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Author(s)	Strzalko, J.; Grabski, J.; Perlikowski, P.; Stefanski, A.; Kapitaniak, T.
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HOW CHAOTIC (OR RANDOM) IS THE DICE THROW?

J. Strzalko, J. Grabski, P.Perlikowski A. Stefanski, T. Kapitaniak

Division of Dynamics, Technical University of Lodz, Stefanowskiego 1/15, 90-924 Lodz, Poland

ABSTRACT

During the mathematical lessons in the primary school we are taught that the outcome of the coin tossing experiment is random and that probability that the tossed coin lands heads (tails) up is equal to the *1/2*. Approximately at the same time during the physics lessons we are told that the motion of the rigid body (coin is an example of such a body) is fully deterministic. Typically students are not given the answer to the question: *Why this duality in the interpretation of the simple mechanical experiment is possible?*

Trying to answer this question we describe the dynamics of gambling games based on the coin toss, the throw of the die and the roulette run. Dynamics of this type of gambling can be described in the terms of the Newtonian mechanics so one can expect that the outcome can be predicted. We give evidence that from the point of view of dynamical systems this dynamics is predictable. However due to the high (but finite) sensitivity to the initial conditions the very precise devises are necessary to predict the outcome so practically this outcome is pseudorandom. Our studies do not give the general answer to the famous Albert Einstein's question: *'Does the God play dice?'* which is connected with the all events in the whole universe but give evidence to the negative answer to the simpler question: *Does the God play dice in the casino?*

A throw of a fair die is commonly considered as a paradigm for chance and gambling. The die is usually a cube of a homogeneous material. The symmetry suggests that such a die has the same chance of landing on each of its six faces after a vigorous roll so it is considered to be fair. Generally, a die with a shape of convex polyhedron is fair by symmetry if and only if it is symmetric with respect to all its faces [1]. The polyhedra with this property are called the isohedra. Every isohedron has an even number of faces. The commonly known examples of isohedra are: tetrahedron, octahedron, dodecahedron and icosahedron which are also used as the shapes for dice. Diaconis and Keller [1] show that there are not symmetric polyhedra which are fair by continuity. As an example they consider the dual of *n*-prism which is a di-pyramid with *2n* identical triangular faces from which two tips have been cut with two planes parallel to the base and equidistant from it. If the cuts are close to the tips, the solid has a very small probability of landing on one of two tiny new faces. However, if the cuts are near the base, the probability of landing on them is high. Therefore by continuity there must be the cuts for which new and old faces have equal probability. It is suggested that the locations of these cuts depend upon the mechanical properties of the die and the table and can be found either experimentally or by the analysis based on the classical mechanics.

However, these definitions are not considering the dynamics of the die motion during the throw. This dynamics is described by perfectly deterministic laws of classical mechanics which map initial conditions (position, configuration, momentum and angular momentum) at the beginning of the motion into one of the final configurations defined by the number on the face on which the die lands. From the point of view of the dynamical systems the outcome from the die throw is deterministic, but as the initial condition--final configuration mapping is strongly nonlinear, one can expect deterministic unpredictability due to the sensitive dependence on the initial conditions and fractal boundaries between the basins of different final configurations. The implementation of this idea has been carried out in a few papers which deal mainly with the coin tossing problem [2-4]. In these works the consideration of the various simplified models of the coin motion (for the detailed discussion see [5,6]) lead to the conclusion that the uncertainties of the outcome are due to the inability of setting the initial conditions in a sufficiently accurate way (for the precise initial conditions the coin tossing is predictable). In our previous work [5] we study the full 3-dimensional model of the coin which considers the non-homogeneous material of the coin, the air resistance, the elasticity and the friction of the floor on which the coin bounces. We show that the process of the coin bouncing on the floor has a significant influence on the final configuration (heads or tails) as the successive impacts introduce sensitive dependence on initial conditions leading to the transient chaotic behavior. With the increase of the number of bounces the structure of the basin boundaries becomes more complex so the distance of a typical initial condition from a basin boundary is so small that practically any uncertainty in initial conditions can lead to the uncertainty of the results of tossing. In theory, the result of the coin tossing is unpredictable only in the case of infinite bounces. However, for the realistic material of the coin and the floor only a few bounces can be observed (usually less than ten) so the transient chaos cannot be well developed.

In this paper, we consider the dynamics of 3-dimensional model of the die. We show that the probability of the die landing on the face, which is the lowest one at the beginning is larger than on any other face. The probabilities of landing on any face approach the same value *1/n* only for the large values of the initial rotational energy and great number of die bounces on the table. This case cannot be realized in the experiment due to the limitations of the initial energy (particularly when the die is thrown from the hand or cup) and dissipation of the energy during the bounces. This allows us to draw the conclusion that dice are not fair by dynamics.

