1 \end{pmatrix}.$$

  (1.3) *If Q is symmetric, then $B = {}^tA$.*
(2) *The vectors $e_1, \ldots, e_n$ are linearly independent in*

$$W/\{x \in W \mid \phi_1(x) = \cdots = \phi_n(x) = 0\}.$$

Proof. (1) Let us begin with the following claim.

Claim 1.2.3.1. $\phi_i(e_i) < 0$ *for all i.*

Proof. If $\phi_i(e_i) \geq 0$, then $e_i \in \{x \in \mathbb{R}_{\geq 0} e_1 + \cdots + \mathbb{R}_{\geq 0} e_n \mid \phi_j(x) \geq 0$ for all $j\}$. This is a contradiction because $e_i \neq 0$. □

The above claim proves (1) in the case where $n = 1$. Here we set

$$\phi'_i = -\phi_1(e_1)\phi_i + \phi_i(e_1)\phi_1 \ (i \geq 2), \quad e'_j = -\phi_1(e_1)e_j + \phi_1(e_j)e_1 \ (j \geq 2).$$

We claim the following:

Claim 1.2.3.2. (i) *$\phi'_i(e_1) = 0$ and $\phi_1(e'_j) = 0$ for all $i \geq 2$ and $j \geq 2$.*

(ii) *$e'_2, \ldots, e'_n$ and $\phi'_2, \ldots, \phi'_n$ satisfy all assumptions* (a) $\sim$ (c) *of the lemma.*

(iii) *Let Q' be the matrix given by $(\phi'_i(e'_j))_{2 \leq i,j \leq n}$. Then*

$$A_1 Q B_1 = \begin{pmatrix} \phi_1(e_1) & 0 \\ 0 & Q' \end{pmatrix},$$

where A_1 and B_1 are matrices given by

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \phi_2(e_1) & -\phi_1(e_1) & 0 & \cdots & 0 \\ \phi_3(e_1) & 0 & -\phi_1(e_1) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_n(e_1) & 0 & 0 & \cdots & -\phi_1(e_1) \end{pmatrix}, \begin{pmatrix} 1 & \phi_1(e_2) & \phi_1(e_3) & \cdots & \phi_1(e_n) \\ 0 & -\phi_1(e_1) & 0 & \cdots & 0 \\ 0 & 0 & -\phi_1(e_1) & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & -\phi_1(e_1) \end{pmatrix}$$

respectively. Note that if Q is symmetric, then $B_1 = {}^tA_1$ and Q' is also symmetric.

Proof. (i) is obvious.

(ii) It is easy to see (a) for $e'_2, \ldots, e'_n$ by using Claim 1.2.3.1. For $i, j \geq 2$ with $i \neq j$, by Claim 1.2.3.1,

$$\phi'_i(e'_j) = \phi_1(e_1)^2\phi_i(e_j) + (-\phi_1(e_1))\phi_i(e_1)\phi_1(e_j) \geq 0.$$

Finally let $x \in \sum_{j \geq 2} \mathbb{R}_{\geq 0} e'_j$ such that $\phi'_i(x) \geq 0$ for all $i \geq 2$. Note that $\phi'_i(x) = (-\phi_1(e_1))\phi_i(x)$ for $i \geq 2$. Therefore, $\phi_i(x) \geq 0$ for all $i \geq 1$, and hence $x = 0$ because $\sum_{j \geq 2} \mathbb{R}_{\geq 0} e'_j \subseteq \sum_{j \geq 1} \mathbb{R}_{\geq 0} e_j$.

(iii) is a straightforward calculation. □

We prove (1) by induction on n. By hypothesis of induction, there are matrices A' and B' satisfying (1.1), (1.2) and (1.3) for Q', that is,

$$A'Q'B' = \begin{pmatrix} -1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -1 \end{pmatrix}.$$

Therefore,

$$\begin{pmatrix} \frac{1}{\sqrt{-\phi_1(e_1)}} & 0 \\ 0 & A' \end{pmatrix} A_1 Q B_1 \begin{pmatrix} \frac{1}{\sqrt{-\phi_1(e_1)}} & 0 \\ 0 & B' \end{pmatrix} = \begin{pmatrix} -1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & -1 \end{pmatrix}.$$

Thus (1) follows.

(2) Let $a_1 e_1 + \cdots + a_n e_n = 0$ be a linear relation on

$$W/\{x \in W \mid \phi_1(x) = \cdots = \phi_n(x) = 0\}.$$

Then there is $x \in W$ such that $x = a_1 e_1 + \cdots + a_n e_n$ and $\phi_1(x) = \cdots = \phi_n(x) = 0$. Thus $0 = \phi_i(x) = \sum \phi_i(e_j) a_j$. Hence (1) yields (2). □

Proof of Proposition 1.1.1. (1) Clearly (1.1) implies (1.2). If we assume (1.2), then (1.1) follows from Lemma 1.2.2.

(2) Let $x = y + z$ be a Zariski decomposition of x with respect to $(\boldsymbol{e}, \boldsymbol{\phi})$ and $y' = \max(\mathrm{Nef}_{\boldsymbol{\phi}} \cap (-\infty, x]_{\boldsymbol{e}})$. Then $y \leq_{\boldsymbol{e}} y'$. As $\phi_\lambda(y) = 0$ for all $\lambda \in \mathrm{Supp}(z; \boldsymbol{e})$,

$$y' - y \in \left\{ x \in \sum\nolimits_{\lambda \in \mathrm{Supp}(z;\boldsymbol{e})} \mathbb{R}_{\geq 0} e_\lambda \mid \phi_\lambda(x) \geq 0 \text{ for all } \lambda \in \mathrm{Supp}(z; \boldsymbol{e}) \right\},$$

and hence $y' = y$.

(3) follows from Lemma 1.2.3. □

Remark 1.2.4. We assume that $\phi_\lambda(e_\mu) \in \mathbb{Q}$ for all $\lambda, \mu \in \Lambda$. Let $x \in \bigoplus_\lambda \mathbb{Q} e_\lambda$ such that $(-\infty, x]_{\boldsymbol{e}} \cap \mathrm{Nef}_{\boldsymbol{\phi}} \neq \emptyset$. Let $x = y + z$ be the Zariski decomposition of x with respect to $(\boldsymbol{e}, \boldsymbol{\phi})$. Then $y, z \in \bigoplus_\lambda \mathbb{Q} e_\lambda$. Indeed, if we set $\mathrm{Supp}(z; \boldsymbol{e}) = \{\lambda_1, \ldots, \lambda_n\}$ and $z = \sum a_i e_{\lambda_i}$, then

$$\sum \phi_{\lambda_i}(e_{\lambda_j}) a_j = \phi_{\lambda_i}(x) \in \mathbb{Q}.$$

On the other hand, by our assumption and (3.1) in Proposition 1.1.1, $(\phi_{\lambda_i}(e_{\lambda_j}))_{1 \leq i,j \leq n} \in \mathrm{GL}_n(\mathbb{Q})$. Thus $(a_1, \ldots, a_n) \in \mathbb{Q}^n$.

2. Green functions for $\mathbb{R}$-Cartier divisors

2.1. Plurisubharmonic functions. Here we recall plurisubharmonic functions and the upper semicontinuous regularization of a function locally bounded above.

Let T be a metric space with a metric d. A function $f : T \to \{-\infty\} \cup \mathbb{R}$ is said to be *upper semicontinuous* if $\{x \in T \mid f(x) < c\}$ is open for any $c \in \mathbb{R}$. In other words,

$$f(a) = \limsup_{x \to a} f(x) \left(:= \inf_{\epsilon > 0} (\sup\{f(y) \mid d(a, y) \leq \epsilon\}) \right)$$

for all $a \in T$. Let $u : T \to \{-\infty\} \cup \mathbb{R}$ be a function such that u is locally bounded above. The *upper semicontinuous regularization* u^* of u is given by

$$u^*(x) = \limsup_{y \to x} u(y).$$

Note that u^* is upper semicontinuous and $u \leq u^*$.

Let D be an open set in $\mathbb{C}$. A function $u : D \to \{-\infty\} \cup \mathbb{R}$ is said to be *subharmonic* if u is upper semicontinuous and

$$u(a) \leq \frac{1}{2\pi} \int_0^{2\pi} u\left(a + r e^{\sqrt{-1}\theta}\right) d\theta$$

holds for any $a \in D$ and $r \in \mathbb{R}_{>0}$ with $\{z \in \mathbb{C} \mid |z - a| \leq r\} \subseteq D$.

Let X be a d-equidimensional complex manifold. A function $u : X \to \{-\infty\} \cup \mathbb{R}$ is said to be *plurisubharmonic* if u is upper semicontinuous and $u \circ \phi$ is subharmonic for any analytic map $\phi : \{z \in \mathbb{C} \mid |z| < 1\} \to X$. We say u is a *real valued plurisubharmonic function* if $u(x) \neq -\infty$ for all $x \in X$. If X is an open set of $\mathbb{C}^d$, then an upper semicontinuous function $u : X \to \mathbb{R} \cup \{-\infty\}$ is plurisubharmonic if and only if

$$u(a) \leq \frac{1}{2\pi} \int_0^{2\pi} u(a + \xi \exp(\sqrt{-1}\theta)) d\theta$$

holds for any $a \in X$ and $\xi \in \mathbb{C}^d$ with $\{a + \xi \exp(\sqrt{-1}\theta) \mid 0 \leq \theta \leq 2\pi\} \subseteq X$. As an example of plurisubharmonic functions, we have the following: if $f_1, \ldots, f_r$ are holomorphic functions on X, then

$$\log(|f_1|^2 + \cdots + |f_r|^2)$$

is a plurisubharmonic function on X. In particular, if

$$x \notin \{z \in X \mid f_1(z) = \cdots = f_r(z) = 0\},$$

then $dd^c(\log(|f_1|^2 + \cdots + |f_r|^2))$ is semipositive around x.

Let $\{u_\lambda\}_{\lambda \in \Lambda}$ be a family of plurisubharmonic functions on X such that $\{u_\lambda\}_{\lambda\in\Lambda}$ is locally uniformly bounded above. If we set $u(x) := \sup_{\lambda\in\Lambda}\{u_\lambda(x)\}$ for $x \in X$, then the upper semicontinuous regularization u^* of u is plurisubharmonic and $u = u^*$ (a.e.) (cf. [9, Theorem 2.9.14 and Proposition 2.6.2]). Moreover, let $\{v_n\}_{n=1}^\infty$ be a sequence of plurisubharmonic functions on X such that $\{v_n\}_{n=1}^\infty$ is locally uniformly bounded above. If we set $v(x) := \limsup_{n\to\infty} v_n(x)$ for $x \in X$, then the upper semicontinuous regularization v^* of v is plurisubharmonic and $v = v^*$ (a.e.) (cf. [9, Proposition 2.9.17 and Theorem 2.6.3]).

2.2. $\mathbb{R}$-Cartier divisors. Let X be either a d-equidimensional smooth algebraic variety over $\mathbb{C}$, or a d-equidimensional complex manifold. Let $\mathrm{Div}(X)$ be the group of Cartier divisors on X. An element D of $\mathrm{Div}(X)_{\mathbb{R}} := \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ is called an $\mathbb{R}$-*Cartier divisor* on X. Let $D = \sum_{i=1}^n a_i D_i$ be the irreducible decomposition of D, that is, $a_1, \ldots, a_n \in \mathbb{R}$ and D_i's are reduced and irreducible divisors on X. For a prime divisor Γ on X (i.e., a reduced and irreducible divisor on X), the coefficient of D at Γ in the above irreducible decomposition is denoted by $\mathrm{mult}_\Gamma(D)$, that is,

$$\mathrm{mult}_\Gamma(D) = \begin{cases} a_i & \text{if } \Gamma = D_i \text{ for some } i, \\ 0 & \text{if } \Gamma \neq D_i \text{ for all } i, \end{cases}$$

and $D = \sum_\Gamma \mathrm{mult}_\Gamma(D)\Gamma$. The support $\mathrm{Supp}(D)$ of D is defined by $\bigcup_{\mathrm{mult}_\Gamma(D)\neq 0} \Gamma$. If $a_i \geq 0$ for all i, then D is said to be *effective* and it is denoted by $D \geq 0$. More generally, for $D_1, D_2 \in \mathrm{Div}(X)_{\mathbb{R}}$,

$$D_1 \leq D_2 \text{ (or } D_2 \geq D_2) \quad \overset{\text{def}}{\Longleftrightarrow} \quad D_2 - D_1 \geq 0.$$

The *round-up* $\lceil D \rceil$ of D and the *round-down* $\lfloor D \rfloor$ of D are defined by

$$\lceil D \rceil = \sum_{i=1}^n \lceil a_i \rceil D_i \quad \text{and} \quad \lfloor D \rfloor = \sum_{i=1}^n \lfloor a_i \rfloor D_i,$$

where $\lceil x \rceil = \min\{a \in \mathbb{Z} \mid x \leq a\}$ and $\lfloor x \rfloor = \max\{a \in \mathbb{Z} \mid a \leq x\}$ for $x \in \mathbb{R}$.

We assume that X is algebraic. Let $\mathrm{Rat}(X)$ be the ring of rational functions on X. Note that X is not necessarily connected, so that $\mathrm{Rat}(X)$ is not necessarily a field. In the case where X is connected, $H^0(X, D)$ is defined to be

$$H^0(X, D) := \{\phi \in \mathrm{Rat}(X)^\times \mid (\phi) + D \geq 0\} \cup \{0\}.$$

In general, let $X = \coprod_\alpha X_\alpha$ be the decomposition into connected components, and let $D_\alpha = D|_{X_\alpha}$. Then $H^0(X, D)$ is defined to be

$$H^0(X, D) := \bigoplus_\alpha H^0(X_\alpha, D_\alpha).$$

Note that if D is effective, then $H^0(X, D)$ is generated by

$$\{\phi \in \mathrm{Rat}(X)^{\times} \mid (\phi) + D \geq 0\}.$$

Indeed, for $\phi_\alpha \in H^0(X_\alpha, D_\alpha)$, if we choose $c \in \mathbb{C}^{\times}$ with $\phi_\alpha + c \neq 0$, then

$$(0, \ldots, 0, \phi_\alpha, 0, \ldots, 0) = (1, \ldots, 1, \phi_\alpha + c, 1, \ldots, 1) - (1, \ldots, 1, c, 1, \ldots, 1),$$

which shows the assertion. Since

$$(\phi_\alpha) + D_\alpha \geq 0 \quad \Longleftrightarrow \quad (\phi_\alpha) + \lfloor D_\alpha \rfloor \geq 0,$$

we have $H^0(X, D) = H^0(X, \lfloor D \rfloor)$.

In the case where X is not necessarily algebraic, the ring of meromorphic functions on X is denoted by $\mathcal{M}(X)$. By using $\mathcal{M}(X)$ instead of $\mathrm{Rat}(X)$, we can define $H^0_{\mathcal{M}}(X, D)$ in the same way as above, that is, if X is connected, then

$$H^0_{\mathcal{M}}(X, D) := \{\phi \in \mathcal{M}(X)^{\times} \mid (\phi) + D \geq 0\} \cup \{0\}.$$

If X is a proper smooth algebraic scheme over $\mathbb{C}$, then $\mathrm{Rat}(X) = \mathcal{M}(X)$ by GAGA, and hence $H^0(X, D) = H^0_{\mathcal{M}}(X, D)$.

2.3. Definition of Green functions for $\mathbb{R}$-Cartier divisors. Let X be a d-equidimensional complex manifold. Let $\mathcal{L}^1_{\mathrm{loc}}$ be the sheaf consisting of locally integrable functions, that is,

$$\mathcal{L}^1_{\mathrm{loc}}(U) := \{g : U \to \mathbb{R} \cup \{\pm\infty\} \mid g \text{ is locally integrable}\}$$

for an open set U of X. Let $\mathscr{T}$ be a subsheaf of $\mathcal{L}^1_{\mathrm{loc}}$ and let S be a subset of $\mathbb{R} \cup \{\pm\infty\}$. Then $\mathscr{T}_S$, $\mathscr{T}^b$ and $-\mathscr{T}$ are defined as follows:

$$\begin{aligned} \mathscr{T}_S(U) &:= \{g \in \mathscr{T}(U) \mid g(x) \in S \text{ for all } x \in U\}, \\ \mathscr{T}^b(U) &:= \{g \in \mathscr{T}(U) \mid g \text{ is locally bounded on } U\}, \\ -\mathscr{T}(U) &:= \{-g \in \mathcal{L}^1_{\mathrm{loc}}(U) \mid g \in \mathscr{T}(U)\}. \end{aligned}$$

Let $\mathscr{T}'$ be another subsheaf of $\mathcal{L}^1_{\mathrm{loc}}$. We assume that $u + u'$ is well-defined as functions for any open set U, $u \in \mathscr{T}(U)$ and $u' \in \mathscr{T}'(U)$. Then $\mathscr{T} + \mathscr{T}'$ is defined to be

$$(\mathscr{T} + \mathscr{T}')(U) := \left\{ g \in \mathcal{L}^1_{\mathrm{loc}}(U) \;\middle|\; \begin{array}{l} \text{For any } x \in U \text{, we can find an open} \\ \text{neighborhood } V_x,\ u \in \mathscr{T}(V_x) \text{ and} \\ u' \in \mathscr{T}'(V_x) \text{ such that } g\big|_{V_x} = u + u'. \end{array} \right\}.$$

Similarly, if $u - u'$ is well-defined as functions for any open set U, $u \in \mathscr{T}(U)$ and $u' \in \mathscr{T}'(U)$, then $\mathscr{T} - \mathscr{T}'$ is defined to be

$$(\mathscr{T} - \mathscr{T}')(U) := \left\{ g \in \mathcal{L}^1_{\mathrm{loc}}(U) \;\middle|\; \begin{array}{l} \text{For any } x \in U \text{, we can find an open} \\ \text{neighborhood } V_x,\ u \in \mathscr{T}(V_x) \text{ and} \\ u' \in \mathscr{T}'(V_x) \text{ such that } g\big|_{V_x} = u - u'. \end{array} \right\}.$$

Note that $\mathscr{T} - \mathscr{T}' = \mathscr{T} + (-\mathscr{T}')$. A subsheaf $\mathscr{T}$ of $\mathcal{L}^1_{\mathrm{loc}}$ is called a *type for Green functions* on X if the following conditions are satisfied (in the following (1), (2) and (3), U is an arbitrary open set of X):

(1) If $u, v \in \mathscr{T}(U)$ and $a \in \mathbb{R}_{\geq 0}$, then $u + v \in \mathscr{T}(U)$ and $au \in \mathscr{T}(U)$.
(2) If $u, v \in \mathscr{T}(U)$ and $u \leq v$ (a.e.), then $u \leq v$.
(3) If $\phi \in \mathscr{O}_X^{\times}(U)$ (i.e., ϕ is a nowhere vanishing holomorphic function on U), then $\log|\phi|^2 \in \mathscr{T}(U)$.

Note that, for $u, v \in \mathscr{T}(U)$, $u = v$ if $u = v$ (a.e.). If $\mathscr{T} = \mathscr{T}_{\mathbb{R}}$, that is, $u(x) \in \mathbb{R}$ for any open set U, $u \in \mathscr{T}(U)$ and $x \in U$, then $\mathscr{T}$ is called a *real valued type*. As examples of types for Green functions on X, we have the following C^0, C^∞ and PSH:

C^0	:	the sheaf consisting of continuous functions.
C^∞	:	the sheaf consisting of C^∞-functions.
PSH	:	the sheaf consisting of plurisubharmonic functions.

Note that

$$\mathrm{PSH}_{\mathbb{R}}(U) = \{g \in \mathrm{PSH}(U) \mid g(x) \neq -\infty \text{ for all } x \in U\}$$

for an open set U of X. Let $\mathscr{T}$ and $\mathscr{T}'$ be types for Green functions on X. We say $\mathscr{T}'$ is a *subjacent type* of $\mathscr{T}$ if the following property holds for any open set U of X:

$$u' \leq u \text{ (a.e.) on } U \text{ for } u' \in \mathscr{T}'(U) \text{ and } u \in \mathscr{T}(U) \quad \Longrightarrow \quad u' \leq u \text{ on } U.$$

Lemma 2.3.1. *Let $\mathscr{T}$ be either $C^0 + \mathrm{PSH}$ or $C^0 + \mathrm{PSH}_{\mathbb{R}} - \mathrm{PSH}_{\mathbb{R}}$. Then $\mathscr{T}$ is a type for Green functions on X. Moreover,* PSH *is a subjacent type of $\mathscr{T}$.*

Proof. The conditions (1) and (3) are obvious for $\mathscr{T}$. Let us see (2). For $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$, we set $||z|| = \sqrt{|z_1|^2 + \cdots + |z_d|^2}$. Moreover, for $a \in \mathbb{C}^d$ and $r > 0$,

$$\{z \in \mathbb{C}^d \mid ||z - a|| < r\}$$

is denoted by $B^d(a; r)$.

The assertion of (2) is local, so that we may assume that $X = B^d((0, \ldots, 0); 1)$. It is sufficient to see that, for $u_1, u_2 \in \mathscr{T}(X)$, if $u_1 \leq u_2$ (a.e.), then $u_1 \leq u_2$. Let us fix $a \in B^d((0, \ldots, 0); 1)$. There are a sufficiently small $r > 0$ and $v_{ij} \in \mathscr{L}^1_{\mathrm{loc}}(B^d(a; r))$ ($i = 1, 2$ and $j = 1, 2, 3$) with the following properties:

(a) $u_1 = v_{11} + v_{12} - v_{13}$ and $u_2 = v_{21} + v_{22} - v_{23}$.
(b) $v_{11}, v_{21} \in C^0(B^d(a; r))$.
(c) $v_{12}, v_{22} \in \mathrm{PSH}(B^d(a; r))$ in the case $\mathscr{T} = C^0 + \mathrm{PSH}$.
(c)′ $v_{12}, v_{22} \in \mathrm{PSH}_{\mathbb{R}}(B^d(a; r))$ in the case $\mathscr{T} = C^0 + \mathrm{PSH}_{\mathbb{R}} - \mathrm{PSH}_{\mathbb{R}}$.
(d) $v_{13} = v_{23} = 0$ in the case $\mathscr{T} = C^0 + \mathrm{PSH}$.
(d)′ $v_{13}, v_{23} \in \mathrm{PSH}_{\mathbb{R}}(B^d(a; r))$ in the case $\mathscr{T} = C^0 + \mathrm{PSH}_{\mathbb{R}} - \mathrm{PSH}_{\mathbb{R}}$.

Let χ_ϵ ($\epsilon > 0$) be the standard smoothing kernels on $\mathbb{C}^d$ (cf. [9, Section 2.5]). It is well known that $v_{ij}(a) = \lim_{\epsilon \to 0}(v_{ij} * \chi_\epsilon)(a)$ for $i = 1, 2$ and $j = 1, 2, 3$ (cf. [9, Proposition 2.5.2 and Theorem 2.9.2]). In the case $\mathscr{T} = C^0 + \mathrm{PSH}$, since $v_{11}(a), v_{21}(a) \in \mathbb{R}$, $v_{12}(a), v_{22}(a) \in \mathbb{R} \cup \{-\infty\}$ and $v_{13} = v_{23} = 0$,

$$\begin{aligned}
\lim_{\epsilon \to 0}(u_i * \chi_\epsilon)(a) &= \lim_{\epsilon \to 0}\left((v_{i1} * \chi_\epsilon)(a) + (v_{i2} * \chi_\epsilon)(a) - (v_{i3} * \chi_\epsilon)(a)\right) \\
&= \lim_{\epsilon \to 0}(v_{i1} * \chi_\epsilon)(a) + \lim_{\epsilon \to 0}(v_{i2} * \chi_\epsilon)(a) - \lim_{\epsilon \to 0}(v_{i3} * \chi_\epsilon)(a) \\
&= v_{i1}(a) + v_{i2}(a) - v_{i3}(a) = u_i(a).
\end{aligned}$$

If $\mathscr{T} = C^0 + \mathrm{PSH}_{\mathbb{R}} - \mathrm{PSH}_{\mathbb{R}}$, then, in the same way as above, we can also see $u_i(a) = \lim_{\epsilon \to 0}(u_i * \chi_\epsilon)(a)$ for $i = 1, 2$ because $v_{ij}(a) \in \mathbb{R}$ for $i = 1, 2$ and $j = 1, 2, 3$. Therefore, (2) follows from inequalities $(u_1 * \chi_\epsilon)(a) \leq (u_2 * \chi_\epsilon)(a)$ ($\forall \epsilon > 0$). The last assertion can be checked similarly. □

Let $\mathscr{T}$ be a type for Green functions on X. Let g be a locally integrable function on X and let $D = \sum_{i=1}^{l} a_i D_i$ be an $\mathbb{R}$-Cartier divisor on X, where D_i's are reduced and irreducible divisors on X. We say g is a *D-Green function of $\mathscr{T}$-type* (or a *Green function of $\mathscr{T}$-type for D*) if, for each point $x \in X$, g has a local expression

$$g = u + \sum_{i=1}^{l} (-a_i) \log |f_i|^2 \quad \text{(a.e.)}$$

over an open neighborhood U_x of x such that $u \in \mathscr{T}(U_x)$, where $f_1, \ldots, f_l$ are local equations of $D_1, \ldots, D_l$ on U_x respectively. Note that this definition does not depend on the choice of local equations $f_1, \ldots, f_l$ on U_x by the properties (1) and (3) of $\mathscr{T}$. The set of all D-Green functions of $\mathscr{T}$-type is denoted by $G_{\mathscr{T}}(X; D)$.

Let g be a D-Green function of $\mathscr{T}$-type. We say g is *of upper bounded type* (resp. *of lower bounded type*) if, in the above local expression $g = u + \sum_{i=1}^{l}(-a_i) \log |f_i|^2$ (a.e.) around each point of X, u is locally bounded above (resp. locally bounded below). If g is of upper and lower bounded type, then g is said to be *of bounded type*. These definitions also do not depend on the choice of local equations. Note that the set of all D-Green functions of $\mathscr{T}$-type and of bounded type is nothing more than $G_{\mathscr{T}^b}(X; D)$.

We assume $x \notin \operatorname{Supp}(D)$. Let g be a D-Green function of $\mathscr{T}$-type. Let $f_1, \ldots, f_l$ and $f_1', \ldots, f_l'$ be two sets of local equations of $D_1, \ldots, D_l$ on an open neighborhood U_x of x. Let

$$g = u + \sum_{i=1}^{l} (-a_i) \log |f_i|^2 \text{ (a.e.)} \quad \text{and} \quad g = u' + \sum_{i=1}^{l} (-a_i) \log |f_i'|^2 \text{ (a.e.)}$$

be two local expressions of g over U_x, where $u, u' \in \mathscr{T}(U_x)$. Since $x \notin \operatorname{Supp}(D)$, there is an open neighborhood V_x of x such that $V_x \subseteq U_x$ and $f_1, \ldots, f_l, f_1', \ldots, f_l' \in \mathscr{O}_X^{\times}(V_x)$. Thus, by the properties (1) and (3) of $\mathscr{T}$,

$$u + \sum_{i=1}^{l} (-a_i) \log |f_i|^2, \quad u' + \sum_{i=1}^{l} (-a_i) \log |f_i'|^2 \in \mathscr{T}(V_x),$$

and hence

$$u + \sum_{i=1}^{l} (-a_i) \log |f_i|^2 = u' + \sum_{i=1}^{l} (-a_i) \log |f_i'|^2 \in \mathscr{T}(V_x)$$

over V_x by the second property of $\mathscr{T}$. This observation shows that

$$u(x) + \sum_{i=1}^{l} (-a_i) \log |f_i(x)|^2$$

does not depend on the choice of the local expression of g. In this sense, the value

$$u(x) + \sum_{i=1}^{l} (-a_i) \log |f_i(x)|^2$$

is called the *canonical value of g at x* and it is denoted by $g_{\mathrm{can}}(x)$. Note that $g_{\mathrm{can}} \in \mathscr{T}(X \setminus \operatorname{Supp}(D))$ and $g = g_{\mathrm{can}}$ (a.e.) on $X \setminus \operatorname{Supp}(D)$. Moreover, if $\mathscr{T}$ is real valued, then $g_{\mathrm{can}}(x) \in \mathbb{R}$. It is easy to see the following propositions.

Proposition 2.3.2. *Let g be a D-Green function of C^{∞}-type. Then the current $dd^c([g]) + \delta_D$ is represented by a unique C^{∞}-form α, that is, $dd^c([g]) + \delta_D = [\alpha]$. We often identifies the current $dd^c([g]) + \delta_D$ with α, and denote it by $c_1(D, g)$.*

Proposition 2.3.3. *Let $\mathscr{T}'$ and $\mathscr{T}''$ be two types for Green functions on X such that $\mathscr{T}', \mathscr{T}'' \subseteq \mathscr{T}$. Then $G_{\mathscr{T}' \cap \mathscr{T}''}(X; D) = G_{\mathscr{T}'}(X; D) \cap G_{\mathscr{T}''}(X; D)$.*

Proposition 2.3.4. (1) *If g is a D-Green function of $\mathscr{T}$-type and $a \in \mathbb{R}_{\geq 0}$, then ag is an (aD)-Green function of $\mathscr{T}$-type. Moreover, if $x \notin \operatorname{Supp}(D)$, then $(ag)_{\mathrm{can}}(x) = a g_{\mathrm{can}}(x)$.*

(2) *If g_1 (resp. g_2) is a D_1-Green function of $\mathscr{T}$-type (resp. D_2-Green function of $\mathscr{T}$-type), then $g_1 + g_2$ is a $(D_1 + D_2)$-Green function of $\mathscr{T}$-type. Moreover, if $x \notin \operatorname{Supp}(D_1) \cup \operatorname{Supp}(D_2)$, then $(g_1 + g_2)_{\mathrm{can}}(x) = (g_1)_{\mathrm{can}}(x) + (g_2)_{\mathrm{can}}(x)$.*

(3) *We assume that $-\mathscr{T} \subseteq \mathscr{T}$. If g is a D-Green function of $\mathscr{T}$-type, then $-g$ is a $(-D)$-Green function of $\mathscr{T}$-type. Moreover, if $x \notin \operatorname{Supp}(D)$, then $(-g)_{\mathrm{can}}(x) = -g_{\mathrm{can}}(x)$.*

(4) *Let g be a D-Green function of $\mathscr{T}$-type. If $g \geq 0$ (a.e.) and $x \notin \operatorname{Supp}(D)$, then $g_{\mathrm{can}}(x) \geq 0$.*

Finally let us consider the following three propositions.

Proposition 2.3.5. *Let $D = b_1 E_1 + \cdots + b_r E_r$ be an $\mathbb{R}$-Cartier divisor on X such that $b_1, \ldots, b_r \in \mathbb{R}$ and E_i's are Cartier divisors on X. Let g be a D-Green function of $\mathscr{T}$-type on X. Let U be an open set of X and let $\phi_1, \ldots, \phi_r$ be local equations of $E_1, \ldots, E_r$ over U respectively. Then there is a unique expression*

$$g = u + \sum_{i=1}^{r} (-b_i) \log |\phi_i|^2 \quad \text{(a.e.)} \qquad (u \in \mathscr{T}(U))$$

on U modulo null functions. This expression is called the local expression of g over U with respect to $\phi_1, \ldots, \phi_r$.

Proof. Let us choose reduced and irreducible divisors $D_1, \ldots, D_l$ and $\alpha_{ij} \in \mathbb{Z}$ such that $E_i = \sum_{j=1}^{l} \alpha_{ij} D_j$ for each i. If we set $a_j = \sum_{i=1}^{r} b_i \alpha_{ij}$, then $D = \sum_{j=1}^{l} a_j D_j$. For each point $x \in U$, there are an open neighborhood U_x of x, local equations $f_{1,x}, \ldots, f_{l,x}$ of $D_1, \ldots, D_l$ on U_x and $u_x \in \mathscr{T}(U_x)$ such that $U_x \subseteq U$ and

$$g = u_x + \sum_{j=1}^{l} (-a_j) \log |f_{j,x}|^2 \quad \text{(a.e.)}$$

on U_x. Note that

$$g = u_x + \sum_{i=1}^{r} (-b_i) \log \left| \prod_{j=1}^{l} f_{j,x}^{\alpha_{ij}} \right|^2 \quad \text{(a.e.)}$$

and $\prod_{j=1}^{l} f_{j,x}^{\alpha_{ij}}$ is a local equation of E_i over U_x, so that we can find nowhere vanishing holomorphic functions $e_{1,x}, \ldots, e_{r,x}$ on U_x such that $\prod_{j=1}^{l} f_{j,x}^{\alpha_{ij}} = e_{i,x} \phi_i$ on U_x for all $i = 1, \ldots, r$. Then

$$g = u_x + \sum_{i=1}^{r} (-b_i) \log |e_{i,x}|^2 + \sum_{i=1}^{r} (-b_i) \log |\phi_i|^2 \quad \text{(a.e.)}$$

on U_x. Thus, for $x, x' \in U$,

$$u_x + \sum_{i=1}^{r} (-b_i) \log |e_{i,x}|^2 = u_{x'} + \sum_{i=1}^{r} (-b_i) \log |e_{i,x'}|^2 \quad \text{(a.e.)}$$

on $U_x \cap U_{x'}$, and hence

$$u_x + \sum_{i=1}^{r} (-b_i) \log |e_{i,x}|^2 = u_{x'} + \sum_{i=1}^{r} (-b_i) \log |e_{i,x'}|^2$$

on $U_x \cap U_{x'}$. This means that there is $u \in \mathscr{T}(U)$ such that u is locally given by $u_x + \sum_{i=1}^{r}(-b_i) \log |e_{i,x}|^2$. Therefore, $g = u + \sum_{i=1}^{r}(-b_i) \log |\phi_i|^2$ (a.e.) on U. The uniqueness of the expression modulo null functions is obvious by the second property of $\mathscr{T}$. □

Proposition 2.3.6. *Let g be a D-Green function of $\mathscr{T}$-type. Then we have the following:*

1. *If g is of lower bounded type, then locally $|\phi| \exp(-g/2)$ is essentially bounded above for $\phi \in H^0_{\mathcal{M}}(X, D)$.*
2. *If g is of upper bounded type, then there is a D-Green function g' of C^∞-type such that $g \leq g'$* (a.e.).

Proof. We set $D = \sum_{i=1}^{l} a_i D_i$ such that $a_1, \ldots, a_l \in \mathbb{R}$ and D_i's are reduced and irreducible divisors on X.

(1) Clearly we may assume that X is connected. For $x \in X$, let

$$g = u + \sum_{i=1}^{l} (-a_i) \log |f_i|^2 \quad \text{(a.e.)}$$

be a local expression of g around x, where $f_1, \ldots, f_l$ are local equations of $D_1, \ldots, D_l$. For $\phi \in H^0_{\mathcal{M}}(X, D)$, we set $\phi = f_1^{b_1} \cdots f_l^{b_l} \cdot v$ around x such that v has no factors of $f_1, \ldots, f_l$. Then, as $(\phi) + D \geq 0$, we can see that $a_i + b_i \geq 0$ for all i, and that v is a holomorphic function around x. On the other hand,

$$\exp(-g/2)|\phi| = \exp(-u/2)|f_1|^{a_1+b_1} \cdots |f_n|^{a_n+b_n}|v| \quad \text{(a.e.)},$$

as required.

(2) By our assumption, there is a locally finite open covering $\{U_\lambda\}_{\lambda \in \Lambda}$ with the following properties:

(a) There are local equations $f_{\lambda,1}, \ldots, f_{\lambda,n}$ of $D_1, \ldots, D_n$ on U_λ.
(b) There is a constant C_λ such that $g \leq C_\lambda - \sum a_i \log |f_{\lambda,i}|^2$ (a.e.) on U_λ.

Let $\{\rho_\lambda\}_{\lambda \in \Lambda}$ be a partition of unity subordinate to the covering $\{U_\lambda\}_{\lambda \in \Lambda}$. We set

$$g' = \sum_{\lambda \in \lambda} \rho_\lambda \left(C_\lambda - \sum a_i \log |f_{\lambda,i}|^2 \right).$$

Clearly $g \leq g'$ (a.e.). Moreover, by Lemma 2.4.1, g' is a D-Green function of C^∞-type. □

Proposition 2.3.7. *Let g be a D-Green function of* ($\mathrm{PSH} + C^\infty$)*-type. Let A be an $\mathbb{R}$-Cartier divisor on X, and let h be an A-Green function of C^∞-type. Let $\alpha = c_1(A, h)$, that is, α is a C^∞ $(1,1)$-form on X such that $dd^c([h]) + \delta_A = [\alpha]$ (cf. Proposition* 2.3.2*). If X is compact and α is positive, then there is a positive number t_0 such that $g + th$ is a $(D + tA)$-Green function of* PSH*-type for all $t \in \mathbb{R}_{\geq t_0}$.*

Proof. For each $x \in X$, let

$$g = u_x + \sum_i (-a_i) \log |f_i|^2 \quad \text{(a.e.)}, \qquad h = v_x + \sum_i (-b_i) \log |f_i|^2 \quad \text{(a.e.)}$$

be local expressions of g and h respectively over an open neighborhood U_x of x. By our assumption, shrinking U_x if necessarily, there are a plurisubharmonic function p_x and a C^∞-function q_x such that $u_x = p_x + q_x$. Moreover, since α is positive, shrinking U_x if necessarily, we can find a positive number t_x such that $dd^c(q_x) + t\alpha$ is positive for all $t \geq t_x$. Because of the compactness of X, we can choose finitely many $x_1, \ldots, x_r \in X$ such that $X = U_{x_1} \cup \cdots \cup U_{x_r}$. If we set $t_0 = \max\{t_{x_1}, \ldots, t_{x_r}\}$, then, for $t \geq t_0$,

$$g + th = p_{x_j} + (q_{x_j} + tv_{x_j}) + \sum_i -(a_i + tb_i) \log |f_i|^2 \quad \text{(a.e.)}$$

over U_{x_j}. Note that $dd^c(q_{x_j} + tv_{x_j}) = dd^c(q_{x_j}) + t\alpha$ is positive, which means that $q_{x_j} + tv_{x_j}$ is a C^∞-plurisubharmonic function. Thus $g + th$ is of PSH-type. □

2.4. Partitions of Green functions. Let X be a d-equidimensional complex manifold. Let $\mathscr{T}$ be a type for Green functions. Besides the properties (1), (2) and (3) as in Subsection 2.3, we assume the following additional property (4):

(4) For an open set U, if $u \in \mathscr{T}(U)$ and $v \in C^\infty(U)$, then $vu \in \mathscr{T}(U)$.

As examples, C^0 and C^∞ satisfy the property (4).

Lemma 2.4.1. *Let D be an $\mathbb{R}$-Cartier divisor on X. Let $\{U_\lambda\}$ be a locally finite covering of X and let $\{\rho_\lambda\}_{\lambda\in\Lambda}$ be a partition of unity subordinate to the covering $\{U_\lambda\}_{\lambda\in\Lambda}$. Let g_λ be a $(D|_{U_\lambda})$-Green function of $\mathscr{T}$-type on U_λ for each λ. Then $g := \sum_\lambda \rho_\lambda g_\lambda$ is a D-Green function of $\mathscr{T}$-type on X.*

Proof. We set $D = a_1 D_1 + \cdots + a_r D_r$. Let $f_{i,x}$ be a local equation of D_i on an open neighborhood U_x of x. As g_λ is a $(D|_{U_\lambda})$-Green function of $\mathscr{T}$-type on U_λ, for λ with $x \in U_\lambda$,

$$g_\lambda = v_{\lambda,x} - \sum a_i \log |f_{i,x}|^2 \quad \text{(a.e.)}$$

around x, where $v_{\lambda,x} \in \mathscr{T}(U_\lambda \cap U_x)$. Thus

$$\begin{aligned} g &= \sum_\lambda \rho_\lambda \Big(v_{\lambda,x} - \sum a_i \log |f_{i,x}|^2\Big) \quad \text{(a.e.)} \\ &= \left(\sum_\lambda \rho_\lambda v_{\lambda,x}\right) - \sum a_i \log |f_{i,x}|^2 \end{aligned}$$

around x, as required. □

The main result of this subsection is the following proposition.

