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# QP-Structures of Degree 3 and 4D Topological Field Theory

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## Abstract

A BV algebra and a QP-structure of degree 3 is formulated. A QP-structure of degree 3 gives rise to Lie algebroids up to homotopy and its algebraic and geometric structure is analyzed. A new algebroid is constructed, which derives a new topological field theory in 4 dimensions by the AKSZ construction.

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# 1 Introduction

A BV algebra and a QP-structure has been motivated by the structure of the Batalin-Vilkovisky formalism of a gauge theory[1] and is its mathematical formulation [2]. In case of a topological field theory of Schwarz type, a BV formalism has been reformulated to the AKSZ formulation, which is a clear construction using geometry of a graded manifold [3][4]. Application to higher  $n + 1$  dimensions has been formulated and new topological field theories in higher dimensions have been founded by applying this construction [5][6][7].

In  $n = 1$ , a classical QP-structure is a Poisson structure on a manifold  $M$  and is also a Lie algebroid on  $T^*M$  from the explicit construction. This is equivalent to the construction of a Poisson structure by the Schouten-Nijenhuis bracket in a classical limit. The topological field theory in two dimensions constructed by the AKSZ formulation [4] is the Poisson sigma model [8][9] and the quantization of this model on disc derives the Kontsevich formula of the deformation quantization on a Poisson manifold [11][12].

In  $n = 2$ , a classical QP-structure is a Courant algebroid [13][14]. The topological field theory derived in three dimensions is the Courant sigma model [15][16][17].

However structures for higher  $n$ , more than 2, have not been understood enough apart from BF theories.

In this paper, we analyze  $n = 3$  case. A QP-structure of degree 3 leads us to a new type of algebroid, which is called a **Lie algebroid up to homotopy**. The notion of this algebroid is defined as a homotopy deformation of a Lie algebroid satisfying some integrability conditions. We will prove that a QP-structure of degree 3 on a N-manifold (nonnegatively graded manifold) is equivalent to a Lie algebroid up to homotopy. This QP-structure defines a new natural 4-dimensional topological field theory via the AKSZ construction.

The paper is organized as follows. In section 2, a BV algebra and a QP-structure of degree 3 are formulated. In section 3, a QP-structure of degree 3 is constructed and analyzed. In section 4, examples of QP-structures of degree 3 are listed. In section 5, the AKSZ construction of a topological field theory in four dimensions is formulated and examples are listed. <sup>c</sup>

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<sup>c</sup>Very recently, Grützmann's paper appears which has overlaps with our paper [18].

## 2 QP-manifolds and BV Algebras

### 2.1 Classical QP-manifold

**Definition 2.1** *A graded manifold  $\mathcal{M}$  is by definition a sheaf of a graded commutative algebra over an ordinary smooth manifold  $M$ .*

In the following, we assume the degrees are nonnegative.

The structure sheaf of  $\mathcal{M}$  is locally isomorphic to a graded commutative algebra  $C^\infty(U) \otimes S(V)$ , where  $U$  is an ordinary local chart of  $M$ ,  $S(V)$  is the polynomial algebra over  $V$  and where  $V := \sum_{i \geq 1} V_i$  is a graded vector space such that the dimension of  $V_i$  is finite for each  $i$ . For example, when  $V = V_1$ ,  $\mathcal{M}$  is a vector bundle whose fiber is  $V_1^*$ : the dual space of  $V_1$ .

**Definition 2.2** *A graded manifold  $(\mathcal{M}, \omega)$  equipped with a graded symplectic structure  $\omega$  of degree  $n$  is called a **P-manifold** of degree  $n$ .*

In the next section, we will study a concrete P-manifold of degree 3.

The structure sheaf  $C^\infty(\mathcal{M})$  of a P-manifold becomes a graded Poisson algebra. The Poisson bracket is defined in the usual manner,

$$\{F, G\} = (-1)^{|F|+1} \iota_{X_F} \iota_{X_G} \omega, \quad (2.1)$$

where  $F, G \in C^\infty(\mathcal{M})$ ,  $|F|$  is the degree of  $F$  and  $X_F := \{F, -\}$  is the Hamiltonian vector field of  $F$ . We recall the basic properties of the Poisson bracket,

$$\begin{aligned} \{F, G\} &= -(-1)^{(|F|-n)(|G|-n)} \{G, F\}, \\ \{F, GH\} &= \{F, G\}H + (-1)^{(|F|-n)|G|} G\{F, H\}, \\ \{F, \{G, H\}\} &= \{\{F, G\}, H\} + (-1)^{(|F|-n)(|G|-n)} \{G, \{F, H\}\}, \end{aligned}$$

where  $n$  is the degree of the symplectic structure and  $F, G, H \in C^\infty(\mathcal{M})$ . We remark that the degree of the Poisson bracket is  $-n$ .

**Definition 2.3** *Let  $(\mathcal{M}, \omega)$  be a P-manifold of degree  $n$ . A function  $\Theta \in C^\infty(\mathcal{M})$  of degree  $n+1$  is called a **Q-structure**, if it is a solution of the **classical master equation**,*

$$\{\Theta, \Theta\} = 0. \quad (2.2)$$

*The triple  $(\mathcal{M}, \omega, \Theta)$  is called a **QP-manifold**.*

We define an operator  $Q := \{\Theta, -\}$ , which is called a homological vector field. From (2.2) we have the cocycle condition,

$$Q^2 = 0,$$

which says that the homological vector field is a coboundary operator on  $C^\infty(\mathcal{M})$  and defines a cohomology called the classical BRST cohomology.

## 2.2 Quantum QP-manifold

**Definition 2.4** *A graded manifold is called a quantum BV-algebra if it has an odd Laplace operator  $\Delta$ , which is a linear operator on  $C^\infty(\mathcal{M})$  satisfying  $\Delta^2 = 0$ , and the graded Poisson bracket is given by*

$$\{F, G\} = (-1)^{|F|} \Delta(FG) - (-1)^{|F|} \Delta(F)G - F\Delta(G), \quad (2.3)$$

where  $F, G \in C^\infty(\mathcal{M})$ .

