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Non-holomorphic Modular Forms and $SL(2,\mathbb{R})/U(1)$ Superconformal Field Theory

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Abstract

We study the torus partition function of the $SL(2,\mathbb{R})/U(1)$ SUSY gauged WZW model coupled to $\mathcal{N}=2$ $U(1)$ current. Starting from the path-integral formulation of the theory, we introduce an infra-red regularization which preserves good modular properties and discuss the decomposition of the partition function in terms of the $\mathcal{N}=2$ characters of discrete (BPS) and continuous (non-BPS) representations. Contrary to our naive expectation, we find a non-holomorphic dependence (dependence on $\bar{\tau}$) in the expansion coefficients of continuous representations. This non-holomorphicity appears in such a way that the anomalous modular behaviors of the discrete (BPS) characters are compensated by the transformation law of the non-holomorphic coefficients of the continuous (non-BPS) characters. Discrete characters together with the non-holomorphic continuous characters combine into real analytic Jacobi forms and these combinations exactly agree with the "modular completion" of discrete characters known in the theory of Mock theta functions [9].

We consider this to be a general phenomenon: we expect to encounter "holomorphic anomaly" ($\bar{\tau}$-dependence) in string partition function on non-compact target manifolds. The anomaly occurs due to the incompatibility of holomorphy and modular invariance of the theory. Appearance of non-holomorphicity in $SL(2,\mathbb{R})/U(1)$ elliptic genus has recently been observed by Troost [10].

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1 Introduction

The torus partition function of 2D conformal field theory plays a basic role in unraveling the physical spectrum of the theory, and the modular invariance plays a key role in this analysis.

Roughly speaking, we have the next three possibilities for the Hilbert space of a sensible conformal theory;

- The space of normalizable states only includes a discrete set of primary states.
- The space of normalizable states only includes a continuous family of primary states.
- The space of normalizable states includes both discrete and continuous spectra.

The first case is most familiar, and corresponds to compact backgrounds of string theory. The partition function of a closed string theory is schematically written in a holomorphically factorized form,

$$Z = \sum_{j,\tilde{j}} N_{j,\tilde{j}}\, \chi_j(\tau)\, \chi_{\tilde{j}}(\tau)^*,$$

where the conformal blocks $\chi_j(\tau)$ are identified with characters of some chiral algebra. $\chi_j(\tau)$ usually possesses simple modular transformation law and the modular invariance of the partition function is achieved in a relatively easy way.

The second case describes non-compact backgrounds. This class of models still has a simple structure of modular invariant partition functions, which includes integrals over parameters of continuous representations. The bosonic or $\mathcal{N}=1$ Liouville theories are non-trivial examples of this class. These theories also contain a discrete set of *non-normalizable* primary fields, which correspond to degenerate representations of (super-) Virasoro algebra. These states, however, do not appear in the torus partition function[1].

The third one is the most intriguing class. The partition function is expected to be decomposed into a sum of the 'discrete' and 'continuous' parts;

$$Z(\tau) = Z_{\text{dis}}(\tau) + Z_{\text{con}}(\tau). \tag{1.1}$$

These models are expected to describe non-compact backgrounds with the continuous part $Z_{\text{con}}(\tau)$ representing the propagating degrees of freedom. However, in contrast to the second case, we also have the discrete part $Z_{\text{dis}}(\tau)$ which represents normalizable bound states localized in some curved region of space-time. A non-trivial issue in this class of conformal field theories is how to achieve the modular invariance of the theory. In fact, in these models, non-trivial mixing of discrete and continuous representations often take place under the modular

[1]This feature is often phrased as the lack of operator-state correspondence.

S-transformation, which makes it difficult to assure modular invariance in a simple manner. Such an anomalous modular behavior has been first observed in the massless (BPS) representation of $\mathcal{N}=4$ superconformal algebra [1]. Similar behavior appears in the discrete (BPS) characters in the $\mathcal{N}=2$ Liouville or the $SL(2,\mathbb{R})/U(1)$ SUSY gauged WZW model [2, 3] (see also [4, 5]). In our previous attempts [3, 6] (see also [7, 8]), these models were assumed to have the partition function of the form (1.1), and we have identified the discrete spectra $Z_{\rm dis}(\tau)$ of the theory. However, the issue of modular invariance remained unsolved due to the complexity of modular property of discrete (BPS) representations mentioned above.