Proposition 2.4.2. *Let g be a D-Green function of $\mathscr{T}$-type on X and let*

$$D = b_1 E_1 + \cdots + b_r E_r$$

be a decomposition such that $E_1, \ldots, E_r \in \mathrm{Div}(X)$ and $b_1, \ldots, b_r \in \mathbb{R}$. Note that E_i is not necessarily a prime divisor. Then we have the following:

(1) *There are locally integrable functions $g_1, \ldots, g_r$ such that g_i is an E_i-Green function of $\mathscr{T}$-type for each i and $g = b_1 g_1 + \cdots + b_r g_r$ (a.e.).*
(2) *If $E_1, \ldots, E_r$ are effective, $b_1, \ldots, b_r \in \mathbb{R}_{\geq 0}$, $g \geq 0$ (a.e.) and g is of lower bounded type, then there are locally integrable functions $g_1, \ldots, g_r$ such that g_i is a non-negative E_i-Green function of $\mathscr{T}$-type for each i and $g = b_1 g_1 + \cdots + b_r g_r$ (a.e.).*

Proof. (1) Clearly we may assume that $b_i \neq 0$ for all i. Let g'_i be an E_i-Green function of C^∞-type. Then there is $f \in \mathscr{T}(X)$ such that $f = g - (b_1 g'_1 + \cdots + b_r g'_r)$ (a.e.). Thus

$$g = b_1(g'_1 + f/b_1) + b_2 g'_2 + \cdots + b_r g'_r \quad \text{(a.e.)}.$$

(2) Clearly we may assume that $b_i > 0$ for all i. First let us see the following claim:

Claim 2.4.2.1. *For each $x \in X$, there are locally integrable functions $g_{1,x}, \ldots, g_{r,x}$ and an open neighborhood U_x of x such that $g_{i,x}$ is a non-negative E_i-Green function of $\mathscr{T}$-type on U_x for every i, and that $g = b_1 g_{1,x} + \cdots + b_r g_{r,x}$* (a.e.) *on U_x.*

Proof. Let U_x be a sufficiently small open neighborhood of x and let $f_{i,x}$ be a local equation of E_i on U_x for every i. Let $g = v_x + \sum_{i=1}^r (-b_i)\log|f_{i,x}|^2$ (a.e.) be the local expression of g on U_x with respect to $f_{1,x}, \ldots, f_{r,x}$. We set $I = \{i \mid f_{i,x}(x) = 0\}$ and $J = \{i \mid f_{i,x}(x) \neq 0\}$.

First we assume $I = \emptyset$. Then, shrinking U_x if necessarily, we may assume that

$$v_x + \sum_{i=1}^r (-b_i)\log|f_{i,x}|^2 \in \mathscr{T}(U_x)$$

and $E_i = 0$ on U_x for all i. Thus if we set

$$g_{i,x} = (1/rb_i)\left(v_x + \sum_{i=1}^r (-b_i)\log|f_{i,x}|^2\right)$$

for each i, then we have our assertion.

Next we consider the case where $I \neq \emptyset$. We put $f = v_x + \sum_{j\in J}(-b_j)\log|f_{j,x}|^2$. Then, shrinking U_x if necessarily, we may assume that $f \in \mathscr{T}(U_x)$ and is bounded below. We set

$$g_{i,x} = \begin{cases} f/(b_i\#(I)) - \log|f_{i,x}|^2 & \text{if } i \in I, \\ 0 & \text{if } i \in J. \end{cases}$$

Note that $g = \sum_{i=1}^r b_i g_{i,x}$ (a.e.) and that $g_{i,x} \geq 0$ around x for $i \in I$. Thus, shrinking U_x if necessarily, we have our assertion. □

By using the above claim, we can construct an open covering $\{U_\lambda\}_{\lambda\in\Lambda}$ and locally integrable functions $g_{1,\lambda}, \ldots, g_{r,\lambda}$ on U_λ with the following properties:

(i) $\{U_\lambda\}_{\lambda\in\Lambda}$ is locally finite and the closure of U_λ is compact for every λ.
(ii) $g_{i,\lambda}$ is a non-negative E_i-Green function of $\mathscr{T}$-type on U_λ for every i.
(iii) $g = b_1 g_{1,\lambda} + \cdots + b_r g_{r,\lambda}$ (a.e.) on U_λ.

Let $\{\rho_\lambda\}_{\lambda\in\Lambda}$ be a partition of unity subordinate to the covering $\{U_\lambda\}_{\lambda\in\Lambda}$. We set $g_i = \sum_\lambda \rho_\lambda g_{i,\lambda}$. Clearly $g_i \geq 0$ and

$$g = \sum_\lambda \rho_\lambda\, g\big|_{U_\lambda} \overset{\text{(a.e.)}}{=} \sum_\lambda \rho_\lambda \sum_{i=1}^r b_i g_{i,\lambda} = \sum_{i=1}^r b_i g_i.$$

Moreover, by Lemma 2.4.1, g_i is an E_i-Green function of $\mathscr{T}$-type. □

2.5. Norms arising from Green functions. Let X be a d-equidimensional complex manifold. Let g be a locally integral function on X. For $\phi \in \mathcal{M}(X)$, we define $|\phi|_g$ to be

$$|\phi|_g := \exp(-g/2)|\phi|.$$

Moreover, the essential supremum of $|\phi|_g$ is denoted by $\|\phi\|_g$, that is,

$$\|\phi\|_g := \operatorname{ess\,sup}\left\{|\phi|_g(x) \mid x \in X\right\}.$$

Lemma 2.5.1. (1) $\|\cdot\|_g$ *satisfies the following properties:*

(1.1) $\|\lambda\phi\|_g = |\lambda|\|\phi\|_g$ *for all* $\lambda \in \mathbb{C}$ *and* $\phi \in \mathcal{M}(X)$.

(1.2) $\|\phi + \psi\|_g \leq \|\phi\|_g + \|\psi\|_g$ *for all* $\phi, \psi \in \mathcal{M}(X)$.

(1.3) *For* $\phi \in \mathcal{M}(X)$, $\|\phi\|_g = 0$ *if and only if* $\phi = 0$.

(2) *Let* V *be a vector subspace of* $\mathcal{M}(X)$ *over* $\mathbb{C}$. *If* $\|\phi\|_g < \infty$ *for all* $\phi \in V$, *then* $\|\cdot\|_g$ *yields a norm on* V. *In particular, if* D *is an* $\mathbb{R}$*-Cartier divisor,* g *is a* D*-Green function of* $\mathcal{T}$*-type and* g *is of lower bounded type, then* $\|\cdot\|_g$ *is a norm of* $H^0_{\mathcal{M}}(X, D)$ *(cf. Proposition* 2.3.6*), where* $\mathcal{T}$ *is a type for Green functions.*

Proof. (1) (1.1) and (1.2) are obvious. If $\|\phi\|_g = 0$, then $|\phi|_g = 0$ (a.e.). Moreover, as g is integrable, the measure of $\{x \in X \mid g(x) = \infty\}$ is zero. Thus $|\phi| = 0$ (a.e.), and hence $\phi = 0$.

(2) follows from (1). □

Let Φ be a continuous volume form on X. For $\phi, \psi \in \mathcal{M}(X)$, if $\phi\bar{\psi}\exp(-g)$ is integrable, then we denote its integral

$$\int_X \phi\bar{\psi}\exp(-g)\Phi$$

by $\langle\phi, \psi\rangle_g$.

We assume that g is a D-Green function of C^0-type. We set

$$D = a_1D_1 + \cdots + a_lD_l,$$

where D_i's are reduced and irreducible divisors on X and $a_1, \ldots, a_r \in \mathbb{R}$. Let us fix $x \in X$. Let $f_1, \ldots, f_l$ be local equations of $D_1, \ldots, D_l$ around x, and let

$$g = u + \sum_{i=1}^{l}(-a_i)\log|f_i|^2 \quad \text{(a.e.)}$$

be the local expression of g around x with respect to $f_1, \ldots, f_l$. For $\phi \in H^0_{\mathcal{M}}(X, D)$, we set $\phi = f_1^{b_1}\cdots f_l^{b_l}v$ around x, where v has no factors of $f_1, \ldots, f_l$. Note that $b_1, \ldots, b_l$ do not depend on the choice of $f_1, \ldots, f_l$. Since $(\phi) + D \geq 0$, we have $a_i + b_i \geq 0$ for all i and v is holomorphic around x. Then

$$|\phi|_g = |f_1|^{a_1+b_1}\cdots|f_l|^{a_l+b_l}|v|\exp(-u/2) \quad \text{(a.e.)}.$$

Let us choose another local equations $f'_1, \ldots, f'_l$ of $D_1, \ldots, D_l$ around x, and let

$$g = u' + \sum_{i=1}^{l}(-a_i)\log|f'_i|^2 \quad \text{(a.e.)}$$

be the local expression of g around x with respect to $f'_1, \ldots, f'_l$. Moreover, we set $\phi = f_1'^{b_1}\cdots f_l'^{b_l}v'$ around x as before. Then

$$|\phi|_g = |f'_1|^{a_1+b_1}\cdots|f'_l|^{a_l+b_l}|v'|\exp(-u'/2) \quad \text{(a.e.)}.$$

Note that

$$|f_1|^{a_1+b_1}\cdots|f_l|^{a_l+b_l}|v|\exp(-u/2) \quad\text{and}\quad |f_1'|^{a_1+b_1}\cdots|f_l'|^{a_l+b_l}|v'|\exp(-u'/2)$$

are continuous, so that

$$|f_1|^{a_1+b_1}\cdots|f_l|^{a_l+b_l}|v|\exp(-u/2) = |f_1'|^{a_1+b_1}\cdots|f_l'|^{a_l+b_l}|v'|\exp(-u'/2)$$

around x. This observation shows that there is a unique continuous function h on X such that $|\phi|_g = h$ (a.e.). In this sense, in the case where g is of C^0-type, we always assume that $|\phi|_g$ means the above continuous function h. Then we have the following proposition.

Proposition 2.5.2. *Let g be a D-Green function of C^0-type.*

1. *For $\phi \in H^0_{\mathcal{M}}(X, D)$, $|\phi|_g$ is locally bounded above.*
2. *If X is compact, then $\langle\phi,\psi\rangle_g$ exists for $\phi,\psi \in H^0_{\mathcal{M}}(X, D)$. Moreover, $\langle\ ,\ \rangle_g$ yields a hermitian inner product on $H^0_{\mathcal{M}}(X, D)$.*

3. Gromov's inequality and distorsion functions for $\mathbb{R}$-Cartier divisors

Let X be a d-equidimensional compact complex manifold. Let D be an $\mathbb{R}$-Cartier divisor on X and let g be a D-Green function of C^0-type. Let us fix a continuous volume form Φ on X. Recall that $|\phi|_g$, $\|\phi\|_g$ and $\langle\phi,\psi\rangle_g$ for $\phi,\psi \in H^0_{\mathcal{M}}(X, D)$ are given by

$$\begin{cases} |\phi|_g := |\phi|\exp(-g/2), \\ \|\phi\|_g := \operatorname{ess\,sup}\{|\phi|_g(x) \mid x \in X\}, \\ \langle\phi,\psi\rangle_g = \displaystyle\int_X \phi\bar{\psi}\exp(-g)\Phi. \end{cases}$$

As described in Subsection 2.5, we can view $|\phi|_g$ as a continuous function, so that $|\phi|_g$ is always assumed to be continuous.

In this section, let us consider Gromov's inequality and distorsion functions for $\mathbb{R}$-Cartier divisors.

3.1. Gromov's inequality for $\mathbb{R}$-Cartier divisors.

Here we observe Gromov's inequality for $\mathbb{R}$-Cartier divisors.

Proposition 3.1.1 (Gromov's inequality for an $\mathbb{R}$-Cartier divisor)**.** *Let $D_1,\ldots,D_l$ be $\mathbb{R}$-Cartier divisors on X and let $g_1,\ldots,g_l$ be locally integrable functions on X such that g_i is a D_i-Green function of C^∞-type for each i. Then there is a positive constant C such that*

$$\|\phi\|^2_{a_1g_1+\cdots+a_lg_l} \leq C(1+|a_1|+\cdots+|a_l|)^{2d}\langle\phi,\phi\rangle_{a_1g_1+\cdots+a_lg_l}$$

holds for all $\phi \in H^0_{\mathcal{M}}(X, a_1D_1+\cdots+a_lD_l)$ and $a_1,\ldots,a_l \in \mathbb{R}$.

Proof. We can find distinct prime divisors $\Gamma_1,\ldots,\Gamma_r$ on X, locally integrable functions $\gamma_1,\ldots,\gamma_r$ on X, C^∞-functions $f_1,\ldots,f_l$ and real numbers α_{ij} such that γ_j is a Γ_j-Green functions of C^∞-type for each $j = 1,\ldots,r$,

$$D_i = \sum_{j=1}^r \alpha_{ij}\Gamma_j \quad\text{and}\quad g_i = f_i + \sum_{j=1}^r \alpha_{ij}\gamma_j \text{ (a.e.)}.$$

Then

$$a_1D_1 + \cdots + a_lD_l = \sum_{j=1}^{r}\left(\sum_{i=1}^{l} a_i\alpha_{ij}\right)\Gamma_j + \sum_{i=1}^{l} a_i(\text{the zero divisor}),$$
$$a_1g_1 + \cdots + a_lg_l = \sum_{j=1}^{r}\left(\sum_{i=1}^{l} a_i\alpha_{ij}\right)\gamma_j + \sum_{i=1}^{l} a_i f_i \quad (\text{a.e.}).$$

Moreover, if we set $A = \max\{|\alpha_{ij}|\}$, then

$$1 + \sum_{i=1}^{l}|a_i| + \sum_{j=1}^{r}\left|\sum_{i=1}^{l} a_i\alpha_{ij}\right| \leq 1 + (Ar+1)\sum_{i=1}^{l}|a_i| \leq (Ar+1)\left(1 + \sum_{i=1}^{l}|a_i|\right).$$

Thus we may assume that $D_1, \ldots D_r$ are distinct prime divisors and

$$D_{r+1} = \cdots = D_l = 0.$$

Let U be an open set of X over which there are local equations $f_1, \ldots, f_r$ of $D_1, \ldots, D_r$ respectively.

Claim 3.1.1.1. *For all $\phi \in H^0_{\mathcal{M}}(X, a_1D_1 + \cdots + a_lD_l)$ and $a_1, \ldots, a_l \in \mathbb{R}$,*

$$\phi f_1^{\lfloor a_1 \rfloor} \cdots f_r^{\lfloor a_r \rfloor}$$

is holomorphic over U, that is, there are $b_1, \ldots, b_r \in \mathbb{Z}$ and a holomorphic function f on U such that $\phi = f_1^{b_1} \cdots f_r^{b_r} f$ and $b_1 + a_1 \geq 0, \ldots, b_r + a_r \geq 0$.

Proof. Fix $x \in U$. Let $f_i = e_i f_{i1} \cdots f_{ic_i}$ be the prime decomposition of f_i in $\mathcal{O}_{X,x}$, where $e_i \in \mathcal{O}_{X,x}^{\times}$ and f_{ij}'s are distinct prime elements of $\mathcal{O}_{X,x}$. Let D_{ij} be the prime divisor given by f_{ij} around x. Since $\phi \in H^0_{\mathcal{M}}(X, a_1D_1 + \cdots + a_lD_l)$, we have

$$(\phi) + a_1D_1 + \cdots + a_lD_l = (\phi) + a_1D_{11} + \cdots + a_1D_{1c_1} + \cdots + a_rD_{r1} + \cdots + a_rD_{rc_r} \geq 0$$

around x. Note that $D_{11}, \ldots, D_{1c_1}, \ldots, D_{r1}, \ldots, D_{rc_r}$ are distinct prime divisors around x. Thus $\phi f_{11}^{\lfloor a_1 \rfloor} \cdots f_{1c_1}^{\lfloor a_1 \rfloor} \cdots f_{r1}^{\lfloor a_r \rfloor} \cdots f_{rc_r}^{\lfloor a_r \rfloor}$ is holomorphic around x. Therefore, as

$$f_1^{\lfloor a_1 \rfloor} \cdots f_r^{\lfloor a_r \rfloor} = e_1^{\lfloor a_1 \rfloor} \cdots e_r^{\lfloor a_r \rfloor} f_{11}^{\lfloor a_1 \rfloor} \cdots f_{1c_1}^{\lfloor a_1 \rfloor} \cdots f_{r1}^{\lfloor a_r \rfloor} \cdots f_{rc_r}^{\lfloor a_r \rfloor},$$

$\phi f_1^{\lfloor a_1 \rfloor} \cdots f_r^{\lfloor a_r \rfloor}$ is holomorphic around x. □

By the above observation, the assertion of the proposition follows from the following local version. □

Lemma 3.1.2. *Let a, b, c be real numbers with $a > b > c > 0$. We set*

$$U = \{z \in \mathbb{C}^d \mid |z| < a\},\ V = \{z \in \mathbb{C}^d \mid |z| < b\} \text{ and } W = \{z \in \mathbb{C}^d \mid |z| < c\}.$$

Let Φ be a continuous volume form on U, $f_1, \ldots, f_l \in \mathcal{O}_U(U)$, $v_1, \ldots, v_l \in C^\infty(U)$ and

$$g_i = v_i - \log|f_i|^2$$

for $i = 1, \ldots, l$. For $a_1, \ldots, a_l \in \mathbb{R}$, we set

$$V(a_1, \ldots, a_l) = \left\{ f_1^{b_1} \cdots f_l^{b_l} f \;\middle|\; \begin{array}{l} f \in \mathcal{O}_U(U) \text{ and } b_1, \ldots, b_l \in \mathbb{Z} \text{ with} \\ b_1 + a_1 \geq 0, \ldots, b_l + a_l \geq 0 \end{array} \right\}.$$

(*Note that $V(a_1, \ldots, a_l)$ is a complex vector space.*) *Then there is a positive constant C such that*

$$\max_{z \in \overline{W}} \{|\phi|^2 \exp(-a_1 g_1 - \cdots - a_l g_l)(z)\} \leq C(|a_1| + \cdots + |a_l| + 1)^{2d} \int_V |\phi|^2 \exp(-a_1 g_1 - \cdots - a_l g_l)\Phi$$

holds for all $\phi \in V(a_1, \ldots, a_l)$ and all $a_1, \ldots, a_l \in \mathbb{R}$.

Proof. We set

$$u_1 = \exp(-v_1), \ldots, u_l = \exp(-v_l), u_{l+1} = \exp(v_1), \ldots, u_{2l} = \exp(v_l).$$

Then in the same way as [14, Lemma 1.1.1], we can find a positive constant D with the following properties:

(a) For $x_0, x \in \overline{V}$, $u_i(x) \geq u_i(x_0)(1 - D|x - x_0|')$ for all $i = 1, \ldots, 2l$, where $|z|' = |z_1| + \cdots + |z_d|$ for $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$.
(b) If $x_0 \in \overline{W}$, then $B(x_0, 1/D) \subseteq \overline{V}$, where

$$B(x_0, 1/D) = \{x \in \mathbb{C}^d \mid |x - x_0|' \leq 1/D\}.$$

We set

$$\Phi_{can} = \left(\frac{\sqrt{-1}}{2}\right)^d dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_d \wedge d\bar{z}_d.$$

Then we can choose a positive constant e with $\Phi \geq e\Phi_{can}$. For

$$\phi = f_1^{b_1} \cdots f_l^{b_l} f \in V(a_1, \ldots, a_l),$$

we assume that the continuous function

$$|\phi|^2 \exp(-a_1 g_1 - \cdots - a_l g_l) = |f_1|^{2(b_1+a_1)} \cdots |f_l|^{2(b_l+a_l)} |f|^2 \exp(-a_1 v_1 - \cdots - a_l v_l)$$

on $\overline{W}$ takes the maximal value at $x_0 \in \overline{W}$. Let us choose $\epsilon_i \in \{\pm 1\}$ such that $a_i = \epsilon_i |a_i|$. Note that

$$\begin{aligned}
\exp(-a_1 v_1(x) - \cdots - a_l v_l(x)) &= \prod_{i=1}^{l} \exp(-\epsilon_i v_i(x))^{|a_i|} \\
&\geq \left(\prod_{i=1}^{l} \exp(-\epsilon_i v_i(x_0))^{|a_i|}\right)(1 - D|x - x_0|')^{|a_1| + \cdots + |a_l|} \\
&= \exp(-a_1 v_1(x_0) - \cdots - a_l v_l(x_0))(1 - D|x - x_0|')^{|a_1| + \cdots + |a_l|}
\end{aligned}$$

on $B(x_0, 1/D)$. Therefore,

$$\int_V |\phi|^2 \exp(-a_1 g_1 - \cdots - a_l g_l)\Phi \geq e \exp(-a_1 v_1(x_0) - \cdots - a_l v_l(x_0)) \times \int_{B(x_0, 1/D)} |f_1|^{2(b_1+a_1)} \cdots |f_l|^{2(b_l+a_l)} |f|^2 (1 - D|x - x_0|')^{|a_1| + \cdots + |a_l|} \Phi_{can}.$$

If we set $x - x_0 = (r_1 \exp(\sqrt{-1}\theta_1), \ldots, r_d \exp(\sqrt{-1}\theta_d))$, then, by using [8, Theorem 4.1.3] and the pluriharmonicity of $|f_1|^{2(b_1+a_1)} \cdots |f_l|^{2(b_l+a_l)}|f|^2$,

$$
\begin{aligned}
&\int_{B(x_0,1/D)} |f_1|^{2(b_1+a_1)} \cdots |f_l|^{2(b_l+a_l)}|f|^2(1 - D|x - x_0|')^{|a_1|+\cdots+|a_l|}\Phi_{can} \\
&\qquad = \int_{\substack{r_1+\cdots+r_d \le 1/D \\ r_1\ge 0,\ldots,r_d\ge 0}} \left(\int_0^{2\pi} \cdots \int_0^{2\pi} |f_1|^{2(b_1+a_1)} \cdots |f_l|^{2(b_l+a_l)}|f|^2 d\theta_1 \cdots d\theta_d \right) \\
&\qquad\qquad\qquad \times r_1 \cdots r_d (1 - D(r_1 + \cdots + r_d))^{|a_1|+\cdots+|a_l|} dr_1 \cdots dr_d \\
&\qquad \ge (2\pi)^d |f_1(x_0)|^{2(b_1+a_1)} \cdots |f_l(x_0)|^{2(b_l+a_l)}|f(x_0)|^2 \\
&\qquad\qquad\qquad \times \int_{[0,1/(dD)]^d} r_1 \cdots r_d (1 - D(r_1 + \cdots + r_d))^{|a_1|+\cdots+|a_l|} dr_1 \cdots dr_d.
\end{aligned}
$$

Therefore, we have

$$
\begin{aligned}
&\int_V |\phi|^2 \exp(-a_1 g_1 - \cdots - a_l g_l)\Phi \\
&\qquad \ge \frac{e(2\pi)^d}{(dD)^{2d}} \max_{z \in \overline{W}} \{|\phi|^2 \exp(-a_1 g_1 - \cdots - a_l g_l)(z)\} \\
&\qquad\qquad\qquad \times \int_{[0,1]^d} t_1 \cdots t_d (1 - (1/d)(t_1 + \cdots + t_d))^{|a_1|+\cdots+|a_l|} dt_1 \cdots dt_d.
\end{aligned}
$$

Hence our assertion follows from [14, Claim 1.1.1.1 in Lemma 1.1.1]. □

3.2. Distorsion functions for $\mathbb{R}$-Cartier divisors. Let D be an $\mathbb{R}$-Cartier divisor on X and let g be a D-Green function of C^0-type. Let V be a complex vector subspace of $H^0_{\mathcal{M}}(X, D)$. Let $\phi_1, \ldots, \phi_l$ be an orthonormal basis of V with respect to $\langle\ ,\ \rangle_g$. It is easy to see that

$$
|\phi_1|_g^2 + \cdots + |\phi_l|_g^2
$$

does not depend on the choice of the orthonormal basis $\phi_1, \ldots, \phi_l$ of V, so that it is denoted by $\operatorname{dist}(V; g)$ and it is called the *distorsion function* of V with respect to g.

Proposition 3.2.1. *Let V be a complex vector subspace of $H^0(X, D)$. Then an inequality*

$$
|s|_g^2(x) \le \langle s, s\rangle_g \operatorname{dist}(V; g)(x) \quad (\forall x \in X)
$$

holds for all $s \in V$. In particular,

$$
|s|_g(x) \le \left(\int_X \Phi \right)^{1/2} \|s\|_g \sqrt{\operatorname{dist}(V; g)(x)} \quad (\forall x \in X).
$$

Proof. Let $e_1, \ldots, e_N$ be an orthonormal basis of V with respect to $\langle\ ,\ \rangle_g$. If we set $s = a_1 e_1 + \cdots + a_N e_N$ for $s \in V$, then

$$
\langle s, s\rangle_g = |a_1|^2 + \cdots + |a_N|^2.
$$

Therefore, by the Cauchy-Schwarz inequality,

$$\begin{aligned}|s|_g(x) &\leq |a_1||e_1|_g(x) + \cdots + |a_N||e_N|_g(x)\\ &\leq \sqrt{|a_1|^2 + \cdots + |a_N|^2}\sqrt{|e_1|_g^2(x) + \cdots + |e_N|_g^2(x)}\\ &= \sqrt{\langle s, s\rangle_g}\sqrt{\operatorname{dist}(V;\overline{L})(x)}.\end{aligned}$$

□

Lemma 3.2.2. *Let g' be another D-Green function of C^0-type such that $g \leq g'$* (a.e.). *Let V be a complex vector subspace of $H^0(X, D)$. Then* $\operatorname{dist}(V; g) \leq \exp(g' - g)\operatorname{dist}(V; g')$.

Proof. We can find a continuous function u on X such that $u \geq 0$ on X and $g' = g + u$ (a.e.). Let $\phi_1, \ldots, \phi_l$ be an orthonormal basis of V with respect to $\langle\ ,\ \rangle_{g'}$ such that $\phi_1, \ldots, \phi_l$ are orthogonal with respect to $\langle\ ,\ \rangle_g$. This is possible because any hermitian matrix can be diagonalizable by a unitary matrix. Then

$$\frac{\phi_1}{\sqrt{\langle\phi_1, \phi_1\rangle_g}}, \ldots, \frac{\phi_l}{\sqrt{\langle\phi_l, \phi_l\rangle_g}}$$

form an orthonormal basis of V with respect to $\langle\ ,\ \rangle_g$. Thus

$$\operatorname{dist}(V; g) = \frac{|\phi_1|_g^2}{\langle\phi_1, \phi_1\rangle_g} + \cdots + \frac{|\phi_l|_g^2}{\langle\phi_l, \phi_l\rangle_g}.$$

On the other hand, as $|\phi_i|_g^2 = |\phi_i|_{g'}^2\exp(u)$,

$$\langle\phi_i, \phi_i\rangle_g = \int_X |\phi_i|_{g'}^2\exp(u)\Phi \geq \int_X |\phi_i|_{g'}^2\Phi = 1$$

Therefore the lemma follows. □

Let us consider the following fundamental estimate.

Theorem 3.2.3. *Let $R = \bigoplus_{n\geq 0} R_n$ be a graded subring of $\bigoplus_{n\geq 0} H^0_{\mathcal{M}}(X, nD)$. If g is a D-Green function of C^∞-type, then there is a positive constant C with the following properties:*

(1) $\operatorname{dist}(R_n; ng) \leq C(n+1)^{3d}$ *for all* $n \geq 0$.

(2) $\dfrac{\operatorname{dist}(R_n; ng)}{C(n+1)^{3d}} \cdot \dfrac{\operatorname{dist}(R_m; mg)}{C(m+1)^{3d}} \leq \dfrac{\operatorname{dist}(R_{n+m}; (n+m)g)}{C(n+m+1)^{3d}}$ *for all* $n, m \geq 0$.

Proof. Let us begin with the following claim:

Claim 3.2.3.1. *There is a positive constant C_1 such that* $\operatorname{dist}(R_n; ng) \leq C_1(n+1)^{3d}$ *for all $n \geq 0$*

Proof. First of all, by Gromov's inequality for an $\mathbb{R}$-Cartier divisor (cf. Proposition 3.1.1), there is a positive constant C' such that

$$\|\phi\|_{ng}^2 \leq C'(n+1)^{2d}\langle\phi, \phi\rangle_{ng}$$

for all $\phi \in H^0_{\mathcal{M}}(X, nD)$ and $n \geq 0$. Let $\phi_1, \ldots, \phi_{l_n}$ be an orthonormal basis of R_n. Then

$$\begin{aligned}\operatorname{dist}(R_n; ng) &\leq \|\phi_1\|_{ng}^2 + \cdots + \|\phi_{l_n}\|_{ng}^2\\ &\leq C'(n+1)^{2d}(\langle\phi_1, \phi_1\rangle_{ng} + \cdots + \langle\phi_{l_g}, \phi_{l_g}\rangle_{ng}) \leq C'(n+1)^{2d}\dim R_n,\end{aligned}$$

as required. □

Claim 3.2.3.2. *There is a positive constant C_2 such that*

$$\operatorname{dist}(R_n; ng) \cdot \operatorname{dist}(R_m; mg) \leq C_2(m+1)^{3d} \operatorname{dist}(R_{n+m}; (n+m)g)$$

for $n \geq m \geq 0$.

Proof. Let $t_1, \ldots, t_l$ be an orthonormal basis of R_m. For each $j = 1, \ldots, l$, we choose an orthonormal basis $s_1, \ldots, s_r$ of R_n such that $s_1 t_j, \ldots, s_r t_j$ are orthogonal in R_{n+m}. Note that the above $s_1, \ldots, s_r$ depend on j. We set $I = \{1 \leq i \leq r \mid s_i t_j \neq 0\}$. As

$$\left\{ \frac{s_i t_j}{\sqrt{\langle s_i t_j, s_i t_j \rangle_{(n+m)g}}} \right\}_{i \in I}$$

can be extended to an orthonormal basis of R_{n+m}, we have

$$\sum_{i \in I} \frac{|s_i t_j|^2_{(n+m)g}}{\langle s_i t_j, s_i t_j \rangle_{(n+m)g}} \leq \operatorname{dist}(R_{n+m}; (n+m)g).$$

By using Gromov's inequality as in the previous claim,

$$\langle s_i t_j, s_i t_j \rangle_{(n+m)g} \leq \langle s_i, s_i \rangle_{ng} \|t_j\|^2_{mg} \leq C'(m+1)^{2d} \langle t_j, t_j \rangle_{mg} = C'(m+1)^{2d}.$$

Hence

$$\begin{aligned} \operatorname{dist}(R_n; ng)|t_j|^2_{mg} = \sum_{i=1}^{r} |s_i t_j|_{(n+m)g} &= \sum_{i \in I} |s_i t_j|_{(n+m)g} \\ &\leq \sum_{i \in I} \frac{C'(m+1)^{2d}}{\langle s_i t_j, s_i t_j \rangle_{(n+m)g}} |s_i t_j|^2_{(n+m)g} \\ &\leq C'(m+1)^{2d} \operatorname{dist}(R_{n+m}; (n+m)g), \end{aligned}$$

which implies

$$\operatorname{dist}(R_n; ng) \cdot \operatorname{dist}(R_m; mg) \leq \dim(R_m) C'(m+1)^{2d} \operatorname{dist}(R_{n+m}; (n+m)g),$$

as required. □

We set $C = \max\{C_1, 8^d C_2\}$. Then, for $n \geq m \geq 0$,

$$\begin{aligned} \frac{C(n+1)^{3d} C(m+1)^{3d}}{C(n+m+1)^{3d}} &\geq C_2(m+1)^{3d} 8^d \left(\frac{n+1}{n+m+1} \right)^{3d} \\ &\geq C_2(m+1)^{3d} 8^d \left(\frac{n+1}{2n+1} \right)^{3d} \\ &> C_2(m+1)^{3d} 8^d \left(\frac{1}{2} \right)^{3d} = C_2(m+1)^{3d}. \end{aligned}$$

Thus the proposition follows from the above claims. □

4. Plurisubharmonic upper envelopes

The main result of this section is the continuity of the upper envelope of a family of Green functions of $\mathrm{PSH}_{\mathbb{R}}$-type bounded above by a Green function of C^0-type. This will give the continuity of the positive part of the Zariski decomposition.

Throughout this section, let X be a d-equidimensional complex manifold. Let us begin with the following fundamental estimate.

Lemma 4.1. *Let $f_1, \ldots, f_r$ be holomorphic functions on X such that $f_1, \ldots, f_r$ are not zero on each connected component of X. Let $a_1, \ldots, a_r \in \mathbb{R}_{\geq 0}$ and $M \in \mathbb{R}$. We denote by $\mathrm{PSH}(X; f_1, \ldots, f_r, a_1, \ldots, a_r, M)$ the set of all plurisubharmonic functions u on X such that*

$$u \leq M - \sum_{i=1}^{r} a_i \log |f_i|^2 \quad \text{(a.e.)}$$

holds over X. Then, for each point $x \in X$, there are an open neighborhood U_x of x and a constant M'_x depending only on $f_1, \ldots, f_r$ and x such that

$$u \leq M + M'_x(a_1 + \cdots + a_r)$$

on U_x for any $u \in \mathrm{PSH}(X; f_1, \ldots, f_r, a_1, \ldots, a_r, M)$.

Proof. Let us begin with the following claim:

Claim 4.1.1. *For any $u \in \mathrm{PSH}(X; f_1, \ldots, f_r, a_1, \ldots, a_r, M)$,*

$$u \leq M - \sum_{i=1}^{r} a_i \log |f_i|^2$$

holds over X .

Proof. Clearly we may assume that $a_i > 0$ for all i. Let us fix $x \in X$. If $f_i(x) = 0$ for some i, then the right hand side is ∞, so that the assertion is obvious. We assume that $f_i(x) \neq 0$ for all i. Then the right hand side is continuous around x. Thus it follows from Lemma 2.3.1. □

Claim 4.1.2. *Let $\epsilon \in \mathbb{R}_{>0}$, $a_1, \ldots, a_d \in \mathbb{R}_{\geq 0}$, and $M \in \mathbb{R}$. Then*

$$u \leq M - 2\log(\epsilon/4)(a_1 + \cdots + a_d)$$

holds on $\Delta^d_{\epsilon/4}$ for any $u \in \mathrm{PSH}(\Delta^d_{\epsilon}; z_1, \ldots, z_d, a_1, \ldots, a_d, M)$, where $(z_1, \ldots, z_d)$ is the coordinate of $\mathbb{C}^d$ and

$$\Delta^d_t = \{(z_1, \ldots, z_d) \in \mathbb{C}^d \mid |z_1| < t, \ldots, |z_d| < t\}.$$

for $t \in \mathbb{R}_{>0}$.

Proof. Note that if $(z_1, \ldots, z_d) \in \Delta^d_{\epsilon/4}$, then

$$\{(z_1 + (\epsilon/2)e^{2\pi i\theta_1}, \ldots, z_d + (\epsilon/2)e^{2\pi i\theta_d}) \mid \theta_1, \ldots, \theta_d \in [0, 1]\} \subseteq \Delta^d_{\epsilon}.$$

Moreover, as

$$\begin{aligned}\epsilon/2 = |(\epsilon/2)e^{2\pi i\theta_j}| &= |z_j + (\epsilon/2)e^{2\pi i\theta_j} - z_j| \\ &\leq |z_j + (\epsilon/2)e^{2\pi i\theta_j}| + |z_j| < |z_j + (\epsilon/2)e^{2\pi i\theta_j}| + \epsilon/4,\end{aligned}$$

we have $|z_j + (\epsilon/2)e^{2\pi i\theta_j}| > \epsilon/4$ for $j = 1, \ldots, d$. Thus, by [8, Theorem 4.1.3],

$$\begin{aligned} u(z_1, \ldots, z_d) &\leq \int_0^1 \cdots \int_0^1 u(z_1 + (\epsilon/2)e^{2\pi i\theta_1}, \ldots, z_d + (\epsilon/2)e^{2\pi i\theta_d}) d\theta_1 \cdots d\theta_d \\ &\leq \int_0^1 \cdots \int_0^1 \left(M - \sum_{j=1}^n a_j \log |z_j + (\epsilon/2)e^{2\pi i\theta_j}|^2 \right) d\theta_1 \cdots d\theta_d \\ &= M - \sum_{j=1}^d a_j \int_0^1 \log |z_j + (\epsilon/2)e^{2\pi i\theta_j}|^2 d\theta_j \\ &\leq M - \sum_{j=1}^d a_j \int_0^1 \log(\epsilon/4)^2 d\theta_j = M - 2\log(\epsilon/4) \sum_{j=1}^d a_j. \end{aligned}$$

□

Next we observe the following claim:

Claim 4.1.3. *If* $\operatorname{Supp}\{x \in X \mid f_1(x) \cdots f_r(x) = 0\}$ *is a normal crossing divisor on* X*, then the lemma holds.*

Proof. We choose an open neighborhood V_x such that $V_x = \Delta_1^d$ and

$$\operatorname{Supp}\{x \in X \mid f_1(x) \cdots f_r(x) = 0\}$$

is given by $\{z_1 \cdots z_l = 0\}$. Then there are $b_{ij} \in \mathbb{Z}_{\geq 0}$ and nowhere vanishing holomorphic functions $v_1, \ldots, v_r$ on Δ_1^d such that

$$f_1 = z_1^{b_{11}} \cdots z_l^{b_{1l}} v_1, \ldots, f_r = z_1^{b_{r1}} \cdots z_l^{b_{rl}} v_r.$$

Thus

$$M - \sum_{i=1}^r a_i \log |f_i|^2 = M - \sum_{i=1}^r a_i \log |v_i|^2 - \sum_{j=1}^l \left(\sum_{i=1}^r a_i b_{ij} \right) \log |z_j|^2.$$

We choose $M_1, M_2 \in \mathbb{R}$ such that $M_1 = \max\{b_{ij} \mid i = 1, \ldots, r, j = 1, \ldots, l\}$ and $M_2 \geq \max_{z \in \Delta_{1/2}^d} \{-\log |v_i(z)|^2\}$ for all i. Then

$$M - \sum_{i=1}^r a_i \log |f_i|^2 \leq M + M_2(a_1 + \cdots + a_r) - \sum_{j=1}^l M_1(a_1 + \cdots + a_r) \log |z_j|^2$$

on $\Delta_{1/2}^d$. Thus, by the previous claim, for any $u \in \operatorname{PSH}(X; f_1, \ldots, f_r, a_1, \ldots, a_r, M)$,

$$u \leq M + (M_2 - 2\log(1/8) l M_1)(a_1 + \cdots + a_r)$$

on $\Delta_{1/8}^d$. □

Let us start a general case. Let $\pi : X' \to X$ be a proper bimeromorphic map such that $\operatorname{Supp}(\{\pi^*(f_1) \cdots \pi^*(f_r) = 0\})$ is a normal crossing divisor on X'. Note that if u is a plurisubharmonic function on X, then $\pi^*(u)$ is also plurisubharmonic on X' (cf. [9, Corollary 2.9.5]). By the above claim, for each point $y \in \pi^{-1}(x)$, there is an open neighborhood U_y of y and a constant M'_y depending only on $f_1, \ldots, f_r$ and y such that, for any $u \in \operatorname{PSH}(X; f_1, \ldots, f_r, a_1, \ldots, a_r, M)$,

$$f^*(u) \leq M + M'_y(a_1 + \cdots + a_r)$$

on U_y. As $\pi^{-1}(x) \subseteq \bigcup_{y \in \pi^{-1}(x)} U_y$ and $\pi^{-1}(x)$ is compact, there are $y_1, \ldots, y_s$ such that $\pi^{-1}(x) \subseteq U_{y_1} \cup \cdots \cup U_{y_s}$. We can choose an open neighborhood U_x of x such that $\pi^{-1}(U_x) \subseteq U_{y_1} \cup \cdots \cup U_{y_s}$. Thus, if we set $M'_x = \max\{M'_{y_1}, \ldots, M'_{y_s}\}$, then

$$f^*(u) \leq M + M'_x(a_1 + \cdots + a_r)$$

on $\pi^{-1}(U_x)$, and hence the lemma follows. □

Let α be a continuous $(1,1)$-form on X. We set

$$\mathrm{PSH}(X;\alpha) := \left\{ \phi \;\middle|\; \begin{array}{ll} \text{(i)} & \phi : X \to \{-\infty\} \cup \mathbb{R}. \\ \text{(ii)} & \phi \in (C^\infty + \mathrm{PSH})(X). \\ \text{(iii)} & [\alpha] + dd^c([\phi]) \geq 0. \end{array} \right\}.$$

First we observe the following lemma.