If  $n$  is odd, a P-manifold  $(\mathcal{M}, \omega)$  has the odd Poisson bracket. If an odd P-manifold  $(\mathcal{M}, \omega)$  has a volume form  $\rho$ , one can define an odd Laplace operator  $\Delta$  (See [19]):

$$\Delta F := \frac{1}{2}(-1)^{|F|} \operatorname{div}_\rho X_F.$$

Here a divergence  $\operatorname{div}_\rho$  is a map from a space of vector fields on  $\mathcal{M}$  to  $C^\infty(\mathcal{M})$  and is defined by

$$\int_{\mathcal{M}} \operatorname{div}_\rho X F dv = - \int_{\mathcal{M}} X(F) dv,$$

for a vector field  $X$  on  $\mathcal{M}$ . The pair  $(\mathcal{M}, \Delta)$  is called a **quantum P-structure**. An odd Laplace operator has degree  $-n$ .

**Definition 2.5** *A function  $\Theta \in C^\infty(\mathcal{M})$  with the degree  $n + 1$  is called a **quantum Q-structure**, if it satisfies a **quantum master equation***

$$\Delta(e^{\frac{i}{\hbar}\Theta}) = 0, \quad (2.4)$$

where  $\hbar$  is a formal parameter. The triple  $(\mathcal{M}, \Delta, \Theta)$  is called a **quantum QP-manifold**.

From the definition of an odd Laplace operator, the equation (2.4) is equivalent to

$$\{\Theta, \Theta\} - 2i\hbar\Delta\Theta = 0. \quad (2.5)$$

If we take the limit of  $\hbar \rightarrow 0$  in (2.5), which is called a classical limit, the classical master equation  $\{\Theta, \Theta\} = 0$  is derived. Since  $\Delta^2 = 0$ ,  $\Delta$  is also a coboundary operator. This defines a quantum BRST cohomology. Let  $\mathcal{O}' = \mathcal{O}e^{\frac{i}{\hbar}\Theta} \in C^\infty(\mathcal{M})$  be a cocycle with respect to  $\Delta$ . The cocycle condition  $\Delta(\mathcal{O}') = \Delta(\mathcal{O}e^{\frac{i}{\hbar}\Theta}) = 0$  is equivalent to

$$\{\Theta, \mathcal{O}\} - i\hbar\Delta\mathcal{O} = 0. \quad (2.6)$$

The solutions of (2.6) are called *observables* in physics. In the classical limit, (2.6) is  $\{\Theta, \mathcal{O}\} = Q\mathcal{O} = 0$ .  $\mathcal{O}$  reduces to an element of a classical BRST cohomology.

### 3 Structures and homotopy algebroids

In this section, we construct and analyze a classical QP-structure of degree 3 explicitly.

#### 3.1 P-structures

Let  $E \rightarrow M$  be a vector bundle over an ordinary smooth manifold  $M$ . The shifted bundle  $E[1] \rightarrow M$  is a graded manifold whose fiber space has the degree +1. We consider the shifted cotangent bundle  $\mathcal{M} := T^*[3]E[1]$ . It is a P-manifold of the degree 3 over  $M$ ,

$$T^*[3]E[1] \rightarrow \mathcal{M}_2 \rightarrow E[1] \rightarrow M,$$

where  $\mathcal{M}_2$  is a certain graded manifold <sup>d</sup>. The structure sheaf  $C^\infty(\mathcal{M})$  of  $\mathcal{M}$  is decomposed into the homogeneous subspaces,

$$C^\infty(\mathcal{M}) = \sum_{i \geq 0} C^i(\mathcal{M}),$$

where  $C^i(\mathcal{M})$  is the space of functions of degree  $i$ . In particular,  $C^0(\mathcal{M}) = C^\infty(M)$ : the algebra of smooth functions on the base manifold and  $C^1(\mathcal{M}) = \Gamma E^*$ : the space of sections of the dual bundle of  $E$ .

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<sup>d</sup>In fact,  $\mathcal{M}_2$  is  $E[1] \oplus E^*[2]$ , which is derived from the result in the previous sentence of Remark 3.2.

Let us denote by  $(x, q, p, \xi)$  a canonical (Darboux) coordinate on  $\mathcal{M}$ , where  $x$  is a smooth coordinate on  $M$ ,  $q$  is a fiber coordinate on  $E[1] \rightarrow M$ ,  $(\xi, p)$  is the momentum coordinate on  $T^*[3]E[1]$  for  $(x, q)$ . The degrees of the variables  $(x, q, p, \xi)$  are respectively  $(0, 1, 2, 3)$ .

Two directions of counting the degree of functions on  $T^*[3]E[1]$  are introduced. Roughly speaking, these are the fiber direction and the base direction.

**Definition 3.1** (*Bidegree, see also Remark 3.3.3 in [14]*) Consider a monomial  $\xi^i p^j q^k$  on a local chart  $(U; x, q, p, \xi)$  of  $\mathcal{M}$ , of which the total degree is  $3i + 2j + k$ . The **bidegree** of the monomial is, by definition,  $(2(i + j), i + k)$ .

This definition is invariant under the natural coordinate transformation,

$$\begin{aligned} x'_i &= x'_i(x_1, x_2, \dots, x_{\dim(M)}), \\ q'_i &= \sum_j t_{ij} q_j, \\ p'_i &= \sum_j t_{ij}^{-1} p_j, \\ \xi'_i &= \sum_j \frac{\partial x_j}{\partial x'_i} \xi_j + \sum_{jkl} \left( \frac{\partial t_{jl}^{-1}}{\partial x'_i} t_{lk} + \frac{\partial t_{kl}}{\partial x'_i} t_{lj}^{-1} \right) p_j q_k, \end{aligned}$$

where  $t$  is a transition function. Since  $T^*[3]E[1]$  is covered by the natural coordinates, the bidegree is globally well-defined (See also Remark 3.2 below.)

