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Dirichlet spaces on H-convex sets in Wiener space

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Dedicated to the memory of Professor Paul Malliavin

Abstract

We consider the $(1,2)$-Sobolev space $W^{1,2}(U)$ on subsets U in an abstract Wiener space, which is regarded as a canonical Dirichlet space on U. We prove that $W^{1,2}(U)$ has smooth cylindrical functions as a dense subset if U is H-convex and H-open. For the proof, the relations between H-notions and quasi-notions are also studied.

Keywords: Dirichlet space, convex set, Wiener space
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1. Introduction

In Euclidean space, extension operators related to Sobolev spaces are useful tools. Their existence is stated as follows: Given a domain U of $\mathbb{R}^n$ with a sufficiently regular boundary, $p \geq 1$, and $r \in \mathbb{N}$, there exists a bounded linear map $T\colon W^{r,p}(U) \to W^{r,p}(\mathbb{R}^n)$ such that $Tf = f$ on U for all $f \in W^{r,p}(U)$. Here, $W^{r,p}(X)$ denotes the Sobolev space on domain X, with differentiability index r and integrability index p. In particular, the above statement implies that $W^{r,p}(\mathbb{R}^n)|_U = W^{r,p}(U)$, where the left-hand side denotes a function space on U that is defined by restricting the defining sets of the functions in $W^{r,p}(\mathbb{R}^n)$ to U. Hereafter, we use the standard notation described above. Such properties can reduce many problems on U to those on $\mathbb{R}^n$, which are often easier to resolve.

In this paper, we discuss a related problem in infinite-dimensional spaces. To the best of the author's knowledge, there are no nontrivial examples that involve the existence of the extension operators described above: Some useful techniques such as covering arguments and a Whitney decomposition in Euclidean space are not directly available in infinite dimensions; this complicates the problem. In this paper, we consider a reduced version of the problem as follows: Let (E,H,μ) be an abstract Wiener space (the definition of which is provided in Section 2) and U, a measurable subset of E with positive μ-measure. Find sufficient conditions on U such that

$$W^{1,2}(E)|_U \text{ is dense in } W^{1,2}(U) \text{ in the topology induced by the Sobolev norm.} \tag{1.1}$$

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The well-definedness of the space $W^{1,2}(U)$ is explained in the next section. Here, we note that $W^{1,2}(U)$ is regarded as the domain of a canonical Dirichlet form on $L^2(U, \mu|_U)$, where $\mu|_U(\cdot) := \mu(\cdot \cap U)$. Since the space $\mathcal{F}C_b^1(E)$ of smooth cylindrical functions on E is known to be dense in $W^{1,2}(E)$, (1.1) is equivalent to the following:

$$\mathcal{F}C_b^1(E)|_U \text{ is dense in } W^{1,2}(U) \text{ in the topology induced by the Sobolev norm.} \tag{1.2}$$

The closure of $\mathcal{F}C_b^1(E)|_U$ in $W^{1,2}(U)$ is often regarded as the minimal domain. Therefore, the problem under consideration is to determine whether the canonical domain and the minimal domain coincide. Even for this weaker property, few examples of non-smooth sets are known to satisfy it. The following is the known result.

Theorem 1.1 ([13, Theorem 2.2])**.** *If U is convex and has a nonempty interior, then* (1.1) *is true.*

We may assume that U is open in this theorem without loss of generality because the topological boundary of U is a μ-null set under these assumptions (see Remark 3.3 (ii)).

In this paper, we provide a refinement of Theorem 1.1. Although Theorem 1.1 has been proved within a more general framework [13], we consider only an abstract Wiener space in order to avoid inessential technical issues. Theorem 1.1 is not satisfactory in that the assumptions involve the vector space structure and topological structure of E. It is desirable to impose assumptions depending only on the structures of the Cameron–Martin space H. Accordingly, we prove the following theorem. Let $\mathfrak{M}(E)$ denote the completion of the Borel σ-field $\mathfrak{B}(E)$ of E by μ.

Theorem 1.2. *Suppose that $U \in \mathfrak{M}(E)$ with positive μ-measure is H-convex and H-open. Then,* (1.1) *holds.*

Here, U is called H-convex if $(U - z) \cap H$ is convex in H for all $z \in E$, and U is called H-open if $(U - z) \cap H$ has 0 as an interior point in H for every $z \in U$. Since convexity in E implies H-convexity, and open sets in E are H-open, the assumptions of Theorem 1.2 are essentially weaker than those of Theorem 1.1.

If E is finite-dimensional, Theorem 1.2 is easy to prove as follows: For simplicity, we further assume that U is bounded and contains 0. For a small positive number γ, we consider a contraction map $T_\gamma \colon E \ni z \mapsto (1-\gamma)z \in E$ and let $U_\gamma := T_\gamma^{-1}(U) \supset U$. For each $f \in W^{1,2}(U)$, the function $f \circ T_\gamma$, denoted by f_γ, is defined on U_γ and $f_\gamma|_U$ approximates f in $W^{1,2}(U)$. Since there is a positive distance between $E \setminus U_{\gamma/2}$ and the closure of U, we can take a Lipschitz function φ on E such that $\varphi = 1$ on U and $\varphi = 0$ on $E \setminus U_{\gamma/2}$. Then, φf_γ is well-defined as a function in $W^{1,2}(E)$ and $(\varphi f_\gamma)|_U = f_\gamma|_U$, which deserves to be an approximating function of f.

This proof breaks down when E is infinite-dimensional, since measures μ and $\mu \circ T_\gamma^{-1}$ are mutually singular. Therefore, our strategy is decomposing E into a finite-dimensional space and an auxiliary space, and applying the procedure stated above for each finite-dimensional section. Theorem 1.1 was proved in this way. The proof of Theorem 1.2 is similar to that of Theorem 1.1, but more technically involved. This is because we have to treat the topology of E induced by the H-distance, which is neither metrizable nor second-countable; we cannot utilize the general theory of good topological spaces. In order to overcome this difficulty, we firstly study the relations between H-notions and quasi-notions, and we use them for removing a suitable set with small capacity to adopt a method utilized in the proof of Theorem 1.1.

The remainder of this paper is organized as follows. In Section 2, we introduce a framework, and we prove some preliminary results that are of contextual interest. Some results may be known to experts; nonetheless, we provide proofs of the claims for which the author could not find suitable references. In Section 3, we prove Theorem 1.2. In Section 4, we discuss some applications.

2. Framework and preliminary propositions

Let (E, H, μ) be an abstract Wiener space. That is, E is a real Banach space with norm $|\cdot|_E$, H is a real separable Hilbert space that is continuously embedded in E, and μ is a Gaussian measure on E such that

$$\int_E \exp\left(\sqrt{-1}\, l(z)\right) \mu(dz) = \exp\left(-|l|_H^2/2\right) \quad \text{for all } l \in E^* \subset H^* \simeq H \subset E.$$

Here, we denote the topological duals of E and H by E^* and H^*, respectively, and we adopt the inclusions and identification stated above. We always assume that E is infinite-dimensional. The inner product and norm of H are denoted by $\langle \cdot, \cdot \rangle$ and $|\cdot|_H$, respectively. For $l \in E^*$ and $z \in E$, $l(z)$ also denotes $\langle l, z \rangle$. This terminology is consistent with the inner product of H when the inclusions $E^* \subset H \subset E$ are taken into consideration. For $s \in \mathbb{R}$, $z \in E$, $A \subset E$, and $B \subset E$, we set $sA = \{sa \mid a \in A\}$ and $A \pm B = \{a \pm b \mid a \in A,\ b \in B\}$, and we denote $\{z\} \pm A$ by $z \pm A$. The following is a basic property of μ (see, e.g., [3, Corollary 2.5.4] for the proof).

