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Author(s)	Tsutaya, Mitsunobu
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A note on homotopy types of connected components of $\text{Map}(S^4, BSU(2))$

Mitsunobu Tsutaya*

Abstract: Gottlieb has shown that connected components of $\text{Map}(S^4, BSU(2))$ are the classifying spaces of gauge groups of principal $SU(2)$ -bundles over S^4 . Tsukuda has investigated the homotopy types of connected components of $\text{Map}(S^4, BSU(2))$. But unfortunately, his proof is not complete for $p = 2$. In this paper, we give a complete proof. Moreover, we investigate the further divisibility of ϵ_i defined in Tsukuda's paper. We apply this to classification problem of gauge groups as A_n -spaces.

1 Introduction

For a principal G -bundle P , the gauge group $\mathcal{G}(P)$ of P is defined as the group of all bundle maps $P \rightarrow P$ covering the identity map of the base space. It is topologized by compact open topology and becomes a topological group. Particularly, we consider the gauge groups of principal $SU(2)$ -bundles over the 4-dimensional sphere S^4 .

Gottlieb [2] has constructed the universal bundles and the classifying spaces of gauge groups. The classifying space of the gauge group of a principal G -bundle P over B with classifying map $\alpha : B \rightarrow BG$ is the connected component of the mapping space $\text{Map}(B, BG)$ containing α .

Kono [4] has investigated the homotopy types of the gauge groups of principal $SU(2)$ -bundles over S^4 . He showed that they are completely classified by the invariant $\text{GCD}(12, \langle c_2(P), [S^4] \rangle)$, where $\text{GCD}(a, b)$ represents the greatest common divisor of a and b ($\text{GCD}(0, a) = \text{GCD}(a, 0) = |a|$), $c_2(P) \in H^4(S^4; \mathbf{Z})$ is the second Chern class of P and $[S^4] \in H_4(S^4; \mathbf{Z})$ is the fundamental class of S^4 .

Moreover, Crabb and Sutherland [1] classified the H -types (equivalent types as H -spaces (or Hopf spaces)) of them; they are completely classified by the invariant $\text{GCD}(180, \langle c_2(P), [S^4] \rangle)$. Of course, if $\mathcal{G}(P)$ and $\mathcal{G}(P')$ are H -equivalent, then they are homotopy equivalent.

Tsukuda [9] has classified the isomorphism classes of them; they are completely classified by $|\langle c_2(P), [S^4] \rangle|$. Moreover, he shows that $\mathcal{G}(P)$ and $\mathcal{G}(P')$ are isomorphic if the classifying spaces of them are homotopy equivalent. We remark that the converse is true in general. But the argument in the proof of Lemma 2.4 of [9] is not valid for $p = 2$. The first aim of the present paper is to give a correct proof of this.

The author [10] has considered the classification problem of the gauge groups by using Stasheff's " A_n " [7] [8]. Since " H -equivalence" and "homotopy equivalence of classifying spaces" are nothing but " A_2 -equivalence" and " A_∞ -equivalence" respectively, this problem is a natural generalization. The second aim is to apply our main theorem to it.

In §2, we review the definition of ϵ_i and the motivation in homotopy theory. In §3 and 4, we investigate the divisibility of ϵ_i . These sections are purely algebraic. In §5, we apply our main result to A_n -types of gauge groups. We give a lower bound of the number of A_n -types of gauge groups of principal $SU(2)$ -bundles over S^4 .

2 Definition and motivation

We review the definition of $\{\epsilon_i\}$. Let P_k be a principal $SU(2)$ -bundle over S^4 with $\langle c_2(P_k), [S^4] \rangle = k \in \mathbf{Z}$. According to [10], the p -localization $\mathcal{G}^{\text{id}}(P_k)_{(p)}$ of the identity component of the gauge group $\mathcal{G}(P_k)$ is A_n -equivalent to

*Department of Mathematics, Kyoto University, Kyoto 606-8502, Japan

E-mail address: tsutaya@math.kyoto-u.ac.jp

The author is a Research Fellow of the Japan Society for the Promotion of Science.