Lemma 4.2. *We assume that X is compact and that $\alpha + dd^c(\psi_0)$ is either positive or zero for some C^∞-function ψ_0 on X. If $\phi \in \mathrm{PSH}(X;\alpha) \cap C^0(X)$, then there are sequences $\{\phi_n\}_{n=1}^\infty$ and $\{\varphi_n\}_{n=1}^\infty$ in*

$$\mathrm{PSH}(X;\alpha) \cap C^\infty(X)$$

such that $\phi_n \leq \phi \leq \varphi_n$ on X for all $n \geq 1$ and that

$$\lim_{n\to\infty} \|\phi - \phi_n\|_{\sup} = \lim_{n\to\infty} \|\varphi_n - \phi\|_{\sup} = 0.$$

Proof. First we assume that $\alpha = dd^c(-\psi_0)$ for some C^∞-function ψ_0 on X. Then

$$\mathrm{PSH}(X;\alpha) = \{\psi_0 + c \mid c \in \mathbb{R} \cup \{-\infty\}\}$$

because X is compact. Thus the assertion of the lemma is obvious.

Next we assume that α is positive. By [4, Theorem 1], there is a sequence of $\{\varphi_n\}_{n=1}^\infty$ in $\mathrm{PSH}(X;\alpha) \cap C^\infty(X)$ such that

$$\varphi_1(x) \geq \varphi_2(x) \geq \cdots \geq \varphi_n(x) \geq \varphi_{n+1}(x) \geq \cdots \geq \phi(x)$$

and $\phi(x) = \lim_{n\to\infty} \varphi_n(x)$ for all $x \in X$. Since X is compact and ϕ is continuous, it is easy to see that $\lim_{n\to\infty} \|\varphi_n - \phi\|_{\sup} = 0$. We set $\phi_n = \varphi_n - \|\varphi_n - \phi\|_{\sup}$ for all $n \geq 1$. Then $\phi_n \in \mathrm{PSH}(X;\alpha) \cap C^\infty(X)$ and $\phi_n \leq \phi$. Note that $\|\phi - \phi_n\|_{\sup} \leq 2\|\varphi_n - \phi\|_{\sup}$. Thus $\lim_{n\to\infty} \|\phi - \phi_n\|_{\sup} = 0$.

Finally we assume that $\alpha' = \alpha + dd^c(\psi_0)$ is positive for some C^∞-function ψ_0 on X. Then

$$\phi' := \phi - \psi_0 \in \mathrm{PSH}(X;\alpha') \cap C^0(X).$$

Thus, by the previous observation, there are sequences $\{\phi'_n\}_{n=1}^\infty$ and $\{\varphi'_n\}_{n=1}^\infty$in

$$\mathrm{PSH}(X;\alpha') \cap C^\infty(X)$$

such that $\phi'_n \leq \phi' \leq \varphi'_n$ on X for all $n \geq 1$ and that

$$\lim_{n\to\infty} \|\phi' - \phi'_n\|_{\sup} = \lim_{n\to\infty} \|\varphi'_n - \phi'\|_{\sup} = 0.$$

We set $\phi_n := \phi'_n + \psi_0$ and $\varphi_n := \varphi'_n + \psi_0$ for every $n \geq 1$. Then

$$\phi_n, \varphi_n \in \mathrm{PSH}(X;\alpha) \cap C^\infty(X) \quad \text{and} \quad \phi_n \leq \phi \leq \varphi_n$$

for all $n \geq 1$. Moreover, $\lim_{n\to\infty} \|\phi_n - \phi\|_{\sup} = \lim_{n\to\infty} \|\varphi_n - \phi\|_{\sup} = 0$. □

Let A be an $\mathbb{R}$-Cartier divisor and let g_A be an A-Green function of C^∞-type on X. Let $\alpha = c_1(A, g_A)$, that is, α is a C^∞-form such that

$$[\alpha] = dd^c([g_A]) + \delta_A$$

(cf. Proposition 2.3.2). Here let us consider the natural correspondence between $G_{\mathrm{PSH}}(X; A)$ and $\mathrm{PSH}(X; \alpha)$ in terms of g_A.

Proposition 4.3. *If $\phi \in \mathrm{PSH}(X; \alpha)$, then $\phi + g_A \in G_{\mathrm{PSH}}(X; A)$. Moreover, we have the following:*

(1) *For $u \in G_{\mathrm{PSH}}(X; A)$, there is $\phi \in \mathrm{PSH}(X; \alpha)$ such that $\phi + g_A = u$ (a.e.).*

(2) *For $\phi_1, \phi_2 \in \mathrm{PSH}(X; \alpha)$,*

$$\phi_1 \leq \phi_2 \quad \Longleftrightarrow \quad \phi_1 + g_A \leq \phi_2 + g_A \text{ (a.e.)}.$$

(3) *For $\phi \in \mathrm{PSH}(X; \alpha)$,*

$$\phi(x) \neq -\infty \ (\forall x \in X) \quad \Longleftrightarrow \quad \phi + g_A \in G_{\mathrm{PSH}_{\mathbb{R}}}(X; A).$$

(4) *For $\phi \in \mathrm{PSH}(X; \alpha)$,*

$$\phi \in C^\infty(X) \quad \Longleftrightarrow \quad \phi + g_A \in G_{C^\infty}(X; A).$$

(5) *For $\phi \in \mathrm{PSH}(X; \alpha)$,*

$$\phi \in C^0(X) \quad \Longleftrightarrow \quad \phi + g_A \in G_{C^0}(X; A).$$

Proof. We set $A = a_1 D_1 + \cdots + a_l D_l$, where D_i's are reduced and irreducible divisors on X and $a_1, \ldots, a_l \in \mathbb{R}$. Let U be an open set of X and let $f_1, \ldots, f_l$ be local equations of $D_1, \ldots, D_l$ on U respectively. Let

$$g_A = h - \sum_{i=1}^{l} a_i \log |f_i|^2 \quad \text{(a.e.)}$$

be the local expression of g_A with respect to $f_1, \ldots, f_l$, where $h \in C^\infty(U)$. Then

$$g_A + \phi = (h + \phi) - \sum_{i=1}^{l} a_i \log |f_i|^2 \quad \text{(a.e.)}.$$

Since $\alpha = dd^c(h)$ on U, we have

$$dd^c([h + \phi]) = [\alpha] + dd^c([\phi]) \geq 0.$$

Thus $g_A + \phi \in G_{\mathrm{PSH}}(X; A)$ and

$$g_A + \phi = (h + \phi) - \sum_{i=1}^{l} a_i \log |f_i|^2 \quad \text{(a.e.)}.$$

is the local expression of $g_A + \phi$ with respect to $f_1, \ldots, f_l$.

(1) For $u \in G_{\mathrm{PSH}}(X; A)$, let

$$u = p - \sum_{i=1}^{l} a_i \log |f_i|^2 \text{ (a.e.)}$$

be the local expression of u with respect to $f_1, \ldots, f_l$, where p is plurisubharmonic. It is easy to see that $p - h$ does not depend on the choice of the local equations $f_1, \ldots, f_l$. Thus there is a function $\phi : X \to \{-\infty\} \cup \mathbb{R}$ such that ϕ is locally given by $p - h$. Moreover

$$dd^c([p - h]) + [\alpha] = dd^c([p]) \geq 0.$$

Hence $\phi \in \mathrm{PSH}(X;\alpha)$ and $\phi + g_A = u$ (a.e.).

(2) Clearly

$$\phi_1 \leq \phi_2 \text{ (a.e.)} \iff \phi_1 + g_A \leq \phi_2 + g_A \text{ (a.e.)}.$$

On the other hand, by Lemma 2.3.1,

$$\phi_1 \leq \phi_2 \iff \phi_1 \leq \phi_2 \text{ (a.e.)}.$$

(3), (4) and (5) are obvious because

$$\phi + g_A = (h + \phi) - \sum_{i=1}^{l} a_i \log |f_i|^2 \text{ (a.e.)}$$

is a local expression of $\phi + g_A$ and h is C^∞. □

Let $\mathscr{T}$ be a type for Green functions on X such that PSH is a subjacent type of $\mathscr{T}$, that is, the following property holds for an arbitrary open set U of X: if $u \leq v$ (a.e.) on U for $u \in \mathrm{PSH}(U)$ and $v \in \mathscr{T}(U)$, then $u \leq v$ on U.

Proposition 4.4. *Let A and B be $\mathbb{R}$-Cartier divisors on X with $A \leq B$. Let h be a B-Green function of $\mathscr{T}$-type on X such that h is of upper bounded type. Let $\{g_\lambda\}_{\lambda\in\Lambda}$ be a family of A-Green functions of* PSH*-type on X. We assume that $g_\lambda \leq h$* (a.e.) *for all $\lambda \in \Lambda$. Then there is an A-Green function g of* PSH*-type on X with the following properties:*

(a) *Let us fix an A-Green function g_A of C^∞-type. Let α be a unique C^∞-form with $[\alpha] = dd^c([g_A]) + \delta_A$. If we choose $\phi \in \mathrm{PSH}(X;\alpha)$ and $\phi_\lambda \in \mathrm{PSH}(X;\alpha)$ for each $\lambda \in \Lambda$ such that $g = g_A + \phi$* (a.e.) *and $g_\lambda = g_A + \phi_\lambda$* (a.e.) (*cf. Proposition* 4.3), *then ϕ is the upper semicontinuous regularization of the function given by*

$$x \mapsto \sup_{\lambda\in\Lambda}\{\phi_\lambda(x)\}.$$

In particular, g_{can} is the upper semicontinuous regularization of the function given by

$$x \mapsto \sup_{\lambda\in\Lambda}\{(g_\lambda)_{\mathrm{can}}(x)\}$$

over $X \setminus \mathrm{Supp}(A)$.

(b) $g \leq h$ (a.e.).

(c) *If there is g_λ such that $g_\lambda \in G_{\mathrm{PSH}_{\mathbb{R}}}(X;A)$, then $g \in G_{\mathrm{PSH}_{\mathbb{R}}}(X;A)$.*

Proof. Let $A = a_1D_1 + \cdots + a_lD_l$ and $B = b_1D_1 + \cdots + b_lD_l$ be the decompositions of A and B such that D_i's are reduced and irreducible divisors, $a_1,\ldots,a_l,b_1,\ldots,b_l \in \mathbb{R}$ and $D_1 \cup \cdots \cup D_l = \mathrm{Supp}(A) \cup \mathrm{Supp}(B)$. Let U be an open set of X and let $f_1,\ldots,f_l$ be local equations of $D_1,\ldots,D_l$ over U respectively. Let

$$h = v + \sum_{i=1}^{l}(-b_i)\log|f_i|^2 \quad \text{(a.e.)}$$

be the local expression of h with respect to $f_1,\ldots,f_l$. Moreover, let

$$g_\lambda = u_\lambda + \sum_{i=1}^{l}(-a_i)\log|f_i|^2 \quad \text{(a.e.)}$$

be the local expression of g_λ with respect to $f_1, \dots, f_l$. Then

$$u_\lambda \leq v - \sum_{i=1}^{l} (b_i - a_i) \log |f_i|^2 \quad \text{(a.e.)}$$

holds for every $\lambda \in \Lambda$. Note that v is locally bounded above. Thus $\{u_\lambda\}_{\lambda \in \Lambda}$ is uniformly locally bounded above by Lemma 4.1. Let u be the function on U given by

$$u(x) = \sup\{u_\lambda(x) \mid \lambda \in \Lambda\}.$$

Let $\tilde{u}$ be the upper semicontinuous regularization of u. Then $\tilde{u}$ is plurisubharmonic on U (cf. Subsection 2.1). Let $f'_1, \dots, f'_l$ be another local equations of $D_1, \dots, D_l$. Then there are $e_1, \dots, e_n \in \mathscr{O}_U^\times(U)$ such that $f'_i = e_i f_i$ for all i, so that

$$g_\lambda = \left(u_\lambda + \sum_{i=1} a_i \log |e_i|^2 \right) + \sum_{i=1}^{l} (-a_i) \log |f'_i|^2 \quad \text{(a.e.)}$$

is the local expression of g_λ with respect to $f'_1, \dots, f'_l$. Thus, if we denote the plurisubharmonic function arising from $f'_1, \dots, f'_l$ by $\tilde{u}'$, then, by Lemma 2.3.1,

$$\tilde{u}' = \tilde{u} + \sum_{i=1}^{l} a_i \log |e_i|^2.$$

This means that

$$\tilde{u} + \sum_{i=1}^{l} (-a_i) \log |f_i|^2$$

does not depend on the choice of $f_1, \dots, f_l$ over $U \setminus \operatorname{Supp}(A)$. Thus there is $g \in G_{\mathrm{PSH}}(X; A)$ such that

$$g\big|_U = \tilde{u} + \sum_{i=1}^{l} (-a_i) \log |f_i|^2 \quad \text{(a.e.)}.$$

Let $g_A = u_A + \sum_{i=1}^{l} (-a_i) \log |f_i|^2$ (a.e.) be the local expression of g_A with respect to $f_1, \dots, f_l$. Then $\phi_\lambda = u_\lambda - u_A$ and $\phi = \tilde{u} - u_A$. Thus (a) follows.

By (a), g_{can} is the upper semicontinuous regularization of the function g' given by $g'(x) = \sup_{\lambda \in \Lambda}\{(g_\lambda)_{\mathrm{can}}(x)\}$ over $X \setminus \operatorname{Supp}(A)$. As PSH is a subjacent type of $\mathscr{T}$, we have $(g_\lambda)_{\mathrm{can}} \leq h_{\mathrm{can}}$ on $X \setminus (\operatorname{Supp}(A) \cup \operatorname{Supp}(B))$ for all $\lambda \in \Lambda$. Note that $g = g'$ (a.e.) (cf. Subsection 2.1). Thus we have $g \leq h$ (a.e.).

Finally we assume that $g_\lambda \in G_{\mathrm{PSH}_{\mathbb{R}}}(X; A)$ for some λ. Then $u_\lambda \leq \tilde{u}$ (a.e.), so that $u_\lambda \leq \tilde{u}$ by Lemma 2.3.1. Thus $\tilde{u}(x) \neq -\infty$. Therefore, $g \in G_{\mathrm{PSH}_{\mathbb{R}}}(X; A)$. □

Let A be an $\mathbb{R}$-Cartier divisor on X and let g be a locally integrable function on X. We set

$$G_{\mathscr{T}}(X; A)_{\leq g} := \{u \in G_{\mathscr{T}}(X; A) \mid u \leq g \text{ (a.e.)}\},$$

where $G_{\mathscr{T}}(X; A)$ is the set of all A-Green functions of $\mathscr{T}$-type on X.

Lemma 4.5. *Let A and B be $\mathbb{R}$-Cartier divisors on X with $A \leq B$. Let g_B be a B-Green function of C^∞-type (resp. C^0-type). There is an A-Green function g_A of C^∞-type (resp. C^0-type) such that*

$$g_A \leq g_B \text{ (a.e.)} \quad \textit{and} \quad G_{\mathrm{PSH}}(X; A)_{\leq g_A} = G_{\mathrm{PSH}}(X; A)_{\leq g_B}.$$

Proof. We set $A = a_1D_1 + \cdots + a_nD_n$ and $B = b_1D_1 + \cdots + b_nD_n$, where D_i's are reduced and irreducible divisors on X and $a_1, \ldots, a_n, b_1, \ldots, b_n \in \mathbb{R}$. For $x \in X$, let U_x be a small open neighborhood of x and let $f_1, \ldots, f_n$ be local equations of $D_1, \ldots, D_n$ on U_x respectively. Note that if $x \notin D_i$, then we take f_i as the constant function 1. Let $g_B = h_x - \sum_i b_i \log |f_i|^2$ (a.e.) be the local expression of g_B on U_x with respect to $f_1, \ldots, f_n$. Shrinking U_x if necessarily, we may assume that there is a constant M_x such that $|h_x| \leq M_x$ on U_x.

Claim 4.5.1. *There are an open neighborhood V_x of x and a positive constant C_x such that $V_x \subseteq U_x$,*

$$h_x + C_x - \sum_i a_i \log |f_i|^2 \leq g_B \quad \text{(a.e.)}$$

on V_x and that

$$u \leq h_x + C_x - \sum_i a_i \log |f_i|^2 \quad \text{(a.e.)}$$

on V_x for all $u \in G_{\mathrm{PSH}}(X; A)_{\leq g_B}$.

Proof. For $u \in G_{\mathrm{PSH}}(X; A)_{\leq g_B}$, let $u = p_x(u) - \sum_i a_i \log |f_i|^2$ (a.e.) be the local expression of u on U_x with respect to $f_1, \ldots, f_n$. Then $u \leq g_B$ (a.e.) is nothing more than

$$p_x(u) \leq h_x - \sum_i (b_i - a_i) \log |f_i|^2 \quad \text{(a.e.)}.$$

If either $a_i = b_i$ or $x \notin D_i$ for all i, then $\sum_i (b_i - a_i) \log |f_i|^2 = 0$ on U_x. Thus our assertion is obvious by taking $C_x = 0$, so that we may assume that $a_i < b_i$ and $x \in D_i$ for some i. By Lemma 4.1, there are an open neighborhood U'_x of x and a positive constant M'_x such that $U'_x \subseteq U_x$ and $p_x(u) \leq M'_x$ on U'_x for all $u \in G_{\mathrm{PSH}}(X; A)_{\leq g_B}$. Note that

$$M'_x = -M_x + (M'_x + M_x) \leq h_x + (M'_x + M_x)$$

on U_x. Thus if we set $C_x = M'_x + M_x$, then $p_x(u) \leq h_x + C_x$ on U'_x for all $u \in G_{\mathrm{PSH}}(X; A)_{\leq g_B}$. As $\lim_{y \to x} \sum_i (b_i - a_i) \log |f_i|^2(y) = -\infty$, we can find an open neighborhood V_x of x such that $V_x \subseteq U'_x$ and $C_x \leq -\sum_i (b_i - a_i) \log |f_i|^2$ on V_x. Therefore,

$$p_x(u) \leq h_x + C_x \leq h_x - \sum_i (b_i - a_i) \log |f_i|^2$$

on V_x for all $u \in G_{\mathrm{PSH}}(X; A)_{\leq g_B}$, as required. □

By using Claim 4.5.1, we can find an open covering $\{V_\lambda\}_{\lambda \in \Lambda}$ of X and a family of constants $\{C_\lambda\}_{\lambda \in \Lambda}$ with the following properties:

(1) $\{V_\lambda\}_{\lambda \in \Lambda}$ is a locally finite covering.
(2) There are local equations $f_{\lambda,1}, \ldots, f_{\lambda,n}$ of $D_1, \ldots, D_n$ on V_λ respectively.
(3) Let $g_B = h_\lambda - \sum_i b_i \log |f_{\lambda,i}|^2$ (a.e.) be the local expression of g_B on V_λ with respect to$f_{\lambda,1}, \ldots, f_{\lambda,n}$. Then

$$h_\lambda + C_\lambda - \sum a_i \log |f_{\lambda,i}|^2 \leq g_B \quad \text{(a.e.)}$$

on V_λ and that

$$u \leq h_\lambda + C_\lambda - \sum_i a_i \log |f_{\lambda,i}|^2 \quad \text{(a.e.)}$$

on V_λ for all $u \in G_{\mathrm{PSH}}(X; A)_{\leq g_B}$.

Let $\{\rho_\lambda\}_{\lambda\in\Lambda}$ be a partition of unity subordinate to the covering $\{V_\lambda\}_{\lambda\in\Lambda}$. We set

$$g_A = \sum_\lambda \rho_\lambda \left(h_\lambda + C_\lambda - \sum_i a_i \log |f_{\lambda,i}|^2 \right).$$

By Lemma 2.4.1, g_A is an A-Green function of C^∞-type (resp. C^0-type). Moreover, $g_A \leq g_B$ (a.e.) and $u \leq g_A$ (a.e.) for all $u \in G_{\mathrm{PSH}}(X;A)_{\leq g_B}$. Therefore the lemma follows. □

The following theorem is the main result of this section.

Theorem 4.6. *Let A be an $\mathbb{R}$-Cartier divisor on X. If X is projective and there is an A-Green function h of C^∞-type such that $dd^c([h]) + \delta_A$ is represented by either a positive C^∞-form or the zero form, then we have the following:*

(1) *Let B be an $\mathbb{R}$-Cartier divisor on X with $A \leq B$. Let g_B be a B-Green function of C^0-type. Then there is $g \in G_{C^0\cap\mathrm{PSH}}(X;A)$ such that $g \leq g_B$* (a.e.) *and*

$$u \leq g \text{ (a.e.)} \quad (\forall u \in G_{\mathrm{PSH}}(X;A)_{\leq g_B}).$$

(2) *If $u \in G_{C^0\cap\mathrm{PSH}}(X;A)$, then there are sequences $\{u_n\}_{n=1}^\infty$ and $\{v_n\}_{n=1}^\infty$ of continuous functions on X with the following properties:*

(2.1) *$u_n \geq 0$ and $v_n \geq 0$ for all $n \geq 1$.*

(2.2) $\lim_{n\to\infty} \|u_n\|_{\sup} = \lim_{n\to\infty} \|v_n\|_{\sup} = 0$.

(2.3) *$u - u_n, u + v_n \in G_{C^\infty\cap\mathrm{PSH}}(X;A)$ all $n \geq 1$.*

Proof. (1) Let us begin with the following claim:

Claim 4.6.1. *There is $g \in G_{\mathrm{PSH}_{\mathbb{R}}}(X;A)$ such that $g \leq g_B$* (a.e.) *and*

$$u \leq g \quad \text{(a.e.)} \quad (\forall u \in G_{\mathrm{PSH}}(X;A)_{\leq g_B}).$$

We say g is the greatest element of $G_{\mathrm{PSH}}(X;A)_{\leq g_B}$ modulo null functions.

Proof. Note that PSH is a subjacent type of C^0 by Lemma 2.3.1, and that $h - c \in G_{\mathrm{PSH}_{\mathbb{R}}}(X;A)_{\leq g_B}$ for some constant c. Thus the assertion follows from Proposition 4.4. □

Claim 4.6.2. *If g_B is of C^∞-type, then the assertion of* (1) *holds.*

Proof. By Lemma 4.5, we may assume that $A = B$. Let $\alpha = c_1(A, g_A)$, that is, α is a C^∞-form such that $[\alpha] = dd^c([g_A]) + \delta_A$. We set

$$\mathrm{PSH}(X;\alpha)_{\leq 0} = \{\psi \in \mathrm{PSH}(X;\alpha) \mid \psi \leq 0\}.$$

By our assumption, we can find a C^∞-function ψ_0 such that $g_A + \psi_0 = h$ (a.e.). Note that $[\alpha + dd^c(\psi_0)] = dd^c([h]) + \delta_A$. Thus $\alpha + dd^c(\psi_0)$ is either positive or zero.

First we assume that $\alpha + dd^c(\psi_0)$ is positive. Let g be the greatest element of

$$G_{\mathrm{PSH}}(X;A)_{\leq g_A}$$

modulo null functions (cf. Claim 4.6.1). We choose $\phi \in \mathrm{PSH}(X;\alpha)$ and $\psi_u \in \mathrm{PSH}(X;\alpha)$ for each $u \in G_{\mathrm{PSH}}(X;A)_{\leq g_A}$ such that $g = g_A + \phi$ (a.e.) and $u = g_A + \psi_u$ (a.e.) (cf. Proposition 4.3). Then

$$\{\psi_u \mid u \in G_{\mathrm{PSH}}(X;A)_{\leq g_A}\} = \mathrm{PSH}(X;\alpha)_{\leq 0}.$$

Moreover, by our construction of g (cf. Proposition 4.4 and Claim 4.6.1), ϕ is the upper semicontinuous regularization of the function ϕ' given by

$$\phi'(x) = \sup\{\psi_u(x) \mid u \in G_{\mathrm{PSH}}(X;A)_{\leq g_A}\} (= \sup\{\psi(x) \mid \psi \in \mathrm{PSH}(X;\alpha)_{\leq 0}\})$$

for $x \in X$. On the other hand, by [3, Theorem 1.4], ϕ' is continuous. Thus $\phi = \phi'$ and ϕ is continuous. Therefore the claim follows by Proposition 4.3.

Next we assume that $\alpha + dd^c(\psi_0) = 0$, that is, $\alpha = dd^c(-\psi_0)$. Then

$$\mathrm{PSH}(X;\alpha) = \{\psi_0 + c \mid c \in \mathbb{R} \cup \{-\infty\}\}.$$

Let g be the greatest element of $G_{\mathrm{PSH}}(X;A)_{\leq g_A}$ modulo null functions. Then, by Proposition 4.3, there is $c \in \mathbb{R}$ such that $g = g_A + (\psi_0 + c)$ (a.e.). Thus the claim follows in this case. □

Finally, let us consider a general case. First of all, we may assume $A = B$ as before. We can take a continuous function f on X such that $g_A = h + f$ (a.e.). By using the Stone-Weierstrass theorem, there is a sequence $\{u_n\}_{n=1}^{\infty}$ of continuous functions on X such that $\lim_{n\to\infty} \|u_n\|_{\sup} = 0$ and $f + u_n$ is C^{∞} for every n. Then, as $g_A + u_n = h + (f + u_n)$ (a.e.), $g_A + u_n$ is of C^{∞}-type for all all n. Let g (resp. g_n) be the greatest element of $G_{\mathrm{PSH}}(X;A)_{\leq g_A}$ (resp. $G_{\mathrm{PSH}}(X;A)_{\leq g_A + u_n}$) modulo null functions. Note that the greatest element of $G_{\mathrm{PSH}}(X;A)_{\leq g_A \pm \|u_n\|_{\sup}}$ modulo null functions is given by $g \pm \|u_n\|_{\sup}$. By the previous claim, $g_n \in G_{C^0\cap\mathrm{PSH}}(X;A)$. Moreover, since

$$g_A - \|u_n\|_{\sup} \leq g_A + u_n \leq g_A + \|u_n\|_{\sup} \quad \text{(a.e.)},$$

we have

$$g - \|u_n\|_{\sup} \leq g_n \leq g + \|u_n\|_{\sup} \text{ (a.e.)}$$

for all n. Let $g = v + \sum_{i=1}^{l}(-a_i)\log|f_i|^2$ (a.e.) and $g_n = v_n + \sum_{i=1}^{l}(-a_i)|f_i|^2$ (a.e.) be local expression of g and g_n. Note that v_n is continuous for every n. By Lemma 2.3.1, $v - \|u_n\|_{\sup} \leq v_n \leq v + \|u_n\|_{\sup}$ holds for all n. Thus v_n converges to v uniformly, which implies that v is continuous.

(2) Let α' be a C^{∞}-form such that $[\alpha'] = dd^c([h]) + \delta_A$. By our assumption, α' is either positive or zero. By Proposition 4.3, there is $\psi \in \mathrm{PSH}(X;\alpha')$ such that ψ is continuous and $\psi + h = u$ (a.e.). Thus, by Lemma 4.2, there are sequences $\{u_n\}_{n=1}^{\infty}$ and $\{v_n\}_{n=1}^{\infty}$ of continuous functions on X with the following properties:

(a) $u_n \geq 0$ and $v_n \geq 0$ for all $n \geq 1$.
(b) $\lim_{n\to\infty} \|u_n\|_{\sup} = \lim_{n\to\infty} \|v_n\|_{\sup} = 0$.
(c) $\psi - u_n, \psi + v_n \in \mathrm{PSH}(X;\alpha') \cap C^{\infty}(X)$ for every $n \geq 1$.

Note that $u - u_n = (\psi - u_n) + h$ (a.e.) and $u + v_n = (\psi + v_n) + h$ (a.e.). Therefore, by Proposition 4.3, $u - u_n, u + v_n \in G_{C^{\infty}\cap\mathrm{PSH}}(X;A)$. □

5. Arithmetic $\mathbb{R}$-Cartier divisors

Throughout this section, let X be a d-dimensional generically smooth and normal arithmetic variety, that is, X is a flat and quasi-projective integral scheme over $\mathbb{Z}$ such that X is normal, X is smooth over $\mathbb{Q}$ and the Krull dimension of X is d.

5.1. Definition of arithmetic $\mathbb{R}$-Cartier divisor. Let $\mathrm{Div}(X)$ be the group of Cartier divisors on X. An element of

$$\mathrm{Div}(X)_{\mathbb{R}} := \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R} \quad (\text{resp. } \mathrm{Div}(X)_{\mathbb{Q}} := \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q})$$

is called an $\mathbb{R}$-*Cartier divisor* (resp. $\mathbb{Q}$-*Cartier divisor*) on X. Let D be an $\mathbb{R}$-Cartier divisor on X and let $D = a_1D_1 + \cdots + a_lD_l$ be the unique decomposition of D such that D_i's are prime divisors on X and $a_1, \ldots, a_l \in \mathbb{R}$. Note that D_i's are not necessarily Cartier divisors on X. The support $\mathrm{Supp}(D)$ of D is defined by $\bigcup_{i\in\{i \mid a_i\neq 0\}} D_i$. If $a_i \geq 0$ for all i, then D is said to be *effective* and it is denoted by $D \geq 0$. More generally, for $D, E \in \mathrm{Div}(X)_{\mathbb{R}}$, if $D - E \geq 0$, then it is denoted by $D \geq E$ or $E \leq D$. We define $H^0(X, D)$ to be

$$H^0(X, D) = \{\phi \in \mathrm{Rat}(X)^{\times} \mid (\phi) + D \geq 0\} \cup \{0\},$$

where $\mathrm{Rat}(X)$ is the field of rational functions on X. Let $F_\infty : X(\mathbb{C}) \to X(\mathbb{C})$ be the complex conjugation map on $X(\mathbb{C})$. Let g be a locally integrable function on $X(\mathbb{C})$. We say g is *F_∞-invariant* if $F_\infty^*(g) = g$ (a.e.) on $X(\mathbb{C})$. Note that we do not require that $F_\infty^*(g)$ is identically equal to g on $X(\mathbb{C})$. A pair $\overline{D} = (D, g)$ is called an *arithmetic $\mathbb{R}$-Cartier divisor on X* if g is F_∞-invariant. If $D \in \mathrm{Div}(X)$ (resp. $D \in \mathrm{Div}(X)_{\mathbb{Q}}$), then $\overline{D}$ is called an arithmetic divisor on X (resp. *arithmetic $\mathbb{Q}$-Cartier divisor on X*). For arithmetic $\mathbb{R}$-Cartier divisors $\overline{D}_1 = (D_1, g_1)$ and $\overline{D}_2 = (D_2, g_2)$, $\overline{D}_1 = \overline{D}_2$ and $\overline{D}_1 \leq \overline{D}_2$ (or $\overline{D}_2 \geq \overline{D}_1$) are defined as follows:

$$\begin{cases} \overline{D}_1 = \overline{D}_2 & \overset{\text{def}}{\Longleftrightarrow} \quad D_1 = D_2 \text{ and } g_1 = g_2 \text{ (a.e.)}, \\ \overline{D}_1 \leq \overline{D}_2 & \overset{\text{def}}{\Longleftrightarrow} \quad D_1 \leq D_2 \text{ and } g_1 \leq g_2 \text{ (a.e.)}. \end{cases}$$

If $\overline{D} \geq (0, 0)$, then $\overline{D}$ is said to be *arithmetically effective* (or *effective* for simplicity). For arithmetic $\mathbb{R}$-Cartier divisors $\overline{D}$ and $\overline{E}$ on X, we set $(-\infty, \overline{D}]$, $[\overline{D}, \infty)$ and $[\overline{D}, \overline{E}]$ as follows:

$$\begin{cases} (-\infty, \overline{D}] := \{\overline{M} \mid \overline{M} \text{ is an arithmetic } \mathbb{R}\text{-Cartier divisor on } X \text{ and } \overline{M} \leq \overline{D}\}, \\ [\overline{D}, \infty) := \{\overline{M} \mid \overline{M} \text{ is an arithmetic } \mathbb{R}\text{-Cartier divisor on } X \text{ and } \overline{D} \leq \overline{M}\}, \\ [\overline{D}, \overline{E}] := \{\overline{M} \mid \overline{M} \text{ is an arithmetic } \mathbb{R}\text{-Cartier divisor on } X \text{ and } \overline{D} \leq \overline{M} \leq \overline{E}\}. \end{cases}$$

Let $\mathscr{T}$ be a type for Green functions on X, that is, $\mathscr{T}$ is a type for Green functions on $X(\mathbb{C})$ together with the following extra F_∞-compatibility condition: if $u \in \mathscr{T}(U)$ for an open set U of $X(\mathbb{C})$, then $F_\infty^*(u) \in \mathscr{T}(F_\infty^{-1}(U))$. On arithmetic varieties, we always assume the above F_∞-compatibility condition for a type for Green functions. We denote

$$\{u \in \mathscr{T}(X(\mathbb{C})) \mid u = F_\infty^*(u)\}$$

by $\mathscr{T}(X)$. Note that $\mathscr{T}(X)$ is different from $\mathscr{T}(X(\mathbb{C}))$. Clearly C^0 and C^∞ have F_∞-compatibility. Moreover, by the following lemma, PSH and $\mathrm{PSH}_{\mathbb{R}}$ have also F_∞-compatibility. If two types $\mathscr{T}$ and $\mathscr{T}'$ for Green functions have F_∞-compatibility, then $\mathscr{T} + \mathscr{T}'$ and $\mathscr{T} - \mathscr{T}'$ have also F_∞-compatibility.

Lemma 5.1.1. *Let $f_1, \ldots, f_r \in \mathbb{R}[X_1, \ldots, X_N]$ and*

$$V = \mathrm{Spec}(\mathbb{C}[X_1, \ldots, X_N]/(f_1, \ldots, f_r)).$$

We assume that V is e-equidimensional and smooth over $\mathbb{C}$. Let $F_\infty : V \to V$ be the complex conjugation map. If u is a plurisubharmonic function on an open set U of V, then $F_\infty^*(u)$ is also a plurisubharmonic function on $F_\infty^{-1}(U)$.

Proof. Fix $x \in U$ and choose $i_1 < \cdots < i_e$ such that the projection $p : V \to \mathbb{C}^e$ given by $(x_1, \ldots, x_N) \mapsto (x_{i_1}, \ldots, x_{i_e})$ is étale at x. Note that the following diagram is commutative:

$$\begin{array}{ccc} V & \xrightarrow{F_\infty} & V \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ \mathbb{C}^e & \xrightarrow{F_\infty} & \mathbb{C}^e \end{array}$$

Let U_x be an open neighborhood of x such that $p\big|_{U_x} : U_x \to W_x = p(U_x)$ is an isomorphism as complex manifolds. Then $p\big|_{F_\infty^{-1}(U_x)} : F_\infty^{-1}(U_x) \to F_\infty^{-1}(W_x)$ is also an isomorphism as complex manifolds. This observation indicates that we may assume $V = \mathbb{C}^e$ in order to see our assertion.

Let $y \in F_\infty^{-1}(U) \subseteq \mathbb{C}^e$ and $\xi \in \mathbb{C}^e$ such that $y + \xi \exp(\sqrt{-1}\theta) \in F_\infty^{-1}(U)$ for all $0 \leq \theta \leq 2\pi$. Then

$$\begin{aligned} F_\infty^*(u)(y) = u(\bar{y}) &\leq \frac{1}{2\pi}\int_0^{2\pi} u(\bar{y} + \bar{\xi}\exp(\sqrt{-1}\theta))d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} u(\bar{y} + \bar{\xi}\exp(-\sqrt{-1}\theta))d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} u\left(\overline{y + \xi\exp(\sqrt{-1}\theta)}\right)d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} F_\infty^*(u)\left(y + \xi\exp(\sqrt{-1}\theta)\right)d\theta, \end{aligned}$$

which shows that $F_\infty^*(u)$ is plurisubharmonic on $F_\infty^{-1}(U)$. □

Let D be an $\mathbb{R}$-Cartier divisor on X and let g be a D-Green function on $X(\mathbb{C})$. By the following lemma, $\frac{1}{2}(g + F_\infty^*(g))$ is an F_∞-invariant D-Green function of $\mathscr{T}$-type on $X(\mathbb{C})$.

Lemma 5.1.2. *If g is a D-Green function of $\mathscr{T}$-type, then $F_\infty^*(g)$ is also a D-Green function of $\mathscr{T}$-type.*

Proof. Let $D = a_1D_1 + \cdots + a_lD_l$ be a decomposition of D such that $a_1, \ldots, a_l \in \mathbb{R}$ and D_i's are Cartier divisors on X. Let U be a Zariski open set of X over which D_i can be written by a local equation ϕ_i for each i. Let $g = u + \sum_{i=1}^l (-a_i)\log|\phi_i|^2$ (a.e.) be the local expression of g with respect to $\phi_1, \ldots, \phi_l$ over $U(\mathbb{C})$. Note that $F_\infty^*(\phi_i) = \bar{\phi}_i$ as a function over $U(\mathbb{C})$. Thus $F_\infty^*(g) = F_\infty^*(u) + \sum_{i=1}^l (-a_i)\log|\phi_i|^2$ (a.e.) is a local expression of $F_\infty^*(g)$, as required. □

We define $\widehat{\mathrm{Div}}_{\mathscr{T}}(X)$, $\widehat{\mathrm{Div}}_{\mathscr{T}}(X)_{\mathbb{Q}}$ and $\widehat{\mathrm{Div}}_{\mathscr{T}}(X)_{\mathbb{R}}$ as follows:

$$\begin{cases} \widehat{\mathrm{Div}}_{\mathscr{T}}(X) := \left\{(D,g) \;\middle|\; \begin{array}{l} D \in \mathrm{Div}(X) \text{ and } g \text{ is an } F_\infty\text{-invariant} \\ D\text{-Green function of } \mathscr{T}\text{-type on } X(\mathbb{C}). \end{array}\right\}, \\ \widehat{\mathrm{Div}}_{\mathscr{T}}(X)_{\mathbb{Q}} := \left\{(D,g) \;\middle|\; \begin{array}{l} D \in \mathrm{Div}(X)_{\mathbb{Q}} \text{ and } g \text{ is an } F_\infty\text{-invariant} \\ D\text{-Green function of } \mathscr{T}\text{-type on } X(\mathbb{C}). \end{array}\right\}, \\ \widehat{\mathrm{Div}}_{\mathscr{T}}(X)_{\mathbb{R}} := \left\{(D,g) \;\middle|\; \begin{array}{l} D \in \mathrm{Div}(X)_{\mathbb{R}} \text{ and } g \text{ is an } F_\infty\text{-invariant} \\ D\text{-Green function of } \mathscr{T}\text{-type on } X(\mathbb{C}). \end{array}\right\}. \end{cases}$$

An element of $\widehat{\mathrm{Div}}_{\mathscr{T}}(X)_{\mathbb{R}}$ (resp. $\widehat{\mathrm{Div}}_{\mathscr{T}}(X)_{\mathbb{Q}}$, $\widehat{\mathrm{Div}}_{\mathscr{T}}(X)$) is called an *arithmetic $\mathbb{R}$-Cartier divisor of $\mathscr{T}$-type on X* (resp. *arithmetic $\mathbb{Q}$-Cartier divisor of $\mathscr{T}$-type on X*, *arithmetic Cartier divisor of $\mathscr{T}$-type on X*). Let $\overline{D} = (D, g)$ be an arithmetic $\mathbb{R}$-Cartier divisor of $\mathscr{T}$-type. Then, as $F_\infty^*(g) = g$ (a.e.), we can see that $F_\infty^*(g_{\mathrm{can}}) = g_{\mathrm{can}}$ holds $X(\mathbb{C}) \setminus \mathrm{Supp}(D)(\mathbb{C})$.