The space  $C^n(\mathcal{M})$  is uniquely decomposed into the homogeneous subspaces with respect to the bidegree,

$$C^n(\mathcal{M}) = \sum_{2i+j=n} C^{2i,j}(\mathcal{M}).$$

Since  $C^{2,0}(\mathcal{M}) = \Gamma E$  and  $C^{0,2}(\mathcal{M}) = \Gamma \wedge^2 E^*$ , we have

$$C^2(\mathcal{M}) = \Gamma E \oplus \Gamma \wedge^2 E^*.$$

**Remark 3.2** The  $P$ -manifold  $T^*[3]E[1]$  is regarded as a shifted manifold of  $T^*[2]E[1]$ . The structure sheaf is also a shifted sheaf of the one on  $T^*[2]E[1]$ . In particular, the space  $C^{2i,j}$  is the shifted space of  $C^{i,j}$  on  $T^*[2]E[1]$ .

For the canonical coordinate on  $\mathcal{M}$ , the symplectic structure has the following form:

$$\omega = \delta x^i \delta \xi_i + \delta q^a \delta p_a,$$

and the associated Poisson bracket has the following expression:

$$\{F, G\} = F \frac{\overleftarrow{\partial}}{\partial x^i} \frac{\overrightarrow{\partial}}{\partial \xi_i} G - F \frac{\overleftarrow{\partial}}{\partial \xi_i} \frac{\overrightarrow{\partial}}{\partial x^i} G + F \frac{\overleftarrow{\partial}}{\partial q^a} \frac{\overrightarrow{\partial}}{\partial p_a} G - F \frac{\overleftarrow{\partial}}{\partial p_a} \frac{\overrightarrow{\partial}}{\partial q^a} G,$$

where  $F, G \in C^\infty(\mathcal{M})$  and  $\frac{\overrightarrow{\partial}}{\partial \phi}$  and  $\frac{\overleftarrow{\partial}}{\partial \phi}$  are the right and left differentiations, respectively. Note that the degree of the symplectic structure is +3 and the one of the Poisson bracket is -3. The bidegree of the Poisson bracket is  $(-2, -1)$ , that is,

$$\{(2i, j), (2k, l)\} = (2(i+k) - 2, j+l-1),$$

where  $(2i, j) \dots$  are functions with the bidgree  $(2i, j)$ .

### 3.2 Q-structures

We consider a (classical) Q-structure,  $\Theta$ , on the P-manifold. It is required that  $\Theta$  has degree 4. That is,  $\Theta \in C^4(\mathcal{M})$ . Because  $C^4(\mathcal{M}) = C^{4,0}(\mathcal{M}) \oplus C^{2,2}(\mathcal{M}) \oplus C^{0,4}(\mathcal{M})$ , the Q-structure is uniquely decomposed into

$$\Theta = \theta_2 + \theta_{13} + \theta_4,$$

where the bidegrees of the substructures are  $(4, 0)$ ,  $(2, 2)$  and  $(0, 4)$ , respectively. In the canonical coordinate,  $\Theta$  is the following polynomial:

$$\Theta = f_1^i{}_a(x) \xi_i q^a + \frac{1}{2} f_2^{ab}(x) p_a p_b + \frac{1}{2} f_3^a{}_{bc}(x) p_a q^b q^c + \frac{1}{4!} f_{4abcd}(x) q^a q^b q^c q^d, \quad (3.7)$$

and the substructures are

$$\begin{aligned} \theta_2 &= \frac{1}{2} f_2^{ab}(x) p_a p_b, \\ \theta_{13} &= f_1^i{}_a(x) \xi_i q^a + \frac{1}{2} f_3^a{}_{bc}(x) p_a q^b q^c, \\ \theta_4 &= \frac{1}{4!} f_{4abcd}(x) q^a q^b q^c q^d, \end{aligned}$$

where  $f_1$ - $f_4$  are structure functions on  $M$ . By counting the bidegree, one can easily prove that the classical master equation  $\{\Theta, \Theta\} = 0$  is equivalent to the following three identities:

$$\{\theta_{13}, \theta_2\} = 0, \quad (3.8)$$

$$\frac{1}{2} \{\theta_{13}, \theta_{13}\} + \{\theta_2, \theta_4\} = 0, \quad (3.9)$$

$$\{\theta_{13}, \theta_4\} = 0. \quad (3.10)$$



The conditions (3.8), (3.9) and (3.10) are equivalent to

$$f_1^i{}_b f_2^{ba} = 0, \quad (3.11)$$

$$f_1^k{}_c \frac{\partial f_2^{ab}}{\partial x^k} + f_2^{da} f_3^b{}_{cd} + f_2^{db} f_3^a{}_{cd} = 0, \quad (3.12)$$

$$f_1^k{}_b \frac{\partial f_1^i{}_a}{\partial x^k} - f_1^k{}_a \frac{\partial f_1^i{}_b}{\partial x^k} + f_1^i{}_c f_3^c{}_{ab} = 0, \quad (3.13)$$

$$f_1^k{}_{[d} \frac{\partial f_3^a{}_{bc]}{\partial x^k} + f_2^{ae} f_{4bcde} - f_3^a{}_e [b f_3^e{}_{cd}] = 0, \quad (3.14)$$

$$f_1^k{}_{[a} \frac{\partial f_{4bcde]}{\partial x^k} + f_3^f{}_{[ab} f_{4cde]} f = 0, \quad (3.15)$$

where  $[b\ c\ d\ \dots]$  is a skewsymmetrization with respect to indices  $b, c, d, \dots$ , etc.

### 3.3 Lie algebroid up to homotopy

In this section we study an algebraic structure associated with the QP-structure in 3.1 and 3.2.