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$\mathcal{G}^{\text{id}}(P_0)_{(p)}$ if and only if the map

$$S^4 \vee \mathbf{HP}^n \xrightarrow{k \vee i} \mathbf{HP}^\infty \vee \mathbf{HP}^\infty \xrightarrow{\nabla} \mathbf{HP}^\infty \xrightarrow{\text{localization}} \mathbf{HP}_{(p)}^\infty$$

extends over $S^4 \times \mathbf{HP}^n$, where $k : S^4 \rightarrow \mathbf{HP}^\infty$ is the classifying map of P_k , $i : \mathbf{HP}^n \rightarrow \mathbf{HP}^\infty$ is the inclusion and $\nabla : \mathbf{HP}^\infty \vee \mathbf{HP}^\infty \rightarrow \mathbf{HP}^\infty$ is the folding map.

Now, we assume there exists the following homotopy commutative diagram:

$$\begin{array}{ccc} S^4 \vee \mathbf{HP}^n & \xrightarrow{k \vee i} & \mathbf{HP}^\infty \vee \mathbf{HP}^\infty & \xrightarrow{\nabla} & \mathbf{HP}^\infty \\ \downarrow j & & & & \downarrow \text{localization} \\ S^4 \times \mathbf{HP}^n & \xrightarrow{f} & & & \mathbf{HP}_{(p)}^\infty \end{array}$$

where p is a prime and $j : S^4 \vee \mathbf{HP}^n \rightarrow S^4 \times \mathbf{HP}^n$ is the inclusion.

We denote the localization of the ring of integers by the prime ideal $(p) \subset \mathbf{Z}$ by $\mathbf{Z}_{(p)}$. The (0th) p -local complex K -theory $K_{(p)}(\mathbf{HP}_{(p)}^\infty)$ of $\mathbf{HP}_{(p)}^\infty$ is computed as

$$K_{(p)}(\mathbf{HP}_{(p)}^\infty) = \mathbf{Z}_{(p)}[a].$$

We may assume that the generator $b \in H^4(\mathbf{HP}_{(p)}^\infty; \mathbf{Q})$ satisfies the equality

$$ch a = \sum_{j=1}^{\infty} \frac{2b^j}{(2j)!}.$$

Similarly, take generators $u \in \tilde{K}_{(p)}(S^4)$ and $s \in H^4(S^4; \mathbf{Q})$ such that $chu = s$. Then, $f^*b = ks \times 1 + 1 \times b$ in $H^4(S^4 \times \mathbf{HP}^n; \mathbf{Q})$ and

$$f^*a = ku \times 1 + 1 \times a + \sum_{i=1}^n \epsilon_i(k)u \times a^i$$

in $\tilde{K}_{(p)}(S^4 \times \mathbf{HP}^n)$, where $\epsilon_i(k) \in \mathbf{Z}_{(p)}$. We calculate f^*cha and $ch f^*a$ as follows:

$$f^*cha = f^* \sum_{j=1}^{\infty} \frac{2b^j}{(2j)!} = \sum_{j=1}^{\infty} \frac{2}{(2j)!} (ks \times 1 + 1 \times b)^j = ks \times 1 + \sum_{j=1}^n \left(\frac{k}{(2j+1)!} s \times b^j + \frac{2}{(2j)!} 1 \times b^j \right),$$

$$\begin{aligned} ch f^*a &= ch \left(ku \times 1 + 1 \times a + \sum_{i=1}^n \epsilon_i(k)u \times a^i \right) = ks \times 1 + 1 \times \sum_{j=1}^n \frac{2}{(2j)!} b^j + \sum_{i=1}^n \sum_{j=1}^n \epsilon_i(k) s \times \left(\sum_{j=1}^n \frac{2}{(2j)!} b^j \right)^i \\ &= ks \times 1 + \sum_{j=1}^n \frac{2}{(2j)!} 1 \times b^j + \sum_{i=1}^n \sum_{l=1}^n \sum_{j_1+\dots+j_i=l} \frac{2^i \epsilon_i(k)}{(2j_1)! \cdots (2j_i)!} s \times b^l. \end{aligned}$$