Here we recall $\widehat{\mathrm{Pic}}_{C^0}(X)$, $\widehat{\mathrm{Pic}}_{C^0}(X)_{\mathbb{Q}}$ and $\widehat{\mathrm{Pic}}_{C^0}(X)_{\mathbb{R}}$ (for details, see [15]). First of all, let $\widehat{\mathrm{Pic}}_{C^0}(X)$ be the group of isomorphism classes of F_∞-invariant continuous hermitian invertible sheaves on X and let $\widehat{\mathrm{Pic}}_{C^0}(X)_{\mathbb{Q}} := \widehat{\mathrm{Pic}}_{C^0}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. For an F_∞-invariant continuous function f on $X(\mathbb{C})$, $\overline{\mathscr{O}}(f)$ is given by $(\mathscr{O}_X, \exp(-f)|\cdot|_{\mathrm{can}})$. Then $\widehat{\mathrm{Pic}}_{C^0}(X)_{\mathbb{R}}$ is defined to be

$$\widehat{\mathrm{Pic}}_{C^0}(X)_{\mathbb{R}} := \frac{\widehat{\mathrm{Pic}}_{C^0}(X) \otimes_{\mathbb{Z}} \mathbb{R}}{\left\{ \sum_i \overline{\mathscr{O}}(f_i) \otimes a_i \;\middle|\; \begin{array}{l} f_1, \ldots, f_r \in C^0(X) \text{ and} \\ a_1, \ldots, a_r \in \mathbb{R} \text{ with } \sum_i a_i f_i = 0 \end{array} \right\}},$$

where $C^0(X) = \{ f \in C^0(X(\mathbb{C})) \mid F_\infty^*(f) = f \}$ as before. Note that there is a natural surjective homomorphism $\overline{\mathscr{O}} : \widehat{\mathrm{Div}}_{C^0}(X) \to \widehat{\mathrm{Pic}}_{C^0}(X)$ given by

$$\overline{\mathscr{O}}(D, g) = (\mathscr{O}_X(D), |\cdot|_g),$$

where $|1|_g = \exp(-g/2)$.

5.2. Volume function for arithmetic $\mathbb{R}$-Cartier divisors. We assume that X is projective. Let $\overline{D} = (D, g)$ be an arithmetic $\mathbb{R}$-Cartier divisor on X. We set

$$\hat{H}^0(X, \overline{D}) = \{ \phi \in H^0(X, D) \mid \|\phi\|_g \leq 1 \}$$

and

$$\hat{h}^0(X, \overline{D}) = \begin{cases} \log \#\hat{H}^0(X, \overline{D}) & \text{if } \hat{H}^0(X, \overline{D}) \text{ is finite,} \\ \infty & \text{otherwise,} \end{cases}$$

where $\|\phi\|_g$ is the essential supremum of $|\phi|_g = |\phi| \exp(-g/2)$. Note that

$$\hat{H}^0(X, \overline{D}) = \{ \phi \in \mathrm{Rat}(X)^\times \mid \widehat{(\phi)} + \overline{D} \geq 0 \} \cup \{0\}.$$

The volume $\widehat{\mathrm{vol}}(\overline{D})$ of $\overline{D}$ is defined to be

$$\widehat{\mathrm{vol}}(\overline{D}) = \limsup_{n \to \infty} \frac{\hat{h}^0(X, n\overline{D})}{n^d / d!}.$$

For arithmetic $\mathbb{R}$-Cartier divisors $\overline{D}$ and $\overline{D}'$ on X, if $\overline{D} \leq \overline{D}'$, then $\hat{H}^0(X, \overline{D}) \subseteq \hat{H}^0(X, \overline{D}')$ and $\widehat{\mathrm{vol}}(\overline{D}) \leq \widehat{\mathrm{vol}}(\overline{D}')$ hold.

Proposition 5.2.1. *Let $\mathscr{T}$ be a type for Green functions on X and let $\overline{D} = (D, g)$ be an arithmetic $\mathbb{R}$-Cartier divisor of $\mathscr{T}$-type on X. If g is either of upper bounded type or of lower bounded type, then $\hat{H}^0(X, \overline{D})$ is finite. Moreover, if g is of upper bounded type, then $\widehat{\mathrm{vol}}(\overline{D}) < \infty$.*

Proof. First we assume that g is of lower bounded type. Then, by Lemma 2.5.1, $\|\cdot\|_g$ yields a norm of $H^0(X, D)$, and hence the assertion follows.

Next we assume that g is of upper bounded type. Then, by Proposition 2.3.6, there is an F_∞-invariant D-Green function g' of C^∞-type such that $g \leq g'$ (a.e.). By Proposition 2.4.2, we can choose $a_1, \ldots, a_l \in \mathbb{R}$ and $\overline{D}_1, \ldots, \overline{D}_l \in \widehat{\mathrm{Div}}_{C^\infty}(X)$ such that $(D, g') = a_1\overline{D}_1 + \cdots + a_l\overline{D}_l$. For each i, by using Lemma 5.2.3 and Lemma 5.2.4, we can find effective arithmetic Cartier divisors $\overline{A}_i$ and $\overline{B}_i$ of C^∞-type such that $\overline{D}_i = \overline{A}_i - \overline{B}_i$. As

$$(D, g') = a_1\overline{A}_1 + \cdots + a_l\overline{A}_l + (-a_1)\overline{B}_1 + \cdots + (-a_l)\overline{B}_l,$$

if we set $\overline{D}'' = \lceil a_1 \rceil \overline{A}_1 + \cdots + \lceil a_l \rceil \overline{A}_l + \lceil (-a_1) \rceil \overline{B}_1 + \cdots + \lceil (-a_l) \rceil \overline{B}_l$, then $(D, g') \leq \overline{D}''$ and $\overline{D}'' \in \widehat{\mathrm{Div}}_{C^\infty}(X)$. Note that

$$\hat{H}^0(X, n\overline{D}) \subseteq \hat{H}^0(X, n(D, g')) \subseteq \hat{H}^0(X, n\overline{D}'') = \hat{H}^0(X, \overline{\mathscr{O}}(\overline{D}'')^{\otimes n})$$

for all $n \geq 1$. Thus our assertion follows from [14, Lemma 3.3]. □

Here we consider the fundamental properties of $\widehat{\mathrm{vol}}$ on $\widehat{\mathrm{Div}}_{C^0}(X)_{\mathbb{R}}$.

Theorem 5.2.2. *There is a natural surjective homomorphism*

$$\overline{\mathscr{O}}_{\mathbb{R}} : \widehat{\mathrm{Div}}_{C^0}(X)_{\mathbb{R}} \to \widehat{\mathrm{Pic}}_{C^0}(X)_{\mathbb{R}}$$

such that the following diagram is commutative:

$$\begin{array}{ccc} \widehat{\mathrm{Div}}_{C^0}(X) \otimes_{\mathbb{Z}} \mathbb{R} & \xrightarrow{\overline{\mathscr{O}} \otimes \mathrm{id}} & \widehat{\mathrm{Pic}}_{C^0}(X) \otimes_{\mathbb{Z}} \mathbb{R} \\ \downarrow & & \downarrow \\ \widehat{\mathrm{Div}}_{C^0}(X)_{\mathbb{R}} & \xrightarrow{\overline{\mathscr{O}}_{\mathbb{R}}} & \widehat{\mathrm{Pic}}_{C^0}(X)_{\mathbb{R}}. \end{array}$$

Moreover, we have the following:

(1) *For all $\overline{D} \in \widehat{\mathrm{Div}}_{C^0}(X)_{\mathbb{R}}$,*

$$\widehat{\mathrm{vol}}(\overline{D}) = \lim_{t \to \infty} \frac{\hat{h}^0(t\overline{D})}{t^d / d!} = \widehat{\mathrm{vol}}(\overline{\mathscr{O}}_{\mathbb{R}}(\overline{D})),$$

where $t \in \mathbb{R}_{>0}$ and $\widehat{\mathrm{vol}}(\overline{\mathscr{O}}_{\mathbb{R}}(\overline{D}))$ is the volume defined in [15, Section 4].

(2) $\widehat{\mathrm{vol}}(a\overline{D}) = a^d \widehat{\mathrm{vol}}(\overline{D})$ *for all $a \in \mathbb{R}_{\geq 0}$ and $\overline{D} \in \widehat{\mathrm{Div}}_{C^0}(X)_{\mathbb{R}}$.*

(3) (*Continuity of* $\widehat{\mathrm{vol}}$) *Let $\overline{D}_1, \ldots, \overline{D}_r, \overline{A}_1, \ldots, \overline{A}_{r'} \in \widehat{\mathrm{Div}}_{C^0}(X)_{\mathbb{R}}$. For a compact set B in $\mathbb{R}^r$ and a positive number ϵ, there are positive numbers δ and δ' such that, for all $a_1, \ldots, a_r, \delta_1, \ldots, \delta_{r'} \in \mathbb{R}$ and $\phi \in C^0(X)$ with $(a_1, \ldots, a_r) \in B$, $\sum_{j=1}^{r'} |\delta_j| \leq \delta$ and $\|\phi\|_{\sup} \leq \delta'$, we have*

$$\left| \widehat{\mathrm{vol}} \left(\sum_{i=1}^r a_i \overline{D}_i + \sum_{j=1}^{r'} \delta_j \overline{A}_j + (0, \phi) \right) - \widehat{\mathrm{vol}} \left(\sum_{i=1}^r a_i \overline{D}_i \right) \right| \leq \epsilon.$$

Moreover, if $\overline{D}_1, \ldots, \overline{D}_r$, $\overline{A}_1, \ldots, \overline{A}_{r'}$ are C^∞, then there is a positive constant C depending only on X and $\overline{D}_1, \ldots, \overline{D}_r, \overline{A}_1, \ldots, \overline{A}_{r'}$ such that

$$\left| \widehat{\mathrm{vol}} \left(\sum_{i=1}^r a_i \overline{D}_i + \sum_{j=1}^{r'} \delta_j \overline{A}_j + (0, \phi) \right) - \widehat{\mathrm{vol}} \left(\sum_{i=1}^r a_i \overline{D}_i \right) \right| \\ \leq C \left(\sum_{i=1}^r |a_i| + \sum_{j=1}^{r'} |\delta_j| \right)^{d-1} \left(\|\phi\|_{\sup} + \sum_{j=1}^{r'} |\delta_j| \right)$$

for all $a_1, \ldots, a_r, \delta_1, \ldots, \delta_{r'} \in \mathbb{R}$ and $\phi \in C^0(X)$.

(4) *Let $\overline{D}_1$ and $\overline{D}_2$ be arithmetic $\mathbb{R}$-Cartier divisors of C^0-type. If $\overline{D}_1$ and $\overline{D}_2$ are pseudo-effective* (*for the definition of pseudo-effectivity, see SubSection* 6.1), *then*

$$\widehat{\mathrm{vol}}(\overline{D}_1 + \overline{D}_2)^{1/d} \geq \widehat{\mathrm{vol}}(\overline{D}_1)^{1/d} + \widehat{\mathrm{vol}}(\overline{D}_2)^{1/d}.$$

(5) (*Fujita's approximation theorem for arithmetic* $\mathbb{R}$*-Cartier divisors*) *If* $\overline{D}$ *is an arithmetic* $\mathbb{R}$*-Cartier divisor of* C^0*-type and* $\widehat{\mathrm{vol}}(\overline{D}) > 0$*, then, for any positive number* ϵ*, there are a birational morphism* $\mu : Y \to X$ *of generically smooth and normal projective arithmetic varieties and an ample arithmetic* $\mathbb{Q}$*-Cartier divisor* $\overline{A}$ *of* C^∞*-type on* Y (*cf. Section* 6) *such that* $\overline{A} \leq \mu^*(\overline{D})$ *and* $\widehat{\mathrm{vol}}(\overline{A}) \geq \widehat{\mathrm{vol}}(\overline{D}) - \epsilon$*.*

Let us begin with the following lemmas.

Lemma 5.2.3. *Let* Y *be a normal projective arithmetic variety. Then we have the following:*

(1) *Let* Z *be a Weil divisor on* Y*. Then there is an effective Cartier divisor* A *on* Y *such that* $Z \leq A$*.*
(2) *Let* D *be a Cartier divisor on* Y*. Then there are effective Cartier divisors* A *and* B *on* Y *such that* $D = A - B$*.*
(3) *Let* $x_1, \ldots, x_l$ *be points of* Y *and let* D *be a Cartier divisor on* Y*. Then there are effective Cartier divisors* A *and* B*, and a non-zero rational function* ϕ *on* Y *such that* $D + (\phi) = A - B$ *and* $x_1, \ldots, x_l \notin \mathrm{Supp}(A) \cup \mathrm{Supp}(B)$*.*

Proof. (1) Let $Z = a_1\Gamma_1 + \cdots + a_l\Gamma_l$ be the decomposition such that Γ_i's are prime divisors on Y and $a_1, \ldots, a_l \in \mathbb{Z}$. Let L be an ample invertible sheaf on Y. Then we can choose a positive integer n and a non-zero section $s \in H^0(Y, L^{\otimes n})$ such that $\mathrm{mult}_{\Gamma_i}(s) \geq a_i$ for all i. Thus, if we set $A = \mathrm{div}(s)$, then A is a Cartier divisor and $Z \leq A$.

(2) First of all, we can find effective Weil divisors A' and B' on Y such that $D = A' - B'$. By the previous (1), there is an effective Cartier divisor A such that $A' \leq A$. We set $B = B' + (A - A')$. Then B is effective and $D = A - B$. Moreover, since $B = A - D$, B is a Cartier divisor.

(3) Let L be an ample invertible sheaf on Y as before. Then there are a positive integer n_1 and a non-zero $s_1 \in H^0(Y, L^{\otimes n_1})$ such that $s_1(x_i) \neq 0$ for all i. We set $A' = \mathrm{div}(s_1)$. Similarly we can find a positive integer n_2 and a non-zero $s_2 \in H^0(Y, \mathscr{O}_Y(n_2A' - D))$ such that $s_2(x_i) \neq 0$ for all i. Therefore, if we set $A = n_2A'$ and $B = \mathrm{div}(s_2)$, then there is a non-zero rational function ϕ on Y such that $A - D = B + (\phi)$, as required. □

Lemma 5.2.4. *Let* $\mathscr{T}$ *be either* C^0 *or* C^∞*. Let* A' *and* A'' *be effective* $\mathbb{R}$*-Cartier divisors on* X *and* $A = A' - A''$*. Let* g_A *be an* F_∞*-invariant* A*-Green function of* $\mathscr{T}$*-type on* $X(\mathbb{C})$*. Then there are effective arithmetic* $\mathbb{R}$*-Cartier divisors* $(A', g_{A'})$ *and* $(A'', g_{A''})$ *of* $\mathscr{T}$*-type such that* $(A, g_A) = (A', g_{A'}) - (A'', g_{A''})$*.*

Proof. Let $g_{A''}$ be an F_∞-invariant A''-Green function of $\mathscr{T}$-type such that $g_{A''} \geq 0$ (a.e.). We put $g_{A'} = g_A + g_{A''}$. Then $g_{A'}$ is an F_∞-invariant A'-Green function of $\mathscr{T}$-type. Replacing $g_{A''}$ with $g_{A''}$ + (positive constant) if necessarily, we have $g_{A'} \geq 0$ (a.e.). □

Lemma 5.2.5. *Let* $\mathscr{T}$ *be a type for Green functions such that* $-\mathscr{T} \subseteq \mathscr{T}$ *and* $C^\infty \subseteq \mathscr{T}$*. Then the kernel of the natural homomorphism* $\widehat{\mathrm{Div}}_{\mathscr{T}}(X) \otimes_{\mathbb{Z}} \mathbb{R} \to \widehat{\mathrm{Div}}_{\mathscr{T}}(X)_{\mathbb{R}}$ *coincides with*

$$\left\{ \sum_{i=1}^{l} (0, \phi_i) \otimes a_i \;\middle|\; \begin{array}{l} a_1, \ldots, a_l \in \mathbb{R}, \phi_1, \ldots, \phi_l \in \mathscr{T}(X) \\ \text{and } a_1\phi_1 + \cdots + a_l\phi_l = 0 \end{array} \right\}.$$

Proof. It is sufficient to show that, for $\sum_{i=1}^{l}(D_i, g_i) \otimes a_i \in \widehat{\mathrm{Div}}_{\mathscr{T}}(X) \otimes_{\mathbb{Z}} \mathbb{R}$, if

$$\sum_{i=1}^{l} a_i D_i = 0 \quad \text{and} \quad \sum_{i=1}^{l} a_i g_i = 0 \text{ (a.e.)},$$

then there are $\phi_1, \ldots, \phi_l \in \mathscr{T}(X)$ such that $\sum_{i=1}^{l}(D_i, g_i) \otimes a_i = \sum_{i=1}^{l}(0, \phi_i) \otimes a_i$ and $a_1\phi_1 + \cdots + a_l\phi_l = 0$. Let $E_1, \ldots, E_r$ be a free basis of the $\mathbb{Z}$-submodule of $\mathrm{Div}(X)$ generated by $D_1, \ldots, D_l$. We set $D_i = \sum_{j=1}^{r} b_{ij}E_j$. Since

$$0 = \sum_{i=1}^{l} a_i D_i = \sum_{j=1}^{r}\left(\sum_{i=1}^{l} a_i b_{ij}\right)E_j,$$

we have $\sum_{i=1}^{l} a_i b_{ij} = 0$ for all $j = 1, \ldots, r$. Let h_j be an F_∞-invariant E_j-Green function of C^∞-type. Note that $\sum_{j=1}^{r} b_{ij}h_j$ is an F_∞-invariant D_i-Green function of $\mathscr{T}$-type. Thus we can find $\phi_1, \ldots, \phi_l \in \mathscr{T}(X)$ such that

$$g_i = \sum_{j=1}^{r} b_{ij}h_j + \phi_i \quad \text{(a.e.)}$$

for each i. Then

$$0 \overset{\text{(a.e.)}}{=} \sum_{i=1}^{l} a_i g_i \overset{\text{(a.e.)}}{=} \sum_{j=1}^{r}\left(\sum_{i=1}^{l} a_i b_{ij}\right)h_j + \sum_{i=1}^{l} a_i\phi_i = \sum_{i=1}^{l} a_i\phi_i.$$

Note that $\sum_i a_i\phi_i \in \mathscr{T}(X)$, so that $\sum_i a_i\phi_i = 0$ over $X(\mathbb{C})$. On the other hand,

$$\begin{aligned}\sum_{i=1}^{l}(D_i, g_i) \otimes a_i &= \sum_{i=1}^{l}\sum_{j=1}^{r}(E_j, h_j) \otimes a_i b_{ij} + \sum_{i=1}^{l}(0, \phi_i) \otimes a_i \\ &= \sum_{j=1}^{r}(E_j, h_j) \otimes \left(\sum_{i=1}^{l} a_i b_{ij}\right) + \sum_{i=1}^{l}(0, \phi_i) \otimes a_i = \sum_{i=1}^{l}(0, \phi_i) \otimes a_i,\end{aligned}$$

as required. □

The proof of Theorem 5.2.2. By Proposition 2.4.2, the natural homomorphism

$$\widehat{\mathrm{Div}}_{C^0}(X) \otimes_{\mathbb{Z}} \mathbb{R} \to \widehat{\mathrm{Div}}_{C^0}(X)_{\mathbb{R}}$$

is surjective. Thus the first assertion follows from Lemma 5.2.5.

(1) We set $\overline{D} = a_1\overline{D}_1 + \cdots + a_l\overline{D}_l$, where $a_1, \ldots, a_l \in \mathbb{R}$ and $\overline{D}_1, \ldots, \overline{D}_l \in \widehat{\mathrm{Div}}_{C^0}(X)$. For each $\overline{D}_i$, by using Lemma 5.2.3 and Lemma 5.2.4, we can find effective arithmetic Cartier divisors $\overline{D}_i'$ and $\overline{D}_i''$ of C^0-type such that $\overline{D}_i = \overline{D}_i' - \overline{D}_i''$. Then

$$\overline{D} = a_1\overline{D}_1' + \cdots + a_l\overline{D}_l' + (-a_1)\overline{D}_1'' + \cdots + (-a_l)\overline{D}_l''.$$

Thus, in order to see our assertion, we may assume that $\overline{D}_i$ is effective for every i. We set $I = \{i \mid a_i \geq 0\}$ and $J = \{i \mid a_i < 0\}$. Moreover, we set

$$\begin{cases}\overline{A}_n = \sum_{i\in I}\lfloor na_i\rfloor\overline{D}_i + \sum_{j\in J}\lfloor(n+1)a_j\rfloor\overline{D}_j, \\ \overline{B}_n = \sum_{i\in I}\lceil na_i\rceil\overline{D}_i + \sum_{j\in J}\lceil(n-1)a_j\rceil\overline{D}_j\end{cases}$$

for $n \in \mathbb{Z}_{\geq 1}$. Then, as $\lim_{n\to\infty} \overline{A}_n/n = \lim_{n\to\infty} \overline{B}_n/n = \overline{D}$, by virtue of [15, Theorem 5.1],

$$\lim_{n\to\infty} \frac{\hat{h}^0(X, \overline{A}_n)}{n^d/d!} = \lim_{n\to\infty} \frac{\hat{h}^0(X, \overline{B}_n)}{n^d/d!} = \widehat{\operatorname{vol}}(\overline{\mathscr{O}}_{\mathbb{R}}(\overline{D})).$$

Note that

$$\begin{cases} \lfloor \lfloor t \rfloor a \rfloor \leq ta \leq \lceil \lceil t \rceil a \rceil & \text{if } a \geq 0, \\ \lfloor (\lfloor t \rfloor + 1) a \rfloor \leq ta \leq \lceil (\lceil t \rceil - 1) a \rceil & \text{if } a < 0 \end{cases}$$

for $a \in \mathbb{R}$ and $t \in \mathbb{R}_{\geq 1}$, which yields $\overline{A}_{\lfloor t \rfloor} \leq t\overline{D} \leq \overline{B}_{\lceil t \rceil}$ for $t \in \mathbb{R}_{\geq 1}$. Therefore,

$$\frac{(\lfloor t \rfloor)^d}{t^d} \cdot \frac{h^0(X, \overline{A}_{\lfloor t \rfloor})}{(\lfloor t \rfloor)^d/d!} \leq \frac{h^0(X, t\overline{D})}{t^d/d!} \leq \frac{(\lceil t \rceil)^d}{t^d} \cdot \frac{h^0(X, \overline{B}_{\lceil t \rceil})}{(\lceil t \rceil)^d/d!},$$

and hence (1) follows.

(2) follows from (1).

(3) The first assertion follows from [15, (4) in Proposition 4.6]. Let us see the second assertion. We choose $\overline{E}_1, \ldots, \overline{E}_m, \overline{B}_1, \ldots, \overline{B}_{m'} \in \widehat{\operatorname{Div}}_{C^\infty}(X)$ such that $\overline{D}_i = \sum_{k=1}^m \alpha_{ik}\overline{E}_k$ and $\overline{A}_j = \sum_{l=1}^{m'} \beta_{jl}\overline{B}_l$ for some $\alpha_{ik}, \beta_{jl} \in \mathbb{R}$. Then

$$\sum_{i=1}^r a_i\overline{D}_i = \sum_{k=1}^m \left(\sum_{i=1}^r a_i\alpha_{ik}\right)\overline{E}_k \quad \text{and} \quad \sum_{j=1}^{r'} \delta_j\overline{A}_j = \sum_{l=1}^{m'} \left(\sum_{j=1}^{r'} \delta_j\beta_{jl}\right)\overline{B}_l.$$

Moreover, if we set $C' = \max\left(\{\alpha_{ik}\} \cup \{\beta_{jl}\}\right)$, then

$$\left|\sum_{i=1}^r a_i\alpha_{ik}\right| \leq C' \sum_{i=1}^r |a_i| \quad \text{and} \quad \left|\sum_{j=1}^{r'} \delta_j\beta_{jl}\right| \leq C' \sum_{j=1}^{r'} |\delta_j|.$$

Thus we may assume that $\overline{D}_1, \ldots, \overline{D}_r, \overline{A}_1, \ldots, \overline{A}_{r'} \in \widehat{\operatorname{Div}}_{C^\infty}(X)$. Therefore, the second assertion of (3) follows from [15, Lemma 3.1, Theorem 4.4 and Proposition 4.6].

(4) If $\widehat{\operatorname{vol}}(\overline{D}_1) > 0$ and $\widehat{\operatorname{vol}}(\overline{D}_2) > 0$, then (4) follows from (3) and [22, Theorem B] (or [16, Theorem 6.2]). Let us fix an ample arithmetic Cartier divisor $\overline{A}$ (for the definition of ampleness, see SubSection 6.1). Then $\widehat{\operatorname{vol}}(\overline{D}_1 + \epsilon\overline{A}) > 0$ and $\widehat{\operatorname{vol}}(\overline{D}_2 + \epsilon\overline{A}) > 0$ for all $\epsilon > 0$ by Proposition 6.3.2. Thus, by using (3) and the previous observation, we obtain (4).

(5) By using the continuity of $\widehat{\operatorname{vol}}$ and the Stone-Weierstrass theorem, we can find an arithmetic $\mathbb{Q}$-Cartier divisor $\overline{D}'$ of C^∞-type such that $\overline{D}' \leq \overline{D}$ and

$$\widehat{\operatorname{vol}}(\overline{D}') > \max\{\widehat{\operatorname{vol}}(\overline{D}) - \epsilon/2, 0\}.$$

Then, by virtue of [6], [22] or [16], there are a birational morphism $\mu : Y \to X$ of generically smooth and normal projective arithmetic varieties and an ample arithmetic $\mathbb{Q}$-Cartier divisor $\overline{A}$ of C^∞-type on Y such that $\overline{A} \leq \mu^*(\overline{D}')$ and $\widehat{\operatorname{vol}}(\overline{A}) \geq \widehat{\operatorname{vol}}(\overline{D}') - \epsilon/2$. Thus (5) follows. □

5.3. Intersection number of arithmetic $\mathbb{R}$-Cartier divisors with a 1-dimensional subscheme. We assume that X is projective. Let C be a 1-dimensional closed integral subscheme of X. Let $\overline{\mathcal{L}} = (\mathcal{L}, h)$ be an F_∞-invariant continuous hermitian invertible sheaf on X. Then it is well-known that $\widehat{\deg}(\overline{\mathcal{L}}\big|_C)$ is defined and it has the following property: if s is not zero element of $H^0(X, \mathcal{L})$ with $s|_C \neq 0$, then

$$\widehat{\deg}(\overline{\mathcal{L}}\big|_C) = \log \#\left(\frac{\mathcal{L}|_C}{\mathscr{O}_C \cdot s}\right) - \frac{1}{2}\sum_{x \in C(\mathbb{C})} \log(h(s,s)(x)).$$

In addition, the map

$$\widehat{\operatorname{Pic}}_{C^0}(X) \to \mathbb{R} \quad (\overline{\mathcal{L}} \mapsto \widehat{\deg}(\overline{\mathcal{L}}\big|_C))$$

is a homomorphism of abelian groups, so that it extends to a homomorphism

$$\widehat{\deg}(-|_C) : \widehat{\operatorname{Pic}}_{C^0}(X) \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{R}$$

given by

$$\widehat{\deg}((\overline{\mathcal{L}}_1 \otimes a_1 + \cdots + \overline{\mathcal{L}}_r \otimes a_r)\big|_C) = a_1 \widehat{\deg}(\overline{\mathcal{L}}_1\big|_C) + \cdots + a_r \widehat{\deg}(\overline{\mathcal{L}}_r\big|_C).$$

If $f_1, \ldots, f_r \in C^0(X)$, $a_1, \ldots, a_r \in \mathbb{R}$ and $a_1 f_1 + \cdots + a_r f_r = 0$, then

$$\begin{aligned}\widehat{\deg}((\overline{\mathscr{O}}(f_1) \otimes a_1 + \cdots + \overline{\mathscr{O}}(f_r) \otimes a_r)\big|_C)& \\ &= a_1 \widehat{\deg}(\overline{\mathscr{O}}(f_1)\big|_C) + \cdots + a_r \widehat{\deg}(\overline{\mathscr{O}}(f_r)\big|_C) \\ &= \sum_{i=1}^{r} a_i \left(\sum_{x \in C(\mathbb{C})} f_i(x)\right) = 0.\end{aligned}$$

Therefore, $\widehat{\deg}(-|_C) : \widehat{\operatorname{Pic}}_{C^0}(X) \otimes_{\mathbb{Z}} \mathbb{R} \to \mathbb{R}$ descents to a homomorphism $\widehat{\operatorname{Pic}}_{C^0}(X)_{\mathbb{R}} \to \mathbb{R}$. By abuse of notation, we use the same symbol $\widehat{\deg}(-|_C)$ to denote the homomorphism $\widehat{\operatorname{Pic}}_{C^0}(X)_{\mathbb{R}} \to \mathbb{R}$. Using this homomorphism, we define

$$\widehat{\deg}(-|_C) : \widehat{\operatorname{Div}}_{C^0}(X)_{\mathbb{R}} \to \mathbb{R}$$

to be $\widehat{\deg}(\overline{D}\big|_C) := \widehat{\deg}(\overline{\mathscr{O}}_{\mathbb{R}}(\overline{D})\big|_C)$ for $\overline{D} \in \widehat{\operatorname{Div}}_{C^0}(X)_{\mathbb{R}}$. If there are effective Cartier divisors $D_1, \ldots, D_l$ and $a_1, \ldots, a_l \in \mathbb{R}$ such that $D = a_1 D_1 + \cdots + a_l D_l$ and $C \not\subseteq \operatorname{Supp}(D_i)$ for all i, then we can see that

$$\widehat{\deg}(\overline{D}\big|_C) = \sum_{i=1}^{l} a_i \log \#(\mathscr{O}_C(D_i)/\mathscr{O}_C) + \frac{1}{2}\sum_{x \in C(\mathbb{C})} g_{\mathrm{can}}(x).$$

Let $\mathscr{T}$ be a type for Green functions on X such that $C^0 \subseteq \mathscr{T}$, $\mathscr{T}$ is real valued and $-\mathscr{T} \subseteq \mathscr{T}$. Let $\overline{D} = (D, g)$ be an arithmetic $\mathbb{R}$-Cartier divisor of $\mathscr{T}$-type on X. There is $h \in \mathscr{T}(X)$ such that $g - h$ is an F_∞-invariant D-Green function of C^0-type. We would like to define $\widehat{\deg}(\overline{D}\big|_C)$ by the following quantity:

$$\widehat{\deg}\left((D, g-h)\big|_C\right) + \frac{1}{2}\sum_{x \in C(\mathbb{C})} h(x).$$

Indeed, it does not depend on the choice of h. Let h' be another element of $\mathscr{T}(X)$ such that $g - h'$ is an F_∞-invariant D-Green function of C^0-type. We can find $u \in C^0(X)$ such that $g - h = g - h' + u$ (a.e.), so that $h' = h + u$ over $X(\mathbb{C})$. Therefore,

$$
\begin{aligned}
\widehat{\deg}\left((D, g - h')\big|_C\right) &+ \frac{1}{2}\sum_{x\in C(\mathbb{C})} h'(x) \\
&= \widehat{\deg}((D, (g-h) - u)\big|_C) + \frac{1}{2}\sum_{x\in C(\mathbb{C})} (h+u)(x) \\
&= \widehat{\deg}((D, (g-h))\big|_C) - \frac{1}{2}\sum_{x\in C(\mathbb{C})} u(x) + \frac{1}{2}\sum_{x\in C(\mathbb{C})} (h+u)(x) \\
&= \widehat{\deg}((D, g-h)\big|_C) + \frac{1}{2}\sum_{x\in C(\mathbb{C})} h(x).
\end{aligned}
$$

Note that if there are effective Cartier divisors $D_1, \ldots, D_l$ and $a_1, \ldots, a_l \in \mathbb{R}$ such that $D = a_1D_1 + \cdots + a_lD_l$ and $C \not\subseteq \operatorname{Supp}(D_i)$ for all i, then

$$
\widehat{\deg}\left(\overline{D}\big|_C\right) = \sum_{i=1}^{l} a_i \log \#(\mathscr{O}_C(D_i)/\mathscr{O}_C) + \frac{1}{2}\sum_{x\in C(\mathbb{C})} g_{\mathrm{can}}(x).
$$

Moreover, $\widehat{\deg}(-|_C) : \widehat{\operatorname{Div}}_{\mathscr{T}}(X)_{\mathbb{R}} \to \mathbb{R}$ is a homomorphism.

Let $Z_1(X)$ be the group of 1-cycles on X and $Z_1(X)_{\mathbb{R}} = Z_1(X) \otimes_{\mathbb{Z}} \mathbb{R}$. Let Z be an element of $Z_1(X)_{\mathbb{R}}$. There is a unique expression $Z = a_1C_1 + \cdots + a_lC_l$ such that $a_1, \ldots, a_l \in \mathbb{R}$ and $C_1, \ldots, C_l$ are 1-dimensional closed integral schemes on X. For $\overline{D} \in \widehat{\operatorname{Div}}_{\mathscr{T}}(X)_{\mathbb{R}}$, we define $\widehat{\deg}\left(\overline{D} \mid Z\right)$ to be

$$
\widehat{\deg}\left(\overline{D} \mid Z\right) := \sum_{i=1}^{l} a_i \widehat{\deg}\left(\overline{D}\big|_{C_i}\right).
$$

Note that $\widehat{\deg}\left(\overline{D} \mid C\right) = \widehat{\deg}\left(\overline{D}\big|_C\right)$ for a 1-dimensional closed integral scheme C on X.

6. Positivity of arithmetic $\mathbb{R}$-Cartier divisors

In this section, we will introduce a lot of kinds of positivity for arithmetic $\mathbb{R}$-Cartier divisors and investigate their properties. Throughout this section, let X be a generically smooth projective and normal arithmetic variety.

6.1. Definitions. Let $\overline{D} = (D, g)$ be an arithmetic $\mathbb{R}$-Cartier divisor on X, that is, $D \in \operatorname{Div}(X)_{\mathbb{R}}$ and g is an F_∞-invariant locally integrable function on $X(\mathbb{C})$. The ampleness, adequateness, nefness, bigness and pseudo-effectivity of $\overline{D}$ are defined as follows:

• **ample** : First we recall the ampleness of a C^∞-hermitian invertible sheaf. According to [13], an F_∞-invariant C^∞-hermitian invertible sheaf $\overline{\mathcal{L}} = (\mathcal{L}, h)$ on X is said to be *ample* if $\mathcal{L}$ is ample, $c_1(\overline{\mathcal{L}})$ is a positive form and $H^0(X, \mathcal{L}^{\otimes n})$ is generated by elements of

$$
\{s \in H^0(X, \mathcal{L}^{\otimes n}) \mid \|s\|_{\sup} < 1\}
$$

as a $\mathbb{Z}$-module for $n \gg 1$. Note that our definition is slightly stronger than Zhang's definition [25], in which the semipositivity of $c_1(\overline{\mathcal{L}})$ is assumed instead of positivity.

We say $\overline{D}$ is *ample* if there are $a_1, \ldots, a_r \in \mathbb{R}_{>0}$ and ample arithmetic $\mathbb{Q}$-Cartier divisors $\overline{A}_1, \ldots, \overline{A}_r$ of C^∞-type (i.e., $\overline{\mathscr{O}}(n_i\overline{A}_i)$ is an ample C^∞-hermitian invertible sheaf for some $n_i \in \mathbb{Z}_{>0}$ in the above sense) such that

$$\overline{D} = a_1\overline{A}_1 + \cdots + a_r\overline{A}_r.$$

Note that an ample arithmetic $\mathbb{R}$-Cartier divisor is of C^∞-type. The set of all ample arithmetic $\mathbb{R}$-Cartier divisors on X is denoted by $\widehat{\mathrm{Amp}}(X)_{\mathbb{R}}$. By applying [16, Lemma 1.1.3] to the case where $P = \widehat{\mathrm{Div}}_{C^\infty}(X)_{\mathbb{Q}}$, $m = 1$, $b_1 = 0$, $A = {}^t(0, \ldots, 0)$ and $x_1 = \overline{A}_1, \ldots, x_r = \overline{A}_r$, we can see that

$$\widehat{\mathrm{Amp}}(X)_{\mathbb{R}} \cap \widehat{\mathrm{Div}}_{C^\infty}(X)_{\mathbb{Q}} = \left\{ \overline{D} \;\middle|\; \begin{array}{l} \overline{\mathscr{O}}(n\overline{D}) \text{ is an ample } C^\infty\text{-hermitian} \\ \text{invertible sheaf on } X \text{ for some } n \in \mathbb{Z}_{>0} \end{array} \right\}.$$

• **adequate** : $\overline{D}$ is said to be *adequate* if there are an ample arithmetic $\mathbb{R}$-Cartier divisor $\overline{A}$ and a non-negative F_∞-invariant continuous function f on $X(\mathbb{C})$ such that $\overline{D} = \overline{A} + (0, f)$. Note that an adequate arithmetic $\mathbb{R}$-Cartier divisor is of C^0-type.

• **nef** : We say $\overline{D}$ is *nef* if the following properties holds:

(a) $\overline{D}$ is of $\mathrm{PSH}_{\mathbb{R}}$-type.

(b) $\widehat{\deg}(\overline{D}\big|_C) \geq 0$ for all 1-dimensional closed integral subschemes C of X.

The cone of all nef arithmetic $\mathbb{R}$-Cartier divisors on X is denoted by $\widehat{\mathrm{Nef}}(X)_{\mathbb{R}}$. Moreover, the cone of all nef arithmetic $\mathbb{R}$-Cartier divisors of C^∞-type (resp. C^0-type) on X is denoted by $\widehat{\mathrm{Nef}}_{C^\infty}(X)_{\mathbb{R}}$ (resp. $\widehat{\mathrm{Nef}}_{C^0}(X)_{\mathbb{R}}$).

• **big** : Let us fix a type $\mathscr{T}$ for Green functions. We say $\overline{D}$ is a *big arithmetic* $\mathbb{R}$*-Cartier divisor of* $\mathscr{T}$*-type* if $\overline{D} \in \widehat{\mathrm{Div}}_{\mathscr{T}^b}(X)_{\mathbb{R}}$ (i.e. $\overline{D} \in \widehat{\mathrm{Div}}_{\mathscr{T}}(X)_{\mathbb{R}}$ and g is of bounded type) and $\widehat{\mathrm{vol}}(\overline{D}) > 0$.

• **pseudo-effective** : $\overline{D}$ is said to be *pseudo-effective* if $\overline{D}$ is of C^0-type and there are arithmetic $\mathbb{R}$-Cartier divisors $\overline{D}_1, \ldots, \overline{D}_r$ of C^0-type and sequences $\{a_{n1}\}_{n=1}^{\infty}, \ldots, \{a_{nr}\}_{n=1}^{\infty}$ in $\mathbb{R}$ such that $\lim_{n\to\infty} a_{ni} = 0$ for all $i = 1, \ldots, r$ and $\widehat{\mathrm{vol}}(\overline{D} + a_{n1}\overline{D}_1 + \cdots + a_{nr}\overline{D}_r) > 0$ for all $n \gg 1$.

6.2. Properties of ample arithmetic $\mathbb{R}$-Cartier divisors. In this subsection, we consider several properties of ample arithmetic $\mathbb{R}$-Cartier divisors. Let us begin with the following proposition.

Proposition 6.2.1. (1) *If $\overline{A}$ and $\overline{B}$ are ample* (*resp. adequate*) *arithmetic $\mathbb{R}$-Cartier divisors and $a \in \mathbb{R}_{>0}$, then $\overline{A} + \overline{B}$ and $a\overline{A}$ are also ample* (*resp. adequate*).

(2) *If $\overline{A}$ is an ample arithmetic $\mathbb{R}$-Cartier divisor, then there are an ample arithmetic $\mathbb{Q}$-Cartier divisor $\overline{A}'$ and an ample arithmetic $\mathbb{R}$-Cartier divisor $\overline{A}''$ such that $\overline{A} = \overline{A}' + \overline{A}''$.*

(3) *Let $\overline{A}$ be an ample (resp. adequate) arithmetic $\mathbb{R}$-Cartier divisor and let $\overline{L}_1, \ldots, \overline{L}_n$ be arithmetic $\mathbb{R}$-Cartier divisors of C^∞-type (resp. of C^0-type). Then there is $\delta \in \mathbb{R}_{>0}$ such that $\overline{A} + \delta_1 \overline{L}_1 + \cdots + \delta_n \overline{L}_n$ is ample (resp. adequate) for $\delta_1, \ldots, \delta_n \in \mathbb{R}$ with $|\delta_1| + \cdots + |\delta_n| \leq \delta$.*

(4) *If $\overline{A}$ is an adequate arithmetic $\mathbb{R}$-Cartier divisor, then $\widehat{\mathrm{vol}}(\overline{A}) > 0$.*

Proof. (1) and (2) are obvious.