Proposition 2.1. *Let $A \in \mathfrak{M}(E)$ and F be a dense linear space of H. If $A + F = A$, either $\mu(A) = 0$ or $\mu(E \setminus A) = 0$ holds.*

For $X \in \mathfrak{M}(E)$, the σ-field $\{A \in \mathfrak{M}(E) \mid A \subset X\}$ on X is denoted by $\mathfrak{M}(X)$. Given $X \in \mathfrak{M}(E)$, a separable Hilbert space $\mathscr{H}$, and $p \in [1, +\infty]$, the $\mathscr{H}$-valued L^p-space on the measure space $(X, \mathfrak{M}(X), \mu|_X)$ is denoted by $L^p(X \to \mathscr{H})$. When $\mathscr{H} = \mathbb{R}$, it is simply denoted by $L^p(X)$. Its canonical norm is denoted by $\|\cdot\|_{L^p(X)}$.

The space $\mathcal{F}C_b^1(E)$ of smooth cylindrical functions on E is defined as

$$\mathcal{F}C_b^1(E) = \left\{ u\colon E \to \mathbb{R} \;\middle|\; \begin{array}{l} u(z) = f(\langle l_1, z \rangle, \dots, \langle l_m, z \rangle),\ l_1, \dots, l_m \in E^*, \\ f \in C_b^1(\mathbb{R}^m) \text{ for some } m \in \mathbb{N} \end{array} \right\},$$

where $C_b^1(\mathbb{R}^m)$ is the set of all bounded C^1-functions on $\mathbb{R}^m$ with bounded first order derivatives. Let G be a finite-dimensional subspace of E^*. We define a closed subspace $G^\perp$ of E as $G^\perp = \{z \in E \mid \langle h, z \rangle = 0 \text{ for every } h \in G\}$. The direct sum $G^\perp \dotplus G$ is identified with E. The canonical projection maps from E to $G^\perp$ and G are denoted by P_G and Q_G, respectively. To be precise, they are defined as follows:

$$P_G z = z - Q_G z, \quad Q_G z = \sum_{i=1}^m \langle h_i, z \rangle h_i,$$

where $\{h_i\}_{i=1}^m \subset G \subset H \subset E$ is an orthonormal basis of G in H. The image measures of μ by P_G and Q_G are denoted by $\mu_{G^\perp}$ and μ_G, respectively. Both measures are centered Gaussian measures; in particular, μ_G is described as

$$\mu_G(dy) = (2\pi)^{-m/2} \exp(-|y|_H^2/2)\, \lambda_m(dy),$$

where $m = \dim G$ and λ_m denotes the Lebesgue measure on G. The product measure of $\mu_{G^\perp}$ and μ_G is identified with μ. When $G = \mathbb{R}h$ for some $h \in E^*$, we write $h^\perp$, $\mu_{h^\perp}$, and μ_h for $G^\perp$, $\mu_{G^\perp}$, and μ_G, respectively.

Let $X \in \mathfrak{M}(E)$. For $h \in E^* \setminus \{0\} \subset E$ and $x \in h^\perp$, we define

$$I_{x,h}^X = \{s \in \mathbb{R} \mid x + sh \in X\}.$$

We fix a linear subspace K of E^* that is dense in H. We call X *K-moderate* if for each $h \in K \setminus \{0\}$, the boundary of $I_{x,h}^X$ in $\mathbb{R}$ is a Lebesgue null set for $\mu^\perp$-a.e. $x \in h^\perp$. It is evident that H-convex sets in $\mathfrak{M}(E)$ are K-moderate. For a function f on X, $x \in E$, and $h \in E^* \setminus \{0\}$, we define a function $f_h(x, \cdot)$ on $I_{x,h}^X$ as $f_h(x, s) = f(x + sh)$.

Suppose that X is K-moderate and $\mu(X) > 0$. For $h \in K \setminus \{0\}$, let $\mathrm{Dom}(\mathcal{E}_h^X)$ be the set of all functions f in $L^2(X)$ such that the following hold:

- For $\mu_{h^\perp}$-a.e. $x \in h^\perp$, $f_h(x, \cdot)$ has an absolutely continuous version $\tilde{f}_h(x, \cdot)$ on the interior of the closure of $I_{x,h}^X$ in $\mathbb{R}$.
- There exists an element of $L^2(X)$, denoted by $\partial_h f$, such that for $\mu_{h^\perp}$-a.e. $x \in h^\perp$, $(\partial_h f)(x + sh) = \frac{\partial \tilde{f}_h}{\partial s}(x, s)$ for a.e. $s \in I_{x,h}^X$ with respect to the one-dimensional Lebesgue measure.

Then, the bilinear form $(\mathcal{E}_h^X, \mathrm{Dom}(\mathcal{E}_h^X))$ on $L^2(X)$, defined as

$$\mathcal{E}_h^X(f, g) = \int_X (\partial_h f)(\partial_h g)\, d\mu, \quad f, g \in \mathrm{Dom}(\mathcal{E}_h^X),$$

is a closed form from [2, Theorem 3.2]. The $(1, 2)$-Sobolev space $W^{1,2}(X)$ on X is then defined as

$$W^{1,2}(X) = \left\{ f \in \bigcap_{h \in K \setminus \{0\}} \mathrm{Dom}(\mathcal{E}_h^X) \;\middle|\; \begin{array}{l} \text{there exists } Df \in L^2(X \to H) \text{ such that} \\ \langle Df, h\rangle = \partial_h f \ \mu\text{-a.e. on } X \text{ for every } h \in K \setminus \{0\} \end{array} \right\}.$$

Space $W^{1,2}(X)$ formally corresponds to the maximal domain in the terminology of [1] and the weak Sobolev space in that of [7], even though the validity of these terminologies have not been investigated in our situation because our framework does not satisfy the conditions in the corresponding theorems in [1, 7]. The bilinear form $(\mathcal{E}^X, W^{1,2}(X))$ on $L^2(X)$, defined as

$$\mathcal{E}^X(f, g) = \int_X \langle Df, Dg\rangle\, d\mu, \quad f, g \in W^{1,2}(X),$$

is a local Dirichlet form in terms of [4, Definition I.5.1.2]. We note some properties for future reference.

Proposition 2.2 (cf. [4], [13, Proposition 2.1])**.** *Let Φ be a Lipschitz function on $\mathbb{R}$ and let f and g be functions in $W^{1,2}(X)$. Then:*

(i) *For any Lebesgue null set A of $\mathbb{R}$, $Df = 0$ μ-a.e. on $f^{-1}(A)$. In particular, if $f = 0$ on a measurable set B, then $Df = 0$ μ-a.e. on B.*
(ii) *$\Phi(f) \in W^{1,2}(X)$ and $D(\Phi(f)) = \Phi'(f)Df$ μ-a.e.*
(iii) *In addition, if $f, g \in L^\infty(X)$, then $fg \in W^{1,2}(X)$ and $D(fg) = f(Dg) + g(Df)$ μ-a.e.*

We write $\mathcal{E}$ for $\mathcal{E}^E$. The norm $\|\cdot\|_{W^{1,2}(X)}$ of $W^{1,2}(X)$ is given by $\|f\|_{W^{1,2}(X)} = \left(\mathcal{E}^X(f,f) + \|f\|^2_{L^2(X)}\right)^{1/2}$. In general, although $W^{1,2}(X)$ may depend on the choice of K, we omit the dependency on K from the notation for simplicity. It is known that $W^{1,2}(E)$ does not depend on the choice of K and includes $\mathcal{F}C_b^1(E)$ as a dense subset in the topology induced by $\|\cdot\|_{W^{1,2}(E)}$. Therefore, under the assumptions on U in Theorem 1.2, conclusion (1.2) implies *a posteriori* that $W^{1,2}(U)$ is independent of the choice of K.