In this paper, we study the $SL(2,\mathbb{R})/U(1)$ SUSY gauged WZW model, focusing mainly on this issue. We shall analyze the torus partition function with a modulus coupled to $\mathcal{N}=2$ $U(1)$ current. We begin our analysis based on the approach of path-integration, which automatically ensures the modular invariance. We then introduce a suitable regularization preserving good modular behaviors[2], and discuss the decomposition of regularized partition function into $\mathcal{N}=2$ characters. It turns out that characters of the discrete representations in the partition function are always accompanied by a series of continuous representations with *non-holomorphic* ($\bar{\tau}$-dependent) coefficients in such a way that together they transform like Jacobi forms. This is in fact the structure known as the "modular completion" of discrete representations in the theory of Mock theta functions [9]. Thus the correct partition function is given by (1.1) with discrete part replaced by its modular completion.

We consider that this is a general phenomenon and string theory amplitudes on non-compact manifolds possess an anomaly coming from the incompatibility between holomorphy and modular invariance. When we insist on strict modular invariance, we may loose holomorphy, while if we relax modular invariance, for instance, to subgroups of $SL(2,\mathbb{Z})$ we may possibly keep holomorphy intact. These alternatives will correspond to the choice of boundary conditions at infinity of non-compact manifolds.

In this paper we further study the elliptic genus of $SL(2,\mathbb{Z})/U(1)$ theory based on the character decomposition of partition function mentioned above and also the direct evaluation of path-integral representation of elliptic genus. The latter approach has an advantage of no need of regularization and can be compared closely with the paper [10] which inspired the present work. It turns out that both analyses will lead us to an identical result given again in terms of real analytic Jacobi forms having $\bar{\tau}$ dependence.

Unexpectedly, we find the use of the mathematical theory of Mock theta functions in the

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[2]This in fact means that the regularized partition function slightly breaks the modular S-invariance. However, the violation is under a good control by a small parameter characterizing the regularization. T-invariance is kept intact by the regularization. See subsection 2.3 for the detail.

analysis of non-compact geometry in string theory[3].

2 Partition Function of $SL(2,\mathbb{R})/U(1)$ SUSY Gauged WZW Model

2.1 $SL(2,\mathbb{R})/U(1)$ SUSY Gauged WZW Model

We shall first introduce the model which we study in this paper, summarizing relevant notations. We consider the $SL(2,\mathbb{R})/U(1)$ SUSY gauged WZW model with level k [4], which is well-known [12] to have $\mathcal{N}=2$ superconformal symmetry with central charge;

$$\hat{c} \equiv \frac{c}{3} = 1 + \frac{2}{k}, \quad k \equiv \kappa - 2. \tag{2.1}$$