(3) First we assume that $\overline{A}$ is ample and that $\overline{L}_1, \ldots, \overline{L}_n$ are of C^∞-type. We set $\overline{L}_i = \sum_{j=1}^{l} b_{ij} \overline{M}_j$ such that $\overline{M}_1, \ldots, \overline{M}_l$ are arithmetic $\mathbb{Q}$-Cartier divisors of C^∞-type and $b_{ij} \in \mathbb{R}$. Then, as

$$\overline{A} + \sum_{i=1}^{n} \delta_i \overline{L}_i = \overline{A} + \sum_{j=1}^{l} \left(\sum_{i=1}^{n} \delta_i b_{ij} \right) \overline{M}_j,$$

we may assume that $\overline{L}_1, \ldots, \overline{L}_n$ are arithmetic $\mathbb{Q}$-Cartier divisors of C^∞-type. Moreover, by (1) and (2), we may further assume that $\overline{A}$ is an ample arithmetic $\mathbb{Q}$-Cartier divisor.

Let us choose $\delta \in \mathbb{Q}_{>0}$ such that $\overline{A} \pm \delta \overline{L}_i$ is ample for every $i = 1, \ldots, n$. Note that

$$\sum_{i=1}^{n} \frac{|\delta_i|}{\delta} (\overline{A} + \mathrm{sign}(\delta_i) \delta \overline{L}_i) = \left(\sum_{i=1}^{n} \frac{|\delta_i|}{\delta} \right) \overline{A} + \sum_{i=1}^{n} \delta_i \overline{L}_i,$$

where $\mathrm{sign}(a)$ for $a \in \mathbb{R}$ is given by

$$\mathrm{sign}(a) = \begin{cases} 1 & \text{if } a \geq 0, \\ -1 & \text{if } a < 0. \end{cases}$$

Hence, if $\sum_{i=1} |\delta_i| \leq \delta$, then $\overline{A} + \sum_{i=1}^{n} \delta_i \overline{L}_i$ is ample.

Next we assume that $\overline{A}$ is adequate and that $\overline{L}_1, \ldots, \overline{L}_n$ are of C^0-type. Then there are an ample arithmetic $\mathbb{R}$-Cartier divisor $\overline{A}'$ and $u \in C^0(X)$ such that $u \geq 0$ and $\overline{A} = \overline{A}' + (0, u)$. As $\overline{A}' - (0, \epsilon)$ is ample for $0 < \epsilon \ll 1$ by the previous observation, we may assume that $u \geq \epsilon$ for some positive number ϵ. By virtue of the Stone-Weierstrass theorem, we can find $v_1, \ldots, v_n \in C^0(X)$ such that $v_i \geq 0$ $(\forall i)$, $\epsilon \geq v_1 + \cdots + v_n$ and $\overline{L}'_i := \overline{L}_i + (0, v_i)$ is of C^∞-type for all i. By the previous case, we can find $0 < \delta < 1$ such that $\overline{A}' + \delta_1 \overline{L}'_1 + \cdots + \delta_n \overline{L}'_n$ is ample for $\delta_1, \ldots, \delta_n \in \mathbb{R}$ with $|\delta_1| + \cdots + |\delta_n| \leq \delta$. Note that

$$\overline{A} + \delta_1 \overline{L}_1 + \cdots + \delta_n \overline{L}_n = \overline{A}' + \delta_1 \overline{L}'_1 + \cdots + \delta_n \overline{L}'_n + (0, u - \delta_1 v_1 - \cdots - \delta_n v_n)$$

and

$$u - \delta_1 v_1 - \cdots - \delta_n v_n \geq u - v_1 - \cdots - v_n \geq 0,$$

as required.

(4) Clearly we may assume that $\overline{A}$ is ample, so that the assertion follows from (2) and (4) in Theorem 5.2.2. □

Next we consider the following proposition.

Proposition 6.2.2. (1) *If $\overline{A}$ is an ample arithmetic $\mathbb{R}$-Cartier divisor and $\overline{B}$ is a nef arithmetic $\mathbb{R}$-Cartier divisor of C^∞-type, then $\overline{A} + \overline{B}$ is ample.*

(2) *If $\overline{A}$ is an adequate arithmetic $\mathbb{R}$-Cartier divisor and $\overline{B}$ is a nef arithmetic $\mathbb{R}$-Cartier divisor of C^0-type, then $\overline{A} + \overline{B}$ is adequate.*

Proof. (1) We set $\overline{B} = b_1\overline{B}_1 + \cdots + b_n\overline{B}_n$ such that $b_1, \dots, b_n \in \mathbb{R}$ and $\overline{B}_1, \dots, \overline{B}_n$ are arithmetic $\mathbb{Q}$-Cartier divisors of C^∞-type. We choose an ample arithmetic $\mathbb{Q}$-Cartier divisor $\overline{A}_1$ and an ample arithmetic $\mathbb{R}$-Cartier divisor $\overline{A}_2$ such that $\overline{A} = \overline{A}_1 + \overline{A}_2$. Then, by (3) in Proposition 6.2.1, there are $\delta_1, \dots, \delta_n \in \mathbb{R}_{>0}$ such that

$$\overline{A}_1 + \sum_{i=1}^{n} \delta_i \overline{B}_i \quad \text{and} \quad \overline{A}_2 - \sum_{i=1}^{n} \delta_i \overline{B}_i$$

are ample and $b_i + \delta_i \in \mathbb{Q}$ for all i. Moreover, we can take an ample arithmetic $\mathbb{Q}$-Cartier divisor $\overline{A}_3$ and an ample arithmetic $\mathbb{R}$-Cartier divisor $\overline{A}_4$ such that

$$\overline{A}_2 - \sum_{i=1}^{n} \delta_i \overline{B}_i = \overline{A}_3 + \overline{A}_4.$$

Then, since

$$\overline{A}_1 + \sum_{i=1}^{n} \delta_i \overline{B}_i + \overline{B} = \overline{A}_1 + \sum_{i=1}^{n} (b_i + \delta_i)\overline{B}_i$$

is a nef arithmetic $\mathbb{Q}$-Cartier divisor of C^∞-type, $\overline{A}_3 + \overline{A}_1 + \sum_{i=1}^{n} \delta_i \overline{B}_i + \overline{B}$ is an ample arithmetic $\mathbb{Q}$-Cartier divisor by [14, Lemma 5.6]. Therefore,

$$\overline{A} + \overline{B} = \overline{A}_4 + \overline{A}_3 + \overline{A}_1 + \sum_{i=1}^{n} \delta_i \overline{B}_i + \overline{B}$$

is an ample arithmetic $\mathbb{R}$-Cartier divisor.

(2) Clearly we may assume that $\overline{A}$ is ample. By (3) in Proposition 6.2.1, there is a positive real number δ such that $\frac{1}{2}\overline{A} - (0, \delta)$ is ample. Note that $\frac{1}{2}A + B$ is ample, that is, $\frac{1}{2}A + B$ is a linear combination of ample Cartier divisors with positive coefficients, which can be checked in the same way as above. Thus, by (2) in Theorem 4.6, there is $u \in C^0(X)$ (i.e., u is an F_∞-invariant continuous function in $X(\mathbb{C})$) such that $0 \leq u < \delta$ on $X(\mathbb{C})$ and $\frac{1}{2}\overline{A} + \overline{B} + (0, u)$ is a nef $\mathbb{R}$-Cartier divisor of C^∞-type. Then, by (1),

$$\frac{1}{2}\overline{A} - (0, \delta) + \frac{1}{2}\overline{A} + \overline{B} + (0, u)$$

is ample. Thus

$$\overline{A} + \overline{B} = \frac{1}{2}\overline{A} - (0, \delta) + \frac{1}{2}\overline{A} + \overline{B} + (0, u) + (0, \delta - u)$$

is adequate. □

Finally let us observe the following lemma.

Lemma 6.2.3. *Let $\overline{D}_1 = (D_1, g_1)$ and $\overline{D}_2 = (D_2, g_2)$ be arithmetic $\mathbb{R}$-Cartier divisors of* $\mathrm{PSH}_{\mathbb{R}}$*-type on X. If $D_1 = D_2$, $g_1 \leq g_2$* (a.e.) *and $\overline{D}_1$ is nef, then $\overline{D}_2$ is also nef.*

Proof. Since $D_1 = D_2$, there is a $\phi \in (\mathrm{PSH}_{\mathbb{R}} - \mathrm{PSH}_{\mathbb{R}})(X(\mathbb{C}))$ such that $g_2 = g_1 + \phi$ (a.e.) and $\phi \geq 0$ (a.e.). Note that $\phi \geq 0$ by Lemma 2.3.1. Let C be a 1-dimensional closed integral subscheme of X. Then

$$\widehat{\deg}(\overline{D}_2\big|_C) = \widehat{\deg}(\overline{D}_1\big|_C) + \frac{1}{2}\sum_{y \in C(\mathbb{C})} \phi(y) \geq \widehat{\deg}(\overline{D}_1\big|_C) \geq 0.$$

□

6.3. Criterions of bigness and pseudo-effectivity. The purpose of this subsection is to prove the following propositions.

Proposition 6.3.1. *For $\overline{D} = (D, g) \in \widehat{\mathrm{Div}}_{C^0}(X)_{\mathbb{R}}$, the following are equivalent:*

(1) *$\overline{D}$ is big, that is, $\widehat{\mathrm{vol}}(\overline{D}) > 0$.*
(2) *For any $\overline{A} \in \widehat{\mathrm{Div}}_{C^0}(X)_{\mathbb{R}}$, there are a positive integer n and a non-zero rational function ϕ such that $\overline{A} \leq n\overline{D} + \widehat{(\phi)}$.*

Proof. "(2) $\Longrightarrow$ (1)" is obvious.

Let us consider "(1) $\Longrightarrow$ (2)". By using Lemma 5.2.3 and Lemma 5.2.4, we can find effective arithmetic $\mathbb{R}$-Cartier divisors $\overline{A}'$ and $\overline{A}''$ of C^0-type such that $\overline{A} = \overline{A}' - \overline{A}''$. Note that $\overline{A} \leq \overline{A}'$. Thus we may assume $\overline{A}$ is effective in order to see our assertion. By virtue of the continuity of $\widehat{\mathrm{vol}}$ (cf. Theorem 5.2.2), there is a positive integer m such that

$$\widehat{\mathrm{vol}}(\overline{D} - (1/m)\overline{A}) > 0,$$

that is, $\widehat{\mathrm{vol}}(m\overline{D} - \overline{A}) > 0$, so that there is a positive integer n and a non-zero rational function ϕ such that

$$n(m\overline{D} - \overline{A}) + \widehat{(\phi)} \geq 0.$$

Thus $mn\overline{D} + \widehat{(\phi)} \geq n\overline{A} \geq \overline{A}$. □

Proposition 6.3.2. *For $\overline{D} = (D, g) \in \widehat{\mathrm{Div}}_{C^0}(X)_{\mathbb{R}}$, the following are equivalent:*

(1) *$\overline{D}$ is pseudo-effective.*
(2) *For any big arithmetic $\mathbb{R}$-Cartier divisor $\overline{A}$ of C^0-type, $\widehat{\mathrm{vol}}(\overline{D} + \overline{A}) > 0$.*
(3) *There is a big arithmetic $\mathbb{R}$-Cartier divisor $\overline{A}$ of C^0-type such that $\widehat{\mathrm{vol}}(\overline{D} + (1/n)\overline{A}) > 0$ for all $n \geq 1$.*

Proof. It is sufficient to see that (1) implies (2). As $\overline{D}$ is pseudo-effective, there are arithmetic $\mathbb{R}$-Cartier divisors $\overline{D}_1, \ldots, \overline{D}_r$ of C^0-type and sequences $\{a_{m1}\}_{m=1}^{\infty}, \ldots, \{a_{mr}\}_{m=1}^{\infty}$ in $\mathbb{R}$ such that $\lim_{m\to\infty} a_{mi} = 0$ for all $i = 1, \ldots, r$ and $\widehat{\mathrm{vol}}(\overline{D} + a_{m1}\overline{D}_1 + \cdots + a_{mr}\overline{D}_r) > 0$ for all $m \gg 1$. By the continuity of $\widehat{\mathrm{vol}}$, there is a sufficiently large positive integer m such that $\overline{A} - (a_{m1}\overline{D}_1 + \cdots + a_{mr}\overline{D}_r)$ is big. Thus

$$\widehat{\mathrm{vol}}(\overline{D} + \overline{A}) \geq \widehat{\mathrm{vol}}(\overline{D} + a_{m1}\overline{D}_1 + \cdots + a_{mr}\overline{D}_r) > 0.$$

□

Proposition 6.3.3. *If $\overline{D} = (D, g)$ is a pseudo-effective arithmetic $\mathbb{R}$-Cartier divisor of C^0-type such that D is big on the generic fiber $X_{\mathbb{Q}}$ (i.e., $\mathrm{vol}(D_{\mathbb{Q}}) > 0$ on $X_{\mathbb{Q}}$), then $\overline{D} + (0, \epsilon)$ is big for all $\epsilon \in \mathbb{R}_{>0}$.*

Proof. Let $\overline{A}$ be an ample arithmetic Cartier divisor on X. Since D is big on $X_{\mathbb{Q}}$, by using the continuity of the volume function over $X_{\mathbb{Q}}$ (cf. [10, I, Corollary 2.2.45]), we can see that there are a positive integer m and a non-zero rational function ϕ such that

$$mD - A + (\phi) \geq 0.$$

If we set $(L, h) = m\overline{D} - \overline{A} + \widehat{(\phi)}$, then h is an L-Green function of C^0-type and L is effective. Thus there is a positive number λ such that

$$m\overline{D} - \overline{A} + \widehat{(\phi)} \geq (0, -\lambda),$$

that is, $m\overline{D} + (0, \lambda) \geq \overline{A} - \widehat{(\phi)}$. We choose a sufficiently large positive integer n such that

$$\frac{\lambda}{n+m} \leq \epsilon.$$

Then

$$\begin{aligned}\overline{D} + \frac{1}{n}(\overline{A} - \widehat{(\phi)}) &\leq \overline{D} + \frac{1}{n}(m\overline{D} + (0, \lambda)) \\ &= \left(1 + \frac{m}{n}\right)\left(\overline{D} + \left(0, \frac{\lambda}{n+m}\right)\right) \\ &\leq \left(1 + \frac{m}{n}\right)\left(\overline{D} + (0, \epsilon)\right).\end{aligned}$$

Note that $\overline{A} - \widehat{(\phi)}$ is ample, so that $\overline{D} + (1/n)(\overline{A} - \widehat{(\phi)})$ is big by Proposition 6.3.2, and hence $\overline{D} + (0, \epsilon)$ is also big. □

Remark 6.3.4. It is very natural to ask whether $\hat{H}^0(X, n(\overline{D} + (0, \epsilon))) \neq \{0\}$ for some $n \in \mathbb{Z}_{>0}$ in the case where D is not necessarily big on $X_{\mathbb{Q}}$. This does not hold in general. For example, let $\mathbb{P}^1_{\mathbb{Z}} = \operatorname{Proj}(\mathbb{Z}[T_0, T_1])$ be the projective line over $\mathbb{Z}$ and $\overline{D} = a\widehat{(T_1/T_0)}$ for $a \in \mathbb{R} \setminus \mathbb{Q}$. It is easy to see that $\overline{D}$ is pseudo-effective and $H^0(\mathbb{P}^1_{\mathbb{Z}}, nD) = \{0\}$ for all $n \in \mathbb{Z}_{>0}$. Thus $\hat{H}^0(\mathbb{P}^1_{\mathbb{Z}}, n(\overline{D} + (0, \epsilon))) = \{0\}$ for $\epsilon \in \mathbb{R}_{>0}$ and $n \in \mathbb{Z}_{>0}$.

6.4. Intersection number of arithmetic $\mathbb{R}$-Cartier divisors of C^0-type. Let

$$\widehat{\operatorname{Div}}_{C^\infty}(X) \times \cdots \times \widehat{\operatorname{Div}}_{C^\infty}(X) \to \mathbb{R}$$

be a symmetric multi-linear map over $\mathbb{Z}$ given by

$$(\overline{D}_1, \ldots, \overline{D}_d) \mapsto \widehat{\deg}(\overline{D}_1 \cdots \overline{D}_d) := \widehat{\deg}(\widehat{c}_1(\overline{\mathscr{O}}(\overline{D}_1)) \cdots \widehat{c}_1(\overline{\mathscr{O}}(\overline{D}_d))),$$

which extends to the symmetric multi-linear map

$$(\widehat{\operatorname{Div}}_{C^\infty}(X) \otimes_{\mathbb{Z}} \mathbb{R}) \times \cdots \times (\widehat{\operatorname{Div}}_{C^\infty}(X) \otimes_{\mathbb{Z}} \mathbb{R}) \to \mathbb{R}$$

over $\mathbb{R}$.

Proposition-Definition 6.4.1. *The above multi-linear map*

$$(\widehat{\operatorname{Div}}_{C^\infty}(X) \otimes_{\mathbb{Z}} \mathbb{R}) \times \cdots \times (\widehat{\operatorname{Div}}_{C^\infty}(X) \otimes_{\mathbb{Z}} \mathbb{R}) \to \mathbb{R}$$

descents to the symmetric multi-linear map

$$\widehat{\operatorname{Div}}_{C^\infty}(X)_{\mathbb{R}} \times \cdots \times \widehat{\operatorname{Div}}_{C^\infty}(X)_{\mathbb{R}} \to \mathbb{R}$$

over $\mathbb{R}$, *whose value at* $(\overline{D}_1,\ldots,\overline{D}_d) \in \widehat{\mathrm{Div}}_{C^\infty}(X)_{\mathbb{R}} \times \cdots \times \widehat{\mathrm{Div}}_{C^\infty}(X)_{\mathbb{R}}$ *is also denoted by* $\widehat{\deg}(\overline{D}_1 \cdots \overline{D}_d)$ *by abuse of notation.*

Proof. Let $a_1,\ldots,a_l \in \mathbb{R}$ and $\phi_1,\ldots,\phi_l \in C^\infty(X)$ such that $a_1\phi_1 + \cdots + a_l\phi_l = 0$. By Lemma 5.2.5, it is sufficient to show that

$$\widehat{\deg}\left(((0,\phi_1)\otimes a_1 + \cdots + (0,\phi_l)\otimes a_l)\cdot \overline{D}_2\cdots\overline{D}_d\right) = 0$$

for all $\overline{D}_2,\ldots,\overline{D}_d \in \widehat{\mathrm{Div}}_{C^\infty}(X)$. First of all, note that there are 1-dimensional closed integral subschemes $C_1,\ldots,C_r$, $c_1,\ldots,c_r \in \mathbb{Z}$ and a current T of $(d-2,d-2)$-type such that

$$\overline{D}_2\cdots\overline{D}_d \sim (c_1C_1 + \cdots + c_rC_r, T).$$

Then

$$\begin{aligned}
&\widehat{\deg}\left(((0,\phi_1)\otimes a_1 + \cdots + (0,\phi_l)\otimes a_l)\cdot \overline{D}_2\cdots\overline{D}_d\right)\\
&\quad = \sum_{i=1}^{l} a_i \widehat{\deg}\left((0,\phi_i)\cdot(c_1C_1 + \cdots + c_rC_r, T)\right)\\
&\qquad = \sum_{i=1}^{l} a_i \left(\sum_{j=1}^{r} c_j \sum_{y \in C_j(\mathbb{C})} \phi_i(y) + (1/2)\int_{X(\mathbb{C})} dd^c(\phi_i)\wedge T\right)\\
&\qquad\quad = \sum_{j=1}^{r} c_j \sum_{y\in C_j(\mathbb{C})} \sum_{i=1}^{l} a_i\phi_i(y) + (1/2)\int_{X(\mathbb{C})} dd^c\left(\sum_{i=1}^{l} a_i\phi_i\right)\wedge T = 0,
\end{aligned}$$

as required. □

Let $\widehat{\mathrm{Div}}^{\mathrm{Nef}}_{C^0}(X)_{\mathbb{R}}$ be the vector subspace of $\widehat{\mathrm{Div}}_{C^0}(X)_{\mathbb{R}}$ generated by $\widehat{\mathrm{Nef}}_{C^0}(X)_{\mathbb{R}}$. The purpose of this subsection is to show the following proposition, which gives another construction of the intersection number due to [25, Lemma 6.5], [26, Section 1] and [12, Section 5] (cf Remark 6.4.3).

Proposition 6.4.2. (1)

$$\widehat{\mathrm{Div}}_{C^0\cap \mathrm{PSH}+C^\infty}(X)_{\mathbb{R}} \subseteq \widehat{\mathrm{Div}}^{\mathrm{Nef}}_{C^0}(X)_{\mathbb{R}} \subseteq \widehat{\mathrm{Div}}_{C^0\cap\mathrm{PSH}-C^0\cap\mathrm{PSH}}(X)_{\mathbb{R}}.$$

(2) *The above symmetric multi-linear map*

$$\widehat{\mathrm{Div}}_{C^\infty}(X)_{\mathbb{R}} \times \cdots \times \widehat{\mathrm{Div}}_{C^\infty}(X)_{\mathbb{R}} \to \mathbb{R}$$

given in Proposition-Definition 6.4.1 *extends to a unique symmetric multi-linear map*

$$\widehat{\mathrm{Div}}^{\mathrm{Nef}}_{C^0}(X)_{\mathbb{R}} \times \cdots \times \widehat{\mathrm{Div}}^{\mathrm{Nef}}_{C^0}(X)_{\mathbb{R}} \to \mathbb{R}$$

such that $(\overline{D},\ldots,\overline{D}) \mapsto \widehat{\mathrm{vol}}(\overline{D})$ *for* $\overline{D} \in \widehat{\mathrm{Nef}}_{C^0}(X)_{\mathbb{R}}$. *By abuse of notation, for*

$$(\overline{D}_1,\ldots,\overline{D}_d) \in \widehat{\mathrm{Div}}^{\mathrm{Nef}}_{C^0}(X)_{\mathbb{R}} \times \cdots \times \widehat{\mathrm{Div}}^{\mathrm{Nef}}_{C^0}(X)_{\mathbb{R}},$$

the image of $(\overline{D}_1,\ldots,\overline{D}_d)$ *by the above extension is also denoted by*

$$\widehat{\deg}(\overline{D}_1\cdots\overline{D}_d).$$

Proof. (1) It is obvious that

$$\widehat{\mathrm{Div}}^{\mathrm{Nef}}_{C^0}(X)_{\mathbb{R}} \subseteq \widehat{\mathrm{Div}}_{C^0\cap\mathrm{PSH}-C^0\cap\mathrm{PSH}}(X)_{\mathbb{R}}.$$

Let $\overline{D} \in \widehat{\mathrm{Div}}_{C^0\cap\mathrm{PSH}+C^\infty}(X)_{\mathbb{R}}$. By Proposition 2.3.7, there is an ample arithmetic Cartier divisor $\overline{A}$ with $\overline{D}+\overline{A} \in \widehat{\mathrm{Div}}_{C^0\cap\mathrm{PSH}}(X)_{\mathbb{R}}$. Thus it is sufficient to show the following claim:

Claim 6.4.2.1. *For $\overline{D} \in \widehat{\mathrm{Div}}_{C^0\cap\mathrm{PSH}}(X)_{\mathbb{R}}$, there is an ample arithmetic Cartier divisor $\overline{B}$ such that $\overline{D}+\overline{B} \in \widehat{\mathrm{Nef}}_{C^0}(X)_{\mathbb{R}}$ and $D+B$ is ample.*

Proof. By virtue of the Stone-Weierstrass theorem, there is an F_∞-invariant non-negative continuous function u on $X(\mathbb{C})$ such that $\overline{D}-(0,u) \in \widehat{\mathrm{Div}}_{C^\infty}(X)_{\mathbb{R}}$. Thus, by Proposition 6.2.1, we can find an ample arithmetic Cartier divisor $\overline{B}$ such that

$$\overline{D}-(0,u)+\overline{B}$$

is ample. In particular, $\overline{D}+\overline{B} \in \widehat{\mathrm{Nef}}_{C^0}(X)_{\mathbb{R}}$ and $D+B$ is ample. □

(2) Let us begin with the following claim.

Claim 6.4.2.2. (a) *For $\overline{D} \in \widehat{\mathrm{Nef}}_{C^\infty}(X)_{\mathbb{R}}$, $\widehat{\deg}(\overline{D}^d) = \widehat{\mathrm{vol}}(\overline{D})$.*

(b) $d!X_1\cdots X_d = \displaystyle\sum_{\emptyset\neq I\subseteq\{1,\ldots,d\}} (-1)^{d-\#(I)}\left(\sum_{i\in I} X_i\right)^d$ *in* $\mathbb{Z}[X_1,\ldots,X_d]$.

(c) *For $\overline{D}_1,\ldots,\overline{D}_d \in \widehat{\mathrm{Nef}}_{C^\infty}(X)_{\mathbb{R}}$,*

$$\widehat{\deg}(\overline{D}_1\cdots\overline{D}_d) = \frac{1}{d!}\sum_{\emptyset\neq I\subseteq\{1,\ldots,d\}} (-1)^{d-\#(I)}\widehat{\mathrm{vol}}\left(\sum_{i\in I}\overline{D}_i\right).$$

Proof. (a) First we assume that $\overline{D}$ is ample. We set $\overline{D} = a_1\overline{A}_1+\cdots+a_l\overline{A}_l$ such that $a_1,\ldots,a_l \in \mathbb{R}_{>0}$ and $\overline{A}_i$'s are ample arithmetic Cartier divisors. Let us choose sufficient small positive numbers $\delta_1,\ldots,\delta_l$ such that $a_i+\delta_i \in \mathbb{Q}$ for all i. Then, by [14, Corollary 5.5],

$$\widehat{\deg}(((a_1+\delta_1)\overline{A}_1+\cdots+(a_l+\delta_l)\overline{A}_l)^d) = \widehat{\mathrm{vol}}((a_1+\delta_1)\overline{A}_1+\cdots+(a_l+\delta_l)\overline{A}_l).$$

Thus, using the continuity of $\widehat{\mathrm{vol}}$, the assertion follows.

Next we consider a general case. Let $\overline{A}$ be an ample arithmetic Cartier divisor of C^∞-type. Then, by Proposition 6.2.2, $\overline{D}+\epsilon\overline{A}$ is ample for all $\epsilon>0$. Thus the assertion follows from the previous observation and the continuity of $\widehat{\mathrm{vol}}$.

(b) In general, let us see that

$$\sum_{\emptyset\neq I\subseteq\{1,\ldots,d\}} (-1)^{\#(I)}\left(\sum_{i\in I} X_i\right)^l = \begin{cases} 0 & \text{if } l<d, \\ (-1)^d d!X_1\cdots X_d & \text{if } l=d \end{cases}$$

holds for integers d and l with $1 \leq l \leq d$. This assertion for d and l is denoted by $A(d,l)$. $A(1,1)$ is obvious. Moreover, it is easy to see $A(d,1)$. Note that

$$\int_0^{X_d} \left(\sum_{\emptyset \neq I \subseteq \{1,\ldots,d\}} (-1)^{\#(I)} \left(\sum_{i \in I} X_i \right)^{l-1} \right) dX_d$$

$$= \frac{1}{l} \sum_{\emptyset \neq I \subseteq \{1,\ldots,d\}} (-1)^{\#(I)} \left(\sum_{i \in I} X_i \right)^{l} + X_d \sum_{\emptyset \neq J \subseteq \{1,\ldots,d-1\}} (-1)^{\#(J)} \left(\sum_{j \in J} X_j \right)^{l-1},$$

which shows that $A(d-1,l-1)$ and $A(d,l-1)$ imply $A(d,l)$. Thus (b) follows by double induction on d and l.

(c) follows from (a) and (b). □

The uniqueness of the symmetric multi-linear map follows from (b) in the previous claim. We set

$$\widehat{P} = \left\{ \overline{D} \in \widehat{\mathrm{Nef}}_{C^0}(X)_{\mathbb{R}} \mid D \text{ is ample} \right\}.$$

Note that $\overline{D} + \overline{A} \in \widehat{P}$ for all $\overline{D} \in \widehat{\mathrm{Nef}}_{C^0}(X)_{\mathbb{R}}$ and $\overline{A} \in \widehat{\mathrm{Amp}}(X)_{\mathbb{R}}$. In particular,

$$\widehat{\mathrm{Div}}_{C^0}^{\mathrm{Nef}}(X)_{\mathbb{R}} = \left\{ \overline{D} - \overline{D}' \mid \overline{D}, \overline{D}' \in \widehat{P} \right\}.$$

For $(\overline{D}_1, \ldots, \overline{D}_d) \in \widehat{P} \times \cdots \times \widehat{P}$, we define $\alpha(\overline{D}_1, \ldots, \overline{D}_d)$ to be

$$\alpha(\overline{D}_1, \ldots, \overline{D}_d) := \frac{1}{d!} \sum_{\emptyset \neq I \subseteq \{1,\ldots,d\}} (-1)^{d-\#(I)} \widehat{\mathrm{vol}} \left(\sum_{i \in I} \overline{D}_i \right). \tag{6.4.2.3}$$

Claim 6.4.2.4. *α is symmetric and*

$$\alpha(a\overline{D}_1 + b\overline{D}_1', \overline{D}_2, \ldots, \overline{D}_d) = a\alpha(\overline{D}_1, \overline{D}_2, \ldots, \overline{D}_d) + b\alpha(\overline{D}_1', \overline{D}_2, \ldots, \overline{D}_d)$$

holds for $\overline{D}_1, \overline{D}_1', \overline{D}_2, \ldots, \overline{D}_d \in \widehat{P}$ and $a, b \in \mathbb{R}_{\geq 0}$ with $a + b > 0$.

Proof. Clearly α is symmetric. By Theorem 4.6, for any $\epsilon > 0$, there are non-negative F_∞-invariant continuous functions $u_1, u_1', u_2, \ldots, u_d$ such that

$$\|u_1\|_{\sup} \leq \epsilon, \|u_1'\|_{\sup} \leq \epsilon, \|u_2\|_{\sup} \leq \epsilon, \ldots, \|u_d\|_{\sup} \leq \epsilon$$

and that $\overline{D}_1(\epsilon) := \overline{D}_1 + (0, u_1)$, $\overline{D}_1'(\epsilon) := \overline{D}_1' + (0, u_1')$, $\overline{D}_2(\epsilon) := \overline{D}_2 + (0, u_2)$, $\ldots$, $\overline{D}_d(\epsilon) := \overline{D}_d + (0, u_d)$ are elements of $\widehat{\mathrm{Nef}}_{C^\infty}(X)_{\mathbb{R}}$. Then, by virtue of Claim 6.4.2.2,

$$\alpha(a\overline{D}_1(\epsilon) + b\overline{D}_1'(\epsilon), \overline{D}_2(\epsilon), \ldots, \overline{D}_d(\epsilon))$$
$$= a\alpha(\overline{D}_1(\epsilon), \overline{D}_2(\epsilon), \ldots, \overline{D}_d(\epsilon)) + b\alpha(\overline{D}_1'(\epsilon), \overline{D}_2(\epsilon), \ldots, \overline{D}_d(\epsilon)).$$

Thus, using the continuity of $\widehat{\mathrm{vol}}$, we have the assertion of the claim. □

By the above claim together with the following Lemma 6.4.4, we obtain the existence of the symmetric multi-linear map. Finally we need to see

$$\widehat{\mathrm{vol}}(\overline{D}) = \widehat{\deg}(\overline{D}^d)$$

for $\overline{D} \in \widehat{\mathrm{Nef}}_{C^0}(X)_{\mathbb{R}}$. Let $\overline{A}$ be an ample arithmetic $\mathbb{R}$-Cartier divisor of C^∞-type. As $\overline{D} + \epsilon\overline{A} \in \widehat{P}$ for $\epsilon > 0$, we have

$$\widehat{\mathrm{vol}}(\overline{D} + \epsilon\overline{A}) = \widehat{\deg}((\overline{D} + \epsilon\overline{A})^d) = \sum_{i=0}^{d} \binom{d}{i} \epsilon^i \widehat{\deg}(\overline{D}^{d-i}\overline{A}^{i}),$$

and hence the assertion follows from the continuity of $\widehat{\mathrm{vol}}$. □

Remark 6.4.3. (1) By our construction, $\widehat{\mathrm{vol}}(\overline{D}) = \widehat{\deg}(\overline{D}^d)$ for $\overline{D} \in \widehat{\mathrm{Nef}}_{C^0}(X)_{\mathbb{R}}$. In particular, $\overline{D}$ is big if and only if $\widehat{\deg}(\overline{D}^d) > 0$. This is however a non-trivial fact for $\overline{D} \in \widehat{\mathrm{Nef}}_{C^\infty}(X)_{\mathbb{R}}$ (cf. [14, Corollary 5.5] and Claim 6.4.2.2).

(2) In [25, Lemma 6.5], [26, Section 1] and [12, Section 5], a symmetric multi-linear map

$$\widehat{\mathrm{Div}}_{C^0}^{\mathrm{Nef}}(X) \times \cdots \times \widehat{\mathrm{Div}}_{C^0}^{\mathrm{Nef}}(X) \to \mathbb{R}$$

is constructed as an extension of

$$\widehat{\mathrm{Div}}_{C^\infty}(X) \times \cdots \times \widehat{\mathrm{Div}}_{C^\infty}(X) \to \mathbb{R}.$$

Of course, it extends to a symmetric multi-linear map

$$\widehat{\mathrm{Div}}_{C^0}^{\mathrm{Nef}}(X)_{\mathbb{R}} \times \cdots \times \widehat{\mathrm{Div}}_{C^0}^{\mathrm{Nef}}(X)_{\mathbb{R}} \to \mathbb{R}$$

by using mult-linearity. Our intersection number in Proposition 6.4.2 coincides with the number given by the above multi-linear map. For details, see [18, SubSection 1.2 and SubSection 2.2].

Lemma 6.4.4. *Let V and W be vector spaces over $\mathbb{R}$ and let P be a cone in V, that is, $ax + by \in P$ whenever $x, y \in P$ and $a, b \in \mathbb{R}_{\geq 0}$ with $a + b > 0$. Let $f : P^s \to W$ be a map such that*

$$f(x_1, \ldots, ax_i + by_i, \ldots, x_s) = af(x_1, \ldots, x_i, \ldots, x_s) + bf(x_1, \ldots, y_i, \ldots, x_s)$$

for all $i = 1, \ldots, s$ and all $x_1, \ldots, x_i, y_i, \ldots, x_s \in P$ and $a, b \in \mathbb{R}_{\geq 0}$ with $a + b > 0$. If $V = \{x - x' \mid x, x' \in P\}$, then there is a unique multi-linear map $\tilde{f} : V^s \to W$ such that $\tilde{f}\big|_{P^s} = f$. Moreover, if f is symmetric, then $\tilde{f}$ is also symmetric.

Proof. For $x_1, \ldots, x_s \in V$, we set $x_i = x_{i,1} - x_{i,-1}$ ($x_{i,1}, x_{i,-1} \in P$) for each i. We would like to define $\tilde{f}(x_1, \ldots, x_s)$ to be

$$\tilde{f}(x_1, \ldots, x_s) = \sum_{\epsilon_1, \ldots, \epsilon_s \in \{\pm 1\}} \epsilon_1 \cdots \epsilon_s f(x_{1,\epsilon_1}, \ldots, x_{s,\epsilon_s}).$$

Claim 6.4.4.1. *The above is well-defined, that is, if we choose another $y_{i,1}, y_{i,-1} \in P$ with $x_i = y_{i,1} - y_{i,-1}$ for each i, then*

$$\sum_{\epsilon_1, \ldots, \epsilon_s \in \{\pm 1\}} \epsilon_1 \cdots \epsilon_s f(x_{1,\epsilon_1}, \ldots, x_{s,\epsilon_s}) = \sum_{\epsilon_1, \ldots, \epsilon_s \in \{\pm 1\}} \epsilon_1 \cdots \epsilon_s f(y_{1,\epsilon_1}, \ldots, y_{s,\epsilon_s}).$$

Proof. For simplicity, we denote

$$\sum_{\epsilon_1, \ldots, \epsilon_s \in \{\pm 1\}} \epsilon_1 \cdots \epsilon_s f(x_{1,\epsilon_1}, \ldots, x_{s,\epsilon_s}) \quad \text{and} \quad \sum_{\epsilon_1, \ldots, \epsilon_s \in \{\pm 1\}} \epsilon_1 \cdots \epsilon_s f(y_{1,\epsilon_1}, \ldots, y_{s,\epsilon_s}).$$

by I_x and I_y respectively. We prove it by induction on s. If $s = 1$, then the assertion is obvious, so that we assume $s > 1$. By the hypothesis of induction, for all $x \in P$,

$$\sum_{\epsilon_2,\ldots,\epsilon_s\in\{\pm1\}} \epsilon_2 \cdots \epsilon_s f(x, x_{2,\epsilon_2}, \ldots, x_{s,\epsilon_s}) = \sum_{\epsilon_2,\ldots,\epsilon_s\in\{\pm1\}} \epsilon_2 \cdots \epsilon_s f(x, y_{2,\epsilon_2}, \ldots, y_{s,\epsilon_s}).$$

As $x_{1,1} + y_{1,-1} = x_{1,-1} + y_{1,1}$, we have

$$f(x_{1,1}, x_{2,\epsilon_2}, \ldots, x_{s,\epsilon_s}) + f(y_{1,-1}, x_{2,\epsilon_2}, \ldots, x_{s,\epsilon_s}) = f(x_{1,-1}, x_{2,\epsilon_2}, \ldots, x_{s,\epsilon_s}) + f(y_{1,1}, x_{2,\epsilon_2}, \ldots, x_{s,\epsilon_s}).$$

Therefore,

$$\begin{aligned} I_x &= \sum_{\epsilon_2,\ldots,\epsilon_s\in\{\pm1\}} \epsilon_2 \cdots \epsilon_s \left(f(x_{1,1}, x_{2,\epsilon_2}, \ldots, x_{s,\epsilon_s}) - f(x_{1,-1}, x_{2,\epsilon_2}, \ldots, x_{s,\epsilon_s})\right) \\ &= \sum_{\epsilon_2,\ldots,\epsilon_s\in\{\pm1\}} \epsilon_2 \cdots \epsilon_s \left(f(y_{1,1}, x_{2,\epsilon_2}, \ldots, x_{s,\epsilon_s}) - f(y_{1,-1}, x_{2,\epsilon_2}, \ldots, x_{s,\epsilon_s})\right) \\ &= \sum_{\epsilon_2,\ldots,\epsilon_s\in\{\pm1\}} \epsilon_2 \cdots \epsilon_s \left(f(y_{1,1}, y_{2,\epsilon_2}, \ldots, y_{s,\epsilon_s}) - f(y_{1,-1}, y_{2,\epsilon_2}, \ldots, y_{s,\epsilon_s})\right) = I_y, \end{aligned}$$

as required. □

Clearly, if f is symmetric, then $\tilde{f}$ is also symmetric. The uniqueness and the multi-linearity of $\tilde{f}$ is straightforward consequences. □

6.5. Asymptotic multiplicity. First we recall the multiplicity of Cartier divisors. Let $(R, \mathfrak{m})$ be a d-dimensional noetherian local domain with $d \geq 1$. For a non-zero element a of R, we denote the multiplicity of a local ring $(R/aR, \mathfrak{m}(R/aR))$ by $\mathrm{mult}_{\mathfrak{m}}(a)$, that is,

$$\mathrm{mult}_{\mathfrak{m}}(a) := \begin{cases} \lim\limits_{n\to\infty} \dfrac{\mathrm{length}_R((R/aR)/\mathfrak{m}^{n+1}(R/aR))}{n^{d-1}/(d-1)!} & \text{if } a \notin R^{\times}, \\ 0 & \text{if } a \in R^{\times}. \end{cases}$$

Note that the above limit always exists and $\mathrm{mult}_{\mathfrak{m}}(a) \in \mathbb{Z}_{\geq 0}$. Moreover, if R is regular, then

$$\mathrm{mult}_{\mathfrak{m}}(a) = \max\{i \in \mathbb{Z}_{\geq 0} \mid a \in \mathfrak{m}^i\}.$$

Let a and b be non-zero elements of R. By applying [11, Theorem 14.6] to the following exact sequence:

$$0 \longrightarrow R/aR \xrightarrow{\times b} R/abR \longrightarrow R/bR \longrightarrow 0,$$

we can see that

$$\mathrm{mult}_{\mathfrak{m}}(ab) = \mathrm{mult}_{\mathfrak{m}}(a) + \mathrm{mult}_{\mathfrak{m}}(b).$$

Let K be the quotient field of R. For $\alpha \in K^{\times}$, we set $\alpha = a/b$ $(a, b \in R \setminus \{0\})$. Then $\mathrm{mult}_{\mathfrak{m}}(a) - \mathrm{mult}_{\mathfrak{m}}(b)$ does not depend on the expression $\alpha = a/b$. Indeed, if $\alpha = a/b = a'/b'$, then, by the previous formula,

$$\mathrm{mult}_{\mathfrak{m}}(a) + \mathrm{mult}_{\mathfrak{m}}(b') = \mathrm{mult}_{\mathfrak{m}}(ab') = \mathrm{mult}_{\mathfrak{m}}(a'b) = \mathrm{mult}_{\mathfrak{m}}(a') + \mathrm{mult}_{\mathfrak{m}}(b).$$

Thus we define $\mathrm{mult}_{\mathfrak{m}}(\alpha)$ to be

$$\mathrm{mult}_{\mathfrak{m}}(\alpha) := \mathrm{mult}_{\mathfrak{m}}(a) - \mathrm{mult}_{\mathfrak{m}}(b).$$

Note that the map

$$\mathrm{mult}_{\mathfrak{m}} : K^{\times} \to \mathbb{Z}$$

is a homomorphism, that is, $\mathrm{mult}_{\mathfrak{m}}(\alpha\beta) = \mathrm{mult}_{\mathfrak{m}}(\alpha) + \mathrm{mult}_{\mathfrak{m}}(\beta)$ for $\alpha, \beta \in K^{\times}$.