We now recall the concepts of capacity and the associated quasi-notions. Since we use these terminologies with respect to only $(\mathcal{E}, W^{1,2}(E))$, we define them in this particular case. For open subsets O of E, the capacity of O (with respect to $(\mathcal{E}, W^{1,2}(E))$) is defined as

$$\mathrm{Cap}_{1,2}(O) := \inf\{\mathcal{E}(f,f) + \|f\|^2_{L^2(E)} \mid f \in W^{1,2}(E) \text{ and } f \geq 1 \ \mu\text{-a.e. on } O\}.$$

The infimum stated above is attained by a unique function e_O, known as the equilibrium potential of O. It holds that $0 \leq e_O \leq 1$ μ-a.e. and $e_O = 1$ μ-a.e. on O. For a general subset A of E, its capacity is defined as

$$\mathrm{Cap}_{1,2}(A) := \inf\{\mathrm{Cap}_{1,2}(O) \mid O \text{ is open and } O \supset A\}.$$

We remark that $\mathrm{Cap}_{1,2}$ is countably subadditive. A function f on E is called quasi-continuous if for any $\varepsilon > 0$, there exists an open set O such that $\mathrm{Cap}_{1,2}(O) < \varepsilon$ and $f|_{E\setminus O}$ is continuous on $E \setminus O$. Since $(\mathcal{E}, W^{1,2}(E))$ is quasi-regular (see [16] for the definition), each element of $W^{1,2}(E)$ has a quasi-continuous modification. A subset A of E is called quasi-closed if for any $\varepsilon > 0$, there exists an open set O such that $\mathrm{Cap}_{1,2}(O) < \varepsilon$ and $A \setminus O$ is closed. A subset A is called quasi-open if $E \setminus A$ is quasi-closed. For two functions f and g on E, we write $f = g$ q.e. if $\mathrm{Cap}_{1,2}(\{f \neq g\}) = 0$.

A subset E_0 of E is called H-invariant if $E_0 + H = E_0$.

Definition 2.3. Let f be a $[-\infty, +\infty]$-valued function on E.

(i) f is called H-continuous if there exists an H-invariant set E_0 such that $\mu(E \setminus E_0) = 0$, $|f(z)| < \infty$ for every $z \in E_0$, and the function $f(z + \cdot)$ on H is continuous in the topology of H for every $z \in E_0$.[1]
(ii) f is called H-Lipschitz if there exist an H-invariant set E_0 and a constant $M \geq 0$ such that $\mu(E \setminus E_0) = 0$, $|f(z)| < \infty$ for every $z \in E_0$, and $|f(w) - f(z)| \leq M|w - z|_H$ for all $w, z \in E_0$. In this case, we say that f has H-Lipschitz constant (at most) M.

[1]This definition may slightly differ from those in other literatures.

The following is a variant of Rademacher's theorem.

Theorem 2.4 ([8], cf. [15, Theorem 4.2]). *Let f be an $\mathfrak{M}(E)$-measurable function on E that is H-Lipschitz with H-Lipschitz constant M. Then, $f \in W^{1,2}(E)$ and $|Df|_H \le M$ μ-a.e.*

We introduce some concepts related to the H-distance.

Definition 2.5. For a subset A of E and $z \in E$, we define

$$\mathsf{d}_E(z, A) = \inf\{|z - w|_E \mid w \in A\} \text{ and } \mathsf{d}_H(z, A) = \inf\{|z - w|_H \mid w \in A \cap (z + H)\},$$

where we set $\inf \emptyset = +\infty$. We also define the following sets:

- the H-closure $\overline{A}^H := \{z \in E \mid \mathsf{d}_H(z, A) = 0\}$,
- the H-boundary $\partial^H A := \overline{A}^H \cap \overline{E \setminus A}^H$,
- the H-exterior $A^{H\text{-ext}} := E \setminus \overline{A}^H$,
- the H-interior $A^{H\text{-int}} := A \setminus \partial^H A \, (= (A^{H\text{-ext}})^{H\text{-ext}})$.

For $z \in E$ and $s > 0$, we define

$$B_H(z, s) = \{z + h \mid h \in H,\ |h|_H < s\} \text{ and } \overline{B}_H(z, s) = \{z + h \mid h \in H,\ |h|_H \le s\}.$$

We omit z from the notation if $z = 0$. Note that $\overline{B}_H(z, s)$ is compact in E (see, e.g., [3, Corollary 3.2.4] for the proof.)

Let us recall that a Suslin set in E is a continuous image of a certain Polish space. Suslin sets are universally measurable and closed under countable intersections and countable unions. Borel sets of E are Suslin sets. More precisely speaking, a subset A of E is Borel if and only if both A and $E \setminus A$ are Suslin sets (see, e.g., [5, 6] for further details).

Lemma 2.6. *If A is a Suslin subset of E with $\mu(A) > 0$, then $\mathsf{d}_H(\cdot, A)$ is universally measurable and H-Lipschitz with H-Lipschitz constant 1.*

Proof. Measurability of $\mathsf{d}_H(\cdot, A)$ follows from the identity

$$\{\mathsf{d}_H(\cdot, A) \le r\} = \bigcap_{k=1}^{\infty} \left(A + \overline{B}_H(r + 1/k)\right)$$

for every $r \ge 0$. The set $\{\mathsf{d}_H(\cdot, A) < \infty\}$ has a full μ-measure from Proposition 2.1. The remaining assertions are easy to prove. □

The next proposition is proved in [18] in a more general context. (Similar results are found, e.g., in [20] in different frameworks.) Since our situation is simpler and the proof is shortened, we include the proof for the readers' convenience.

Proposition 2.7. *Suppose that a subset A of E is H-open and $\mu(A) = 0$. Then, $\mathrm{Cap}_{1,2}(A) = 0$.*

Proof. We denote $E \setminus A$ by A^c. Take any $\varepsilon > 0$ and a compact subset C of A^c such that $\mu(E \setminus C) < \varepsilon$. Let $r > 0$ and define $C_r := C + \overline{B}_H(r)$, which is a compact set. Define $f_r(z) = \big(r^{-1}\mathsf{d}_H(z, C)\big) \wedge 1$ for $z \in E$. Then, $f_r \in W^{1,2}(E)$ and $|Df_r|_H \le 1/r$ μ-a.e. from Lemma 2.6, Theorem 2.4, and Proposition 2.2. Moreover, $0 \le f_r \le 1$ on E, $f_r = 1$ on $E \setminus C_r$, and $f_r = 0$ on C. Then,

$$\begin{aligned}\mathrm{Cap}_{1,2}\left(E \setminus \left(A^c + \overline{B}_H(r)\right)\right) &\le \mathrm{Cap}_{1,2}(E \setminus C_r) \le \mathcal{E}(f_r, f_r) + \|f_r\|^2_{L^2(E)} \\ &\le \int_{E\setminus C} r^{-2}\, d\mu + \int_{E\setminus C} d\mu \quad \text{(by Proposition 2.2 (i))} \\ &\le (r^{-2} + 1)\varepsilon.\end{aligned}$$

Since ε is arbitrary, $\mathrm{Cap}_{1,2}\left(E \setminus \left(A^c + \overline{B}_H(r)\right)\right) = 0$. Therefore,

$$\mathrm{Cap}_{1,2}(A) = \mathrm{Cap}_{1,2}\left(\bigcup_{k=1}^{\infty} \left(E \setminus \left(A^c + \overline{B}_H(1/k)\right)\right)\right) = 0. \qquad \square$$

Corollary 2.8 (cf. [22, Theorem 7.3.3])**.** *Let $A \in \mathfrak{M}(E)$ be H-invariant. Then, either* $\mathrm{Cap}_{1,2}(A) = 0$ *or* $\mathrm{Cap}_{1,2}(E \setminus A) = 0$ *holds.*

Proof. Since both A and $E \setminus A$ are H-open, the assertion follows from Propositions 2.1 and 2.7. $\square$

Lemma 2.9. *Let f be an $\mathfrak{M}(E)$-measurable and H-Lipschitz function on E. Then, f is quasi-continuous.*

From Theorem 2.4, f belongs to $W^{1,2}(E)$ under the assumption; thus, f has a quasi-continuous modification. The point of Lemma 2.9 is that f itself is quasi-continuous without modification.