**Definition 3.3** *Let  $Q = \theta_2 + \theta_{13} + \theta_4$  be a  $Q$ -structure on  $T^*[3]E[1]$ , where  $(\theta_2, \theta_{13}, \theta_4)$  is the unique decomposition of  $\Theta$ . We call the quadruple  $(E; \theta_2, \theta_{13}, \theta_4)$  a **Lie algebroid up to homotopy**, in shorthand, *Lie algebroid u.t.h.**

We should study the algebraic properties of the Lie algebroid up to homotopy. Let us define a bracket product by

$$[e_1, e_2] := \{\{\theta_{13}, e_1\}, e_2\}, \quad (3.16)$$

where  $e_1, e_2 \in \Gamma E$ . By the bidegree counting,  $\Gamma E$  is closed under this bracket. The bracket is not necessarily a Lie bracket, but it is still skewsymmetric:

$$\begin{aligned} [e_1, e_2] &= \{\{\theta_{13}, e_1\}, e_2\}, \\ &= \{\theta_{13}, \{e_1, e_2\}\} + \{e_1, \{\theta_{13}, e_2\}\}, \\ &= -\{\{\theta_{13}, e_2\}, e_1\} = -[e_2, e_1], \end{aligned}$$

where  $\{e_1, e_2\} = 0$  is used. A bundle map  $\rho : E \rightarrow TM$  which is called an anchor map is defined by the following identity:

$$\rho(e)(f) := \{\{\theta_{13}, e\}, f\},$$

where  $f \in C^\infty(M)$ . The bracket and the anchor map satisfy the algebroid conditions (A0) and (A1) below:

$$\text{(A0)} \quad \rho[e_1, e_2] = [\rho(e_1), \rho(e_2)],$$

$$\text{(A1)} \quad [e_1, fe_2] = f[e_1, e_2] + \rho(e_1)(f)e_2,$$

where the bracket  $[\rho(e_1), \rho(e_2)]$  is the usual Lie bracket on  $\Gamma TM$ . The bracket (3.16) does not satisfy the Jacobi identity in general. So we should study its Jacobi anomaly, which characterizes the algebraic structure of the Lie algebroid u.t.h. The structures  $\theta_{13}$ ,  $\theta_2$  and  $\theta_4$  define the three operations:

- $\delta(-) := \{\theta_{13}, -\}$ ; a de Rham type derivation on  $\Gamma \wedge E^*$ ,
- $(\alpha_1, \alpha_2) := \{\{\theta_2, \alpha_1\}, \alpha_2\}$ ; a symmetric pairing on  $E^*$ , where  $\alpha_1, \alpha_2 \in \Gamma E^*$ ,
- $\Omega(e_1, e_2, e_3, e_4) := \{\{\{\{\theta_4, e_1\}, e_2\}, e_3\}, e_4\}$ ; a 4-form on  $E$ .

Remark that  $\delta\delta \neq 0$  in general. Because the degree of the pairing is  $-2$ , it is  $C^\infty(M)$ -valued. The pairing induces a symmetric bundle map  $\partial : E^* \rightarrow E$  which is defined by the equation,  $(\alpha_1, \alpha_2) = \langle \partial\alpha_1, \alpha_2 \rangle$ , where  $\langle -, - \rangle$  is the canonical pairing of the duality of  $E$  and  $E^*$ . Since  $\langle \alpha, e \rangle = \{\alpha, e\}$ , we have

$$\partial\alpha = -\{\theta_2, \alpha\}.$$

By direct computation, we obtain

$$\frac{1}{2} \{\{\{\{\theta_{13}, \theta_{13}\}, e_1\}, e_2\}, e_3\} = [[e_1, e_2], e_3] + (\text{cyclic permutations}),$$

and

$$\{\{\{\{\theta_2, \theta_4\}, e_1\}, e_2\}, e_3\} = -\partial\Omega(e_1, e_2, e_3).$$

From Eq. (3.9), we get an explicit formula of the Jacobi anomaly,

$$\text{(A2)} \quad [[e_1, e_2], e_3] + (\text{cyclic permutations}) = \partial\Omega(e_1, e_2, e_3).$$

In a similar way, we obtain the following identities:

$$\text{(A3)} \quad \rho\partial = 0,$$

$$\text{(A4)} \quad \rho(e)(\alpha_1, \alpha_2) = (\mathcal{L}_e\alpha_1, \alpha_2) + (\alpha_1, \mathcal{L}_e\alpha_2),$$

(A5)  $\delta\Omega = 0$ ,

where  $\mathcal{L}_e(-) := \{\{\theta_{13}, e\}, -\}$  is the Lie type derivation which acts on  $E^*$ . Axioms (A3) and (A4) are induced from Eq. (3.8) and (A5) is from Eq. (3.10).

The fundamental relations (3.11)–(3.15) correspond to Axioms (A1)–(A5)<sup>e</sup>. Thus, the notion of the Lie algebroid up to homotopy is characterized by the algebraic properties (A1)–(A5). One concludes that

*The classical algebra associated with the QP-manifold  $(T^*[3]E[1], \Theta)$  is the space of sections of the vector bundle  $E$  with the operations  $([\cdot, \cdot], \rho, \partial, \Omega)$  satisfying (A1)–(A5).*

In the next section, we will study some special examples of Lie algebroid u.t.h.s.

**Remark 3.4** If the pairing is nondegenerate, then the bundle map  $\partial$  is bijective and then from (A3) we have  $\rho = 0$ .

**Remark 3.5** (Higher Courant-Dorfman brackets) We define a bracket on  $C^\infty(\mathcal{M})$  by

$$[-, -]_{CD} := \{\{\Theta, -\}, -\},$$

which is called a Courant-Dorfman (CD) bracket. It is well-known that  $[\cdot, \cdot]_{CD}$  is a Loday bracket ([20]). Since the degree of the CD-bracket is  $-2$ , the total space of degree  $i \leq 2$ ,

$$C^2(\mathcal{M}) \oplus C^1(\mathcal{M}) \oplus C^0(M)$$

is closed under the CD-bracket, in particular, the top space  $C^2(\mathcal{M}) = \Gamma(E \oplus \wedge^2 E^*)$  is a subalgebra. If  $\theta_2 = 0$ , the CD-bracket on  $E \oplus \wedge^2 E^*$  has the following form,

$$[e_1 + \beta_1, e_2 + \beta_2]_{CD} = [e_1, e_2] + \mathcal{L}_{e_1}\beta_2 - i_{e_2}\delta\beta_1 + \Omega(e_1, e_2),$$

where  $\beta_1, \beta_2 \in \Gamma\wedge^2 E^*$ . This CD-bracket is regarded as a higher analogue of Courant-Dorfman's original bracket (cf. [13]). We refer the reader to Hagiwara [21] and Sheng [22] for the detailed study of the higher CD-brackets.