For $x \in X$, we define a homomorphism

$$\mathrm{mult}_x : \mathrm{Div}(X) \to \mathbb{Z}$$

to be $\mathrm{mult}_x(D) := \mathrm{mult}_{\mathfrak{m}_x}(f_x)$, where $\mathfrak{m}_x$ is the maximal ideal of $\mathscr{O}_{X,x}$ and f_x is a local equation of D at x. Note that this definition does not depend on the choice of the local equation f_x. By abuse of notation, the natural extension

$$\mathrm{mult}_x \otimes \mathrm{id}_{\mathbb{R}} : \mathrm{Div}(X)_{\mathbb{R}} \to \mathbb{R}$$

is also denoted by mult_x.

Let $\overline{D}$ be an arithmetic $\mathbb{R}$-Cartier divisor of C^0-type. For $x \in X$, we define $\nu_x(\overline{D})$ to be

$$\nu_x(\overline{D}) := \begin{cases} \inf\{\mathrm{mult}_x(D + (\phi)) \mid \phi \in \hat{H}^0(X, \overline{D}) \setminus \{0\}\} & \text{if } \hat{H}^0(X, \overline{D}) \neq \{0\}, \\ \infty & \text{if } \hat{H}^0(X, \overline{D}) = \{0\} \end{cases}$$

We call $\nu_x(\overline{D})$ the *multiplicity at x* of the complete arithmetic linear series of $\overline{D}$. First let us see the following lemma.

Lemma 6.5.1. *Let $\overline{D}$ and $\overline{E}$ be arithmetic $\mathbb{R}$-Cartier divisors of C^0-type. Then we have the following:*

(1) *If $\overline{D}$ is effective, then $\nu_x(\overline{D}) \leq \mathrm{mult}_x(D)$.*
(2) $\nu_x(\overline{D} + \overline{E}) \leq \nu_x(\overline{D}) + \nu_x(\overline{E})$.
(3) *If $\overline{D} \leq \overline{E}$, then $\nu_x(\overline{E}) \leq \nu_x(\overline{D}) + \mathrm{mult}_x(E - D)$.*
(4) *For $\phi \in \mathrm{Rat}(X)^{\times}$, $\nu_x(\overline{D} + \widehat{(\phi)}) = \nu_x(\overline{D})$.*

Proof. (1) is obvious.

(2) If either $\hat{H}^0(X, \overline{D}) = \{0\}$ or $\hat{H}^0(X, \overline{E}) = \{0\}$, then the assertion is obvious, so that we may assume that $\hat{H}^0(X, \overline{D}) \neq \{0\}$ and $\hat{H}^0(X, \overline{E}) \neq \{0\}$. Let $\phi \in \hat{H}^0(X, \overline{D}) \setminus \{0\}$ and $\psi \in \hat{H}^0(X, \overline{E}) \setminus \{0\}$. Then, as

$$\widehat{(\phi\psi)} + \overline{E} + \overline{D} = \widehat{(\phi)} + \overline{D} + \widehat{(\psi)} + \overline{E} \geq 0,$$

we have $\phi\psi \in \hat{H}^0(X, \overline{D} + \overline{E}) \setminus \{0\}$. Thus

$$\nu_x(\overline{D} + \overline{E}) \leq \mathrm{mult}_x((\phi\psi) + D + E) = \mathrm{mult}_x((\phi) + D) + \mathrm{mult}_x((\psi) + E),$$

which implies (2).

(3) If we set $\overline{F} = \overline{E} - \overline{D}$, then, by (1) and (2),

$$\nu_x(\overline{E}) = \nu_x(\overline{D} + \overline{F}) \leq \nu_x(\overline{D}) + \nu_x(\overline{F}) \leq \nu_x(\overline{D}) + \mathrm{mult}_x(F).$$

(4) Let $\alpha : H^0(X, D + (\phi)) \to H^0(X, D)$ be the natural isomorphism given by $\alpha(\psi) = (\phi\psi)$. Note that $(\overline{D} + \widehat{(\phi)}) + \widehat{(\psi)} = \overline{D} + \widehat{(\alpha(\psi))}$. Thus we have (4). □

We set

$$N(\overline{D}) = \left\{ n \in \mathbb{Z}_{>0} \mid \hat{H}^0(X, n\overline{D}) \neq \{0\} \right\}.$$

Note that $N(\overline{D})$ is a sub-semigroup of $\mathbb{Z}_{>0}$, that is, if $n, m \in N(\overline{D})$, then $n+m \in N(\overline{D})$. We assume that $N(\overline{D}) \neq \emptyset$. For $x \in X$, we define $\mu_x(\overline{D})$ to be

$$\mu_x(\overline{D}) := \inf\left\{ \mathrm{mult}_x(D + (1/n)(\phi)) \mid n \in N(\overline{D}),\ \phi \in \hat{H}^0(X, n\overline{D}) \setminus \{0\} \right\},$$

which is called the *asymptotic multiplicity at* x of the complete arithmetic $\mathbb{Q}$-linear series of $\overline{D}$.

We can see that

$$\mu_x(\overline{D}) = \inf\left\{ \frac{\nu_x(n\overline{D})}{n} \;\middle|\; n \in N(\overline{D}) \right\}.$$

Indeed, an inequality $\mu_x(\overline{D}) \leq \nu_x(n\overline{D})/n$ for $n \in N(\overline{D})$ is obvious, so that $\mu_x(\overline{D}) \leq \inf\left\{\nu_x(n\overline{D})/n \mid n \in N(\overline{D})\right\}$. Moreover, for $n \in N(\overline{D})$ and $\phi \in \hat{H}^0(X, n\overline{D}) \setminus \{0\}$,

$$\inf\left\{ \frac{\nu_x(n\overline{D})}{n} \;\middle|\; n \in N(\overline{D}) \right\} \leq \frac{\nu_x(n\overline{D})}{n} \leq \operatorname{mult}_x(D + (1/n)(\phi))$$

holds, and hence we have the converse inequality.

By the above lemma,

$$\nu_x((n+m)\overline{D}) \leq \nu_x(n\overline{D}) + \nu_x(m\overline{D})$$

for all $n, m \in N(\overline{D})$. Thus, if $\hat{h}^0(\overline{D}) \neq \{0\}$ (i.e., $N(\overline{D}) = \mathbb{Z}_{>0}$), then

$$\lim_{n\to\infty} \frac{\nu_x(n\overline{D})}{n} = \inf\left\{ \frac{\nu_x(n\overline{D})}{n} \;\middle|\; n > 0 \right\}.$$

Proposition 6.5.2. *Let $\overline{D}$ and $\overline{E}$ be arithmetic $\mathbb{R}$-Cartier divisors of C^0-type such that $N(\overline{D}) \neq \emptyset$ and $N(\overline{E}) \neq \emptyset$. Then we have the following:*

(1) $\mu_x(\overline{D} + \overline{E}) \leq \mu_x(\overline{D}) + \mu_x(\overline{E})$.
(2) *If $\overline{D} \leq \overline{E}$, then $\mu_x(\overline{E}) \leq \mu_x(\overline{D}) + \operatorname{mult}_x(E - D)$.*
(3) $\mu_x(\overline{D} + \widehat{(\phi)}) = \mu_x(\overline{D})$ *for $\phi \in \operatorname{Rat}(X)^{\times}$.*
(4) $\mu_x(a\overline{D}) = a\mu_x(\overline{D})$ *for $a \in \mathbb{Q}_{>0}$.*

Proof. First let us see (4). We assume that $a \in \mathbb{Z}_{>0}$. Let $n \in N(\overline{D})$ and $\phi \in \hat{H}^0(n\overline{D}) \setminus \{0\}$. Then $\phi^a \in \hat{H}^0(n(a\overline{D})) \setminus \{0\}$. Thus

$$\mu_x(a\overline{D}) \leq \operatorname{mult}_x(aD + (1/n)(\phi^a)) = a \operatorname{mult}_x(D + (1/n)(\phi)),$$

which yields $\mu_x(a\overline{D}) \leq a\mu_x(\overline{D})$. Conversely let $n \in N(a\overline{D})$ and $\psi \in \hat{H}^0(n(a\overline{D})) \setminus \{0\}$. Then

$$\mu_x(\overline{D}) \leq \operatorname{mult}_x(D + (1/na)(\psi)) = (1/a) \operatorname{mult}_x(aD + (1/n)(\psi)),$$

and hence $\mu_x(\overline{D}) \leq (1/a)\mu_x(a\overline{D})$. Thus (4) follows in the case where $a \in \mathbb{Z}_{>0}$.

In general, we choose a positive integer m such that $ma \in \mathbb{Z}_{>0}$. Then, by the previous observation,

$$m\mu_x(a\overline{D}) = \mu_x(ma\overline{D}) = ma\mu_x(\overline{D}),$$

as required.

By (4), we may assume that $\hat{h}^0(\overline{D}) \neq 0$ and $\hat{h}^0(\overline{E}) \neq 0$ in order to see (1), (2) and (3), so that (1), (2) and (3) follow from (2), (3) and (4) in Lemma 6.5.1 respectively. □

Finally we consider the vanishing result of the asymptotic multiplicity for a nef and big arithmetic $\mathbb{R}$-Cartier divisor.

Proposition 6.5.3. *If $\overline{D}$ is a nef and big arithmetic $\mathbb{R}$-Cartier divisor of C^0-type, then $\mu_x(\overline{D}) = 0$ for all $x \in X$.*

Proof. Step 1 (the case where $\overline{D}$ is an ample arithmetic $\mathbb{R}$-Cartier divisor) : Note that if $\overline{D}$ is an ample arithmetic $\mathbb{Q}$-Cartier divisor, then the assertion is obvious. By using Lemma 5.2.3 and Lemma 5.2.4, there are $a_1, \ldots, a_l \in \mathbb{R}$ and effective arithmetic $\mathbb{Q}$-Cartier divisors

$$\overline{A}_1, \ldots, \overline{A}_l, \overline{B}_1, \ldots, \overline{B}_l$$

of C^∞-type such that

$$\overline{D} = a_1\overline{A}_1 + \cdots + a_l\overline{A}_l - a_1\overline{B}_1 - \cdots - a_l\overline{B}_l.$$

Let us choose sufficiently small arbitrary positive numbers $\delta_1, \ldots, \delta_l, \delta'_1, \ldots, \delta'_l$ such that $a_i - \delta_i, a_i + \delta'_i \in \mathbb{Q}$ for all i. We set

$$\overline{D}' = (a_1 - \delta_1)\overline{A}_1 + \cdots + (a_l - \delta_l)\overline{A}_l - (a_1 + \delta'_1)\overline{B}_1 - \cdots - (a_l + \delta'_l)\overline{B}_l.$$

Then, $\overline{D}' \leq \overline{D}$ and $\overline{D}'$ is an ample arithmetic $\mathbb{Q}$-Cartier divisor by Proposition 6.2.1. By (2) in Proposition 6.5.2,

$$0 \leq \mu_x(\overline{D}) \leq \mu_x(\overline{D}') + \operatorname{mult}_x(D - D') = \sum (\delta_i \operatorname{mult}_x(A_i) + \delta'_i \operatorname{mult}_x(B_i))$$

because $\mu_x(\overline{D}') = 0$. Therefore,

$$0 \leq \mu_x(\overline{D}) \leq \sum (\delta_i \operatorname{mult}_x(A_i) + \delta'_i \operatorname{mult}_x(B_i)),$$

and hence $\mu_x(\overline{D}) = 0$.

Step 2 (the case where $\overline{D}$ is an adequate arithmetic $\mathbb{R}$-Cartier divisor) : In this case, there is an ample arithmetic $\mathbb{R}$-Cartier divisor $\overline{A}$ and a non-negative F_∞-invariant continuous function ϕ on $X(\mathbb{C})$ such that $\overline{D} = \overline{A} + (0, \phi)$. By (2) in Proposition 6.5.2,

$$0 \leq \mu_x(\overline{D}) \leq \mu_x(\overline{A}) = 0,$$

as required.

Step 3 (general case) : Let $\overline{A}$ be an ample arithmetic $\mathbb{Q}$-Cartier divisor. Since $\overline{D}$ is big, by Proposition 6.3.1, there are a positive integer m and $\phi \in \operatorname{Rat}(X)^\times$ such that $\overline{A} \leq m\overline{D} + \widehat{(\phi)}$. We set $\overline{E} = m\overline{D} + \widehat{(\phi)}$. Then $\overline{E}$ is nef. Moreover, for $\delta \in (0, 1]$, by Proposition 6.2.2, $\delta\overline{A} + (1 - \delta)\overline{E}$ is adequate and $\delta\overline{A} + (1 - \delta)\overline{E} \leq \overline{E}$. Hence

$$\mu_x(\overline{E}) \leq \mu_x(\delta\overline{A} + (1 - \delta)\overline{E}) + \delta \operatorname{mult}_x(E - A) \leq \delta \operatorname{mult}_x(E - A),$$

which implies that $\mu_x(\overline{E}) = 0$. Therefore, using (3) and (4) in Proposition 6.5.2,

$$\mu_x(\overline{D}) = \frac{1}{m}\mu_x(m\overline{D}) = \frac{1}{m}\mu_x(\overline{E}) = 0.$$

□

6.6. Generalized Hodge index theorem for an arithmetic $\mathbb{R}$-Cartier divisor. In this subsection, let us consider the following theorem, which is an $\mathbb{R}$-Cartier divisor version of [14, Corollary 6.4]:

Theorem 6.6.1. *Let $\overline{D}$ be an arithmetic $\mathbb{R}$-Cartier divisor of* $(C^0 \cap \mathrm{PSH})$*-type on X. If D is nef on every fiber of $X \to \operatorname{Spec}(\mathbb{Z})$ (i.e., $\deg(D|_C) \geq 0$ for all 1-dimensional closed vertical integral subschemes C on X), then $\widehat{\operatorname{vol}}(\overline{D}) \geq \widehat{\deg}(\overline{D}^d)$.*

Proof. Let us begin with the following claim:

Claim 6.6.1.1. *We set $\overline{D} = (D, g)$. If $\overline{D}$ is of C^{∞}-type, D is ample (that is, there are $a_1, \ldots, a_l \in \mathbb{R}_{>0}$ and ample Cartier divisors $A_1, \ldots, A_l$ such that $D = a_1A_1 + \cdots + a_lA_l$) and $dd^c([g]) + \delta_D$ is positive, then the assertion of the theorem holds.*

Proof. By virtue of Proposition 2.4.2, we can find F_∞-invariant locally integrable functions $h_1, \ldots, h_l$ such that h_i is an A_i-Green function h_i of C^∞-type for each i and $g = a_1h_1 + \cdots + a_lh_l$ (a.e.). Let $\delta_1, \ldots, \delta_l$ be sufficiently small positive real numbers such that $a_1 + \delta_1, \ldots, a_l + \delta_l \in \mathbb{Q}$. We set

$$(D', g') = (a_1 + \delta_1)(A_1, h_1) + \cdots + (a_l + \delta_l)(A_l, h_l).$$

Then D' is an ample $\mathbb{Q}$-Cartier divisor and

$$dd^c([g']) + \delta_{D'} = dd^c([g]) + \delta_D + \sum_{i=1}^{l} \delta_i(dd^c([h_i]) + \delta_{A_i}).$$

is positive because $\delta_1, \ldots, \delta_l$ are sufficiently small. Therefore, by [14, Corollary 6.4], we have $\widehat{\mathrm{vol}}(\overline{D}') \geq \widehat{\deg}(\overline{D}'^d)$, which implies the claim by using the continuity of $\widehat{\mathrm{vol}}$ (cf. Theorem 5.2.2). □

First we assume that $\overline{D}$ is of C^∞-type. Let $\overline{A} = (A, h)$ be an arithmetic Cartier divisor of C^∞-type such that A is ample and $dd^c([h]) + \delta_A$ is positive. Then, by using the same idea as in the proofs of Proposition 6.2.1 and Proposition 6.2.2, we can see that $D + \epsilon A$ is ample for all $\epsilon > 0$, Thus, by the above claim, $\widehat{\mathrm{vol}}(\overline{D} + \epsilon(A, h)) \geq \widehat{\deg}((\overline{D} + \epsilon(A, h))^d)$, and hence the assertion follows by taking $\epsilon \to 0$.

Finally we consider a general case. By Claim 6.4.2.1, there is an ample arithmetic Cartier divisor $\overline{B}$ such that $\overline{A} := \overline{D} + \overline{B} \in \widehat{\mathrm{Nef}}_{C^0}(X)_{\mathbb{R}}$ and A is ample. Let ϵ be an arbitrary positive number. Then, by virtue of Theorem 4.6, we can find an F_∞-invariant continuous function u on $X(\mathbb{C})$ such that $0 \leq u(x) \leq \epsilon$ for all $x \in X(\mathbb{C})$ and $\overline{A}' := \overline{A} + (0, u) \in \widehat{\mathrm{Div}}_{C^\infty \cap \mathrm{PSH}}(X)_{\mathbb{R}}$, which means that $\overline{A}' \in \widehat{\mathrm{Nef}}_{C^\infty}(X)_{\mathbb{R}}$. Note that

$$\begin{cases} \widehat{\deg}(\overline{D}^d) = \sum_{i=0}^{d} (-1)^{d-i}\binom{d}{i}\widehat{\deg}(\overline{A}^i \cdot \overline{B}^{d-i}), \\ \widehat{\deg}(\overline{D}'^d) = \sum_{i=0}^{d} (-1)^{d-i}\binom{d}{i}\widehat{\deg}(\overline{A}'^i \cdot \overline{B}^{d-i}), \end{cases}$$

where $\overline{D}' := \overline{D} + (0, u)$. By (6.4.2.3), $\widehat{\deg}(\overline{A}^i \cdot \overline{B}^{d-i})$ and $\widehat{\deg}(\overline{A}'^i \cdot \overline{B}^{d-i})$ are given by an alternative sum of volumes, so that, by the continuity of $\widehat{\mathrm{vol}}$, there is a constant C such that C does not depend on ϵ and that

$$\left| \widehat{\deg}(\overline{A}'^i \cdot \overline{B}^{d-i}) - \widehat{\deg}(\overline{A}^i \cdot \overline{B}^{d-i}) \right| \leq C\epsilon$$

for all $i = 0, \ldots, d$, and hence

$$\left| \widehat{\deg}(\overline{D}'^d) - \widehat{\deg}(\overline{D}^d) \right| \leq 2^d C\epsilon.$$

On the other hand, by the continuity of $\widehat{\mathrm{vol}}$ again, there is a constant C' such that C' does not depend on ϵ and that

$$\left|\widehat{\mathrm{vol}}(\overline{D}') - \widehat{\mathrm{vol}}(\overline{D})\right| \leq C'\epsilon.$$

Therefore, by using the previous case,

$$\begin{aligned}\widehat{\mathrm{vol}}(\overline{D}) - \widehat{\deg}(\overline{D}^d) &\geq \left(\widehat{\mathrm{vol}}(\overline{D}') - C'\epsilon\right) - \left(\widehat{\deg}(\overline{D}'^d) + 2^d C\epsilon\right)\\ &= \left(\widehat{\mathrm{vol}}(\overline{D}') - \widehat{\deg}(\overline{D}'^d)\right) - (C' + 2^d C)\epsilon \geq -(C' + 2^d C)\epsilon.\end{aligned}$$

Thus the theorem follows because ϵ is an arbitrary positive number. □

7. Limit of nef arithmetic $\mathbb{R}$-Cartier divisors on arithmetic surfaces

Let X be a regular projective arithmetic surface and let $\mathscr{T}$ be a type for Green functions on X such that PSH is a subjacent type of $\mathscr{T}$. The purpose of this section is to prove the following theorem.

Theorem 7.1. *Let $\{\overline{M}_n = (M_n, h_n)\}_{n=0}^{\infty}$ be a sequence of nef arithmetic $\mathbb{R}$-Cartier divisors on X with the following properties:*

(a) *There is an arithmetic Cartier divisor $\overline{D} = (D, g)$ of $\mathscr{T}$-type such that g is of upper bounded type and that $\overline{M}_n \leq \overline{D}$ for all $n \geq 1$.*
(b) *There is a proper closed subset E of X such that* $\mathrm{Supp}(D) \subseteq E$ *and* $\mathrm{Supp}(M_n) \subseteq E$ *for all $n \geq 1$.*
(c) $\lim_{n\to\infty} \mathrm{mult}_C(M_n)$ *exists for all 1-dimensional closed integral subschemes C on X.*
(d) $\limsup_{n\to\infty} (h_n)_{\mathrm{can}}(x)$ *exists in $\mathbb{R}$ for all $x \in X(\mathbb{C}) \setminus E(\mathbb{C})$.*

Then there is a nef arithmetic $\mathbb{R}$-Cartier divisor $\overline{M} = (M, h)$ on X such that $\overline{M} \leq \overline{D}$,

$$M = \sum_C \left(\lim_{n\to\infty} \mathrm{mult}_C(M_n)\right) C$$

and that $h_{\mathrm{can}}|_{X(\mathbb{C})\setminus E(\mathbb{C})}$ is the upper semicontinuous regularization of the function given by $x \mapsto \limsup_{n\to\infty}(h_n)_{\mathrm{can}}(x)$ over $X(\mathbb{C}) \setminus E(\mathbb{C})$. Moreover,

$$\limsup_{n\to\infty} \widehat{\deg}\left(\overline{M}_n\big|_C\right) \leq \widehat{\deg}\left(\overline{M}\big|_C\right)$$

holds for all 1-dimensional closed integral subschemes C on X.

Proof. Let $C_1, \ldots, C_l$ be 1-dimensional irreducible components of E. Then there are $a_1, \ldots, a_l, a_{n1}, \ldots, a_{nl} \in \mathbb{R}$ such that

$$D = a_1 C_1 + \cdots + a_l C_l \quad \text{and} \quad M_n = a_{n1} C_1 + \cdots + a_{nl} C_l.$$

We set $p_i = \lim_{n\to\infty} a_{ni}$ for $i = 1, \ldots, l$ and $M = p_1 C_1 + \cdots + p_l C_l$.

Let U be a Zariski open set of X over which we can find local equations $\phi_1, \ldots, \phi_l$ of $C_1, \ldots, C_l$ respectively. Let

$$h_n = u_n + \sum_{i=1}^{l} (-a_{ni}) \log |\phi_i|^2 \text{ (a.e.)} \quad \text{and} \quad g = v + \sum_{i=1}^{l} (-a_i) \log |\phi_i|^2 \text{ (a.e.)}$$

be the local expressions of h_n and g with respect to $\phi_1, \ldots, \phi_l$, where $u_n \in \mathrm{PSH}_{\mathbb{R}}$ and v is locally bounded above.

Claim 7.1.1. *$\{u_n\}_{n=0}^{\infty}$ is locally uniformly bounded above, that is, for each point $x \in U(\mathbb{C})$, there are an open neighborhood V_x of x and a constant M_x such that $u_n(y) \le M_x$ for all $y \in V_x$ and $n \ge 0$.*

Proof. Since $h_n \le g$ (a.e.), we have

$$u_n \le v - \sum_{i=1}^{n} (a_i - a_{ni}) \log |\phi_i|^2 \text{ (a.e.)}$$

over $U(\mathbb{C})$. If $x \notin C_1(\mathbb{C}) \cup \cdots \cup C_l(\mathbb{C})$, then $\phi_i(x) \neq 0$ for all i. Thus, as

$$v - \sum_{i=1}^{n} (a_i - a_{ni}) \log |\phi_i|^2$$

is locally bounded above, the assertion follows from Lemma 2.3.1.

Next we assume that $x \in C_1(\mathbb{C}) \cup \cdots \cup C_l(\mathbb{C})$. Clearly we may assume $x \in C_1(\mathbb{C})$. Note that $C_i(\mathbb{C}) \cap C_j(\mathbb{C}) = \emptyset$ for $i \neq j$. Thus $\phi_1(x) = 0$ and $\phi_i(x) \neq 0$ for all $i \ge 2$. Therefore, we can find an open neighborhood V_x of x and a constant M'_x such that $|\phi_1| < 1$ on V_x and

$$u_n \le M'_x - (a_1 - a_{n1}) \log |\phi_1|^2 \quad \text{(a.e.)}$$

over V_x for all $n \ge 1$. Moreover, we can also find a positive constant M'' such that $a_1 - a_{n1} \le M''$ for all $n \ge 1$, so that

$$u_n \le M'_x - M'' \log |\phi_1|^2 \quad \text{(a.e.)}$$

holds over V_x. Thus the claim follows from Lemma 4.1. □

We set $u(x) := \limsup_{n\to\infty} u_n(x)$ for $x \in U(\mathbb{C})$. Note that $u(x) \in \{-\infty\} \cup \mathbb{R}$. Let $\tilde{u}$ be the upper semicontinuous regularization of u. Then, as u_n is plurisubharmonic for all $n \ge 1$, by the above claim, $\tilde{u}$ is also plurisubharmonic on $U(\mathbb{C})$ (cf. Subsection 2.1).

Claim 7.1.2. *$\tilde{u}(x) \neq -\infty$ for all $x \in U(\mathbb{C})$.*

Proof. If $x \notin C_1(\mathbb{C}) \cup \cdots \cup C_l(\mathbb{C}) = E(\mathbb{C})$, then $\phi_i(x) \neq 0$ for all i. Note that $\limsup_{n\to\infty} (h_n)_{\mathrm{can}}(x)$ exists in $\mathbb{R}$ and that

$$(h_n)_{\mathrm{can}}(x) = u_n(x) + \sum_{i=1}^{l} (-a_{ni}) \log |\phi_i(x)|^2.$$

Thus $\limsup_{n\to\infty} u_n(x)$ exists in $\mathbb{R}$ and

$$\limsup_{n\to\infty} u_n(x) = \limsup_{n\to\infty} (h_n)_{\mathrm{can}}(x) + \sum_{i=1}^{l} p_i \log |\phi_i(x)|^2.$$

Hence the assertion follows in this case.

Next we assume that $x \in C_1(\mathbb{C}) \cup \cdots \cup C_l(\mathbb{C})$. We may assume $x \in C_1(\mathbb{C})$. As before, $\phi_1(x) = 0$ and $\phi_i(x) \neq 0$ for $i \ge 2$. By using Lemma 5.2.3, let us choose a rational function ψ and effective Cartier divisors A and B such that $C_1 + (\psi) = A - B$ and $C_1 \not\subseteq \operatorname{Supp}(A) \cup \operatorname{Supp}(B)$. We set

$$M'_n = M_n + a_{n1}(\psi), \quad h'_n = h_n - a_{n1} \log |\psi|^2 \quad \text{and} \quad \overline{M}'_n = (M'_n, h'_n).$$

Then $\overline{M}'_n = \overline{M}_n + a_{n1}\widehat{(\psi)}$ and

$$
\begin{aligned}
0 \le \widehat{\deg}(\overline{M}_n\big|_{C_1}) = \widehat{\deg}(\overline{M}'_n\big|_{C_1}) \\
&= a_{n1}\left(\log \#(\mathscr{O}_{C_1}(A)/\mathscr{O}_{C_1}) - \log \#(\mathscr{O}_{C_1}(B)/\mathscr{O}_{C_1})\right) \\
&\qquad + \sum_{i=2}^{l} a_{ni} \log \#(\mathscr{O}_{C_1}(C_i)/\mathscr{O}_{C_1}) + \frac{1}{2}\sum_{y \in C_1(\mathbb{C})} (h'_n)_{\mathrm{can}}(y).
\end{aligned}
$$

Thus we can find a constant T such that

$$\sum_{y \in C_1(\mathbb{C})} (h'_n)_{\mathrm{can}}(y) \ge T$$

for all $n \ge 1$, which yields

$$\sum_{y \in C_1(\mathbb{C})} \limsup_{n\to\infty} (h'_n)_{\mathrm{can}}(y) \ge \limsup_{n\to\infty} \left(\sum_{y \in C_1(\mathbb{C})} (h'_n)_{\mathrm{can}}(y) \right) \ge T.$$

In particular, $\limsup_{n\to\infty} (h'_n)_{\mathrm{can}}(x) \neq -\infty$. On the other hand,

$$h'_n = u_n - a_{n1} \log |\phi_1 \psi|^2 - \sum_{i=2}^{l} a_{ni} \log |\phi_i|^2 \quad \text{(a.e.)}.$$

Note that $(\phi_1\psi)(x) \in \mathbb{C}^\times$. Thus

$$\limsup_{n\to\infty} (u_n(x)) = \limsup_{n\to\infty} (h'_n)_{\mathrm{can}}(x) + p_1 \log |(\phi_1\psi)(x)|^2 + \sum_{i=2}^{l} p_i \log |\phi_i(x)|^2.$$

Therefore we have the assertion of the claim in this case. □

Claim 7.1.3. *$\tilde{u} + \sum_{i=1}^{l}(-p_i)\log|\phi_i|^2$ does not depend on the choice of $\phi_1, \ldots, \phi_l$.*

Proof. Let $\phi'_1, \ldots, \phi'_l$ be another local equations of $C_1, \ldots, C_l$. Then there are $e_1, \ldots, e_l \in \mathscr{O}_U^\times(U)$ such that $\phi'_i = e_i\phi_i$ for all i. Let $g_n = u'_n - \sum_{i=1}^{l} a_{ni} \log|\phi'_i|^2$ (a.e.) be the local expression of g_n with respect to $\phi'_1, \ldots, \phi'_l$. Then $u'_n = u_n + \sum_{i=1}^{l} a_{ni} \log|e_i|^2$ by Lemma 2.3.1. Thus

$$\tilde{u}' = \tilde{u} + \sum_{i=1}^{l} p_i \log |e_i|^2,$$

which implies that

$$\tilde{u} + \sum_{i=1}^{l} (-p_i) \log |\phi_i|^2 = \tilde{u}' + \sum_{i=1}^{l} (-p_i) \log |\phi'_i|^2.$$

□

By the above claim, there is an M-Green function h of $\mathrm{PSH}_{\mathbb{R}}$-type on $X(\mathbb{C})$ such that

$$h|_{U(\mathbb{C})} = \tilde{u} + \sum_{i=1}^{l} (-p_i) \log |\phi_i|^2.$$

By our construction, $h_{\mathrm{can}}|_{X(\mathbb{C}) \setminus E(\mathbb{C})}$ is the upper semicontinuous regularization of the function given by $h^\sharp(x) = \limsup_{n\to\infty} (h_n)_{\mathrm{can}}(x)$ over $X(\mathbb{C}) \setminus E(\mathbb{C})$.

Claim 7.1.4. *h is F_∞-invariant and $h \leq g$* (a.e.).

Proof. As PSH is a subjacent type of $\mathscr{T}$, we have $(h_n)_{\mathrm{can}} \leq g_{\mathrm{can}}$ over $X(\mathbb{C}) \setminus E(\mathbb{C})$, so that $h^\sharp \leq g_{\mathrm{can}}$ over $X(\mathbb{C}) \setminus E(\mathbb{C})$. Note that $h^\sharp = h$ (a.e.) (cf. Subsection 2.1). Thus the claim follows because $h^\sharp$ is F_∞-invariant. □

Finally let us check that

$$\widehat{\deg}\left(\overline{M}\big|_C\right) \geq \limsup_{n\to\infty} \widehat{\deg}\left(\overline{M}_n\big|_C\right) \geq 0$$

holds for all 1-dimensional closed integral subschemes C on X.

By Lemma 5.2.3 again, we can choose non-zero rational functions $\psi_1, \ldots, \psi_l$ on X and effective Cartier divisors

$$A_1, \ldots, A_l, B_1, \ldots, B_l$$

such that $C_i + (\psi_i) = A_i - B_i$ for all i and $C \not\subseteq \mathrm{Supp}(A_i) \cup \mathrm{Supp}(B_i)$ for all i. We set

$$\begin{cases} M''_n = M_n + \sum_{i=1}^l a_{ni}(\psi_i), & h''_n = h_n + \sum_{i=1}^l (-a_{ni}) \log|\psi_i|^2, \quad \overline{M}''_n = (M''_n, h''_n) \\ M'' = M + \sum_{i=1}^l p_i(\psi_i), & h'' = h + \sum_{i=1}^l (-p_i) \log|\psi_i|^2, \quad \overline{M}'' = (M'', h'') \end{cases}$$

First we assume that C is not flat over $\mathbb{Z}$. Then

$$\widehat{\deg}(\overline{M}_n\big|_C) = \widehat{\deg}(\overline{M}''_n\big|_C) = \sum_{i=1}^l a_{in} \left(\log \#(\mathscr{O}_C(A_i)/\mathscr{O}_C) - \log \#(\mathscr{O}_C(B_i)/\mathscr{O}_C)\right)$$

and

$$\widehat{\deg}(\overline{M}\big|_C) = \widehat{\deg}(\overline{M}''\big|_C) = \sum_{i=1}^l p_i \left(\log \#(\mathscr{O}_C(A_i)/\mathscr{O}_C) - \log \#(\mathscr{O}_C(B_i)/\mathscr{O}_C)\right).$$

Thus

$$\widehat{\deg}(\overline{M}\big|_C) = \lim_{n\to\infty} \widehat{\deg}(\overline{M}_n\big|_C) \geq 0$$

Next we assume that C is flat over $\mathbb{Z}$. Then

$$\begin{aligned} \widehat{\deg}(\overline{M}_n\big|_C) &= \widehat{\deg}(\overline{M}''_n\big|_C) \\ &= \sum_{i=1}^l a_{in} \left(\log \#(\mathscr{O}_C(A_i)/\mathscr{O}_C) - \log \#(\mathscr{O}_C(B_i)/\mathscr{O}_C)\right) + \frac{1}{2} \sum_{y \in C(\mathbb{C})} (h''_n)_{\mathrm{can}}(y) \end{aligned}$$

and

$$\begin{aligned} \widehat{\deg}(\overline{M}\big|_C) &= \widehat{\deg}(\overline{M}''\big|_C) \\ &= \sum_{i=1}^l p_i \left(\log \#(\mathscr{O}_C(A_i)/\mathscr{O}_C) - \log \#(\mathscr{O}_C(B_i)/\mathscr{O}_C)\right) + \frac{1}{2} \sum_{y \in C(\mathbb{C})} (h'')_{\mathrm{can}}(y). \end{aligned}$$

Let us consider a Zariski open set U of X with $C \cap U \neq \emptyset$. Let

$$h_n = u_n + \sum (-a_{ni}) \log|\phi_i|^2 \text{ (a.e.)} \quad \text{and} \quad h = \tilde{u} + \sum (-p_i) \log|\phi_i|^2 \text{ (a.e.)}$$

be the local expressions of h_n and h as before. Then

$$h''_n = u_n + \sum (-a_{ni}) \log|\phi_i \psi_i|^2 \text{ (a.e.)} \quad \text{and} \quad h'' = \tilde{u} + \sum (-p_i) \log|\phi_i \psi_i|^2 \text{ (a.e.)}.$$

Moreover, $(\phi_i\psi_i)(y) \in \mathbb{C}^\times$ for all $y \in C(\mathbb{C})$ and i. Thus

$$\limsup_{n\to\infty}(h''_n)_{\mathrm{can}}(y) \leq (h'')_{\mathrm{can}}(y).$$

Therefore,

$$\limsup_{n\to\infty}\sum_{y\in C(\mathbb{C})}(h''_n)_{\mathrm{can}}(y) \leq \sum_{y\in C(\mathbb{C})}\limsup_{n\to\infty}(h''_n)_{\mathrm{can}}(y) \leq \sum_{y\in C(\mathbb{C})}(h'')_{\mathrm{can}}(y),$$

which yields

$$0 \leq \limsup_{n\to\infty}\widehat{\deg}(\overline{M}_n\big|_C) \leq \widehat{\deg}(\overline{M}\big|_C).$$

□

8. σ-DECOMPOSITIONS ON ARITHMETIC SURFACES

In this section, we consider a σ-decomposition of an effective arithmetic $\mathbb{R}$-Cartier divisor of C^0-type. It is necessary to see the property (1) of Theorem 9.3.5.

Let X be a regular projective arithmetic surface. We fix an F_∞-invariant continuous volume form Φ on $X(\mathbb{C})$ with $\int_{X(\mathbb{C})}\Phi = 1$. Let $\overline{D} = (D, g)$ be an effective arithmetic $\mathbb{R}$-Cartier divisor of C^0-type on X. For a 1-dimensional closed integral subscheme C on X, we set

$$\nu_C(\overline{D}) := \min\left\{\mathrm{mult}_C(D + (\phi)) \mid \phi \in \hat{H}^0(X, \overline{D}) \setminus \{0\}\right\}$$

as in Subsection 6.5. Moreover, we set

$$F(\overline{D}) = \mathrm{Fx}(\overline{D}) = \sum_C \nu_C(\overline{D})C \quad \text{and} \quad M(\overline{D}) = \mathrm{Mv}(\overline{D}) = D - \mathrm{Fx}(\overline{D}).$$

Let $V(\overline{D})$ be the complex vector space generated by $\hat{H}^0(X, \overline{D})$ in $H^0(X, D) \otimes_{\mathbb{Z}} \mathbb{C}$, that is, $V(\overline{D}) := \langle \hat{H}^0(X, \overline{D})\rangle_{\mathbb{C}}$.

Lemma 8.1. $\mathrm{dist}(V(\overline{D}); g)$ *is* F_∞*-invariant.*

Proof. First of all, note that, for $\phi \in \mathrm{Rat}(X)$, $F_\infty^*(\phi) = \overline{\phi}$ as a function on $X(\mathbb{C})$. Let us see $\langle\phi, \psi\rangle_g \in \mathbb{R}$ for all $\phi, \psi \in \langle \hat{H}^0(X, \overline{D})\rangle_{\mathbb{R}}$. Indeed,

$$\begin{aligned}\langle\phi, \psi\rangle_g &= \int_{X(\mathbb{C})}\phi\bar{\psi}\exp(-g)\Phi = -\int_{X(\mathbb{C})}F_\infty^*(\phi\bar{\psi}\exp(-g)\Phi)\\ &= -\int_{X(\mathbb{C})}F_\infty^*(\phi)F_\infty^*(\bar{\psi})F_\infty^*(\exp(-g))F_\infty^*(\Phi)\\ &= \int_{X(\mathbb{C})}\bar{\phi}\psi\exp(-g)\Phi = \langle\psi, \phi\rangle_g = \overline{\langle\phi, \psi\rangle_g}.\end{aligned}$$

Thus $\langle\phi, \psi\rangle_g$ yields an inner product of $\langle \hat{H}^0(X, \overline{D})\rangle_{\mathbb{R}}$, so that let $\phi_1, \ldots, \phi_N$ be an orthonormal basis of $\langle \hat{H}^0(X, \overline{D})\rangle_{\mathbb{R}}$ over $\mathbb{R}$. These give rise to an orthonormal basis of $\langle \hat{H}^0(X, \overline{D})\rangle_{\mathbb{C}}$. Therefore,

$$\mathrm{dist}(V(\overline{D}); g) = |\phi_1|_g^2 + \cdots + |\phi_N|_g^2.$$

Note that $F_\infty^*(|\phi_i|_g) = |\bar{\phi}_i|_g = |\phi_i|_g$, and hence the lemma follows. □

Here we define $g_{F(\overline{D})}$, $g_{M(\overline{D})}$, $\overline{M}(\overline{D})$ and $\overline{F}(\overline{D})$ as follows:

$$\begin{cases} g_{F(\overline{D})} = -\log \operatorname{dist}\left(V(\overline{D}); g\right), & g_{M(\overline{D})} = g - g_{F(\overline{D})} = g + \log \operatorname{dist}\left(V(\overline{D}); g\right), \\ \overline{M}(\overline{D}) = \left(M(\overline{D}), g_{M(\overline{D})}\right), & \overline{F}(\overline{D}) = \left(F(\overline{D}), g_{F(\overline{D})}\right). \end{cases}$$

Let us check the following proposition:

Proposition 8.2. (1) $\hat{H}^0(X, \overline{D}) \subseteq \hat{H}^0(X, \overline{M}(\overline{D}))$.