Proof of Lemma 2.9. From Proposition 2.7, $\mathrm{Cap}_{1,2}(E \setminus E_0) = 0$, where E_0 is provided in Definition 2.3. Therefore, by considering $f \cdot 1_{E_0}$ instead of f, we may assume $E_0 = E$ without loss of generality. Let f have H-Lipschitz constant M.

We take an increasing sequence $\{G_n\}_{n=1}^{\infty}$ of finite-dimensional subspaces of E^* such that $\bigcup_{n=1}^{\infty} G_n$ is dense in H. We also define $G_n^{\perp}$, P_{G_n}, Q_{G_n}, $\mu_{G_n^{\perp}}$, and μ_{G_n} as in the first part of this section. For $n \in \mathbb{N}$, let

$$\hat{G}_n = \left\{ y \in G_n \,\middle|\, f(\cdot + y) \text{ is a } \mu_{G_n^{\perp}}\text{-integrable function on } G_n^{\perp} \right\}.$$

From Fubini's theorem, $\mu_{G_n}(G_n \setminus \hat{G}_n) = 0$. Define a function $\hat{f}_n$ on $\hat{G}_n$ by

$$\hat{f}_n(y) = \int_{G_n^{\perp}} f(x + y)\, \mu_{G_n^{\perp}}(dx).$$

Then, it is easy to see that for $y, y' \in \hat{G}_n$,

$$|\hat{f}_n(y) - \hat{f}_n(y')| \le M|y - y'|_H. \tag{2.1}$$

Therefore, $\hat{f}_n$ extends to a continuous function $\hat{\hat{f}}_n$ that is defined on G_n, and (2.1) holds for every $y, y' \in G_n$ with $\hat{f}_n$ replaced by $\hat{\hat{f}}_n$. Define a function f_n on E as $f_n(z) =$

$\hat{\hat{f}}_n(Q_n(z))$ for $z \in E$. Then, f_n is continuous on E and identical to the conditional expectation of f given $\sigma(Q_n)$. Since $\sigma(Q_n; n \in \mathbb{N}) = \mathfrak{B}(E)$, f_n converges to f μ-a.e. by the martingale convergence theorem. Moreover, since $Q_n|_H$ is a contraction operator on H, f_n is also H-Lipschitz with H-Lipschitz constant M. Then, $\{f_n\}_{n=1}^\infty$ is bounded in $W^{1,2}(E)$. From the Banach–Saks theorem, the Cesàro means of a certain subsequence of $\{f_n\}$, denoted by $\{g_n\}$, converge in $W^{1,2}(E)$. Note that g_n is continuous on E as well as H-Lipschitz with H-Lipschitz constant M. From [16, Proposition III.3.5] or [12, Theorem 2.1.4], by taking a subsequence if necessary, g_n converges q.e. to some quasi-continuous function g. Since f_n converges to f μ-a.e., so does g_n. Define $B = \{z \in E \mid \lim_{n\to\infty} g_n(z) = f(z)\}$. Clearly, $\mu(E \setminus B) = 0$. Take $z \in \overline{B}^H$. There exists a sequence $\{z_k\}_{k=1}^\infty$ in B such that $\lim_{k\to\infty} |z_k - z|_H = 0$. Then,

$$\begin{aligned}|g_n(z) - f(z)| &\le |g_n(z) - g_n(z_k)| + |g_n(z_k) - f(z_k)| + |f(z_k) - f(z)| \\ &\le M|z - z_k|_H + |g_n(z_k) - f(z_k)| + M|z - z_k|_H.\end{aligned}$$

Taking $\limsup_{n\to\infty}$ on both sides and letting $k \to \infty$, we obtain $\lim_{n\to\infty} g_n(z) = f(z)$. Therefore, $z \in B$. That is, $\overline{B}^H = B$ and $E \setminus B$ is H-open. From Proposition 2.7, $\mathrm{Cap}_{1,2}(E \setminus B) = 0$. This implies that $f = g$ q.e., in particular, f is quasi-continuous. □

The following proposition, which is of contextual interest, is utilized in the proof of Theorem 1.2 in the next section.

Proposition 2.10. *Let $A \in \mathfrak{M}(E)$.*

(i) *If A is H-open, then $A^{H\text{-ext}}$ is quasi-open; in particular, $A^{H\text{-ext}}, \overline{A}^H, \partial^H A \in \mathfrak{M}(E)$.*
(ii) *If A is H-open, then A is quasi-open.*
(iii) *If A is H-closed, then A is quasi-closed.*

Proof. (i): If $\mathrm{Cap}_{1,2}(\overline{A}^H) = 0$, then the assertion is clear. We assume $\mathrm{Cap}_{1,2}(\overline{A}^H) > 0$. Choose a countable dense subset H_0 of H. From the H-openness of A, $\mathsf{d}_H(z, A) = \inf\{|h|_H \cdot 1_{A-h}(z) \mid h \in H_0\}$ for each $z \in E$. Thus, $\mathsf{d}_H(\cdot, A)$ is $\mathfrak{M}(E)$-measurable. Let $E_0 = \{z \in E \mid \mathsf{d}_H(z, A) < \infty\}$. Since E_0 is H-invariant and $E_0 \supset \overline{A}^H$, $\mathrm{Cap}_{1,2}(E \setminus E_0) = 0$ from Corollary 2.8. In particular, $\mu(E_0) = 1$. Therefore, $\mathsf{d}_H(\cdot, A)$ satisfies the definition of H-Lipschitz functions. From Lemma 2.9, it is quasi-continuous. Since $A^{H\text{-ext}} = \{z \in E \mid \mathsf{d}_H(z, A) > 0\}$, $A^{H\text{-ext}}$ is quasi-open.

(ii): By applying (i) to the H-open set $A^{H\text{-ext}}$ and by using the identity $A = (A^{H\text{-ext}})^{H\text{-ext}}$, we conclude that A is quasi-open.

(iii): It is sufficient to apply (ii) to $E \setminus A$. □

The following is an improvement on Lemma 2.9.

Proposition 2.11. *If an $\mathfrak{M}(E)$-measurable function f on E is H-continuous, then f is quasi-continuous.*

Proof. This is clear from Corollary 2.8 and Proposition 2.10. □

Remark 2.12. From the proof, we can replace $W^{1,2}(E)$ by $W^{1,p}(E)$ in Theorem 2.4 and $\mathrm{Cap}_{1,2}$ by $\mathrm{Cap}_{1,p}$ in Proposition 2.7 and Corollary 2.8 for any $p \in (1, \infty)$. Here, $W^{1,p}(E)$

is the first order L^p-Sobolev space on E in terms of Malliavin calculus, and $\mathrm{Cap}_{1,p}$ is the associated capacity. (See [17] for example, where symbol $\mathbb{D}_1^p$ is used in place of $W^{1,p}$.) Moreover, Lemma 2.9, Proposition 2.10, and Proposition 2.11 are valid, even if quasi-notions are interpreted in terms of $\mathrm{Cap}_{1,p}$.

3. Proof of Theorem 1.2

In this section, we prove Theorem 1.2. We assume that $U \in \mathfrak{M}(E)$ satisfies the assumptions of Theorem 1.2: $\mu(U) > 0$, U is H-open and H-convex. For a subset F of E and subset A of F, we denote the closure, interior, and boundary of A with respect to the relative topology of F by $\overline{A}^F$, $A^{F\text{-int}}$, and $\partial^F A$, respectively. Although these terminologies are slightly inconsistent with the corresponding ones in Definition 2.5, we use them as long as there is no ambiguity.

Let us recall that K was taken and fixed as a dense subspace of H in Section 2. We also fix an increasing sequence $\{G_n\}_{n=1}^\infty$ of finite-dimensional subspaces of K such that $\bigcup_{n=1}^\infty G_n$ is dense in H. We also define $G_n^\perp$, P_{G_n}, Q_{G_n}, $\mu_{G_n^\perp}$, and μ_{G_n} as in the previous section. For a finite-dimensional subspace G of K and $x \in G^\perp$, μ_{x+G} denotes a measure on $x + G$ that is defined as the induced measure of μ_G by the canonical map from G to $x + G$.