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<sup>e</sup>Actually, the axiom (A0) depends on (A1) and (A2).

## 4 Examples and twisting transformations

### 4.1 The cases of $\theta_2 = \theta_4 = 0$

In this case, the bracket (3.16) satisfies (A0), (A1) and the Jacobi identity. Therefore, the bundle  $E \rightarrow M$  becomes a Lie algebroid:

**Definition 4.1** ([23]) *A Lie algebroid over a manifold  $M$  is a vector bundle  $E \rightarrow M$  with a Lie algebra structure on the space of the sections  $\Gamma(E)$  defined by the bracket  $[e_1, e_2]$  for  $e_1, e_2 \in \Gamma(E)$  and an anchor map  $\rho : E \rightarrow TM$  satisfying (A0) and (A1) above.*

We take  $\{e_a\}$  as a local basis of  $\Gamma E$  and let a local expression of an anchor map be  $\rho(e_a) = f^i_{1a}(x) \frac{\partial}{\partial x^i}$  and a Lie bracket be  $[e_b, e_c] = f^a_{bc}(x) e_a$ . The Q-structure  $\Theta$  associated with the Lie algebroid  $E$  is defined as a function on  $T^*[3]E[1]$ ,

$$\Theta := \theta_{13} := f^i_{1a}(x) \xi_i q^a + \frac{1}{2} f^a_{bc}(x) p_a q^b q^c,$$

which is globally well-defined. Conversely, if we consider  $\Theta := \theta_{13}$ , the classical master equation induces the Lie algebroid structure on  $E$ .

Let us consider the case that the bundle is a vector space on a point. A Lie algebroid over a point  $\mathfrak{g} \rightarrow \{pt\}$  is a Lie algebra  $\mathfrak{g}$ . The P-manifold over  $\mathfrak{g} \rightarrow \{pt\}$  is isomorphic to  $\mathfrak{g}^*[2] \oplus \mathfrak{g}[1]$  and the structure sheaf is the polynomial algebra over  $\mathfrak{g}[2] \oplus \mathfrak{g}^*[1]$ ,

$$C^\infty(\mathcal{M}) = S(\mathfrak{g}) \otimes \bigwedge^{\cdot} \mathfrak{g}^*.$$

The bidegree is defined by the natural manner,

$$C^{2i,j}(\mathcal{M}) = S^i(\mathfrak{g}) \otimes \bigwedge^j \mathfrak{g}^*.$$

The Q-structure associated with the Lie bracket on  $\mathfrak{g}$  is

$$\theta_{13} = \frac{1}{2} f^a_{bc} p_a q^b q^c \cong \frac{1}{2} f^a_{bc} p_a \otimes (q^b \wedge q^c), \quad (4.17)$$

where  $p. \in \mathfrak{g}$ ,  $q. \in \mathfrak{g}^*$  and  $f^a_{bc}$  is the structure constant of the Lie algebra.

## 4.2 The cases of $\theta_2 \neq 0$ and $\theta_4 = 0$

In this case, the bracket induced by  $\theta_{13}$  still satisfies the Jacobi identity.

We assume that  $\mathfrak{g}$  is semi-simple. Then the dual space  $\mathfrak{g}^*$  has a metric,  $(\cdot, \cdot)_{K^{-1}}$ , which is the inverse of the Killing form on  $\mathfrak{g}$ . The metric inherits the following invariant condition from the Killing form:

$$(\mathcal{L}_p q_1, q_2)_{K^{-1}} + (q_1, \mathcal{L}_p q_2)_{K^{-1}} = 0, \quad (4.18)$$

where  $\mathcal{L}_p(-)$  is the canonical coadjoint action of  $\mathfrak{g}$  to  $\mathfrak{g}^*$ . Eq. (4.18) is a linear version of (A4). Thus, we obtain a Q-structure,

$$\Theta := k^{ab} p_a p_b + \frac{1}{2} f^a{}_{bc} p_a q^b q^c, \quad (4.19)$$

where  $k^{ab} p_a p_b := (\cdot, \cdot)_{K^{-1}}$ .

## 4.3 Non Lie algebra example

We consider the cases that the Jacobi identity is broken. Let  $(\mathfrak{g}, [\cdot, \cdot], (\cdot, \cdot)_K)$  be a vector space (not necessarily Lie algebra) equipped with a skewsymmetric bracket  $[\cdot, \cdot]$  and an invariant metric  $(\cdot, \cdot)_K$ . The metric induces a bijection  $K : \mathfrak{g} \rightarrow \mathfrak{g}^*$  which is defined by the identity,

$$(p_1, p_2)_K = \langle K p_1, p_2 \rangle.$$

We define a map from  $\mathfrak{g}^*$  to  $\mathfrak{g}$  by  $\partial := K^{-1}$  and define a 4-form by,

$$\Omega(p_1, p_2, p_3, p_4) := \left( [[p_1, p_2], p_3] + \text{cyclic permutations}, p_4 \right)_K.$$