A die is modeled as a rigid body of homogeneous material, i.e., the center of mass and the geometrical center are located in the same point. We consider the dice with isohedral shape and sharp edges and corners. (Precision casino dice have their pips drilled, and them filled flush with a paint of the same density as the acetate used for the dice, so their remains in balance. They also have sharp edges and corners.) The typical examples of such dice are: tetrahedron, cube, octahedron, dodecahedron, icosahedron. We neglected the influence of air resistance as in the work [5] it have been shown that the influence of the air resistance on the motion of the tossed coin is very small. For n face die there are n possible final configurations (the die can land on one of its faces F_i $(i=1,2,...)$. All initial conditions are mapped into one of the final configurations. The initial conditions which are mapped onto the i-th face configuration create i-th face basin of attraction $\beta(F_i)$. The boundaries which separate the basins of different faces consist of initial conditions mapped onto the die standing on its edge configuration which is unstable. For the dice with sharp edges one can assume that the set of initial conditions which are mapped into the boundaries has a zero Lebesque measure.

The possibility that the boundaries between the basins of different faces are fractal, particularly riddled [5] or intermingled ones is worth investigating. Near a given basin boundary, if the initial conditions are given with the uncertainty ϵ, then a fraction $f(\epsilon)$ of initial conditions gives the unpredictable outcome. In the limit $\epsilon \to 0, f(\epsilon) \approx \epsilon^{\alpha}$, where $\alpha<1$ for fractal and $\alpha=1$ for smooth boundary. From the point of view of the predictability of the die throw the possibility of the occurrence of the intermingled basins is the most interesting. Let us briefly explain the term of the intermingled basins of attraction. A basin $\beta(F_i)$ is called the riddled one when: (i) it has a positive Lebesque measure, (ii) for any point in $\beta(F_i)$, a ball in the phase space of arbitrarily small radius has a nonzero fraction of its volume in some other (say $\beta(F_j)$ basin. The basin $\beta(F_j)$ may or may not be riddled by the basin $\beta(F_i)$. If the basin $\beta(F_j)$ is also riddled by the basin $\beta(F_i)$ such basins are called the intermingled ones. In the case of the thrown die the intermingled basins of attraction of all n faces will mean, that in any neighborhood of the initial condition leading to one of F_i, there are initial conditions which are mapped to other faces, i.e., there does not exist an open set of initial conditions which is mapped to one of the final configurations or infinitely small inaccuracy in the initial conditions makes the result of the die throw unpredictable.

Analysis of the structure of the basin boundaries allow us to identify the condition under which the die throw is predictable and fair by dynamics.

Definition 1. *The die throw is predictable if for almost all initial conditions x_0 there exists an open set U ($x_0 \in U$) which is mapped into the given final configuration.*
Assume that the initial condition x_0 is set with the inaccuracy ϵ. Consider a ball B centered at x_0 with a radius ϵ. Definition 1.1 implies that if B is a subset of U then randomizer is predictable.

Definition 2. *The die throw is fair by dynamics if in the neighborhood of any initial condition leading to one of the n final configurations $F_1,...,F_i,...,F_n$, where $i=1,...,n$, there are sets of points $\beta(F_1),...,\beta(F_i),...,\beta(F_n)$, which lead to all other possible configurations and a measures of sets $\beta(F_i)$ are equal.*
Definition 2 implies that for the infinitely small inaccuracy of the initial conditions all final configurations are equally probable.

Our numerical studies allows as to state the following conclusion:

Main result: The result of the die throw is predictable according to the definition 1. In other words, if one can settle the initial condition with appropriate accuracy, the outcome of the coin tossing procedure is predictable and repeatable. In the most cases this accuracy should be very high, i.e., ϵ must be very small but can be positive.

To summarize, in this paper we consider the dynamics of the 3-dimensional model of the dice of different isohedral shapes. We show that the probability that the die lands on the face which is the lowest is larger than on any other face, i.e., the dice is not fair by the dynamics. If an experienced player can reproduce the initial conditions with a small finite uncertainty, there is a good chance that the desired final state will be obtained. The probabilities of landing on any face approach the same value $1/n$ only for the large values of the initial rotational energy and a great number of die bounces on the table. This can be done in computer simulations but not in the real experiment when a die is thrown from the hand or the cup as due to the limitation of the initial energy the die can bounce only a few times. Theoretically probabilities of landing on any face are equal only in the unrealistic Hamiltonian case of infinite number of die bounces on the table when the dynamics is chaotic. The case of the predictability of roulette is considered in [6].

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