Since $f^*cha = ch f^*a$, we have the following formula:

$$\frac{k}{(2\ell+1)!} = \sum_{i=1}^{\ell} \sum_{\substack{j_1+\dots+j_i=\ell \\ j_1, \dots, j_i \geq 1}} \frac{2^i \epsilon_i(k)}{(2j_1)! \cdots (2j_i)!}.$$

For $\ell = 1$ and $k \neq 0$, we obtain $\epsilon_1(k)/k = 1/6$. Moreover, using this formula and induction on i , one can see that $\epsilon_i(k)/k \in \mathbf{Q}$ is independent of $k \neq 0$. Let $\epsilon_i = \epsilon_i(k)/k$. Then $\epsilon_i(k) = \epsilon_i$ for any $k \in \mathbf{Z}$. Of course, the sequence $\{\epsilon_i\}_{i=1}^{\infty}$ satisfy the following formula for each ℓ :

$$\frac{1}{(2\ell+1)!} = \sum_{i=1}^{\ell} \sum_{\substack{j_1+\dots+j_i=\ell \\ j_1, \dots, j_i \geq 1}} \frac{2^i \epsilon_i}{(2j_1)! \cdots (2j_i)!}.$$

For example, $\epsilon_1 = 1/6$, $\epsilon_2 = -1/180$, $\epsilon_3 = 1/1512$ etc. From the above argument, if the map (localization) $\nabla(k \vee i) : S^4 \vee \mathbf{HP}^n \rightarrow \mathbf{HP}_{(p)}^\infty$ extends over $S^4 \times \mathbf{HP}^n$, then $\epsilon_1 k, \dots, \epsilon_n k \in \mathbf{Z}_{(p)}$.

Tsukuda [9] defines a non-negative integer (or infinity) $d_p(k)$ for a prime p and an integer k as the largest n such that there exists an extension of

$$S^4 \vee \mathbf{HP}^n \xrightarrow{k \vee i} \mathbf{HP}^\infty \vee \mathbf{HP}^\infty \xrightarrow{\nabla} \mathbf{HP}^\infty \xrightarrow{\text{localization}} \mathbf{HP}_{(p)}^\infty$$

over $S^4 \times \mathbf{HP}^n$. Remark $d_p(0) = \infty$. Clearly, if we define $\epsilon_0 = 1$, then

$$d_p(k) \leq d'_p(k) := \min\{n \in \mathbf{Z}_{\geq 0} \mid \epsilon_{n+1} k \notin \mathbf{Z}_{(p)}\}.$$

It is shown in Lemma 2.5 of [10] that $d_p(k) = d_p(k')$ for any prime p if the classifying spaces $B\mathcal{G}(P_k)$ and $B\mathcal{G}(P_{k'})$ are homotopy equivalent. Lemma 2.4 in [9] asserts that $d'_p(k) < \infty$ (therefore $d_p(k) < \infty$) for $k \neq 0$ and any prime p . In the proof of it, he has shown that $k/(2n+1) \in \mathbf{Z}_{(p)}$ if $n \leq d'_p(k)$. For an odd prime p , if $p^r \nmid k$, this implies that $d'_p(k) < (p^r - 1)/2$. But for $p = 2$, this gives no information about $d'_2(k)$. So this proof is invalid for $p = 2$. We will give a correct proof for this case in §4.

We also state the result of [10]. If $\mathcal{G}(P_k)$ and $\mathcal{G}(P_{k'})$ are A_n -equivalent, then $\min\{n, d_p(k)\} = \min\{n, d_p(k')\}$ for any prime p . Using this, we give a lower bound for the number of A_n -types of the gauge groups.