**Remark 4.2** *The 4-form above is considered to be a higher analogue of the Cartan 3-form  $([p_1, p_2], p_3)_K$ .*

Axioms (A0)–(A4) obviously hold on  $\mathfrak{g}$ . We check (A5). It suffices to show (3.10). Let us denote by  $\{-, p_1, p_2, \dots, p_n\}$  the n-fold bracket  $\{\dots\{-, p_1\}, p_2\}, \dots, p_n\}$ . We already have (3.8) and (3.9). From  $\{\theta_{13}, \{\theta_{13}, \theta_{13}\}\} = 0$  and (3.9), we have  $\{\theta_{13}, \{\theta_2, \theta_4\}\} = 0$ . Since  $\{\theta_{13}, \theta_2\} = 0$ , this is equal to  $\{\theta_2, \{\theta_3, \theta_4\}\} = 0$  up to sign. This gives  $\{\{\theta_2, \{\theta_3, \theta_4\}\}, p_1, \dots, p_5\} = 0$  for any  $p_1, \dots, p_5$ . From  $\{\theta_2, p\} = 0$ , we have

$$\{\theta_2, \{\{\theta_3, \theta_4\}, p_1, \dots, p_5\}\} = 0.$$

Since  $K^{-1} = -\{\theta_2, -\}$  is bijective, we get

$$\{\{\theta_3, \theta_4\}, p_1, \dots, p_5\} = 0,$$

which yields the desired relation  $\{\theta_3, \theta_4\} = 0$ .

**Proposition 4.3** *The triple  $(\mathfrak{g}, \partial, \Omega)$  is a Lie algebra(oid) up to homotopy.*

#### 4.4 Twisting by 3-form and the cases of $\theta_2 = 0$ and $\theta_4 \neq 0$

We introduce the notion of twisting transformation by 3-form before studying the cases of  $\theta_2 = 0$ . Given a Q-structure  $\Theta$  and a 3-form  $\phi \in C^{0,3}(\mathcal{M})$ , there exists the second Q-structure which is defined by the canonical transformation,

$$\Theta^\phi := \exp(X_\phi)(\Theta), \quad (4.20)$$

where  $X_\phi := \{\phi, -\}$  is the Hamiltonian vector field of  $\phi$ . The transformation (4.20) is called a **twisting by 3-form**, or simply twisting. By a direct computation, we obtain

$$\begin{aligned} \theta_2^\phi &= \theta_2, \\ \theta_{13}^\phi &= \theta_{13} - \{\theta_2, \phi\}, \\ \theta_4^\phi &= \theta_4 - \{\theta_{13}, \phi\} + \frac{1}{2}\{\{\theta_2, \phi\}, \phi\}, \end{aligned}$$

where  $\Theta^\phi = \theta_2^\phi + \theta_{13}^\phi + \theta_4^\phi$  and  $X_\phi^{i \geq 3}(\Theta) = 0$ . The twisting by 3-form defines an equivalence relation on the Q-structures.

We notice that  $\theta_2$  is an invariant for the twisting. If  $\theta_2 = 0$ , then  $\theta_{13}$  is an invariant and

$$\theta_4^\phi = \theta_4 - \delta\phi,$$

where  $\delta\phi = \{\theta_{13}, \phi\}$ . This leads us

**Proposition 4.4** *The class of Q-structures which have no  $\theta_2$  is classified into  $H_{dR}^4(\wedge E^*, \delta)$  by the twisting by 3-form.*

# 5 AKSZ Construction of Topological Field Theory in 4 Dimensions

## 5.1 General Theory

In this section, we consider the AKSZ construction of a topological field theory in 4 dimensions.

For a graded manifold  $\mathcal{N}$ , let  $\mathcal{N}|_0$  be the degree zero part.

Let  $X$  be a manifold in 4 dimensions and  $M$  be a manifold in  $d$  dimensions. Let  $(\mathcal{X}, D)$  be a differential graded (dg) manifold  $\mathcal{X}$  with a  $D$ -invariant nondegenerate measure  $\mu$ , such that  $\mathcal{X}|_0 = X$ , where  $D$  is a differential on  $\mathcal{X}$ .  $(\mathcal{M}, \omega, \Theta)$  is a QP-manifold of degree 3 and  $\mathcal{M}|_0 = M$ . A degree  $\deg(-)$  on  $\mathcal{X}$  is called the *form degree* and a degree  $\text{gh}(-)$  on  $\mathcal{M}$  is called the *ghost number*<sup>f</sup>. Let  $\text{Map}(\mathcal{X}, \mathcal{M})$  be a space of smooth maps from  $\mathcal{X}$  to  $\mathcal{M}$ .  $|-| = \deg(-) + \text{gh}(-)$  is the degree on  $\text{Map}(\mathcal{X}, \mathcal{M})$  and called the *total degree*. A QP-structure on  $\text{Map}(\mathcal{X}, \mathcal{M})$  is constructed from the above data.

Since  $\text{Diff}(\mathcal{X}) \times \text{Diff}(\mathcal{M})$  naturally acts on  $\text{Map}(\mathcal{X}, \mathcal{M})$ ,  $D$  and  $Q$  induce homological vector fields on  $\text{Map}(\mathcal{X}, \mathcal{M})$ ,  $\hat{D}$  and  $\check{Q}$ .

Two maps are introduced. An *evaluation map*  $\text{ev} : \mathcal{X} \times \mathcal{M}^{\mathcal{X}} \rightarrow \mathcal{M}$  is defined as

$$\text{ev} : (z, \Phi) \mapsto \Phi(z),$$

where  $z \in \mathcal{X}$  and  $\Phi \in \mathcal{M}^{\mathcal{X}}$ .

A *chain map*  $\mu_* : \Omega^\bullet(\mathcal{X} \times \mathcal{M}) \rightarrow \Omega^\bullet(\mathcal{M})$  is defined as  $\mu_* F = \int_{\mathcal{X}} \mu F$  where  $F \in \Omega^\bullet(\mathcal{X} \times \mathcal{M})$  and  $\int_{\mathcal{X}} \mu$  is an integration on  $\mathcal{X}$  by the  $D$ -invariant measure  $\mu$ . It is an usual integral for the even degree parts and the Berezin integral for the odd degree parts.