The following is a consequence of the basic theory of convex analysis; it is proved in the same way as in [14, Lemma 4.7].

Lemma 3.1. *Let G be a finite-dimensional subspace of H and $a \in E$. Define $F = a+G$. If $U \cap F \neq \emptyset$, then $U^{H\text{-int}} \cap F = (U \cap F)^{F\text{-int}}$, $\overline{U}^H \cap F = \overline{U \cap F}^F$, and $(\partial^H U) \cap F = \partial^F (U \cap F)$.*

Proof. Select y from $U^{H\text{-int}} \cap F$. There exists $\delta > 0$ such that $B_H(y, \delta) \subset U^{H\text{-int}}$.

First, we show that $U^{H\text{-int}} \cap F \supset (U \cap F)^{F\text{-int}}$. Take x from $(U \cap F)^{F\text{-int}}$. There exists $s > 0$ such that $w := (1 + s)x - sy \in (U \cap F)^{F\text{-int}}$. Then, $B_H(x, \frac{s\delta}{1+s}) = \frac{1}{1+s} w + \frac{s}{1+s} B_H(y, \delta) \subset U$, that is, $x \in U^{H\text{-int}}$. Since $x \in F$, we obtain $x \in U^{H\text{-int}} \cap F$.

Next, we show that $\overline{U}^H \cap F \subset \overline{U \cap F}^F$. Take $x \in \overline{U}^H \cap F$. Then,

$$\bigcup_{t\in(0,1]} \big((1 - t)x + t(B_H(y, \delta) \cap F)\big) \subset (U \cap F)^{F\text{-int}}$$

(cf. [19, Theorem 6.1]), and x is an accumulation point in F of the left-hand side. Therefore, $x \in \overline{U \cap F}^F$.

Both the converse inclusions are evident. The last identity follows from the first two identities. □

Lemma 3.2. *There exist a compact subset V_0 of U and $r > 0$ such that $\mu(V_0) > 0$ and $V_0 + \overline{B}_H(4r) \subset U$.*

Proof. In the proof, we do not use the H-convexity of U. By taking an open set O of E with $0 < \mu(\overline{O}^H) < \mu(U)$ and considering $U \setminus \overline{O}^H$ instead of U, we may assume $\mu(U^{H\text{-ext}}) > 0$. Define $\varphi(z) = \mathsf{d}_H(z, U^{H\text{-ext}})$ for $z \in E$. Then, φ is $\mathfrak{M}(E)$-measurable from the proof of Proposition 2.10 (i). Since $U = \{\varphi > 0\}$, we can take $r > 0$ such that

$\mu(\{\varphi \geq 5r\}) > 0$. Take a compact subset V_0 of $\{\varphi \geq 5r\}$ such that $\mu(V_0) > 0$. These satisfy the required conditions. □

Hereafter, V_0 and r always denote those in Lemma 3.2. We define $V = V_0 + \overline{B}_H(r)$. Note that V is compact and

$$V + \overline{B}_H(3r) \subset U. \tag{3.1}$$

Remark 3.3. (i) We have $\mu(\partial^H U) = 0$. Indeed, let $x \in P_{G_n}(V)$. From Lemma 3.1,

$$(\partial^H U) \cap (x + G_n) = \partial^{x+G_n}(U \cap (x + G_n)). \tag{3.2}$$

Since $U \cap (x+G_n)$ is convex in $x+G_n$, the right-hand side of (3.2) is a null set with respect to the Lebesgue measure on $x + G_n$, i.e., μ_{x+G_n}-null. By integrating over $P_{G_n}(V)$, $(\partial^H U) \cap (V + G_n)$ is proved to be a μ-null set. Since $\mu(E \setminus (V + G_n)) \to 0$ as $n \to \infty$ from Proposition 2.1, we obtain $\mu(\partial^H U) = 0$.

(ii) Similarly, we can prove that if U is a convex set with nonempty interior in E, then the topological boundary of U is a μ-null set.

Definition 3.4. Let G be a subspace of H. For $z \in E$ and $s > 0$, we define

$$B_G(z, s) = \{z + h \mid h \in G,\ |h|_H < s\} \text{ and } \overline{B}_G(z, s) = \{z + h \mid h \in G,\ |h|_H \leq s\}.$$

We often omit z from the notation if $z = 0$.

Let W_0 be a subspace of $W^{1,2}(U)$, defined as follows:

$$W_0 = \left\{ f \in W^{1,2}(U) \;\middle|\; \begin{array}{l} f \text{ is bounded on } U \text{ and } f = 0\ \mu\text{-a.e. on} \\ U \setminus \big(V + \overline{B}_{G_R}(R)\big) \text{ for some } R \in \mathbb{N} \end{array} \right\}. \tag{3.3}$$

Lemma 3.5. *Space W_0 is dense in $W^{1,2}(U)$.*

Proof. Since $W^{1,2}(U) \cap L^\infty(U)$ is dense in $W^{1,2}(U)$, it is sufficient to prove that each function in $W^{1,2}(U) \cap L^\infty(U)$ can be approximated by functions in W_0. Take $f \in W^{1,2}(U) \cap L^\infty(U)$ and let $M = \|f\|_{L^\infty(U)}$. From Proposition 2.1, $\lim_{n\to\infty} \mu\left(V_0 + \overline{B}_{G_n}(n)\right) = 1$. For each $n \in \mathbb{N}$, define

$$\xi_n(z) = \big(1 - r^{-1}\mathsf{d}_H(z, V_0 + \overline{B}_{G_n}(n))\big) \vee 0, \quad z \in E.$$

From Theorem 2.4 and Proposition 2.2, $\xi_n \in W^{1,2}(E)$ and $|D\xi_n|_H \leq 1/r$ μ-a.e. In addition, $0 \leq \xi_n \leq 1$ on E, $\xi_n = 0$ on $E \setminus \big(V + \overline{B}_{G_n}(n)\big)$, and $\xi_n = 1$ on $V_0 + \overline{B}_{G_n}(n)$. From Proposition 2.2, $f\xi_n \in W_0$ and

$$\begin{aligned} \|f\xi_n\|^2_{W^{1,2}(U)} &\leq 2M^2\|\xi_n\|^2_{W^{1,2}(U)} + 2\|f\|^2_{W^{1,2}(U)} + M^2 \\ &\leq 2M^2(r^{-2} + 1) + 2\|f\|^2_{W^{1,2}(U)} + M^2, \end{aligned}$$

which is bounded in n. Therefore, the Cesàro means of a certain subsequence of $\{f\xi_n\}_{n=1}^\infty$ converges in $W^{1,2}(U)$. Since $\xi_n \to 1$ μ-a.e. as $n \to \infty$, the limit function is f. □

Hereafter, we fix a function f in W_0 and write G for G_R in (3.3). For the proof of Theorem 1.2, it is sufficient to prove that f is approximated by elements in $W^{1,2}(E)|_U$. For this purpose, we first construct a partition of unity.

Since $Q_G(V)$ is compact in G, we can take a finite number of points $a_1, a_2, \dots, a_S$ from $Q_G(V)$ such that $Q_G(V) \subset \bigcup_{i=1}^S B_G(a_i, r)$ for some $S \in \mathbb{N}$. For $i = 1, \dots, S$, define

$$A_i = Q_G^{-1}(\overline{B}_G(a_i, r)) \cap V$$

and

$$\psi_i(z) = \bigl(1 - r^{-1}\mathsf{d}_H(z, A_i + G)\bigr) \vee 0, \quad z \in E.$$

Then, each A_i is compact in E, $V + G \subset \bigcup_{i=1}^S (A_i + G)$, $\sum_{i=1}^S \psi_i(z) \geq 1$ for $z \in V + G$, and $\psi_i(z) = 0$ for $z \in E \setminus (A_i + G + B_H(r))$ for each i.