The world-sheet action in the present convention is written as

$$\begin{aligned}
S(g, A, \psi^{\pm}, \tilde{\psi}^{\pm}) &= \kappa S_{\text{gWZW}}(g, A) + S_{\psi}(\psi^{\pm}, \tilde{\psi}^{\pm}, A), &(2.2)\\
\kappa S_{\text{gWZW}}(g, A) &= \kappa S^{SL(2,\mathbb{R})}_{\text{WZW}}(g) + \frac{\kappa}{\pi}\int_{\Sigma} d^2v \left\{ \text{Tr}\left(\frac{\sigma_2}{2} g^{-1}\partial_{\bar{v}} g\right) A_v + \text{Tr}\left(\frac{\sigma_2}{2}\partial_v g g^{-1}\right) A_{\bar{v}} \right.\\
&\qquad \left. + \text{Tr}\left(\frac{\sigma_2}{2} g \frac{\sigma_2}{2} g^{-1}\right) A_{\bar{v}} A_v + \frac{1}{2} A_{\bar{v}} A_v \right\}, &(2.3)\\
S^{SL(2,\mathbb{R})}_{\text{WZW}}(g) &= -\frac{1}{8\pi}\int_{\Sigma} d^2v\, \text{Tr}\left(\partial_\alpha g^{-1} \partial_\alpha g\right) + \frac{i}{12\pi}\int_B \text{Tr}\left((g^{-1}dg)^3\right), &(2.4)\\
S_{\psi}(\psi^{\pm}, \tilde{\psi}^{\pm}, A) &= \frac{1}{2\pi}\int d^2v \left\{ \psi^+ (\partial_{\bar{v}} + A_{\bar{v}})\psi^- + \psi^- (\partial_{\bar{v}} - A_{\bar{v}})\psi^+ \right.\\
&\qquad \left. + \tilde{\psi}^+ (\partial_v + A_v)\tilde{\psi}^- + \tilde{\psi}^- (\partial_v - A_v)\tilde{\psi}^+ \right\}, &(2.5)
\end{aligned}$$

where the complex fermions $\psi^{\pm}$ (and $\tilde{\psi}^{\pm}$) have charge ± 1 with respect to the $U(1)$-gauge group. The bosonic part $\kappa S_{\text{gWZW}}(g, A)$ is the gauged WZW action for the coset $SL(2,\mathbb{R})_\kappa/U(1)_A$ [13, 14], where $U(1)_A$ indicates the gauging of axial $U(1)$-symmetry; $g \rightarrow \Omega\, g\, \Omega$, $\Omega(v,\bar{v}) = e^{iu(v,\bar{v})\sigma_2}$ ($u(v,\bar{v}) \in \mathbb{R}$, σ_2 is the Pauli matrix)[5]. It is well-known that this model describes the string theory on 2D Euclidean black-hole with the cigar geometry [16]. The WZW action $\kappa S^{SL(2,\mathbb{R})}_{\text{WZW}}(g)$ is formally equal to $-\kappa S^{SU(2)}_{\text{WZW}}(g)$, and has a negative signature in $i\sigma_2$-direction. Since we have $H^3(SL(2,\mathbb{R})) = 0$, the action $\kappa S^{SL(2,\mathbb{R})}_{\text{WZW}}(g)$ can be rewritten in a purely two dimensional form and the level κ need not be an integer.

[3] Other applications of mock theta functions to studies of superconformal field theories are given in recent papers [11].

[4] k is the level of the total $SL(2,\mathbb{R})$-current, whose bosonic part has the level $\kappa \equiv k+2$.

[5] We take $\{\frac{i\sigma_2}{2},\ \frac{\sigma_3}{2},\ \frac{\sigma_1}{2}\}$ for the basis of $SL(2,\mathbb{R}) \cong SU(1,1)$.

According to the familiar treatment of gauged WZW models, one can easily separate the anomalous degrees of freedom originating from chiral gauge transformations (the compact scalar field Y given in the next subsection), which makes the relevant field contents to be free and chiral [13, 14, 15]. After this procedure, the chiral currents defined by

$$j^A(v) = \kappa \mathrm{Tr}\left(T^A \partial_v g g^{-1}\right), \quad \tilde{j}^A(\bar{v}) = -\kappa \mathrm{Tr}\left(\overline{T^A} g^{-1}\partial_{\bar{v}} g\right), \tag{2.6}$$

$$T^3 = \frac{1}{2}\sigma_2 , \quad T^{\pm} = \pm\frac{1}{2}\left(\sigma_3 \pm i\sigma_1\right) \tag{2.7}$$

satisfy the affine $\widehat{SL}(2,\mathbb{R})_\kappa$ current algebra (we write the left-mover only);

$$\begin{cases} j^3(v)j^3(0) & \sim -\dfrac{\kappa/2}{v^2} \\ j^3(v)j^{\pm}(0) & \sim \dfrac{\pm 1}{v} j^{\pm}(0) \\ j^+(v)j^-(0) & \sim \dfrac{\kappa}{v^2} - \dfrac{2}{v} j^3(0) \end{cases} \tag{2.8}$$