A (classical) P-structure on  $\text{Map}(\mathcal{X}, \mathcal{M})$  is defined as follows:

**Definition 5.1** For a graded symplectic form  $\omega$  on  $\mathcal{M}$ , a graded symplectic form  $\omega$  on  $\text{Map}(\mathcal{X}, \mathcal{M})$  is defined as  $\omega := \mu_* \text{ev}^* \omega$ .

We can confirm that  $\omega$  satisfies the definition of a graded symplectic form because  $\mu_* \text{ev}^*$  preserves nondegeneracy and closedness. Thus  $\omega$  is a P-structure on  $\text{Map}(\mathcal{X}, \mathcal{M})$  and induces

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<sup>f</sup>The ghost number  $\text{gh}(-)$  is the degree  $|-|$  on  $\mathcal{M}$  in section 2.

a graded Poisson bracket  $\{-, -\}$  on  $\text{Map}(\mathcal{X}, \mathcal{M})$ . Since  $|\mu_*\text{ev}^*| = -4$ ,  $|\omega| = -1$  and  $\{-, -\}$  on  $\text{Map}(\mathcal{X}, \mathcal{M})$  has degree 1 and an odd Poisson bracket.

Next we define a Q-structure  $S$  on  $\text{Map}(\mathcal{X}, \mathcal{M})$ .  $S$  is called a *BV action* and consists of two parts  $S = S_0 + S_1$ .  $S_0$  is constructed as follows: Let  $\omega$  be the odd symplectic form on  $\mathcal{M}$ . We take a fundamental form  $\vartheta$  such that  $\omega = -d\vartheta$  and define  $S_0 := \iota_{\hat{D}}\mu_*\text{ev}^*\vartheta$ .  $|S_0| = 0$  because  $\mu_*\text{ev}^*$  has degree  $-4$ .  $S_1$  is constructed as follows: We take a Q-structure  $\Theta$  on  $\mathcal{M}$  and define  $S_1 := \mu_*\text{ev}^*\Theta$ .  $S_1$  also has degree 0.

We can prove that  $S$  is a Q-structure on  $\text{Map}(\mathcal{X}, \mathcal{M})$ , since

$$\{\Theta, \Theta\} = 0, \iff \{S, S\} = 0 \quad (5.21)$$

from the definition of  $S_0$  and  $S_1$ .

A quantum version is

$$\Delta(e^{\frac{i}{\hbar}\Theta}) = 0 \iff \hat{\Delta}(e^{\frac{i}{\hbar}S}) = 0, \quad (5.22)$$

where  $\hat{\Delta}$  is an odd Laplace operator on  $\text{Map}(\mathcal{X}, \mathcal{M})$ . The infinitesimal form of the right hand side in (5.22) is  $\{S, S\} - 2i\hbar\hat{\Delta}S = 0$ , which is called a *quantum master equation*.<sup>9</sup>

The following theorem has been confirmed [3]:

**Theorem 5.2** *If  $\mathcal{X}$  is a dg manifold and  $\mathcal{M}$  is a QP-manifold, the graded manifold  $\text{Map}(\mathcal{X}, \mathcal{M})$  has a QP-structure.*

**Definition 5.3** *A topological field theory in 4 dimensions is a triple  $(\mathcal{X}, \mathcal{M}, S)$ , where  $\mathcal{X}$  is a dg manifold with  $\dim \mathcal{X}|_0 = 4$ ,  $\mathcal{M}$  is a QP-manifold with the degree 3, and  $S$  is a BV action with the total degree 0.*

In order to interpret this theory as a ‘physical’ topological field theory, we must take  $\mathcal{X} = T[1]X$ . Then we can confirm that a QP-structure on  $\text{Map}(\mathcal{X}, \mathcal{M})$  is equivalent to the AKSZ formulation of a topological field theory [4][10]. We set  $\mathcal{X} = T[1]X$  from now.

In ‘physics’, a quantum field theory is constructed by quantizing a classical field theory. First we consider a Q-structure  $\{\cdot, \cdot\}$  and a classical P-structure  $S$  such that

$$\{S, S\} = 0.$$

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<sup>9</sup>Discussion for an odd Laplace operator is too naive. In general, the quantum master equation has an obstruction expressed by the modular class [24]. We must regularize an odd Laplace operator and a quantum BV action.



Next we define a quantum P-structure  $\hat{\Delta}$  and confirm that

$$\tilde{\Delta}(e^{\frac{i}{\hbar}S}) = 0.$$

Finally we calculate a partition function

$$Z = \int_{\mathcal{L}} e^{\frac{i}{\hbar}S},$$

on a Lagrangian submanifold  $\mathcal{L} \subset \text{Map}(\mathcal{X}, \mathcal{M})$ . Quantization is not discussed in this paper.

## 5.2 Local Coordinate Expression and Examples

A general theory in the previous subsection is applied to the local coordinate expression in section 3.1 and a known topological field theory in 4 dimensions is obtained as a special case and a new nontrivial topological field theory is constructed. Let us take a manifold  $X$  in 4 dimensions and a manifold  $M$  in  $d$  dimensions. Let  $E[1]$  is a graded vector bundle on  $M$ . We take  $\mathcal{X} = T[1]X$  and  $\mathcal{M} = T^*[3]E[1]$ .

Let  $(\sigma^\mu, \theta^\mu)$  be a local coordinate on  $T[1]X$ .  $\sigma^\mu$  is a local coordinate on the base manifold  $X$  and  $\theta^\mu$  is one on the fiber of  $T[1]X$ , respectively. Let  $\mathbf{x}^i$  be a smooth map  $\mathbf{x}^i : X \rightarrow M$  and  $\xi_i$  be a section of  $T^*[1]X \otimes \mathbf{x}^*(T^*[3]M)$ ,  $\mathbf{q}^a$  be a section of  $T^*[1]X \otimes \mathbf{x}^*(E[1])$  and  $\mathbf{p}_a$  be a section of  $T^*[1]X \otimes \mathbf{x}^*(T^*[3]E\mathbf{x}[1])$ . These are called *superfields*. The exterior derivative  $d$  is taken as a differential  $D$  on  $X$ . From  $d$ , a differential  $\mathbf{d} = \theta^\mu \frac{\partial}{\partial \sigma^\mu}$  on  $\mathcal{X}$  is induced.