(2) $g_{M(\overline{D})}$ *is an* $M(\overline{D})$*-Green function of* $(C^\infty \cap \mathrm{PSH})$*-type on* $X(\mathbb{C})$.

(3) $g_{F(\overline{D})}$ *is an* $F(\overline{D})$*-Green function of* $(C^0 - C^\infty \cap \mathrm{PSH})$*-type over* $X(\mathbb{C})$.

(4) $\overline{M}(\overline{D})$ *is nef.*

Proof. (1) If $\phi \in \hat{H}^0(X, \overline{D}) \setminus \{0\}$, then $(\phi) + D \geq F(\overline{D})$, and hence $(\phi) + M(\overline{D}) \geq 0$. Note that $|\phi|_g^2 = \operatorname{dist}(V(\overline{D}); g)|\phi|^2_{g_{M(\overline{D})}}$ for $\phi \in \hat{H}^0(X, M(\overline{D}))$. Thus, as $\|\phi\|_g \leq 1$, by Proposition 3.2.1,

$$|\phi|^2_{g_{M(\overline{D})}} = |\phi|_g^2 / \operatorname{dist}(V(\overline{D}); g) \leq \|\phi\|_g^2 \leq 1.$$

Therefore, $\phi \in \hat{H}^0(X, \overline{M}(\overline{D}))$.

(2), (3) Let us fix $x \in X(\mathbb{C})$. We set

$$\nu_x := \min\{\operatorname{mult}_x(D + (\phi)) \mid \phi \in V(\overline{D}) \setminus \{0\}\}.$$

Note that $\operatorname{mult}_x(D + (\phi)) = \operatorname{mult}_x(D) + \operatorname{ord}_x(\phi)$. First let us see the following claim:

Claim 8.2.1. (a) *If* $\phi_1, \ldots, \phi_n \in V(\overline{D}) \setminus \{0\}$ *and* $V(\overline{D})$ *is generated by* $\phi_1, \ldots, \phi_n$, *then* $\nu_x = \min\{\operatorname{mult}_x(D + (\phi_1)), \ldots, \operatorname{mult}_x(D + (\phi_n))\}$.

(b) $\nu_x = \operatorname{mult}_x(F(\overline{D}))$.

Proof. (a) is obvious. Let us consider the natural homomorphism

$$\langle \hat{H}^0(X, \overline{D}) \rangle_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathscr{O}_X \to \mathscr{O}_X(\lfloor D \rfloor),$$

which is surjective on $X \setminus \operatorname{Supp}(D)$ because $0 \leq \lfloor D \rfloor \leq D$. In particular,

$$V(\overline{D}) \otimes_{\mathbb{C}} \mathscr{O}_{X(\mathbb{C})} \to \mathscr{O}_{X(\mathbb{C})}(\lfloor D \rfloor),$$

is surjective on $X(\mathbb{C}) \setminus \operatorname{Supp}(D)(\mathbb{C})$, so that if $x \in X(\mathbb{C}) \setminus \operatorname{Supp}(D)(\mathbb{C})$, then $\nu_x = 0$. On the other hand, if $x \in X(\mathbb{C}) \setminus \operatorname{Supp}(D)(\mathbb{C})$, then $\operatorname{mult}_x(F(\overline{D})) = 0$ because $0 \leq F(\overline{D}) \leq D$. Therefore, we may assume that $x \in \operatorname{Supp}(D)(\mathbb{C})$, so that there is a 1-dimensional closed integral subscheme C of X with $x \in C(\mathbb{C})$. Let $\psi_1, \ldots, \psi_n$ be all elements of $\hat{H}^0(X, \overline{D}) \setminus \{0\}$. Let η be the generic point of C. Then

$$\operatorname{mult}_C(F(\overline{D})) = \min\{\operatorname{mult}_C(D) + \operatorname{ord}_\eta(\psi_1), \ldots, \operatorname{mult}_C(D) + \operatorname{ord}_\eta(\psi_n)\}.$$

Thus, by using (a),

$$\begin{aligned} \operatorname{mult}_x(F(\overline{D})) &= \operatorname{mult}_C(F(\overline{D})) \\ &= \min\{\operatorname{mult}_C(D) + \operatorname{ord}_\eta(\psi_1), \ldots, \operatorname{mult}_C(D) + \operatorname{ord}_\eta(\psi_n)\} \\ &= \min\{\operatorname{mult}_x(D + (\psi_1)), \ldots, \operatorname{mult}_x(D + (\psi_n))\} = \nu_x. \end{aligned}$$

□

Let $\phi_1, \ldots, \phi_N$ be an orthonormal basis of $V(\overline{D})$ with respect to $\langle\ ,\ \rangle_g$. Let $g = u_x + (-a)\log|z|^2$ (a.e.) be a local expression of g around x, where z is a local chart around x with $z(x) = 0$. For every i, we set $\phi_i = z^{a_i} v_i$ around x with $v_i \in \mathscr{O}^{\times}_{X(\mathbb{C}),x}$. Then $|\phi_i|^2_g = |z|^{2(a_i+a)} \exp(-u_x)|v_i|^2$. By the above claim,

$$\nu_x = \min\{a_1 + a, \ldots, a_N + a\} = \operatorname{mult}_x(F(\overline{D})).$$

Thus

$$\operatorname{dist}(V(\overline{D}); g) = |z|^{2\nu_x} \exp(-u_x) \sum_{i=1}^{N} |z|^{2(a_i+a-\nu_x)} |v_i|^2.$$

Therefore,

$$\begin{cases} g_{F(\overline{D})} = u_x - \log\left(\sum_{i=1}^N |z|^{2(a_i+a-\nu_x)}|v_i|^2\right) - \nu_x \log|z|^2, \\ g_{M(\overline{D})} = \log\left(\sum_{i=1}^N |z|^{2(a_i+a-\nu_x)}|v_i|^2\right) - (a - \nu_x)\log|z|^2. \end{cases}$$

Note that $\log\left(\sum_{i=1}^N |z|^{2(a_i+a-\nu_x)}|v_i|^2\right)$ is a subharmonic C^∞-function. Thus we get (2) and (3).

(4) For $\phi \in \hat{H}^0(X, \overline{D}) \setminus \{0\}$ and a 1-dimensional closed integral subscheme C on X, as

$$\operatorname{mult}_C(M(\overline{D}) + (\phi)) = \operatorname{mult}_C(D + (\phi)) - \nu_C(\overline{D}),$$

there is a $\psi \in \hat{H}^0(X, \overline{D}) \setminus \{0\}$ such that $\operatorname{mult}_C(M(\overline{D}) + (\psi)) = 0$. This means that

$$C \not\subseteq \operatorname{Supp}(M(\overline{D}) + (\psi)).$$

Then, by Proposition 3.2.1, $0 < |\psi|_{g_{M(\overline{D})}}(x) \leq 1$ for all $x \in C(\mathbb{C})$ as before. Hence

$$\widehat{\deg}\left(\overline{M}(\overline{D})\big|_C\right) = \log \#\mathscr{O}_C((\psi) + M(\overline{D}))/\mathscr{O}_C - \sum_{x \in C(\mathbb{C})} \log|\psi|_{g_{M(\overline{D})}}(x) \geq 0.$$

□

For $n \geq 1$, we set

$$\begin{cases} M_n(\overline{D}) := \dfrac{1}{n} M(n\overline{D}), & g_{M_n(\overline{D})} := \dfrac{1}{n} g_{M(n\overline{D})}, \\[2ex] F_n(\overline{D}) := \dfrac{1}{n} F(n\overline{D}), & g_{F_n(\overline{D})} := \dfrac{1}{n} g_{F(n\overline{D})}. \end{cases}$$

In addition,

$$\overline{M}_n(\overline{D}) := \left(M_n(\overline{D}), g_{M_n(\overline{D})}\right) \quad \text{and} \quad \overline{F}_n(\overline{D}) := \left(F_n(\overline{D}), g_{F_n(\overline{D})}\right).$$

Then we have the following proposition, which guarantees a decomposition

$$\overline{D} = \overline{M}_\infty(\overline{D}) + \overline{F}_\infty(\overline{D})$$

as described in the proposition. This decomposition $\overline{D} = \overline{M}_\infty(\overline{D}) + \overline{F}_\infty(\overline{D})$ is called the *σ-decomposition of $\overline{D}$*. Moreover, $\overline{M}_\infty(\overline{D})$ (resp. $\overline{F}_\infty(\overline{D})$) is called the *asymptotic movable part* (resp. the *asymptotic fixed part*) of $\overline{D}$. The σ-decomposition is an arithmetic analog of the σ-decomposition introduced by Nakayama [21].

Proposition 8.3. *There is a nef arithmetic $\mathbb{R}$-Cartier divisor $\overline{M}_\infty(\overline{D}) = \left(M_\infty(\overline{D}), g_{M_\infty(\overline{D})}\right)$ on X with the following properties:*

(1) $\mathrm{mult}_C(M_\infty(\overline{D})) = \lim_{n\to\infty} \mathrm{mult}_C(M_n(\overline{D}))$ *for all* 1*-dimensional closed integral subschemes* C *on* X.

(2) $\left(g_{M_\infty(\overline{D})}\right)_{\mathrm{can}}$ *is the upper semicontinuous regularization of the function given by*

$$x \mapsto \limsup_{n\to\infty} \left(g_{M_n(\overline{D})}\right)_{\mathrm{can}}(x)$$

over $X(\mathbb{C}) \setminus \mathrm{Supp}(D)(\mathbb{C})$. *In particular,*

$$\left(g_{M_\infty(\overline{D})}\right)_{\mathrm{can}}(x) = \limsup_{n\to\infty} \left(\left(g_{M_n(\overline{D})}\right)_{\mathrm{can}}(x)\right) \quad \text{(a.e.)}.$$

Moreover, if $\overline{D}$ *is of* C^∞*-type, then* $\lim_{n\to\infty}\left(\left(g_{M_n(\overline{D})}\right)_{\mathrm{can}}(x)\right)$ *exists.*

(3) $\widehat{\deg}\left(\overline{M}_\infty(\overline{D})\big|_C\right) \geq \limsup_{n\to\infty} \widehat{\deg}\left(\overline{M}_n(\overline{D})\big|_C\right)$ *holds for all* 1*-dimensional closed integral subschemes* C *on* X.

(4) *If we set* $F_\infty(\overline{D})$, $g_{F_\infty(\overline{D})}$ *and* $\overline{F}_\infty(\overline{D})$ *as follows:*

$$\begin{cases} F_\infty(\overline{D}) := D - M_\infty(\overline{D}), \\ g_{F_\infty(\overline{D})} := g - g_{M_\infty(\overline{D})}, \\ \overline{F}_\infty(\overline{D}) := \left(F_\infty(\overline{D}), g_{F_\infty(\overline{D})}\right) \ (= \overline{D} - \overline{M}_\infty(\overline{D})), \end{cases}$$

then $\mu_C(\overline{D}) = \mathrm{mult}_C(F_\infty(\overline{D}))$ *for all* 1*-dimensional closed integral subschemes* C *on* X *and* $\overline{F}_\infty(\overline{D})$ *is an effective arithmetic* $\mathbb{R}$*-Cartier divisor of* $(C^0 - \mathrm{PSH}_{\mathbb{R}})$*-type. In addition, if* $\overline{D}$ *is of* C^∞*-type, then there is a constant* e *such that*

$$n g_{F_\infty(\overline{D})} \leq g_{F(n\overline{D})} + 3\log(n+1) + e \quad \text{(a.e.)}$$

for all $n \geq 1$.

(5) *If* $\overline{D}$ *is of* C^∞*-type, then there is a constant* e' *such that*

$$\hat{h}^0(X, n\overline{M}_\infty(\overline{D})) \leq \hat{h}^0(X, n\overline{D}) \leq \hat{h}^0(X, n\overline{M}_\infty(\overline{D})) + e' n \log(n+1)$$

for all $n \geq 1$.

Proof. It is easy to see that

$$\mathrm{mult}_C(F((n+m)\overline{D})) \leq \mathrm{mult}_C(F(n\overline{D})) + \mathrm{mult}_C(F(m\overline{D}))$$

for all $n, m \geq 1$ and 1-dimensional closed integral subschemes C. Thus

$$\lim_{n\to\infty} \mathrm{mult}_C(F_n(\overline{D}))$$

exists and

$$\lim_{n\to\infty} \mathrm{mult}_C(F_n(\overline{D})) = \inf_{n\geq 1}\{\mathrm{mult}_C(F_n(\overline{D}))\}.$$

Therefore $\lim_{n\to\infty} \mathrm{mult}_C(M_n(\overline{D}))$ exists because $M_n(\overline{D}) = D - F_n(\overline{D})$. Note that $\mu_C(\overline{D}) = \lim_{n\to\infty} \mathrm{mult}_C(F_n(\overline{D}))$ as $\mathrm{mult}_C(F_n(\overline{D})) = \nu_C(\overline{D})/n$ (cf. Subsection 6.5).

Claim 8.3.1. *Let* h *be a* D*-Green function of* C^∞*-type. Then there is a positive constant* A *such that, for* $x \in X(\mathbb{C}) \setminus \mathrm{Supp}(D)(\mathbb{C})$,

$$\lim_{n\to\infty} \frac{\log\left(\mathrm{dist}(V(n\overline{D}); nh)(x)\right)}{n}$$

exists in $\mathbb{R}_{\leq 0}$ *and*

$$\lim_{n\to\infty} \frac{\log\left(\operatorname{dist}(V(n\overline{D}); nh)(x)\right) - \log(A(n+1)^3)}{n} = \sup_{n\geq 1}\left\{\frac{\log\left(\operatorname{dist}(V(n\overline{D}); nh)(x)\right) - \log(A(n+1)^3)}{n}\right\}.$$

Proof. First of all, note that $\bigoplus_{n=0}^{\infty} V(n\overline{D})$ is a graded subring of $\bigoplus_{n=0}^{\infty} H^0(X, nD)$. By Theorem 3.2.3, there is a positive constant A such that

$$\operatorname{dist}(V(n\overline{D}); nh) \leq A(n+1)^3$$

and

$$\frac{\operatorname{dist}(V(n\overline{D}); nh)}{A(n+1)^3} \cdot \frac{\operatorname{dist}(V(m\overline{D}); mh)}{A(m+1)^3} \leq \frac{\operatorname{dist}(V((n+m)\overline{D}); (n+m)h)}{A(n+m+1)^3}$$

for all $n, m \geq 1$. Moreover, $\operatorname{dist}(V(n\overline{D}); nh)(x) \neq 0$ for $x \in X(\mathbb{C}) \setminus \operatorname{Supp}(D)(\mathbb{C})$. Thus the claim follows. □

By using the Stone-Weierstrass theorem, for a positive number ϵ, we can find continuous functions u and v with the following properties:

$$\begin{cases} u \geq 0,\ \|u\|_{\sup} \leq \epsilon,\ h := g + u \text{ is of } C^\infty\text{-type}, \\ v \geq 0,\ \|v\|_{\sup} \leq \epsilon,\ h' := g - v \text{ is of } C^\infty\text{-type}. \end{cases}$$

By Lemma 3.2.2,

$$\exp(-n\epsilon)\operatorname{dist}(V(n\overline{D}); nh') \leq \operatorname{dist}(V(n\overline{D}); ng) \leq \exp(n\epsilon)\operatorname{dist}(V(n\overline{D}); nh).$$

Thus, by the above claim, for $x \in X(\mathbb{C}) \setminus \operatorname{Supp}(D)(\mathbb{C})$,

$$\limsup_{n\to\infty} \frac{\log\left(\operatorname{dist}(V(n\overline{D}); ng)(x)\right)}{n}$$

exists in $\{a \in \mathbb{R} \mid a \leq \epsilon\}$. Since ϵ is arbitrary positive number, we actually have

$$\limsup_{n\to\infty} \frac{\log\left(\operatorname{dist}(V(n\overline{D}); ng)(x)\right)}{n} \leq 0. \tag{8.3.2}$$

This observation shows that $\limsup_{n\to\infty}\left(g_{M_n(\overline{D})}\right)_{\mathrm{can}}(x)$ exists in $\mathbb{R}$ for $x \in X(\mathbb{C}) \setminus \operatorname{Supp}(D)(\mathbb{C})$. Therefore, by Theorem 7.1, there is a nef arithmetic $\mathbb{R}$-Cartier divisor

$$\overline{M}_\infty(\overline{D}) = \left(M_\infty(\overline{D}), g_{M_\infty(\overline{D})}\right)$$

satisfying (1), (2) and (3). Further the last assertion of (2) is a consequence of the above claim.

Let us see (4). Obviously $\mu_C(\overline{D}) = \operatorname{mult}_C(F_\infty(\overline{D}))$ because

$$\mu_C(\overline{D}) = \lim_{n\to\infty} \operatorname{mult}_C(F_n(\overline{D})).$$

Note that

$$\left(g_{F_\infty(\overline{D})}\right)_{\mathrm{can}}(x) = -\limsup_{n\to\infty} \frac{\log\left(\operatorname{dist}(V(n\overline{D}); ng)(x)\right)}{n} \quad \text{(a.e.)}.$$

on $X(\mathbb{C}) \setminus \mathrm{Supp}(D)(\mathbb{C})$. Thus (8.3.2) yields $\left(g_{F_\infty(\overline{D})}\right)_{\mathrm{can}}(x) \geq 0$ (a.e.). Hence $\overline{F}_\infty(\overline{D})$ is effective. Moreover, it is obvious that $g_{F_\infty(\overline{D})}$ is of $(C^0 - \mathrm{PSH}_{\mathbb{R}})$-type because g is of C^0-type and $g_{M_\infty(\overline{D})}$ is of $\mathrm{PSH}_{\mathbb{R}}$-type.

We assume that $\overline{D}$ is of C^∞-type. By the above claim, there is a positive constant A' such that

$$
\begin{aligned}
&-\lim_{n\to\infty} \frac{\log\left(\mathrm{dist}(V(n\overline{D}); ng)(x)\right)}{n} \\
&\qquad = \lim_{n\to\infty} \frac{-\log\left(\mathrm{dist}(V(n\overline{D}); ng)(x)\right) + \log(A'(n+1)^3)}{n} \\
&\qquad\qquad = \inf_{n\geq 1}\left\{\frac{-\log\left(\mathrm{dist}(V(n\overline{D}); ng)(x)\right) + \log(A'(n+1)^3)}{n}\right\}
\end{aligned}
$$

on $X(\mathbb{C}) \setminus \mathrm{Supp}(D)(\mathbb{C})$. Thus, for $n \geq 1$,

$$
g_{F_\infty(\overline{D})} \leq \frac{-\log\left(\mathrm{dist}(V(n\overline{D}); ng)(x)\right) + \log(A'(n+1)^3)}{n} \quad \text{(a.e.)}.
$$

which implies the last assertion of (4).

Finally let us check (5). By (4), we have $\overline{M}_\infty(\overline{D}) \leq \overline{D}$, so that

$$
\hat{h}^0(X, n\overline{M}_\infty(\overline{D})) \leq \hat{h}^0(X, n\overline{D})
$$

holds for $n \geq 1$. Moreover, by (4) again,

$$
n\overline{M}_\infty(\overline{D}) + (0, 3\log(n+1) + \log(A')) \geq \overline{M}(n\overline{D})
$$

for all $n \geq 1$. Thus, by using (1) in Proposition 8.2,

$$
\hat{h}^0\left(X, n\overline{D}\right) \leq \hat{h}^0\left(X, \overline{M}(n\overline{D})\right) \leq \hat{h}^0\left(X, n\overline{M}_\infty(\overline{D}) + (0, 3\log(n+1) + \log(A'))\right).
$$

Note that there is a positive constant e' such that

$$
\hat{h}^0\left(X, n\overline{M}_\infty(\overline{D}) + (0, 3\log(n+1) + \log(A'))\right) \leq \hat{h}^0(X, n\overline{M}_\infty(\overline{D})) + e' n \log(n+1)
$$

for all $n \geq 1$ (cf. [14, (3) in Proposition 2,1] and [16, Lemma 1.2.2]). Thus (5) follows. □

9. Zariski decompositions and their properties on arithmetic surfaces

Throughout this section, let X be a regular projective arithmetic surface and let $\mathscr{T}$ be a type for Green functions on X. We always assume that PSH is a subjacent type of $\mathscr{T}$.

9.1. Preliminaries. In this subsection, we prepare several lemmas for the proof of Theorem 9.2.1.

Lemma 9.1.1. *We assume that $\mathscr{T}$ is either C^0 or $\mathrm{PSH}_{\mathbb{R}}$. Let M be a 1-equidimensional complex manifold and let $D_1, \ldots, D_n$ be $\mathbb{R}$-Cartier divisors on M. Let $g_1, \ldots, g_n$ be locally integrable functions on M such that g_i is a D_i-Green functions of $\mathscr{T}$-type for each i. We set*

$$
g(x) = \max\{g_1(x), \ldots, g_n(x)\} \quad (x \in M)
$$

and

$$D = \sum_{x \in M} \max\{\mathrm{mult}_x(D_1), \ldots, \mathrm{mult}_x(D_n)\} x.$$

Then g is a D-Green function of $\mathscr{T}$-type.

Proof. For $x \in M$, let z be a local chart of an open neighborhood U_x of x with $z(x) = 0$, and let

$$g_1 = u_1 - a_1 \log |z|^2 \text{ (a.e.)}, \ \ldots, \ g_n = u_n - a_n \log |z|^2 \text{ (a.e.)}$$

be local expressions of $g_1, \ldots, g_n$ respectively over U_x, where $a_i = \mathrm{mult}_x(D_i)$ and $u_i \in \mathscr{T}(U_x)$ for $i = 1, \ldots, n$. Clearly we may assume that $a_1 = \max\{a_1, \ldots, a_n\}$. First of all, we have

$$g = \max\{u_i + (a_1 - a_i) \log |z|^2 \mid i = 1, \ldots, n\} - a_1 \log |z|^2 \quad \text{(a.e.)}$$

over U_x. In addition, the value of

$$u := \max\{u_i + (a_1 - a_i) \log |z|^2 \mid i = 1, \ldots, n\}$$

at $y \in U_x$ is finite

First we consider the case where $\mathscr{T} = \mathrm{PSH}_{\mathbb{R}}$. Then $u_1, \ldots, u_n$ are subharmonic over U_x, so that $u_i + (a_1 - a_i) \log |z|^2$ is also subharmonic over U_x for every i. Therefore, u is subharmonic over U_x.

Next let us see the case where $\mathscr{T} = C^0$. We set $I = \{i \mid a_i = a_1\}$. Then, shrinking U_x if necessarily, we may assume that $u_1 > u_j + (a_1 - a_j) \log |z|^2$ on U_x for all $j \notin I$. Thus $u = \max\{u_i \mid i \in I\}$, and hence u is continuous. □

Lemma 9.1.2. *We assume that $\mathscr{T}$ is either C^0 or $\mathrm{PSH}_{\mathbb{R}}$. Let*

$$\overline{D}_1 = (D_1, g_1), \ldots, \overline{D}_n = (D_n, g_n)$$

be arithmetic $\mathbb{R}$-Cartier divisors of $\mathscr{T}$-type on X. We set

$$\begin{cases} \max\{D_1, \ldots, D_n\} := \sum_C \max\{\mathrm{mult}_C(D_1), \ldots, \mathrm{mult}_C(D_n)\} C, \\ \max\{\overline{D}_1, \ldots, \overline{D}_n\} := (\max\{D_1, \ldots, D_n\}, \max\{g_1, \ldots, g_n\}). \end{cases}$$

Then we have the following:

(1) $\max\{\overline{D}_1, \ldots, \overline{D}_n\}$ *is an arithmetic $\mathbb{R}$-Cartier divisor of $\mathscr{T}$-type for D.*
(2) *If $\mathscr{T} = \mathrm{PSH}_{\mathbb{R}}$ and $\overline{D}_1, \ldots, \overline{D}_n$ are nef, then* $\max\{\overline{D}_1, \ldots, \overline{D}_n\}$ *is nef.*

Proof. (1) It is obvious that $\max\{g_1, \ldots, g_n\}$ is F_∞-invariant, so that (1) follows from Lemma 9.1.1.

(2) For simplicity, we set $D = \max\{D_1, \ldots, D_n\}$, $g = \max\{g_1, \ldots, g_n\}$ and $\overline{D} = \max\{\overline{D}_1, \ldots, \overline{D}_n\}$. Let C be a 1-dimensional closed integral subscheme of X. Let γ be the generic point of C. Since the codimension of

$$\mathrm{Supp}(D - D_1) \cap \cdots \cap \mathrm{Supp}(D - D_n)$$

is greater than or equal to 2, there is i such that $\gamma \notin \mathrm{Supp}(D - D_i)$. By Proposition 2.3.4, $g - g_i$ is a $(D - D_i)$-Green function of $(\mathrm{PSH}_{\mathbb{R}} - \mathrm{PSH}_{\mathbb{R}})$-type and $g - g_i \geq 0$ (a.e.). Moreover, as $x \notin \mathrm{Supp}(D - D_i)$ for $x \in C(\mathbb{C})$, by Proposition 2.3.4,

$$(g - g_i)_{\mathrm{can}}(x) \geq 0.$$

Therefore, $\widehat{\deg}\left(\overline{D} - \overline{D}_i\big|_C\right) \geq 0$, and hence

$$\widehat{\deg}\left(\overline{D}\big|_C\right) \geq \widehat{\deg}\left(\overline{D}_i\big|_C\right) \geq 0.$$

□

Lemma 9.1.3. *Let (D, g) be an effective arithmetic $\mathbb{R}$-Cartier divisor of C^0-type on X and let E be an $\mathbb{R}$-Cartier divisor on X with $0 \leq E \leq D$. Then there is an F_∞-invariant E-Green function h of* $(C^0 \cap \mathrm{PSH})$*-type such that*

$$0 \leq (E, h) \leq (D, g).$$

Proof. Let h_1 be an F_∞-invariant E-Green function of $(C^\infty \cap \mathrm{PSH})$-type. There is a constant C_1 such that $h_1 + C_1 \leq g$ (a.e.). We set $h = \max\{h_1 + C_1, 0\}$. Then, by Lemma 9.1.1, h is an F_∞-invariant E-Green function of $(C^0 \cap \mathrm{PSH})$-type and $0 \leq h \leq g$ (a.e.). □

9.2. The existence of Zariski decompositions. Let $\overline{D} = (D, g)$ be an arithmetic $\mathbb{R}$-Cartier divisor of $\mathscr{T}$-type on X such that g is of upper bounded type. Let us consider

$$(-\infty, \overline{D}] \cap \widehat{\mathrm{Nef}}(X)_{\mathbb{R}} = \{\overline{M} \mid \overline{M} \text{ is nef and } \overline{M} \leq \overline{D}\}.$$

The following theorem is one of the main theorems of this paper, which guarantees the greatest element $\overline{P}$ of $(-\infty, \overline{D}] \cap \widehat{\mathrm{Nef}}(X)_{\mathbb{R}}$ under the assumption $(-\infty, \overline{D}] \cap \widehat{\mathrm{Nef}}(X)_{\mathbb{R}} \neq \emptyset$. If we set $\overline{N} = \overline{D} - \overline{P}$, then we have a decomposition $\overline{D} = \overline{P} + \overline{N}$. It is called the *Zariski decomposition of* $\overline{D}$, and $\overline{P}$ (resp. $\overline{N}$) is called the *positive part* (resp. *negative part*) of $\overline{D}$.

Theorem 9.2.1 (Zariski decomposition on an arithmetic surface)**.** *If*

$$(-\infty, \overline{D}] \cap \widehat{\mathrm{Nef}}(X)_{\mathbb{R}} \neq \emptyset,$$

then there is $\overline{P} = (P, p) \in (-\infty, \overline{D}] \cap \widehat{\mathrm{Nef}}(X)_{\mathbb{R}}$ such that $\overline{P}$ is greatest in $(-\infty, \overline{D}] \cap \widehat{\mathrm{Nef}}(X)_{\mathbb{R}}$, that is, $\overline{M} \leq \overline{P}$ for all $\overline{M} \in (-\infty, \overline{D}] \cap \widehat{\mathrm{Nef}}(X)_{\mathbb{R}}$. Moreover, if $\overline{D}$ is of C^0-type, then $\overline{P}$ is also of C^0-type.

Proof. For a 1-dimensional closed integral subscheme C of X, we put

$$a(C) = \sup\{\mathrm{mult}_C(M) \mid (M, g_M) \in (-\infty, \overline{D}] \cap \widehat{\mathrm{Nef}}(X)_{\mathbb{R}}\}.$$

We choose $\overline{M_0} = (M_0, g_0) \in (-\infty, \overline{D}] \cap \widehat{\mathrm{Nef}}(X)_{\mathbb{R}}$. Then $\mathrm{mult}_C(M_0) \leq a(C) \leq \mathrm{mult}_C(D)$. Let $\{C_1, \ldots, C_l\}$ be the set of all 1-dimensional closed integral subschemes in $\mathrm{Supp}(D) \cup \mathrm{Supp}(M_0)$. Note that if $C \notin \{C_1, \ldots, C_l\}$, then $a(C) = 0$. Thus we set $P = \sum_C a(C) C$.

Claim 9.2.1.1. *There is a sequence $\{\overline{M}_n = (M_n, g_n)\}_{n=0}^{\infty}$ in $(-\infty, \overline{D}] \cap \widehat{\mathrm{Nef}}(X)_{\mathbb{R}}$ such that $\overline{M}_n \leq \overline{M}_{n+1}$ for all $n \geq 0$ and that*

$$\lim_{n\to\infty} \mathrm{mult}_{C_i}(M_n) = a(C_i)$$

for all $i = 1, \ldots, n$.

Proof. For each i, let $\{\overline{M}_{i,n}\}_{n=1}^{\infty}$ be a sequence in $(-\infty, \overline{D}] \cap \widehat{\mathrm{Nef}}(X)_{\mathbb{R}}$ such that

$$\lim_{n\to\infty} \mathrm{mult}_{C_i}(M_{i,n}) = a(C_i).$$

We set $\overline{M}_n = \max\left\{\{\overline{M}_0\} \cup \{\overline{M}_{i,j}\}_{1 \leq i \leq l, 1 \leq j \leq n}\right\}$ for $n \geq 1$. By Lemma 9.1.2, $\overline{M}_n \in (-\infty, \overline{D}] \cap \widehat{\mathrm{Nef}}(X)_{\mathbb{R}}$. Moreover, $\overline{M}_n \leq \overline{M}_{n+1}$ and

$$\lim_{n\to\infty} \mathrm{mult}_{C_i}(M_n) = a(C_i)$$

for all i. □

Since PSH is a subjacent type of $\mathscr{T}$, by using Lemma 2.3.1,

$$(g_0)_{\text{can}} \leq \cdots \leq (g_n)_{\text{can}} \leq (g_{n+1})_{\text{can}} \leq \cdots \leq g_{\text{can}}$$

holds on $X(\mathbb{C}) \setminus (\operatorname{Supp}(D) \cup \operatorname{Supp}(M_0))(\mathbb{C})$, which means that $\lim_{n\to\infty}(g_n)_{\text{can}}(x)$ exists for $x \in X(\mathbb{C}) \setminus (\operatorname{Supp}(D) \cup \operatorname{Supp}(M_0))(\mathbb{C})$. Therefore, by Theorem 7.1, there is an F_∞-invariant P-Green function h of $\mathrm{PSH}_{\mathbb{R}}$-type on $X(\mathbb{C})$ such that $(P, h) \leq \overline{D}$ and (P, h) is nef. Here we consider

$$[(P, h), \overline{D}] \cap \widehat{\mathrm{Nef}}(X)_{\mathbb{R}} = \{(M, g_M) \mid (M, g_M) \text{ is nef and } (P, h) \leq (M, g_M) \leq \overline{D}\}.$$

Note that $M = P$ for all $(M, g_M) \in [(P, h), \overline{D}] \cap \widehat{\mathrm{Nef}}(X)_{\mathbb{R}}$.

Claim 9.2.1.2. *If $\overline{P} = (P, p)$ is the greatest element of $[(P, h), \overline{D}] \cap \widehat{\mathrm{Nef}}(X)_{\mathbb{R}}$, then $\overline{P}$ is also the greatest element of $(-\infty, \overline{D}] \cap \widehat{\mathrm{Nef}}(X)_{\mathbb{R}}$.*

Proof. For $(N, g_N) \in (-\infty, \overline{D}] \cap \widehat{\mathrm{Nef}}(X)_{\mathbb{R}}$, we set $(M, g_M) = (\max\{P, N\}, \max\{h, g_N\})$. Then

$$(M, g_M) \in [(P, h), \overline{D}] \cap \widehat{\mathrm{Nef}}(X)_{\mathbb{R}} \quad \text{and} \quad (N, g_N) \leq (M, g_M).$$

Thus the claim follows. □

By Proposition 4.4, there is a P-Green function p of $\mathrm{PSH}_{\mathbb{R}}$-type such that $p \leq g$ (a.e.) and p_{can} is the upper semicontinuous regularization of the function p' given by

$$p'(x) := \sup\{(g_M)_{\text{can}}(x) \mid \overline{M} \in [(P, h), \overline{D}] \cap \widehat{\mathrm{Nef}}(X)_{\mathbb{R}}\}$$

over $X(\mathbb{C}) \setminus \operatorname{Supp}(P)(\mathbb{C})$. Since $(g_M)_{\text{can}}$ is F_∞-invariant on $X(\mathbb{C}) \setminus \operatorname{Supp}(P)(\mathbb{C})$, p' is also F_∞-invariant, and hence p is F_∞-invariant because $p = p'$ (a.e.) on $X(\mathbb{C}) \setminus \operatorname{Supp}(P)(\mathbb{C})$ (cf. Subsection 2.1). We set $\overline{P} = (P, p)$. Then $(P, h) \leq \overline{P} \leq \overline{D}$ and hence $\overline{P}$ is nef by Lemma 6.2.3. In addition, $\overline{P}$ is the greatest element of $[(P, h), \overline{D}] \cap \widehat{\mathrm{Nef}}(X)_{\mathbb{R}}$.

Finally we assume that $\overline{D}$ is of C^0-type. Let e be the degree of P on the generic fiber of $X \to \operatorname{Spec}(\mathbb{Z})$. As $\overline{P}$ is nef, we have $e \geq 0$. Let $X(\mathbb{C}) = X_1 \cup \cdots \cup X_r$ be the decomposition into connected components of $X(\mathbb{C})$. We set $P = \sum_{i=1}^r \sum_j a_{ij} P_{ij}$ on $X(\mathbb{C})$, where $P_{ij} \in X_i$ for all i and j. Note that $e = \sum_j a_{ij}$ for all i. Let us fix a C^∞-volume form ω_i on X_i with $\int_{X_i} \omega_i = 1$. Let p_{ij} be a P_{ij}-Green function of C^∞-type on X_i such that $dd^c([p_{ij}]) + \delta_{P_{ij}} = [\omega_i]$. We set $p' = \sum_{i=1}^r \sum_j a_{ij} p_{ij}$. Then p' is a P-Green function of C^∞-type and

$$dd^c([p']) + \delta_P = \sum_{i=1}^r \left(\sum\nolimits_j a_{ij}\right)[\omega_i] = e \sum_{i=1}^r [\omega_i].$$

Thus, if $e > 0$, then $dd^c([p']) + \delta_P$ is represented by a positive C^∞-form $e \sum_{i=1}^r \omega_i$. Moreover, if $e = 0$, then $dd^c([p']) + \delta_P = 0$. Let us consider

$$\left\{ \varphi \;\middle|\; \begin{array}{l} \varphi \text{ is a } P\text{-Green function of PSH-type} \\ \text{on } X(\mathbb{C}) \text{ with } \varphi \leq g \text{ (a.e.)} \end{array} \right\}.$$

By Theorem 4.6, the above set has the greatest element $\tilde{p}$ modulo null functions such that $\tilde{p}$ is a P-Green function of $(C^0 \cap \mathrm{PSH})$-type. Since g is F_∞-invariant, we have $F_\infty^*(\tilde{p}) \leq F_\infty^*(g) = g$ (a.e.). Moreover, by Lemma 5.1.1 and Lemma 5.1.2, $F_\infty^*(\tilde{p})$ is a P-Green function of PSH-type. Thus $F_\infty^*(\tilde{p}) \leq \tilde{p}$ (a.e.), and hence

$$\tilde{p} = F_\infty^*(F_\infty^*(\tilde{p})) \leq F_\infty^*(\tilde{p}) \quad \text{(a.e.)}.$$

Therefore, $\tilde{p}$ is F_∞-invariant. Note that $(P, \tilde{p})$ is nef because $p \leq \tilde{p}$ (a.e.). Hence $p = \tilde{p}$ (a.e.). □

9.3. Properties of Zariski decompositions. Let $\overline{D} = (D, g)$ be an arithmetic $\mathbb{R}$-Cartier divisor of $\mathscr{T}$-type on X such that g is of upper bounded type. First of all, let us observe the following three properties of the Zariski decompositions:

Proposition 9.3.1. *We assume $(-\infty, \overline{D}] \cap \widehat{\mathrm{Nef}}(X)_{\mathbb{R}} \neq \emptyset$. Let $\overline{D} = \overline{P} + \overline{N}$ be the Zariski decomposition of $\overline{D}$. Then we have the following:*

(1) *For a non zero rational function ϕ on X, $\overline{D} + \widehat{(\phi)} = (\overline{P} + \widehat{(\phi)}) + \overline{N}$ is the Zariski decomposition of $\overline{D} + \widehat{(\phi)}$.*
(2) *For $a \in \mathbb{R}_{>0}$, $a\overline{D} = a\overline{P} + a\overline{N}$ is the Zariski decomposition of $a\overline{D}$.*

Proof. Note that $\pm\widehat{(\phi)}$ is nef and that

$$\overline{D}_1 \leq \overline{D}_2 \quad \iff \quad \overline{D}_1 + \widehat{(\phi)} \leq \overline{D}_2 + \widehat{(\phi)}$$

and

$$\overline{D}_1 \leq \overline{D}_2 \quad \iff \quad a\overline{D}_1 \leq a\overline{D}_2$$

for arithmetic $\mathbb{R}$-Cartier divisors $\overline{D}_1, \overline{D}_2$, a non-zero rational function ϕ and $a \in \mathbb{R}_{>0}$. Thus the assertions of this proposition are obvious. □

Proposition 9.3.2. (1) *If $\hat{h}^0(X, a\overline{D}) \neq 0$ for some $a \in \mathbb{R}_{>0}$, then*

$$(-\infty, \overline{D}] \cap \widehat{\mathrm{Nef}}(X)_{\mathbb{R}} \neq \emptyset.$$

(2) *If $\overline{D}$ is of C^0-type and $(-\infty, \overline{D}] \cap \widehat{\mathrm{Nef}}(X)_{\mathbb{R}} \neq \emptyset$, then $\overline{D}$ is pseudo-effective.*

Proof. (1) We choose $\phi \in \hat{H}^0(X, a\overline{D}) \setminus \{0\}$. Then $a\overline{D} + \widehat{(\phi)} \geq 0$, which implies $\overline{D} \geq (-1/a)\widehat{(\phi)}$. Note that $(-1/a)\widehat{(\phi)}$ is nef, so that $(-1/a)\widehat{(\phi)} \in (-\infty, \overline{D}] \cap \widehat{\mathrm{Nef}}(X)_{\mathbb{R}}$, as required.