We take a real-valued nondecreasing smooth function Φ on $\mathbb{R}$ such that $\Phi(0) = 0$ and $\Phi(t) = 1$ for $t \geq 1$. Define

$$\varphi_1 = \Phi(\psi_1), \ \varphi_j = \Phi\left(\sum_{i=1}^j \psi_i\right) - \Phi\left(\sum_{i=1}^{j-1} \psi_i\right) \ \text{for } j = 2, \dots, S.$$

For each j, φ_j is H-Lipschitz, $0 \leq \varphi_j \leq 1$ on E, and $\varphi_j = 0$ on $E \setminus (A_j + G + B_H(r))$. Moreover, $\sum_{j=1}^S \varphi_j = 1$ on $V + G$. Thus, $f\varphi_j|_U \in W^{1,2}(U) \cap L^\infty(U)$ for each j and $f = \sum_{j=1}^S f\varphi_j$ on U. Therefore, in order to prove Theorem 1.2, it is sufficient to prove that each $f\varphi_j|_U$ can be approximated by elements in $W^{1,2}(E)|_U$.

We fix j and write g for $f\varphi_j|_U$.

Lemma 3.6. *We have $\{g \neq 0\} \subset A_j + B_H(r) + \overline{B}_G(R')$, where $R' > 0$ is taken such that it is large enough to satisfy $Q_G(V - A_j) + B_G(R + r) \subset \overline{B}_G(R')$.*

Proof. By the definition of g, we have $\{g \neq 0\} \subset (V + \overline{B}_G(R)) \cap (A_j + G + B_H(r))$. Take an element z from the right-hand side. Then, z is described as

$$z = z_1 + y_1 = z_2 + y_2 + h,$$

where $z_1 \in V$, $y_1 \in \overline{B}_G(R)$, $z_2 \in A_j$, $y_2 \in G$, and $h \in B_H(r)$. Then,

$$y_2 = Q_G(z_1 - z_2 + y_1 - h) \in Q_G(V - A_j) + B_G(R + r) \subset \overline{B}_G(R').$$

This completes the proof. $\square$

We set

$$Y = A_j + B_H(r) + \overline{B}_G(R' + 1), \tag{3.4}$$

which belongs to $\mathfrak{M}(E)$ and is relatively compact as well as H-open. (See Figure 1.) We define

$$Y' = (Y + B_H(r)) \cap U \quad \text{and} \quad X = \bigl((Q_G^{-1}(a_j) \cap U) + B_G(R'')\bigr) \cap U$$

with $R'' = R' + 1 + 3r$.

Lemma 3.7. *It holds that $Y \cap U \subset Y' \subset X$.*

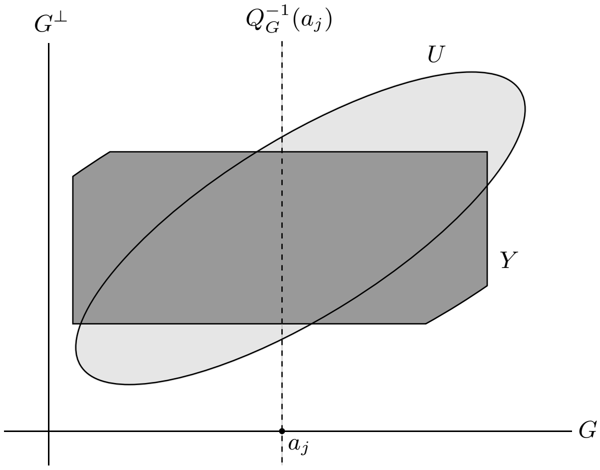

Figure 1: Illustration of U and Y etc.

Proof. The first inclusion is evident. To prove the second inclusion, choose z from Y'. Then, we can write $z = z_1 + h_1 + y_1$ for some $z_1 \in A_j = Q_G^{-1}(\overline{B}_G(a_j, r)) \cap V$, $h_1 \in B_H(2r)$, and $y_1 \in \overline{B}_G(R'+1)$. There exists $y_2 \in \overline{B}_G(r)$ such that $z_1 - y_2 \in Q_G^{-1}(a_j)$. Since $Q_G(P_G h_1) = 0$, $|P_G h_1|_H < 2r$, and $|Q_G h_1|_H < 2r$, z is decomposed as

$$z = (z_1 - y_2 + P_G h_1) + (y_2 + Q_G h_1 + y_1),$$

where

$$z_1 - y_2 + P_G h_1 \in Q_G^{-1}(a_j) \cap \bigl(V + \overline{B}_G(r) + B_H(2r)\bigr) \subset Q_G^{-1}(a_j) \cap U \quad \text{(from (3.1))}$$

and

$$y_2 + Q_G h_1 + y_1 \in \overline{B}_G(r) + B_G(2r) + \overline{B}_G(R'+1) \subset B_G(R'+1+3r).$$

Since $z \in U$, we conclude that $z \in X$. $\square$

Let $\gamma \in (0, 1/2]$. We define a map $T_\gamma \colon E \to E$ as

$$\begin{aligned} T_\gamma(z) &:= P_G(z) + (1-\gamma) Q_G(z) + \gamma a_j \\ &= z + \gamma(a_j - Q_G(z)). \end{aligned}$$

Then, for any $w \in E$, $T_\gamma(w+G) = w+G$ and $T_\gamma|_{w+G}$ is a homothety on $w+G$ that is centered at $P_G(w) + a_j$ with a magnification ratio $1-\gamma$.

From a simple calculation, the induced measure of μ by the map T_γ, denoted by $\mu \circ T_\gamma^{-1}$, is absolutely continuous with respect to μ, and the Radon–Nikodym derivative $d(\mu \circ T_\gamma^{-1})/d\mu$ is uniformly bounded in γ on $Q_G^{-1}(C)$ for any compact set C of G.

Let $X_\gamma := T_\gamma^{-1}(X)$. From the definitions, X and X_γ are H-convex and belong to $\mathfrak{M}(E)$. Therefore, X and X_γ are moderate, and we can consider the function spaces $W^{1,2}(X)$ and $W^{1,2}(X_\gamma)$. We also note that $X \subset X_\beta \subset X_\gamma$ if $0 < \beta < \gamma$. We define a function g_γ on X_γ by

$$g_\gamma(z) = g(T_\gamma(z)) \quad \text{for } z \in X_\gamma.$$

Then, for a sufficiently small γ,

$$\{g_\gamma \neq 0\} \subset Y \tag{3.5}$$

by Lemma 3.6 and (3.4). Hereafter, we consider only such a small γ, say, in the interval $(0, \gamma_0]$ for some $\gamma_0 > 0$.

The following lemma is intuitively evident; nonetheless, we have provided the proof.

Lemma 3.8. *Function g_γ belongs to $W^{1,2}(X_\gamma)$. Moreover, $g_\gamma|_X$ converges to $g|_X$ in $W^{1,2}(X)$ as $\gamma \downarrow 0$.*

Proof. First, we prove that $g_\gamma \in W^{1,2}(X_\gamma)$ and

$$Dg_\gamma = (I - \gamma Q_G)\big((Dg) \circ T_\gamma\big), \tag{3.6}$$

where I denotes the identity operator on H. Since

$$\int_{X_\gamma} g_\gamma^2 \, d\mu = \int_X g^2 \, d(\mu \circ T_\gamma^{-1}) \le \int_Y g^2 \left\| \frac{d(\mu \circ T_\gamma^{-1})}{d\mu} \right\|_{L^\infty(Y)} d\mu < \infty, \tag{3.7}$$

we obtain $g_\gamma \in L^2(X_\gamma)$. Similarly, we have $(I - \gamma Q_G)\big((Dg) \circ T_\gamma\big) \in L^2(X_\gamma \to H)$. Take any $h \in K \setminus \{0\}$ and define $k = (I - \gamma Q_G)h \in K \setminus \{0\}$. For $\mu_{k^\perp}$-a.e. $x \in k^\perp$, there exists an absolutely continuous version $\tilde{g}_k(x, \cdot)$ of $g_k(x, \cdot)$ such that

$$\langle (Dg)(x + sk), k \rangle = \frac{\partial \tilde{g}_k}{\partial s}(x, s) \quad \text{for a.e. } s \in I^X_{x,k}.$$