3 An explicit formula for ϵ_i

As we have seen, algebraically, the sequence $\{\epsilon_i\}_{i=0}^\infty$ of rational numbers is defined inductively by the following formula:

$$\frac{1}{(2\ell+1)!} = \sum_{i=1}^{\ell} \sum_{\substack{j_1+\dots+j_i=\ell \\ j_1, \dots, j_i \geq 1}} \frac{2^i \epsilon_i}{(2j_1)! \cdots (2j_i)!}$$

and $\epsilon_0 = 1$. Equivalently, $\{\epsilon_i\}$ is defined by the equality

$$\sum_{\ell=0}^{\infty} \frac{x^\ell}{(2\ell+1)!} = \sum_{i=0}^{\infty} \epsilon_i \left(\sum_{j=1}^{\infty} \frac{2x^j}{(2j)!} \right)^i$$

in the ring of formal power series $\mathcal{Q}[[x]]$.

Proposition 3.1. *The rational number ϵ_i is the i -th coefficient of the Taylor expansion of $1/F'(x)$ at $0 \in \mathbf{C}$ for*

$$F(x) = \left(\cosh^{-1} \left(1 + \frac{x}{2} \right) \right)^2,$$

where F is holomorphic in a neighborhood of 0.

Proof. Define a holomorphic function H by

$$H(x) = 2 \cosh \sqrt{x} - 2 = \sum_{i=1}^{\infty} \frac{2}{(2i)!} x^i$$

in a neighborhood of 0. Then F given by the above formula is the inverse function of H . We also define G by

$$G(x) = \sum_{i=0}^{\infty} \epsilon_i x^i.$$

Then, formally, $H'(x) = G(H(x))$ by the definition of ϵ_i . Therefore, we have $G(x) = 1/F'(x)$. \square

Since

$$\frac{d}{dx} \cosh^{-1} x = \frac{1}{\sqrt{x^2+1}},$$

the next proposition is seen by easy computation.

Proposition 3.2. *The holomorphic function F satisfies the following differential equation:*

$$x(x+4)F''(x) + (x+2)F'(x) - 2 = 0.$$

If the power series

$$\sum_{i=1}^{\infty} a_i x^i$$

satisfies the above equation, then

$$a_1 = 1, a_{i+1} = -\frac{i^2}{(2i+2)(2i+1)} a_i \quad (i \geq 1).$$

From these equations,

$$a_i = (-1)^{i-1} \frac{2((i-1)!)^2}{(2i)!}$$

for $i \geq 1$. Hence,

$$F(x) = \sum_{i=1}^{\infty} (-1)^{i-1} \frac{2((i-1)!)^2}{(2i)!} x^i$$

and

$$F'(x) = \sum_{i=0}^{\infty} (-1)^i \frac{(i!)^2}{(2i+1)!} x^i.$$

Therefore,

$$G(x) = \frac{1}{F'(x)} = \sum_{j=0}^{\infty} (-1)^j \left(\sum_{i=1}^{\infty} (-1)^i \frac{(i!)^2}{(2i+1)!} x^i \right)^j = 1 + \sum_{j=1}^{\infty} \sum_{i_1, \dots, i_j \geq 1} (-1)^{j+i_1+\dots+i_j} \frac{(i_1!)^2 \cdots (i_j!)^2}{(2i_1+1)! \cdots (2i_j+1)!} x^{i_1+\dots+i_j}.$$

This implies the following formula.

Theorem 3.3.

$$\epsilon_\ell = \sum_{j=1}^{\ell} \sum_{\substack{i_1+\dots+i_j=\ell \\ i_1, \dots, i_j \geq 1}} (-1)^{j+\ell} \frac{(i_1!)^2 \cdots (i_j!)^2}{(2i_1+1)! \cdots (2i_j+1)!}$$

4 Divisibility of ϵ_i

For a prime p and a rational number n , we denote the p -adic valuation of n by $v_p(n)$. Equivalently, if

$$n = \frac{p^a t}{p^b s}$$

where s and t are integers prime to p , then $v_p(n) = a - b$. First, we review the divisibility of factorials. For a positive integer n and each prime p , the following formula is known (see, for example, [5]):

$$v_p(n!) = \frac{n - S_p(n)}{p-1},$$

where $S_p(n) = n_0 + \dots + n_r$ for $n = n_r p^r + n_{r-1} p^{r-1} + \dots + n_0$ and $0 \leq n_i < p$.