(2) Let $\overline{D} = \overline{P} + \overline{N}$ be the Zariski decomposition of $\overline{D}$ and let $\overline{A}$ be an ample arithmetic $\mathbb{R}$-Cartier divisor. For $n \in \mathbb{Z}_{>0}$, by Proposition 6.2.2, $\overline{P} + (1/n)\overline{A}$ is adequate. In particular, $\widehat{\mathrm{vol}}(\overline{P} + (1/n)\overline{A}) > 0$, and hence

$$\widehat{\mathrm{vol}}(\overline{D} + (1/n)\overline{A}) \geq \widehat{\mathrm{vol}}(\overline{P} + (1/n)\overline{A}) > 0,$$

which shows that $\overline{D}$ is pseudo-effective. □

Remark 9.3.3. It is expected that the converse of (2) in Proposition 9.3.2 holds, that is, if $\overline{D}$ is of C^0-type and $\overline{D}$ is pseudo-effective, then $(-\infty, \overline{D}] \cap \widehat{\mathrm{Nef}}(X)_{\mathbb{R}} \neq \emptyset$ (cf. [18]).

Proposition 9.3.4. *We assume that $\overline{D}$ is of C^∞-type and $\overline{D}$ is effective. Let $\overline{P}$ be the positive part of the Zariski decomposition of $\overline{D}$. Then there is a constant e' such that*

$$\hat{h}^0(X, n\overline{P}) \leq \hat{h}^0(X, n\overline{D}) \leq \hat{h}^0(X, n\overline{P}) + e' n \log(n+1)$$

for all $n \geq 1$. In particular, $\widehat{\mathrm{vol}}(\overline{P}) = \widehat{\mathrm{vol}}(\overline{D})$.

Proof. The assertion is a consequence of Proposition 8.3 because $\overline{M}_{\infty}(\overline{D}) \leq \overline{P}$. □

The following theorem is also one of the main theorems of this paper.

Theorem 9.3.5. *We assume that $\overline{D}$ is of C^0-type and $(-\infty, \overline{D}] \cap \widehat{\mathrm{Nef}}(X)_{\mathbb{R}} \neq \emptyset$. Let $\overline{P}$ (resp. $\overline{N}$) be the positive part (resp. negative part) of the Zariski decomposition of $\overline{D}$. Then we have the following:*

(1) $\widehat{\mathrm{vol}}(\overline{P}) = \widehat{\mathrm{vol}}(\overline{D}) = \widehat{\deg}(\overline{P}^2)$.

(2) $\widehat{\deg}(\overline{P}|_C) = 0$ *for all* 1*-dimensional closed integral subschemes C with $C \subseteq$* Supp(N).

(3) *If $\overline{M}$ is an arithmetic $\mathbb{R}$-Cartier divisor of* $\mathrm{PSH}_{\mathbb{R}}$*-type on X such that $0 \leq \overline{M} \leq \overline{N}$ and* $\deg(\overline{M}|_C) \geq 0$ *for all* 1*-dimensional closed integral subschemes C with $C \subseteq$* Supp(N), *then* $\overline{M} = 0$.

(4) *We assume $N \neq 0$. Let $N = c_1 C_1 + \cdots + c_l C_l$ be the decomposition such that $c_1, \ldots, c_l \in \mathbb{R}_{>0}$ and $C_1, \ldots, C_l$ are distinct* 1*-dimensional closed integral subschemes on X. Then the following hold:*

  (4.1) *There are effective arithmetic Cartier divisors $(C_1, h_1), \ldots, (C_l, h_l)$ of $(C^0 \cap$* PSH)*-type such that $c_1(C_1, h_1) + \cdots + c_l(C_l, h_l) \leq \overline{N}$.*

  (4.2) *If $(C_1, k_1), \ldots, (C_l, k_l)$ are effective arithmetic Cartier divisors of* $\mathrm{PSH}_{\mathbb{R}}$*-type such that $\alpha_1(C_1, k_1) + \cdots + \alpha_l(C_l, k_l) \leq \overline{N}$ for some $\alpha_1, \ldots, \alpha_l \in \mathbb{R}_{>0}$, then*

$$(-1)^l \det\left(\widehat{\deg}\left((C_i, k_i)|_{C_j}\right)\right) > 0.$$

Proof. (1) It follows from Proposition 6.4.2 that $\widehat{\mathrm{vol}}(\overline{P}) = \widehat{\deg}(\overline{P}^2)$. We need to show $\widehat{\mathrm{vol}}(\overline{P}) = \widehat{\mathrm{vol}}(\overline{D})$. If $\widehat{\mathrm{vol}}(\overline{D}) = 0$, then the assertion is obvious, so that we may assume that $\widehat{\mathrm{vol}}(\overline{D}) > 0$.

First we consider the case where $\overline{D}$ is of C^∞-type. We choose a positive integer n and a non-zero rational function ϕ such that $n\overline{D} + \widehat{(\phi)}$ is effective. By Proposition 9.3.1, the positive part of the Zariski decomposition $n\overline{D} + \widehat{(\phi)}$ is $n\overline{P} + \widehat{(\phi)}$. Thus, by using Proposition 9.3.4,

$$n^2\widehat{\mathrm{vol}}(\overline{P}) = \widehat{\mathrm{vol}}(n\overline{P}) = \widehat{\mathrm{vol}}(n\overline{P} + \widehat{(\phi)}) = \widehat{\mathrm{vol}}(n\overline{D} + \widehat{(\phi)}) = \widehat{\mathrm{vol}}(n\overline{D}) = n^2\widehat{\mathrm{vol}}(\overline{D}),$$

as required.

Let us consider a general case. By the Stone-Weierstrass theorem, there is a sequence $\{u_n\}_{n=1}^{\infty}$ of non-negative F_∞-invariant continuous functions such that $\lim_{n\to\infty} \|u_n\|_{\sup} = 0$ and $\overline{D}_n := \overline{D} - (0, u_n)$ is of C^∞-type for every $n \geq 1$. By the continuity of $\widehat{\mathrm{vol}}$ (cf. Theorem 5.2.2),

$$\lim_{n\to\infty} \widehat{\mathrm{vol}}(\overline{D}_n) = \widehat{\mathrm{vol}}(\overline{D}).$$

In particular, $\overline{D}_n$ is big for $n \gg 1$. Let $\overline{P}_n$ be the positive part of the Zariski decomposition of $\overline{D}_n$. Since $\overline{P}_n \leq \overline{D}_n \leq \overline{D}$ and $\overline{P}_n$ is nef, we have $\overline{P}_n \leq \overline{P}$, and hence

$$\widehat{\mathrm{vol}}(\overline{D}_n) = \widehat{\mathrm{vol}}(\overline{P}_n) \leq \widehat{\mathrm{vol}}(\overline{P}) \leq \widehat{\mathrm{vol}}(\overline{D}).$$

Thus the assertion follows by taking $n \to \infty$.

(4.1) Before starting the proofs of (2), (3) and (4.2), let us see (4.1) first. By Proposition 2.4.2, there are effective arithmetic Cartier divisors $(C_1, h'_1), \ldots, (C_l, h'_l)$

of C^0-type such that $c_1(C_1, h'_1)+\cdots+c_l(C_l, h'_l) = \overline{N}$. For each i, by using Lemma 9.1.3, we can find an effective arithmetic Cartier divisor (C_i, h_i) of $(C^0 \cap \mathrm{PSH})$-type such that $(C_i, h_i) \leq (C_i, h'_i)$, as required.

(2) We may assume $N \neq 0$. We assume $\deg(\overline{P}\big|_{C_i}) > 0$ for some i. By (4.1),

$$0 \leq c_i(C_i, h_i) \leq \overline{N}.$$

Note that if C' is a 1-dimensional closed integral subscheme with $C' \neq C_i$, then

$$\deg((C_i, h_i)|_{C'}) \geq 0.$$

Thus, since $\deg(\overline{P}\big|_{C_i}) > 0$, we can find a sufficiently small positive number ϵ such that $\overline{P} + \epsilon(C_i, h_i)$ is nef and $\overline{P} + \epsilon(C_i, h_i) \leq \overline{D}$. This is a contradiction.

(3) Since $0 \leq \overline{M} \leq \overline{N}$, if C' is a 1-dimensional closed integral subscheme with $C' \not\subseteq \mathrm{Supp}(N)$, then $\widehat{\deg}(\overline{M}\big|_{C'}) \geq 0$. Thus $\overline{M}$ is nef, and hence $\overline{P} + \overline{M}$ is nef and $\overline{P} + \overline{M} \leq \overline{D}$. Therefore, $\overline{M} = 0$.

(4.2) By Lemma 1.2.3, it is sufficient to see the following: if $\beta_1, \ldots, \beta_l \in \mathbb{R}_{\geq 0}$ and

$$\widehat{\deg}\left((\beta_1(C_1, k_1) + \cdots + \beta_l(C_l, k_l))\big|_{C_i}\right) \geq 0$$

for all i, then $\beta_1 = \cdots = \beta_l = 0$. Replacing $\beta_1, \ldots, \beta_l$ with $t\beta_1, \ldots, t\beta_l$ ($t > 0$), we may assume that $0 \leq \beta_i \leq \alpha_i$ for all i. Thus the assertion follows from (3). □

Theorem 9.3.6 (Asymptotic orthogonality of σ-decomposition)**.** *If $\overline{D}$ is of C^0-type, effective and big, then*

$$\lim_{n\to\infty} \widehat{\deg}\left(\overline{M}_n(\overline{D}) \mid F_n(\overline{D})\right) = 0.$$

(*For the definition of $\overline{M}_n(\overline{D})$ and $F_n(\overline{D})$, see Section* 8.)

Proof. Let us begin with the following claim:

Claim 9.3.6.1. *$P = M_\infty(\overline{D})$ and $N = F_\infty(\overline{D})$.*

Proof. First of all, note that $\overline{M}_\infty(\overline{D}) \leq \overline{P}$ and $\overline{F}_\infty(\overline{D}) \geq \overline{N}$. Since $\overline{D}$ is effective, $(0, 0) \in (-\infty, \overline{D}] \cap \widehat{\mathrm{Nef}}(X)_{\mathbb{R}}$, so that $\overline{P}$ is effective. Then, by (2) of Proposition 6.5.2,

$$\mu_C(\overline{D}) \leq \mu_C(\overline{P}) + \mathrm{mult}_C(N).$$

Moreover, by Proposition 6.5.3, $\mu_C(\overline{P}) = 0$ because $\overline{P}$ is nef and big. Thus we have

$$\mathrm{mult}_C(F_\infty(\overline{D})) = \mu_C(\overline{D}) \leq \mathrm{mult}_C(N),$$

which implies $F_\infty(\overline{D}) \leq N$. Therefore, $N = F_\infty(\overline{D})$, and hence $P = M_\infty(\overline{D})$. □

Claim 9.3.6.2. $\widehat{\deg}\left(\overline{M}_\infty(\overline{D})\big|_C\right) = 0$ *for any* 1*-dimensional closed integral subscheme C with $C \subseteq \mathrm{Supp}(N)$.*

Proof. Since $\overline{M}_\infty(\overline{D}) \leq \overline{P}$ and $P = M_\infty(\overline{D})$, there is $\phi \in (C^0 - \mathrm{PSH}_{\mathbb{R}})(X(\mathbb{C}))$ such that $\phi \geq 0$ and $\overline{P} = \overline{M}_\infty(\overline{D}) + (0, \phi)$. Thus, for a 1-dimensional closed integral subscheme C with $C \subseteq \mathrm{Supp}(N)$, by (3) in Theorem 9.3.5,

$$0 \leq \widehat{\deg}\left(\overline{M}_\infty(\overline{D})\big|_C\right) \leq \widehat{\deg}\left(\overline{P}\big|_C\right) = 0,$$

as required. □

Let $C_1, \ldots, C_l$ be irreducible components of $\operatorname{Supp}(D)$. We set $F_n(\overline{D}) = \sum_{i=1}^{l} a_{ni} C_i$ and $F_\infty(\overline{D}) = \sum_{i=1}^{l} a_i C_i$. Then $\lim_{n\to\infty} a_{ni} = a_i$. Moreover, if we set $I = \{i \mid a_i > 0\}$, then $\bigcup_{i \in I} C_i = \operatorname{Supp}(N)$. Therefore, by the above claim and (3) in Proposition 8.3,

$$
\begin{aligned}
0 &\leq \liminf_{n\to\infty} \widehat{\deg}\left(\overline{M}_n(\overline{D}) \mid F_n(\overline{D})\right) \leq \limsup_{n\to\infty} \widehat{\deg}\left(\overline{M}_n(\overline{D}) \mid F_n(\overline{D})\right) \\
&\leq \sum_{i=1}^{l} \limsup_{n\to\infty} a_{ni} \widehat{\deg}\left(\overline{M}_n(\overline{D})\big|_{C_i}\right) = \sum_{i=1}^{l} a_i \limsup_{n\to\infty} \widehat{\deg}\left(\overline{M}_n(\overline{D})\big|_{C_i}\right) \\
&= \sum_{i \in I} a_i \limsup_{n\to\infty} \widehat{\deg}\left(\overline{M}_n(\overline{D})\big|_{C_i}\right) \leq \sum_{i \in I} a_i \widehat{\deg}\left(\overline{M}_\infty(\overline{D})\big|_{C_i}\right) = 0.
\end{aligned}
$$

Hence the theorem follows. □

Finally let us consider Fujita's approximation theorem on an arithmetic surface.

Proposition 9.3.7. *We assume that $\overline{D}$ is C^0-type and $\widehat{\operatorname{vol}}(\overline{D}) > 0$. Then, for any $\epsilon > 0$, there is $\overline{A} \in \widehat{\operatorname{Div}}_{C^\infty}(X)_{\mathbb{R}}$ such that*

$$\overline{A} \text{ is nef}, \quad \overline{A} \leq \overline{D} \quad \text{and} \quad \widehat{\operatorname{vol}}(\overline{A}) \geq \widehat{\operatorname{vol}}(\overline{D}) - \epsilon.$$

Proof. By using the continuity of $\widehat{\operatorname{vol}}$, we can find a sufficiently small positive number δ such that

$$\widehat{\operatorname{vol}}(\overline{D} - (0, \delta)) > \max\{\widehat{\operatorname{vol}}(\overline{D}) - \epsilon, 0\}.$$

Let $\overline{D} - (0, \delta) = \overline{P}_\delta + \overline{N}_\delta$ be the Zariski decomposition of $\overline{D} - (0, \delta)$. Since $\overline{P}_\delta$ is a big arithmetic $\mathbb{R}$-Cartier divisor of C^0-type, by Theorem 4.6, there is an F_∞-invariant continuous function u on $X(\mathbb{C})$ such that $0 \leq u < \delta$ on $X(\mathbb{C})$ and $\overline{P}_\delta + (0, u)$ is nef and of C^∞-type. If we set $\overline{A} = \overline{P}_\delta + (0, u)$, then $\overline{A} \leq \overline{D}$ and

$$\widehat{\operatorname{vol}}(\overline{D}) - \epsilon < \widehat{\operatorname{vol}}(\overline{D} - (0, \delta)) \leq \widehat{\operatorname{vol}}(\overline{A}).$$

□

Remark 9.3.8. We assume that $\overline{D}$ is of C^0-type, big and not nef. Let $\overline{D} = \overline{P} + \overline{N}$ be the Zariski decomposition of $\overline{D}$ and let $N = c_1 C_1 + \cdots + c_l C_l$ be the decomposition such that $c_1, \ldots, c_l \in \mathbb{R}_{>0}$ and $C_1, \ldots, C_l$ are distinct 1-dimensional closed integral subschemes on X. Then $C_1, \ldots, C_l$ are not necessarily linearly independent in $\operatorname{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ (cf. Remark 9.4.2).

Remark 9.3.9. After writing this paper, several significant progresses were made on Zariski decompositions. Here we would like to report them. Let $\overline{D}$ and $\overline{P}$ be arithmetic $\mathbb{R}$-Cartier divisors of C^0-type on X.

(1) A generalization of Proposition 9.3.4 was found, that is, if $\overline{P}$ is the greatest element of $(-\infty, \overline{D}] \cap \widehat{\operatorname{Nef}}_{C^0}(X)_{\mathbb{R}}$, then $\hat{h}^0(X, n\overline{P}) = \hat{h}^0(X, n\overline{D})$ for all $n \geq 0$ (cf. [20, Appendix B]). It can be proved as follows: If $\phi \in \hat{H}^0(X, n\overline{D}) \setminus \{0\}$, then $(-1/n)\widehat{(\phi)} \in (-\infty, \overline{D}] \cap \widehat{\operatorname{Nef}}_{C^0}(X)_{\mathbb{R}}$ because $n\overline{D} + \widehat{(\phi)} \geq 0$ and $-\widehat{(\phi)}$ is nef. Thus $(-1/n)\widehat{(\phi)} \leq \overline{P}$, and hence $\phi \in \hat{H}^0(X, n\overline{P})$.

(2) A numerical characterizations of the greatest element of $(-\infty, \overline{D}] \cap \widehat{\operatorname{Nef}}_{C^0}(X)_{\mathbb{R}}$ was obtained, that is, the following are equivalent (cf [20]):

(a) $\overline{P}$ is the greatest element of $(-\infty, \overline{D}] \cap \widehat{\operatorname{Nef}}_{C^0}(X)_{\mathbb{R}}$.

(b) $\overline{P}$ is an element of $(-\infty, \overline{D}] \cap \widehat{\mathrm{Nef}}_{C^0}(X)_{\mathbb{R}}$ with the following property: if $\overline{B}$ is an arithmetic $\mathbb{R}$-Cartier divisor of C^0-type such that $(0,0) \lneqq \overline{B} \leq \overline{D} - \overline{P}$ and $\overline{P} + \overline{B}$ is of ($C^0 \cap$ PSH)-type, then $\widehat{\deg}(\overline{P} \cdot \overline{B}) = 0$ and $\widehat{\deg}(\overline{B}^2) < 0$.

(3) In the case where $\overline{D}$ is big, the greatest element of $(-\infty, \overline{D}] \cap \widehat{\mathrm{Nef}}_{C^0}(X)_{\mathbb{R}}$ is characterized by $\widehat{\mathrm{vol}}(\overline{D}) = \widehat{\mathrm{vol}}(\overline{P})$. Namely, if $\overline{D}$ is big, $\overline{P} \in (-\infty, \overline{D}] \cap \widehat{\mathrm{Nef}}_{C^0}(X)_{\mathbb{R}}$ and $\widehat{\mathrm{vol}}(\overline{D}) = \widehat{\mathrm{vol}}(\overline{P})$, then $\overline{P}$ gives the greatest element of $(-\infty, \overline{D}] \cap \widehat{\mathrm{Nef}}_{C^0}(X)_{\mathbb{R}}$ (cf. [19, Theorem 4.2.1]).

9.4. Examples of Zariski decompositions on $\mathbb{P}^1_{\mathbb{Z}}$. Let $\mathbb{P}^1_{\mathbb{Z}} = \mathrm{Proj}(\mathbb{Z}[x,y])$, $C_0 = \{x = 0\}$, $C_\infty = \{y = 0\}$ and $z = x/y$. Let α and β be positive real numbers. We set

$$D = C_0, \quad g = -\log|z|^2 + \log\max\{\alpha^2|z|^2, \beta^2\} \quad \text{and} \quad \overline{D} = (D, g).$$

The purpose of this subsection is to show the following fact:

Proposition 9.4.1. *The Zariski decomposition of $\overline{D}$ exists if and only if either $\alpha \geq 1$ or $\beta \geq 1$. Moreover, we have the following:*

(1) *If $\alpha \geq 1$ and $\beta \geq 1$, then $\overline{D}$ is nef.*
(2) *If $\alpha \geq 1$ and $\beta < 1$, then the positive part of $\overline{D}$ is given by*

$$(\theta C_0, -\theta\log|z|^2 + \log\max\{\alpha^2|z|^{2\theta}, 1\}),$$

where $\theta = \log\alpha/(\log\alpha - \log\beta)$.
(3) *If $\alpha < 1$ and $\beta \geq 1$, then the positive part of $\overline{D}$ is given by*

$$(C_0 - (1-\theta')C_\infty, -\log|z|^2 + \log\max\{|z|^{2\theta'}, \beta^2\}),$$

where $\theta' = \log\beta/(\log\beta - \log\alpha)$.

Proof. Let us begin with the following claim:

Claim 9.4.1.1. *For $a, b, \lambda \in \mathbb{R}_{>0}$, we set*

$$L = \lambda C_0, \quad h = -\lambda\log|z|^2 + \log\max\{a^2|z|^{2\lambda}, b^2\} \quad \text{and} \quad \overline{L} = (L, h).$$

Then we have the following:

(a) *$\overline{L}$ is an arithmetic $\mathbb{R}$-Cartier divisor of ($C^0 \cap$PSH)-type. In additions, $\overline{L}$ is effective if and only if $a \geq 1$.*
(b) $H^0(\mathbb{P}^1_{\mathbb{Z}}, L) = \bigoplus_{i \in \mathbb{Z}, 0 \leq i \leq \lambda} \mathbb{Z}z^{-i}$.
(c) *For $i \in \mathbb{Z}$ with $0 \leq i \leq \lambda$, $\|z^{-i}\|_h = \dfrac{1}{a^{1-i/\lambda}b^{i/\lambda}}$.*
(d) *For $s = \sum_{0 \leq i \leq \lambda} c_i z^{-i} \in H^0(\mathbb{P}^1_{\mathbb{Z}}, L)$,*

$$\|s\|_h \geq \sqrt{\sum_{0 \leq i \leq \lambda}\left(\frac{c_i}{a^{1-i/\lambda}b^{i/\lambda}}\right)^2}.$$

(e) $\hat{H}^0(\mathbb{P}^1_{\mathbb{Z}}, \overline{L}) = \{0\}$ *if $a < 1$ and $b < 1$.*
(f) *$\overline{L}$ is nef if and only if $a \geq 1$ and $b \geq 1$.*
(g) *$\overline{L}$ is adequate if $a^2 > 2^\lambda$ and $b^2 > 2^\lambda$.*

Proof. (a) and (b) are obvious. (c) is a straightforward calculation. (e) follows from (d). Let us see (d) , (f) and (g).

(d) Indeed,

$$\begin{aligned}
\|s\|_h &\geq \sup_{|\zeta|=(b/a)^{\frac{1}{\lambda}}} \{|s|_h(\zeta)\} = \frac{1}{a} \sup_{|\zeta|=(b/a)^{\frac{1}{\lambda}}} \left\{ \left| \sum_{0\leq i\leq\lambda} c_i \zeta^{-i} \right| \right\} \\
&\geq \frac{1}{a} \sqrt{\int_0^1 \left| \sum_{0\leq i\leq\lambda} c_i \left((b/a)^{\frac{1}{\lambda}} \exp(2\pi\sqrt{-1}t) \right)^{-i} \right|^2 dt} \\
&= \frac{1}{a} \sqrt{\sum_{0\leq i,j\leq\lambda} \int_0^1 c_i c_j (b/a)^{\frac{-i-j}{\lambda}} \exp(2\pi\sqrt{-1}(j-i)t) dt} \\
&= \sqrt{\sum_{0\leq i\leq\lambda} \left(\frac{c_i}{a^{1-i/\lambda} b^{i/\lambda}} \right)^2}.
\end{aligned}$$

(f) It is easy to see that $\widehat{\deg}(\overline{L}\big|_{C_0}) = \log(b)$ and $\widehat{\deg}(\overline{L}\big|_{C_\infty}) = \log(a)$. For $\gamma \in \overline{\mathbb{Q}}$, let C_γ be the 1-dimensional closed integral subscheme of $\mathbb{P}^1_{\mathbb{Z}}$ given by the Zariski closure of $\{(\gamma : 1)\}$. Then

$$\widehat{\deg}(\overline{L}\big|_{C_\gamma}) \geq \sum_{\sigma \in C_\gamma(\mathbb{C})} \left(-\lambda \log|\sigma(\gamma)| + \log\max\{a|\sigma(\gamma)|^\lambda, b\} \right).$$

Thus (f) follows.

(g) We choose $\delta \in \mathbb{R}_{>0}$ such that $a^2 \geq (2(1+\delta))^\lambda$ and $b^2 \geq (2(1+\delta))^\lambda$. Then, as

$$\lambda \log((1+\delta)|z|^2 + (1+\delta)) \leq \lambda \log\max\{2(1+\delta)|z|^2, 2(1+\delta)\} \leq \log\max\{a^2|z|^{2\lambda}, b^2\},$$

we have

$$\lambda(C_0, -\log|z|^2 + \log((1+\delta)|z|^2 + (1+\delta))) \leq \overline{L}.$$

Note that $(C_0, -\log|z|^2 + \log((1+\delta)|z|^2 + (1+\delta)))$ is ample. Thus (g) follows. □

Next we claim the following:

Claim 9.4.1.2. *If $\alpha < 1$ and $\beta < 1$, then the Zariski decomposition of $\overline{D}$ does not exists.*

Proof. For $t > 0$, we set

$$\overline{D}_t = (C_0, -\log|z|^2 + \log\max\{t^2\alpha^2|z|^2, t^2\beta^2\}).$$

It is easy to see that

$$a\overline{D}_{t_1} + b\overline{D}_{t_2} = (a+b)\overline{D}_{(t_1^a t_2^b)^{\frac{1}{a+b}}}$$

for $t_1, t_2 \in \mathbb{R}_{>0}$ and $a, b \in \mathbb{R}_{>0}$. Moreover, by (g) in Claim 9.4.1.1, $\overline{D}_{t_0}$ is adequate if $t_0 \gg 1$. We assume that the Zariski decomposition of $\overline{D}$ exists. Let $\overline{P}$ be the positive part of $\overline{D}$. We choose $\epsilon > 0$ such that $t_0^{\frac{\epsilon}{1+\epsilon}}\alpha < 1$ and $t_0^{\frac{\epsilon}{1+\epsilon}}\beta < 1$. $\overline{P} + \epsilon\overline{D}_{t_0}$ is adequate by Proposition 6.2.2. Thus, by Proposition 6.2.1,

$$\widehat{\mathrm{vol}}\left(\overline{D}_{t_0^{\frac{\epsilon}{1+\epsilon}}}\right) = \frac{\widehat{\mathrm{vol}}\left((1+\epsilon)\overline{D}_{(t_0^\epsilon)^{\frac{1}{1+\epsilon}}}\right)}{(1+\epsilon)^2} = \frac{\widehat{\mathrm{vol}}(\overline{D} + \epsilon\overline{D}_{t_0})}{(1+\epsilon)^2} \geq \frac{\widehat{\mathrm{vol}}(\overline{P} + \epsilon\overline{D}_{t_0})}{(1+\epsilon)^2} > 0,$$

which yields a contradiction by virtue of (e) in Claim 9.4.1.1. □

By the above claim, it is sufficient to see (1), (2) and (3). (1) follows from (f) in Claim 9.4.1.1.

(2) In this case, $\overline{D}$ is effective. Thus the Zariski decomposition of $\overline{D}$ exists. First we assume that $\alpha > 1$, so that $0 < \theta < 1$ and $\alpha^{1-\theta}\beta^{\theta} = 1$. Let us see the following claim:

Claim 9.4.1.3. $\langle \hat{H}^0(\mathbb{P}^1_{\mathbb{Z}}, n\overline{D})\rangle_{\mathbb{Z}} = \bigoplus_{i\in\mathbb{Z}, 0\leq i\leq n\theta} \mathbb{Z}z^{-i}$.

Proof. By (c) in Claim 9.4.1.1, $\|z^{-i}\|_{ng} = \beta^{\frac{n\theta-i}{1-\theta}}$. Thus $z^{-i} \in \hat{H}^0(\mathbb{P}^1_{\mathbb{Z}}, n\overline{D})$ for $0 \leq i \leq n\theta$. For $s = \sum_{i=0}^n a_i z^{-i} \in H^0(\mathbb{P}^1_{\mathbb{Z}}, nD)$, by (d) in Claim 9.4.1.1,

$$\|s\|_{ng} \geq \sqrt{\sum_{i=0}^{n} \left(|a_i|\beta^{\frac{n\theta-i}{1-\theta}}\right)^2}$$

Thus, if $\|s\|_{ng} \leq 1$, then $a_i = 0$ for $i > n\theta$, which means that $s \in \bigoplus_{0\leq i\leq n\theta} \mathbb{Z}z^{-i}$. □

Claim 9.4.1.4. *$\overline{D}$ is big and*

$$\mu_C(\overline{D}) = \begin{cases} 1-\theta & \text{if } C = C_0, \\ 0 & \text{if } C \neq C_0 \end{cases}$$

for a 1-dimensional closed integral subscheme C of $\mathbb{P}^1_{\mathbb{Z}}$.

Proof. Note that $(z^{-i}) + nD = (n-i)C_0 + iC_\infty$. Thus the second assertion follows from Claim 9.4.1.3. Let us see that $\overline{D}$ is big. We set

$$S_n = \left\{ \sum_{0\leq i\leq n\theta/3} a_i z^{-i} \;\middle|\; |a_i| \leq \beta^{\frac{-i}{1-\theta}} \right\}.$$

It is easy to see that $S_n \subseteq \hat{H}^0(\mathbb{P}^1_{\mathbb{Z}}, n\overline{D})$ for $n \gg 1$. Note that, for $M \in \mathbb{R}_{\geq 0}$,

$$\#\{a \in \mathbb{Z} \mid |a| \leq M\} = 2\lfloor M\rfloor + 1 \geq \lfloor M\rfloor + 1 \geq M.$$

Therefore

$$\#(S_n) \geq \prod_{0\leq i\leq n\theta/3} \beta^{\frac{-i}{1-\theta}} = \beta^{\frac{-1}{1-\theta}\frac{\lfloor n\theta/3\rfloor(\lfloor n\theta/3\rfloor+1)}{2}},$$

which implies

$$\hat{h}^0(X, n\overline{D}) \geq \log\#(S_n) \geq \frac{-\log\beta}{1-\theta}\frac{\lfloor n\theta/3\rfloor(\lfloor n\theta/3\rfloor+1)}{2}$$

for $n \gg 1$, and hence $\widehat{\mathrm{vol}}(\overline{D}) > 0$. □

We set

$$P' = \theta C_0, \quad p' = -\theta\log|z|^2 + \log\max\{\alpha^2|z|^{2\theta}, 1\} \quad \text{and} \quad \overline{P}' = (P', p').$$

By Claim 9.4.1.4 and Claim 9.3.6.1 in the proof of Theorem 9.3.6, if $\overline{P} = (P, p)$ is the positive part of the Zariski decomposition, then $P = \theta C_0$. Let us see that $\overline{P}' = \overline{P}$. First of all, $\overline{P}' \leq \overline{D}$ and $\overline{P}'$ is nef by (f) in Claim 9.4.1.1. Thus $\overline{P}' \leq \overline{P}$, and hence there is a continuous function u such that $u \geq 0$ and $\overline{P} = \overline{P}' + (0, u)$. Note that

$$0 \leq u \leq -(1-\theta)\log|z|^2 + \log\max\{\alpha^2|z|^2, \beta^2\} - \log\max\{\alpha^2|z|^{2\theta}, 1\}.$$

In particular, if $|z| \geq \beta^{\frac{1}{1-\theta}}$, then $u(z) = 0$. As $p = -\theta \log |z|^2 + u$ on $\{z \mid |z| < \beta^{\frac{1}{1-\theta}}\}$, u is subharmonic on $\{z \mid |z| < \beta^{\frac{1}{1-\theta}}\}$. Thus, by the maximal principle,

$$u(z) \leq \sup_{|\zeta|=\beta^{\frac{1}{1-\theta}}} \{u(\zeta)\} = 0,$$

which implies that $u(z) = 0$ on $\{z \mid |z| < \beta^{\frac{1}{1-\theta}}\}$. Therefore $\overline{P}' = \overline{P}$.

Finally let us consider the case where $\alpha = 1$. Let $\overline{P}$ be the positive part of $\overline{D}$. For $t \in (1, 1/\beta)$, we set

$$\overline{D}_t = (C_0, -\log |z|^2 + \log \max\{t^2|z|^2, t^2\beta^2\})$$

as in the proof of Claim 9.4.1.2. Then $\overline{D} \leq \overline{D}_t$ and, by the previous observation, the positive part $\overline{P}_t$ of $\overline{D}_t$ is given by

$$\overline{P}_t = (\theta_t C_0, -\theta_t \log |z|^2 + \log \max\{t^2|z|^{2\theta_t}, 1\}),$$

where $\theta_t = \log t/(-\log \beta)$. Therefore, $(0, 0) \leq \overline{P} \leq \overline{P}_t$, and hence $\overline{P} = (0, 0)$ as $t \to 1$.

(3) If we set $\overline{D}'' = \overline{D} - \widehat{(z)}$, then $\overline{D}'' = (C_\infty, -\log |w|^2 + \log \max\{\beta^2|w|^2, \alpha^2\})$, where $w = y/x$. Thus, in the same way as (2), we can see that the positive part of $\overline{D}''$ is

$$(\theta' C_\infty, -\theta' \log |w|^2 + \log \max\{\beta^2|w|^{2\theta'}, 1\}),$$

where $\theta' = \log \beta/(\log \beta - \log \alpha)$, so that the positive part of $\overline{D} = \overline{D}'' + \widehat{(z)}$ is

$$(C_0 - (1 - \theta')C_\infty, -\log |z|^2 + \log \max\{|z|^{2\theta'}, \beta^2\})$$

by Proposition 9.3.1. □

Remark 9.4.2. Let us choose $\alpha, \alpha', \beta, \beta' \in \mathbb{R}_{>0}$ such that $\alpha \geq 1$, $\alpha' \geq 1$, $\alpha\beta' < 1$ and $\alpha'\beta < 1$. We set

$$M = C_0 + C_\infty, \quad \varphi = -\log |z|^2 + \log \max\{\alpha^2|z|^2, \beta^2\} + \log \max\{\alpha'^2, \beta'^2|z|^2\}$$

and $\overline{M} = (M, \varphi)$, that is,

$$\varphi = \begin{cases} -\log |z|^2 + \log(\alpha'\beta)^2 & \text{if } |z| \leq \beta/\alpha, \\ \log(\alpha\alpha')^2 & \text{if } \beta/\alpha \leq |z| \leq \alpha'/\beta', \\ \log |z|^2 + \log(\alpha\beta')^2 & \text{if } |z| \geq \alpha'/\beta'. \end{cases}$$

It is easy to see that $\overline{M}$ is an effective arithmetic Cartier divisor of $(C^0 \cap \mathrm{PSH})$-type and that

$$\widehat{\deg}(\overline{M}\big|_{C_0}) = \log(\alpha'\beta) \quad \text{and} \quad \widehat{\deg}(\overline{M}\big|_{C_\infty}) = \log(\alpha\beta').$$

If we set

$$\vartheta = \frac{\log \alpha + \log \alpha'}{\log \alpha - \log \beta}, \quad \vartheta' = \frac{\log \alpha + \log \alpha'}{\log \alpha' - \log \beta'}$$

and

$$\psi = -\vartheta \log |z|^2 + \log \max\{\alpha^2|z|^{2\vartheta}, \alpha'^{-2}\} + \log \max\{\alpha'^2, \alpha^{-2}|z|^{2\vartheta'}\},$$

that is,

$$\psi = \begin{cases} -\vartheta \log |z|^2 & \text{if } |z| \leq \beta/\alpha, \\ \log(\alpha\alpha')^2 & \text{if } \beta/\alpha \leq |z| \leq \alpha'/\beta', \\ \vartheta' \log |z|^2 & \text{if } |z| \geq \alpha'/\beta', \end{cases}$$

then the positive part of $\overline{M}$ is

$$(\vartheta C_0 + \vartheta' C_\infty, \psi).$$

This can be checked in the similar way as Proposition 9.4.1. For details, we leave it to the readers. In the case where $\alpha = \alpha' = 1$, the negative part of $\overline{M}$ is $\overline{M}$ itself, which means that the support of the negative part contains C_0 and C_∞ despite $C_0 - C_\infty = (z)$. This example also show that if the positive parts of $\overline{D}$ and $\overline{D}'$ are $\overline{P}$ and $\overline{P}'$ respectively, then the positive part of $\overline{D} + \overline{D}'$ is not necessarily $\overline{P} + \overline{P}'$.

Remark 9.4.3. Let λ be a positive real number. We set

$$\phi_\lambda = -\log|z|^2 + \log(|z|^2 + \lambda) \quad \text{and} \quad \overline{M}_\lambda = (C_0, \phi_\lambda).$$

We denote $\overline{M}_1$ by $\overline{L}$, that is, $\overline{L} = (C_0, -\log|z|^2 + \log(|z|^2 + 1))$. It is easy to see that $\overline{M}_\lambda$ is an arithmetic Cartier divisor of $(C^\infty \cap \mathrm{PSH})$-type, $\widehat{\deg}(\overline{M}_\lambda^2) = (\log(\lambda) + 1)/2$ and that $\overline{M}_\lambda$ is nef for $\lambda \geq 1$. In particular, $\overline{M}_\lambda$ is big for $\lambda \geq 1$.

From now on, we fix λ with $0 < \lambda < 1$. By using an inequality:

$$\log(1 + \lambda x) \geq \lambda \log(1 + x) \quad (x \in \mathbb{R}_{\geq 0}),$$

we can see that $\lambda\overline{L} \leq \overline{M}_\lambda$, which means that $\overline{M}_\lambda$ is big. On the other hand,

$$\widehat{\deg}(\overline{M}_\lambda\big|_{C_0}) = \log(\lambda) < 0,$$

so that $\overline{M}_\lambda$ is not nef. We set

$$\Phi_\lambda = dd^c(\log(|z|^2 + \lambda)) = \frac{\lambda}{2\pi\sqrt{-1}(|z|^2 + \lambda)^2} dz \wedge d\bar{z},$$

which gives rise to an F_∞-invariant volume form on $\mathbb{P}^1(\mathbb{C})$ with $\int_{\mathbb{P}^1(\mathbb{C})} \Phi_\lambda = 1$. Moreover, we set

$$\widehat{\mathrm{Div}}_{\Phi_\lambda}(\mathbb{P}^1_{\mathbb{Z}})_{\mathbb{R}} = \left\{ (A, g_A) \;\middle|\; \begin{array}{l} (1)\ A \text{ is an } \mathbb{R}\text{-Cartier divisor on } \mathbb{P}^1_{\mathbb{Z}}. \\ (2)\ g_A \text{ is an } F_\infty\text{-invariant } A\text{-Green function of } C^\infty\text{-type} \\ \quad\ \text{on } \mathbb{P}^1(\mathbb{C}) \text{ such that } dd^c([g_A]) + \delta_A = (\deg(A))\Phi_\lambda. \end{array} \right\},$$

which is the Arakelov Chow group consisting of admissible metrics with respect to Φ_λ due to Arakelov-Faltings [7]. Let us see that the set

$$\{(A, g_A) \in \widehat{\mathrm{Div}}_{\Phi_\lambda}(\mathbb{P}^1_{\mathbb{Z}})_{\mathbb{R}} \mid (A, g_A) \text{ is nef and } (0,0) \leq (A, g_A) \leq \overline{M}_\lambda\}$$

have only one element $(0, 0)$.

Indeed, let $\overline{A} = (A, g_A)$ be an element of the above set. Then there are constants a, b such that $0 \leq a \leq 1$ and $\overline{A} = a\overline{M}_\lambda + (0, b)$. Since $g_A \leq \phi_\lambda$, we have $b \leq (1 - a)\phi_\lambda$. Thus $b \leq 0$ because $\phi_\lambda(\infty) = 0$. In addition,

$$\widehat{\deg}(\overline{A}\big|_{C_0}) = a\log(\lambda) + b \geq 0.$$

In particular, $b \geq 0$, so that $b = 0$, and hence $a\log(\lambda) \geq 0$. Thus $a = 0$.

This example shows that the Arakelov Chow group consisting of admissible metrics is insufficient to get the Zariski decomposition.

Finally note that $\lambda\overline{L}$ is not necessarily the positive part of $\overline{M}_\lambda$ because $\widehat{\mathrm{vol}}(\overline{M}_\lambda) \geq (\log(\lambda) + 1)/2$ (cf. Theorem 6.6.1), $\widehat{\mathrm{vol}}(\lambda\overline{L}) = \lambda^2/2$ and $(\log(\lambda) + 1)/2 > \lambda^2/2$ for $0 < 1 - \lambda \ll 1$.

Remark 9.4.4. Let n be a positive integer and $f \in \mathbb{R}[T]$ such that $\deg(f) = 2n$ and $f(t) > 0$ for all $t \in \mathbb{R}_{\geq 0}$. It seems to be not easy to find the positive part of

$$\left(nC_0, -n \log |z|^2 + \log f(|z|)\right)$$

on $\mathbb{P}^1_{\mathbb{Z}}$.

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