Then a BV action  $S$  has the following expression:

$$\begin{aligned} S &= S_0 + S_1, \\ S_0 &= \int_{\mathcal{X}} \mu (\xi_i \mathbf{d}\mathbf{x}^i - \mathbf{p}_a \mathbf{d}\mathbf{q}^a), \\ S_1 &= \int_{\mathcal{X}} \mu (f_1^i{}_a(\mathbf{x}) \xi_i \mathbf{q}^a + \frac{1}{2} f_2^{ab}(\mathbf{x}) \mathbf{p}_a \mathbf{p}_b + \frac{1}{2} f_3^a{}_{bc}(\mathbf{x}) \mathbf{p}_a \mathbf{q}^b \mathbf{q}^c + \frac{1}{4!} f_{4abcd}(\mathbf{x}) \mathbf{q}^a \mathbf{q}^b \mathbf{q}^c \mathbf{q}^d). \end{aligned}$$

**Nonabelian BF theory.** Let  $\Theta$  be a Q-structure (4.17) for a Lie algebra  $\mathfrak{g}$ .  $\xi_i \mathbf{d}\mathbf{x}^i = 0$  since  $M = \{pt\}$ . If we define a curvature  $\mathbf{F}^a = \mathbf{d}\mathbf{q}^a - \frac{1}{2} f^a{}_{bc} \mathbf{q}^b \mathbf{q}^c$ , a Q-structure is

$$S = \int_{\mathcal{X}} \mu (-\mathbf{p}_a \mathbf{F}^a),$$

which is equivalent to a BV formalism for a nonabelian BF theory in 4 dimensions.

**Topological Yang-Mills Theory.** We take a nondegenerate Killing form  $(\cdot, \cdot)_K$  for a Lie algebra  $\mathfrak{g}$  and consider the Q-structure (4.19). A topological field theory constructed from (4.19) is

$$S = \int_{\mathcal{X}} \mu (-\mathbf{p}_a \mathbf{F}^a + k^{ab} \mathbf{p}_a \mathbf{p}_b).$$

This is equivalent to a topological Yang-Mills theory,

$$S = -\frac{1}{4} \int_{\mathcal{X}} \mu k_{ab} \mathbf{F}^a \mathbf{F}^b,$$

if we delete  $\mathbf{p}_a$  by the equations of motion.

**Nonassociative BF Theory.** Let us take a non Lie algebra  $(\mathfrak{g}, [\cdot, \cdot], (\cdot, \cdot)_K)$  in section 4.3. If we take  $M = \{pt\}$  and  $\mathcal{M} = \mathfrak{g}^*[2] \oplus \mathfrak{g}[1]$ ,  $(\mathfrak{g}, [\cdot, \cdot], (\cdot, \cdot)_K)$  leads a QP-structure with degree 3. In the canonical basis, it is expressed as

$$\begin{aligned} f_1^i{}_a(x) &= 0, & f_2^{ab}(x) &= K^{ab}, \\ f_3^a{}_{bc}(x) &= f^a{}_{bc}, & f_{4abcd}(x) &= K_{ae}^{-1} f^e{}_{f[b} f^f{}_{cd]}, \end{aligned}$$

where  $K^{ab} = (p_a, p_b)$  is nondegenerate and  $[p_a, p_b] = f^c{}_{ab} p_c$  is a nonassociative bracket and does not satisfy the Jacobi identity. The AKSZ construction derives a new nontrivial topological field theory in 4 dimensions. A BV action  $S$  has the following expression:

$$\begin{aligned} S &= \int_{\mathcal{X}} \mu (-\mathbf{p}_a d\mathbf{q}^a + \frac{1}{2} K^{ab} \mathbf{p}_a \mathbf{p}_b + \frac{1}{2} f^a{}_{bc} \mathbf{p}_a \mathbf{q}^b \mathbf{q}^c + \frac{1}{4!} K_{ae}^{-1} f^e{}_{f[b} f^f{}_{cd]} \mathbf{q}^a \mathbf{q}^b \mathbf{q}^c \mathbf{q}^d) \\ &= -\frac{1}{4} \int_{\mathcal{X}} \mu (K_{ab} \mathbf{F}^a \mathbf{F}^b + \frac{1}{3!} K_{ae}^{-1} f^e{}_{f[b} f^f{}_{cd]} \mathbf{q}^a \mathbf{q}^b \mathbf{q}^c \mathbf{q}^d). \end{aligned}$$

It is easily confirmed that  $\{S, S\} = 0$ .

**Topological 3-brane on  $Spin(7)$ -structure.** Let  $(M, \Omega)$  be an 8-dimensional  $Spin(7)$ -manifold. Here  $\Omega$  is a  $Spin(7)$  4-form, which satisfies  $d\Omega = 0$  and the selfdual condition  $\Omega = *\Omega$ . A  $Spin(7)$  structure is defined as the subgroup of  $GL(8)$  to preserve  $\Omega$ . The Q-structure on  $(TM, \Omega)$  is given by

$$\Theta = \xi_i q^i + \frac{1}{4!} \Omega_{ijkl}(x) q^i q^j q^k q^l. \quad (5.23)$$

The BV action  $S$  for (5.23) defines the same theory as the topological 3-brane analyzed in [25].

## 6 Conclusions and Discussion

We have defined a BV algebra and a QP-structure of degree 3. A QP-structure of degree 3 has been constructed explicitly and a Lie algebroid u.t.h. has been defined as its algebraic and geometric structure. A general theory of the AKSZ construction of a topological field theory has been expressed and a new topological field theory in four dimensions has been constructed from a QP-structure.

Quantization of this theory and analysis of a Lie algebroid u.t.h. will shed light on a super Poisson geometry and a quantum field theory. They are future problems.

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