Lemma 4.1. *For a positive integer n*

$$v_2 \left(\frac{(n!)^2}{(2n+1)!} \right) \geq -n,$$

where the equality holds if and only if $n = 1$.

Proof. Using the above formula, we obtain

$$v_2\left(\frac{(n!)^2}{(2n+1)!}\right) = 2(n - S_2(n)) - (2n+1 - S_2(2n+1)) = S_2(n) \geq -n.$$

As easily seen, this equality holds if and only if $n = 1$. □

Now, we observe the divisibility of ϵ_i by 2.

Proposition 4.2.

$$v_2(\epsilon_\ell) = -\ell$$

Proof. By Theorem 3.3,

$$\epsilon_\ell = \sum_{j=1}^{\ell} \sum_{\substack{i_1+\dots+i_j=\ell \\ i_1, \dots, i_j \geq 1}} (-1)^{j+\ell} \frac{(i_1!)^2 \cdots (i_j!)^2}{(2i_1+1)! \cdots (2i_j+1)!}.$$

Now, from the previous lemma, we have

$$v_2\left(\frac{(i_1!)^2 \cdots (i_j!)^2}{(2i_1+1)! \cdots (2i_j+1)!}\right) \geq -i_1 - \cdots - i_j = -\ell,$$

where the equality holds if and only if $i_1 = \cdots = i_j = 1$. Then we obtain

$$\epsilon_\ell \equiv 6^{-\ell} \pmod{2^{-\ell+1}\mathbf{Z}_{(2)}}.$$

Therefore, $v_2(\epsilon_\ell) = -\ell$. □

In general, the divisibility of ϵ_i by an odd prime p is more complicated than by 2 because the interval between a multiple of p and the next one is longer. But for $p = 3$, we will have a similar result.

Lemma 4.3. *Let p be an odd prime and n a positive integer. Then*

$$v_p\left(\frac{(n!)^2}{(2n+1)!}\right) \geq -\frac{2n}{p-1},$$

where the equality holds if and only if $n = (p-1)/2$.

Proof. Let $n = n_r p^r + n_{r-1} p^{r-1} + \cdots + n_0$ with $0 \leq n_i < p$. Assume $n_0 = \cdots = n_{s-1} = (p-1)/2 \neq n_s$ ($0 \leq s \leq r+1$). Let $m = n_r p^{r-s} + \cdots + n_s$ for $s \leq r$ and $m = 0$ for $s = r+1$. Then we have $2n+1 = p^s(2m+1)$, where $2m+1$ is prime to p since $n_s \neq (p-1)/2$. Thus we obtain

$$v_p\left(\frac{(n!)^2}{(2n+1)!}\right) = v_p\left(\frac{1}{2n+1}\right) + v_p\left(\frac{(n!)^2}{(2n)!}\right) = -s - \frac{2S_p(n) - S_p(2n)}{p-1} = -s - \frac{2S_p(m) - S_p(2m)}{p-1} \geq -s - \frac{2m}{p-1},$$

where the last equality holds if and only if $m = 0$. Moreover,

$$\frac{2(n-m)}{p-1} = \frac{(2n+1) - (2m+1)}{p-1} = \frac{(p^s-1)(2m+1)}{p-1} \geq 1 + p + \cdots + p^{s-1} \geq s,$$

where $2(n-m)/(p-1) = s$ if and only if $s = 1$ and $m = 0$. Therefore, we conclude that

$$v_p\left(\frac{(n!)^2}{(2n+1)!}\right) \geq -\frac{2n}{p-1},$$

where the equality holds if and only if $n = (p-1)/2$. □

Proposition 4.4. *For a positive integer $\ell \leq n(p-1)/2$,*

$$v_p(\epsilon_\ell) \geq -n,$$

where the equality holds if and only if $\ell = n(p-1)/2$. Especially, $v_3(\epsilon_\ell) = -\ell$ for any ℓ .