For $x \in h^\perp$,

$$T_\gamma(x + sh) = x + sh + \gamma(a_j - Q_G(sh)) = x + \gamma a_j + sk$$

and

$$\langle x + \gamma a_j, k \rangle = \langle x + \gamma a_j, h - \gamma Q_G h \rangle = -\gamma \langle Q_G x - (1 - \gamma) a_j, h \rangle.$$

Therefore, by letting

$$b = \gamma \langle Q_G x - (1 - \gamma) a_j, h \rangle / |k|_H^2 \quad \text{and} \quad x' = x + \gamma a_j + bk,$$

we have $x' \in k^\perp$, $I^X_{x',k} + b = I^{X_\gamma}_{x,h}$, and $\tilde{g}_k(x', \cdot - b)$ is an absolutely continuous version of $(g_\gamma)_h(x, \cdot)$ on $I^{X_\gamma}_{x,h}$. Moreover,

$$\begin{aligned}
\frac{\partial}{\partial s} \tilde{g}_k(x', s - b) &= \langle (Dg)(x' + (s - b)k), k \rangle \\
&= \langle (Dg)(T_\gamma(x + sh)), (I - \gamma Q_G)h \rangle \\
&= \big\langle (I - \gamma Q_G)\big((Dg)(T_\gamma(x + sh))\big), h \big\rangle .
\end{aligned}$$

This implies that $g_\gamma \in W^{1,2}(X_\gamma)$ and $Dg_\gamma = (I - \gamma Q_G)\big((Dg) \circ T_\gamma\big)$.

Next, we prove that $g_\gamma|_X$ converges to $g|_X$ in $W^{1,2}(X)$ as $\gamma \downarrow 0$. For $\mu_{G^\perp}$-a.e. $x \in G^\perp$, the convergence of $g_\gamma|_{(x+G)\cap X}$ to $g|_{(x+G)\cap X}$ in $L^2((x + G) \cap X, \mu_{x+G}|_{(x+G)\cap X})$ is proved in a standard way as follows. For $x \in G^\perp$, define

$$g^*(z) = \begin{cases} g(z) & \text{if } z \in (x + G) \cap X, \\ 0 & \text{if } z \in (x + G) \setminus X. \end{cases}$$

For $\mu_{G^\perp}$-a.e. x, g^* belongs to $L^2(x+G, \mu_{x+G})$. Let Ψ be a smooth function on G with compact support and $\int_G \Psi(y)\,\lambda_m(dy) = 1$, where $m = \dim G$ and λ_m denotes the Lebesgue measure on G. For each $\varepsilon > 0$, define a smooth function ψ_ε on $x+G$ by $\psi_\varepsilon(z) = \varepsilon^{-m}\int_G g^*(z-y)\Psi(\varepsilon^{-1}y)\,dy$. Then, denoting $L^2((x+G)\cap X, \mu_{x+G}|_{(x+G)\cap X})$ by $L^2((x+G)\cap X)$, we have

$$\begin{aligned}
&\|g_\gamma|_{(x+G)\cap X} - g|_{(x+G)\cap X}\|_{L^2((x+G)\cap X)}\\
&\le \|g_\gamma|_{(x+G)\cap X} - (\psi_\varepsilon\circ T_\gamma)|_{(x+G)\cap X}\|_{L^2((x+G)\cap X)}\\
&\quad + \|(\psi_\varepsilon\circ T_\gamma)|_{(x+G)\cap X} - \psi_\varepsilon|_{(x+G)\cap X}\|_{L^2((x+G)\cap X)}\\
&\quad + \|\psi_\varepsilon|_{(x+G)\cap X} - g|_{(x+G)\cap X}\|_{L^2((x+G)\cap X)}.
\end{aligned} \tag{3.8}$$

The last term of (3.8) converges to 0 as $\varepsilon \downarrow 0$, as does the first term of (3.8) by using an estimate similar to (3.7). From the dominated convergence theorem, the second term converges to 0 as $\gamma \downarrow 0$. Therefore, by letting $\gamma \downarrow 0$ and $\varepsilon \downarrow 0$, $g_\gamma|_{(x+G)\cap X}$ converges to $g|_{(x+G)\cap X}$ in $L^2((x+G)\cap X, \mu_{x+G}|_{(x+G)\cap X})$. By integrating $\|g_\gamma|_{(x+G)\cap X} - g|_{(x+G)\cap X}\|^2_{L^2((x+G)\cap X)}$ over $G^\perp$ with respect to $\mu_{G^\perp}(dx)$ and by using the dominated convergence theorem, we obtain $\|g_\gamma|_X - g|_X\|^2_{L^2(X)} \to 0$ as $\gamma \downarrow 0$.

Similarly, we can show that

$$\big((Dg)\circ T_\gamma\big)|_X \to (Dg)|_X \quad \text{in } L^2(X\to H) \text{ as } \gamma\downarrow 0. \tag{3.9}$$

From (3.6), $\{Dg_\gamma|_X\}_{\gamma\in(0,\gamma_0]}$ is bounded in $L^2(X\to H)$. Therefore, $\{g_\gamma|_X\}_{\gamma\in(0,\gamma_0]}$ is bounded in $W^{1,2}(X)$, and it is weakly relatively compact. Since any accumulation point should be $g|_X$, $g_\gamma|_X$ converges weakly to $g|_X$ in $W^{1,2}(X)$. Since $\lim_{\gamma\downarrow 0}\|g_\gamma|_X\|_{W^{1,2}(X)} = \|g|_X\|_{W^{1,2}(X)}$ in view of (3.6) and (3.9), we conclude that $g_\gamma|_X$ converges to $g|_X$ in $W^{1,2}(X)$ as $\gamma\downarrow 0$. $\square$

We extend the defining set of g_γ to $X_\gamma\cup U$ by letting $g_\gamma(z) = 0$ for $z\in U\setminus X_\gamma$. Since

$$\{g_\gamma \ne 0\}\cap U \subset Y\cap U \subset (Y + B_H(r))\cap U = Y' \subset X \subset X_\gamma,$$

we have $g_\gamma|_U \in W^{1,2}(U)$.

Lemma 3.9. *It holds that $\overline{Y\cap U}^H \subset X_\gamma$.*

Proof. Let $z\in \overline{Y\cap U}^H$. Then, z is described as

$$z = z_1 + h_1 + y_1 + h_2$$

for some $z_1 \in Q_G^{-1}(\overline{B}_G(a_j, r))\cap V$, $h_1\in B_H(r)$, $y_1\in \overline{B}_G(R'+1)$, and $h_2\in B_H(r)$. Then, $z\in V + B_H(2r) + G \subset U + G$. Therefore, $U\cap(z+G)\ne\emptyset$.