and the pair of free fermions ψ^+, ψ^- satisfy the OPE's $\psi^+(v)\psi^-(0) \sim 1/v$, $\psi^{\pm}(v)\psi^{\pm}(0) \sim 0$. The explicit realization of $\mathcal{N}=2$ SCA is given by

$$\begin{aligned} T(v) &= \frac{1}{k}\eta_{AB} j^A j^B + \frac{1}{k} J^3 J^3 - \frac{1}{2}(\psi^+\partial_v\psi^- - \partial_v\psi^+\psi^-), \quad (\eta_{AB} = \mathrm{diag}(1,1,-1)), \\ J &= \psi^+\psi^- + \frac{2}{k} J^3, \quad G^{\pm} = \frac{1}{\sqrt{k}}\psi^{\pm} j^{\mp}, \end{aligned} \tag{2.9}$$

where we introduced the total $U(1)$-current $J^3 \equiv j^3 + \psi^+\psi^-$, which couples with the gauge field A and commuetes with all the generators of $\mathcal{N}=2$ SCA (2.9).

2.2 Path Integral Evaluation of Torus Partition Function

The main purpose of this subsection is to evaluate the torus partition function of the SUSY gauged WZW model (2.2) by path-integration. We define the world-sheet torus Σ by the identifications $(w,\bar{w}) \sim (w+2\pi, \bar{w}+2\pi) \sim (w+2\pi\tau, \bar{w}+2\pi\bar{\tau})$ ($\tau \equiv \tau_1 + i\tau_2$, $\tau_2 > 0$, and use the convention $v = e^{iw}$, $\bar{v} = e^{-i\bar{w}}$). We call the cycles defined by these two identifications as the α and β-cycles as usual.

The relevant calculation is carried out in a way parallel to that given in [3], although we have the following differences;

- We shall focus on the $\widetilde{\mathrm{R}}$-sector (R-sector with $(-1)^F$ insertion) of the theory. Character formulas in the following sections are those in the $\widetilde{\mathrm{R}}$ sector. Formulas in other sectors are obtained by spectral flow.

- We introduce the complex moduli z, $\bar{z}$ which couple with the zero-modes of $\mathcal{N}=2$ $U(1)$-currents $J(v)$, $\tilde{J}(\bar{v})$ (2.9). In other words, we would like to evaluate the partition sum weighted by $e^{2\pi i\left(zJ_0-\bar{z}\tilde{J}_0\right)}$.

We shall start with the Wick rotated model in order to make the gauged WZW action (2.3) positive definite. This means replacing $g(v,\bar{v})\in SL(2,\mathbb{R})$ with $g(v,\bar{v})\in H_3^+\cong SL(2,\mathbb{C})/SU(2)$, and the gauge field $A\equiv(A_{\bar{v}}d\bar{v}+A_v dv)\dfrac{\sigma_2}{2}$ should be regarded as a hermitian 1-form. It is convenient to reexpress our (axial-like) gauged WZW action $S^{SL(2,\mathbb{R})}_{\text{gWZW}}(g,A)$ (2.3) in terms of $S^{(A)}_{\text{gWZW}}(g,h,h^\dagger)$ given in (B.4) with the identification $A_{\bar{v}}\dfrac{\sigma_2}{2}=\partial_{\bar{v}}hh^{-1}$, $A_v\dfrac{\sigma_2}{2}=\partial_v h^\dagger h^{\dagger\,-1}$. We also parameterize $h\in\exp\left(\mathbb{C}\sigma_2\right)$ as

$$h=\Omega h[u],\quad \Omega\equiv e^{(X+iY)\frac{\sigma_2}{2}},\quad h[u]\equiv e^{i\Phi[u]\frac{\sigma_2}{2}},\tag{2.10}$$

where real scalar fields X, Y correspond to the axial ($\mathbb{R}_A$) and vector ($U(1)_V$) gauge transformations respectively. $\Phi[u](w,\bar{w})$ is associated with the modulus of a holomorphic line bundle; $u\equiv s_1\tau+s_2\in\text{Jac}(\Sigma)\cong\Sigma$, $(0\le s_1,s_2<1)$, conventionally defined as

$$\Phi[u](w,\bar{w})=\frac{i}{2\tau_2}\left\{(\bar{w}\tau-w\bar{\tau})s_1+(\bar{w}-w)s_2\right\}\equiv\frac{1}{\tau_2}\text{Im}(w\bar{u}).\tag{2.11}$$