Proof. Let positive integers i_1, \dots, i_j satisfy $i_1 + \dots + i_j = \ell$. From the previous proposition, we have

$$v_p \left(\frac{(i_1!)^2 \cdots (i_j!)^2}{(2i_1 + 1)! \cdots (2i_j + 1)!} \right) \geq -\frac{2\ell}{p-1} \geq -n,$$

where the left equality holds if and only if $i_1 = \dots = i_j = (p-1)/2$. Thus using Theorem 3.3, we obtain

$$v_p(\epsilon_\ell) \geq -n,$$

where the equality holds if and only if $\ell = n(p-1)/2$. \square

Remark 1. Here we remark that $v_p(\epsilon_\ell)$ is not non-increasing on ℓ in general. For example, $v_5(\epsilon_2) = -1$ but $v_5(\epsilon_3) = 0$, $v_7(\epsilon_9) = -3$ but $v_7(\epsilon_{10}) = -2$, etc.

From Proposition 4.2 and 4.4, we conclude our main result.

Theorem 4.5. Let k be a non-zero integer and p be a prime. Then

$$d'_p(k) = \begin{cases} v_2(k) & (p = 2) \\ v_p(k)(p-1)/2 & (p \text{ is odd}) \end{cases}.$$

Then $d_2(k) \leq d'_2(k) < \infty$ for any non-zero integer k and we have correctly verified Lemma 2.4 of [9].

5 Applications to A_n -types of gauge groups

As in §2, we assume there exists the following homotopy commutative diagram:

$$\begin{array}{ccc} S^4 \vee \mathbf{HP}^n & \xrightarrow{k \vee i} & \mathbf{HP}^\infty \vee \mathbf{HP}^\infty \xrightarrow{\nabla} \mathbf{HP}^\infty \\ \downarrow j & & \downarrow \text{localization} \\ S^4 \times \mathbf{HP}^n & \xrightarrow{f} & \mathbf{HP}_{(p)}^\infty \end{array}$$

where p is a prime and i and j are the inclusions. Let us consider the map

$$S^4 \times \mathbf{HP}^n \cup * \times \mathbf{HP}^{n+1} \xrightarrow{f \cup (\text{localization})i} \mathbf{HP}_{(p)}^\infty.$$

The obstruction to extending this map over $S^4 \times \mathbf{HP}^{n+1}$ lives in $\pi_{4n+7}(\mathbf{HP}_{(p)}^\infty)$. Then, from Theorem of [6], the obstruction to extending the map

$$S^4 \times \mathbf{HP}^n \cup * \times \mathbf{HP}^{n+1} \xrightarrow{(p \times \text{id}) \cup \text{id}} S^4 \times \mathbf{HP}^n \cup * \times \mathbf{HP}^{n+1} \xrightarrow{f \cup (\text{localization})i} \mathbf{HP}_{(p)}^\infty$$

over $S^4 \times \mathbf{HP}^{n+1}$ vanishes for an odd prime p . Here we remark that for the cofibration

$$S^{4n+7} \xrightarrow{\varphi} S^4 \times \mathbf{HP}^n \cup * \times \mathbf{HP}^{n+1} \rightarrow S^4 \times \mathbf{HP}^{n+1},$$

$\varphi : S^{4n+7} \rightarrow S^4 \times \mathbf{HP}^n \cup * \times \mathbf{HP}^{n+1}$ is defined by the restriction to the boundary of the direct product of the characteristic maps $(D^4, S^3) \rightarrow (S^4, *)$ and $(D^{4n+4}, S^{4n+3}) \rightarrow (\mathbf{HP}^{n+1}, \mathbf{HP}^n)$ and the composition of the direct product and the restriction $\pi_4(S^4) \times \pi_{4n+4}(\mathbf{HP}^{n+1}, \mathbf{HP}^n) \rightarrow \pi_{4n+8}(S^4 \times \mathbf{HP}^{n+1}, S^4 \times \mathbf{HP}^n \cup * \times \mathbf{HP}^{n+1}) \rightarrow \pi_{4n+7}(S^4 \times \mathbf{HP}^n \cup * \times \mathbf{HP}^{n+1})$ is bilinear. Hence one can see $d_p(pk) > d_p(k)$ and $d_p(k) \geq v_p(k)$ inductively. For $p = 2$, from [3], $d_2(4k) > d_2(k)$ and $d_2(k) \geq [v_2(k)/2]$ similarly. Then, combining with Theorem 4.5, we have