From Lemma 3.1, we have $\overline{U}^H\cap(z+G) = \overline{U\cap(z+G)}^{z+G}$. Since $z\in\overline{U}^H\cap(z+G)$, we have $z\in\overline{U\cap(z+G)}^{z+G}$. Moreover, there exists $y_2\in\overline{B}_G(r)$ such that $z_1 - y_2 \in Q_G^{-1}(a_j)$. Then,

$$\begin{aligned}
z &= (z_1 - y_2 + P_G(h_1+h_2)) + (Q_G(h_1+h_2) + y_1 + y_2)\\
&\in \big(Q_G^{-1}(a_j)\cap\big(V + \overline{B}_G(r) + B_H(2r)\big)\big) + \big(B_G(2r) + \overline{B}_G(R'+1) + \overline{B}_G(r)\big)\\
&\subset \big(Q_G^{-1}(a_j)\cap U\big) + G \quad \text{(from (3.1))}.
\end{aligned}$$

Therefore, $\big(U\cap(z+G)\big)\cap Q_G^{-1}(a_j)\neq\emptyset$. Combining this with the facts that $U\cap(z+G)$ is convex in $z+G$ and $z\in\overline{U\cap(z+G)}^{z+G}$, we obtain $T_\delta(z)\in U\cap(z+G)\subset U$ for all $\delta\in(0,\gamma]$. Furthermore, if δ is sufficiently small, we have $|T_\delta(z)-z|_H<r$, which implies that $T_\delta(z)\in Y+B_H(r)$. Therefore, $T_\delta(z)\in Y'$ for such δ. This implies that $z\in T_\delta^{-1}(Y')\subset X_\delta\subset X_\gamma$. $\square$

From this lemma, $\overline{Y\cap U}^H\cap(E\setminus X_\gamma)=\emptyset$. Since both $Y\cap U$ and X_γ are H-open and belong to $\mathfrak{M}(E)$, both $\overline{Y\cap U}^H$ $(=E\setminus(Y\cap U)^{H\text{-ext}})$ and $E\setminus X_\gamma$ are quasi-closed from Proposition 2.10.

Let $m\in\mathbb{N}$. We can take open subsets $O_{m,1}$ and $O_{m,2}$ of E such that $\mathrm{Cap}_{1,2}(O_{m,i})<1/m$ $(i=1,2)$ and both $\overline{Y\cap U}^H\setminus O_{m,1}$ and $(E\setminus X_\gamma)\setminus O_{m,2}$ are closed in E. Since $\mathrm{Cap}_{1,2}$ is tight (see, e.g., [21]), there exists an open set $O_{m,3}$ such that $\mathrm{Cap}_{1,2}(O_{m,3})<1/m$ and $E\setminus O_{m,3}$ is compact. Let $O_m=O_{m,1}\cup O_{m,2}\cup O_{m,3}$. We define $C_m=\overline{Y\cap U}^H\setminus O_m$ and $C'_{\gamma,m}=(E\setminus X_\gamma)\setminus O_m$. Since both sets are compact and $C_m\cap C'_{\gamma,m}=\emptyset$, $\mathsf{d}_E(C_m,C'_{\gamma,m})=:\alpha>0$. Define $C''_{\gamma,m}=C'_{\gamma,m}+\{z\in E\mid |z|_E\le\alpha/2\}$ and

$$\rho_{\gamma,m}(z)=\frac{\mathsf{d}_E(z,C''_{\gamma,m})}{\mathsf{d}_E(z,C_m)+\mathsf{d}_E(z,C''_{\gamma,m})},\quad z\in E.$$

Then, $\rho_{\gamma,m}$ is H-Lipschitz, $0\le\rho_{\gamma,m}\le1$ on E, $\rho_{\gamma,m}=1$ on C_m, and $\rho_{\gamma,m}=0$ on $C''_{\gamma,m}$.

We note that $g_\gamma|_{X_\gamma}\in W^{1,2}(X_\gamma)\cap L^\infty(X_\gamma)$ and $1-e_{O_m}=0$ on O_m, where e_{O_m} is the equilibrium potential of O_m. Moreover, X_γ, O_m, and $C'_{\gamma,m}+\{z\in E\mid |z|_E<\alpha/2\}$ are all H-open sets, and their union is equal to E. Therefore, $g_{\gamma,m}:=g_\gamma\cdot(1-e_{O_m})\cdot\rho_{\gamma,m}$ is well-defined as an element of $W^{1,2}(E)\cap L^\infty(E)$ and $g_{\gamma,m}=g_\gamma\cdot(1-e_{O_m})$ on U from (3.5). Then,

$$\|g-g_{\gamma,m}|_U\|_{W^{1,2}(U)}\le\|g-g_\gamma|_U\|_{W^{1,2}(U)}+\|g_\gamma|_U-g_{\gamma,m}|_U\|_{W^{1,2}(U)}$$

and

$$\begin{aligned}
&\|g_\gamma|_U-g_{\gamma,m}|_U\|^2_{W^{1,2}(U)}\\
&=\|g_\gamma|_U-g_\gamma(1-e_{O_m})|_U\|^2_{W^{1,2}(U)}\\
&=\|(g_\gamma e_{O_m})|_U\|^2_{W^{1,2}(U)}\\
&\le 2\|((Dg_\gamma)e_{O_m})|_U\|^2_{L^2(U)}+2M^2\|(De_{O_m})|_U\|^2_{L^2(U)}+M^2\|e_{O_m}|_U\|^2_{L^2(U)}\\
&\to0\quad\text{as } m\to\infty.
\end{aligned}\tag{3.10}$$

Here, we note that the first term of (3.10) converges to 0 since $|(Dg_\gamma)e_{O_m}|_H\le|Dg_\gamma|_H$ and $\big(|(Dg_\gamma)e_{O_m}|_H\big)|_U$ converges to 0 in measure $\mu|_U$.

By combing these estimates with Lemma 3.8, $\lim_{\gamma\downarrow0}\lim_{m\to\infty}\|g-g_{\gamma,m}|_U\|_{W^{1,2}(U)}=0$. That is, g can be approximated in $W^{1,2}(U)$ by elements of $W^{1,2}(E)|_U$. This completes the proof of Theorem 1.2.

4. Concluding remarks

Let U be the same as in Theorem 1.2.

(i) Feyel and Üstünel [9] proved the following logarithmic Sobolev inequality on U.

Theorem 4.1 (cf. [9, Theorem 6.4]). *For any $f \in \mathcal{F}C_b^1(E)$,*

$$\fint_U f^2 \log\left(f^2 \Big/ \fint_U f^2\,d\mu\right) d\mu \le 2 \fint_U |Df|_H^2\,d\mu. \tag{4.1}$$

Here, $\fint_U \cdots d\mu := \mu(U)^{-1} \int_U \cdots d\mu$ denotes the normalized integral on U.

(A more general result is proved in [9, Theorem 6.4].) Theorem 1.2 implies that (4.1) is also true for all $f \in W^{1,2}(U)$ from the approximation procedure.

(ii) We consider a Markov process associated with $(\mathcal{E}^U, W^{1,2}(U))$. From [10, Theorem 2.1] (see also [21]), the closure of $(\mathcal{E}^U, \mathcal{F}C_b^1(E)|_U)$ is a quasi-regular local Dirichlet form on $L^2(\bar{U}, \mu|_U)$, where $\bar{U}$ denotes the closure of U in E. Therefore, there is an associated diffusion process $\{X_t\}$ on $\bar{U}$. Moreover, suppose that the indicator function of U belongs to $BV(E)$, which is defined in [10, 11]. Then, we have the Skorohod-type representation of $\{X_t\}$ ([11, Theorem 4.2]):

$$X_t(\omega) - X_0(\omega) = W_t(\omega) - \frac{1}{2}\int_0^t X_s(\omega)\,ds + \frac{1}{2}\int_0^t \sigma_U(X_s(\omega))\,dA_s^{\|D1_U\|}(\omega), \quad t \ge 0,$$

where $\{W_t\}$ is the E-valued Brownian motion starting at 0, σ_U is an H-valued function on E, and $A^{\|D1_U\|}$ is a positive continuous additive functional. σ_U and $A^{\|D1_U\|}$ formally correspond to the vector field normal to the boundary U^∂ of U and the additive functional induced by the surface measure of U^∂, respectively (see [11] for more precise descriptions). In [10, 11], $\{X_t\}$ is called the *modified* reflecting Ornstein–Uhlenbeck process (for more general U) because the domain of the Dirichlet form is defined by the smallest extension of $\mathcal{F}C_b^1(E)|_U$, and in general, it is not clear whether it really is a natural one. Under the assumptions of Theorem 1.2, the domain is equal to $W^{1,2}(U)$, and $\{X_t\}$ can be aptly called the *true* reflecting Ornstein–Uhlenbeck process on $\bar{U}$.

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