It is a real harmonic function satisfying the twisted boundary conditions;

$$\Phi[u](w+2\pi,\bar{w}+2\pi)=\Phi[u](w,\bar{w})-2\pi s_1,\quad \Phi[u](w+2\pi\tau,\bar{w}+2\pi\bar{\tau})=\Phi[u](w,\bar{w})+2\pi s_2.\tag{2.12}$$

Emphasizing the modulus dependence, we shall denote the corresponding gauge field as $A[u]$, namely,

$$A[u]_{\bar{w}}=\partial_{\bar{w}}X+i\partial_{\bar{w}}Y+a[u]_{\bar{w}},\quad A[u]_w=\partial_w X-i\partial_w Y+a[u]_w,\tag{2.13}$$

$$a[u]_{\bar{w}}\equiv i\partial_{\bar{w}}\Phi[u]\equiv-\frac{u}{2\tau_2},\quad a[u]_w\equiv -i\partial_w\Phi[u]\equiv-\frac{\bar{u}}{2\tau_2}.\tag{2.14}$$

It will be useful to point out that our modulus parameter u is normalized so that the partial derivative of classical action (2.2) with respect to it yields

$$-\frac{\partial}{\partial u}S(g,a[u],\psi^\pm,\tilde{\psi}^\pm)\Big|_{u=0}=2\pi i J_0^3,\qquad -\frac{\partial}{\partial\bar{u}}S(g,a[u],\psi^\pm,\tilde{\psi}^\pm)\Big|_{u=0}=-2\pi i\tilde{J}_0^3.$$

Now, the desired partition function is schematically written as

$$Z(\tau,z)=\int_\Sigma\frac{d^2u}{\tau_2}\int\mathcal{D}[g,A[u],\psi^\pm,\tilde{\psi}^\pm]\times\exp\left[-\kappa S_{\text{gWZW}}\left(g,A[u+\frac{2}{k}z]\right)-S_\psi\left(\psi^\pm,\tilde{\psi}^\pm,A[u+\frac{k+2}{k}z]\right)\right],\tag{2.15}$$

where $\frac{d^2u}{\tau_2} \equiv ds_1 ds_2$ is the modular invariant measure of modulus parameter u, and we work in the $\widetilde{\text{R}}$-sector for world-sheet fermions.

As expected, the inclusion of complex parameter z corresponds to the marginal deformation[6] described by the insertion of $e^{2\pi i\left(zJ_0 - \bar{z}\tilde{J}_0\right)}$. In fact, one can directly confirm from (2.15) that $\frac{\partial}{\partial z} Z(\tau, z)\big|_{z=0}$ yields the insertion of zero-mode $2\pi i J_0$, after making a suitable chiral gauge transformation[7]. Moreover, this deformation is strictly marginal *even* under the u-integration, because the modulus u couples with the zero-mode of current $j^3 + \psi^+\psi^- +$ (anomaly contribution) associated to the $U(1)$-coseting, which has no singular OPE with the $\mathcal{N} = 2$ $U(1)$-current J. These facts imply that the complex parameter z appearing in (2.15) should be captured by the insertion of $e^{2\pi i z J_0}$ in the operator formalism. (The same is true for $\bar{z}$.)

A simple way to evaluate (2.15) is to convert the functional integral of gauge field into those of real scalar fields X, Y and modulus u, as is manipulated in [14]. In doing so, we have to be a bit careful, because different values of moduli couple with bosonic and fermionic parts.