$$v_p(k) \leq d_p(k) \leq d'_p(k) = \frac{p-1}{2} v_p(k)$$

for an odd prime p and

$$\left\lceil \frac{v_2(k)}{2} \right\rceil \leq d_2(k) \leq d'_2(k) = v_2(k).$$

Especially, $d_3(k) = v_3(k)$. From §2, the next theorem follows.

Theorem 5.1. *Let P be a principal $SU(2)$ -bundle over S^4 with $\langle c_2(P), [S^4] \rangle = 3^n$. Then the 3-localization $\mathcal{G}^{\text{id}}(P)_{(3)}$ of the identity component $\mathcal{G}^{\text{id}}(P)$ of $\mathcal{G}(P)$ is A_n -equivalent to $\text{Map}(S^4, SU(2))_{(3)}$ but not A_{n+1} -equivalent.*

This gives an example of A_n -equivalent but not A_{n+1} -equivalent topological monoids for any n .

Now we give the lower bound of the number of A_n -types of gauge groups of principal $SU(2)$ -bundle over S^4 . As stated in §2, if $\mathcal{G}(P_k)$ and $\mathcal{G}(P_{k'})$ are A_n -equivalent, then $\min\{n, d_p(k)\} = \min\{n, d_p(k')\}$ for any prime p . If p is an odd prime, then

$$\#\{\min\{n, d_p(k)\} \mid k \in \mathbf{Z}\} \geq \left\lfloor \frac{2n}{p-1} + 1 \right\rfloor$$

since $0 = d_p(1) < d_p(p) < d_p(p^2) < \dots < d_p(p^{\lfloor 2n/(p-1) \rfloor}) \leq n$. If $p = 2$, then

$$\#\{\min\{n, d_2(k)\} \mid k \in \mathbf{Z}\} \geq \left\lfloor \frac{n}{2} + 1 \right\rfloor$$

since $0 = d_2(1) < d_2(4) < d_2(16) < \dots < d_2(4^{\lfloor n/2 \rfloor}) \leq n$.

Theorem 5.2. *The number of A_n -types of gauge groups of principal $SU(2)$ -bundles over S^4 is greater than*

$$\left\lfloor \frac{n}{2} + 1 \right\rfloor \prod_{p:\text{odd prime}} \left\lfloor \frac{2n}{p-1} + 1 \right\rfloor.$$

We can express the logarithm of this as follows:

$$\begin{aligned} \log \left(\left\lfloor \frac{n}{2} + 1 \right\rfloor \prod_{p:\text{odd prime}} \left\lfloor \frac{2n}{p-1} + 1 \right\rfloor \right) &= \log \left\lfloor \frac{n}{2} + 1 \right\rfloor + \sum_{p:\text{odd prime}} \log \left\lfloor \frac{2n}{p-1} + 1 \right\rfloor \\ &= \log \left\lfloor \frac{n}{2} + 1 \right\rfloor + \sum_{r=2}^{n+1} \left(\pi \left(\frac{2n}{r-1} + 1 \right) - 1 \right) (\log r - \log(r-1)) \\ &= \log \left\lfloor \frac{n}{2} + 1 \right\rfloor + \sum_{r=1}^n \pi \left(\frac{2n}{r} + 1 \right) \log \left(1 + \frac{1}{r} \right) - \log(n+1), \end{aligned}$$

where π is the prime counting function. The second equality is seen by

$$\#\left\{ p : \text{an odd prime} \left| \left\lfloor \frac{2n}{p-1} + 1 \right\rfloor \geq r \right\} = \pi \left(\frac{2n}{r-1} + 1 \right) - 1.$$

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