Namely,

- For the bosonic sector, the modulus parameter takes a value $u + \frac{2}{k}z$. The scalar field of anomaly free direction X decouples, while the contribution from anomalous $U(1)_V$ (Y direction) is extracted by utilizing the identity;

$$\begin{aligned} &-\kappa S^{(A)}_{\text{gWZW}}(\Omega^{-1} g \Omega^{\dagger\,-1},\, \Omega h[u + \frac{2}{k}z],\, h[u + \frac{2}{k}z]^\dagger \Omega^\dagger) \\ &\quad = -\kappa S^{(V)}_{\text{gWZW}}(g, h[u + \frac{2}{k}z], h^\dagger[u + \frac{2}{k}z]) + \kappa S^{(A)}_{\text{gWZW}}(\Omega\Omega^{\dagger\,-1},\, h[u + \frac{2}{k}z],\, h[u + \frac{2}{k}z]^{\dagger\,-1}), \end{aligned} \tag{2.16}$$

  which is readily derived from the definitions (B.3), (B.4). We here assume the gauge invariance of path-integral measure as usual; $\mathcal{D}(\Omega^{-1} g \Omega^{\dagger\,-1}) = \mathcal{D}g$.

- For the fermionic sector, the modulus parameter takes a value $u + \frac{k+2}{k}z$. We should regularize the fermion determinant so as to be gauge invariant along the $\mathbb{R}_A$-direction, and the anomalous $U(1)_V$-direction is again described by the gauged WZW action $-2S^{(A)}_{\text{gWZW}}(\Omega\Omega^{\dagger\,-1},\, h[u + \frac{k+2}{k}z],\, h[u + \frac{k+2}{k}z]^{\dagger\,-1})$.

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[6]Here we use the word "marginal" in the sense that it preserves the exact conformal symmetry (while violating supersymmetry).

[7]A non-trivial point is the contribution from the anomaly for chiral gauge transformations (coupling to the scalar field Y). As we will observe later, the linear couplings of modulus z with the currents $i\partial Y$ cancel out, and the z-dependence appears at a quadratic order, which does not spoil the interpretation of $\frac{\partial}{\partial z} Z\big|_{z=0}$ as the insertion of $2\pi i J_0$.

- We have to also introduce the standard bc-ghosts with spin (1,0) to rewrite the Jacobian for the transformation of path-integral measure $\mathcal{D}A \rightarrow \mathcal{D}X\mathcal{D}Y$.

Explicitly, the partition function of each sector is evaluated as follows;

H_3^+-sector : For the H_+^3-sector, we obtain by using the formulas (B.7) (up to a normalization factor);

$$Z_g\left(\tau, u+\frac{2}{k}z\right) \equiv \int \mathcal{D}g \, \exp\left[-\kappa S^{(V)}_{\mathrm{gWZW}}\left(g,\, h[u+\frac{2}{k}z],\, h[u+\frac{2}{k}z]^{\dagger}\right)\right] = \frac{e^{2\pi\frac{(u_2+\frac{2}{k}z_2)^2}{\tau_2}}}{\sqrt{\tau_2}|\theta_1(\tau, u+\frac{2}{k}z)|^2}. \tag{2.17}$$

fermion and ghost sectors : The path-integration of fermionic and ghost sectors yields the standard fermion determinants with periodic boundary conditions, since we are working in the $\widetilde{\mathrm{R}}$-sector for $\psi^{\pm}$, $\tilde{\psi}^{\pm}$.

$$Z_\psi\left(\tau, u+\frac{k+2}{k}z\right) \equiv \int \mathcal{D}[\psi^{\pm}, \tilde{\psi}^{\pm}] \, \exp\left[-S_\psi\left(\psi^{\pm}, \tilde{\psi}^{\pm}, a[u+\frac{k+2}{k}z]\right)\right] = e^{-2\pi\frac{(u_2+\frac{k+2}{k}z_2)^2}{\tau_2}} \frac{\left|\theta_1(\tau, u+\frac{k+2}{k}z)\right|^2}{|\eta(\tau)|^2}, \tag{2.18}$$

$$Z_{\mathrm{gh}}(\tau) \equiv \int \mathcal{D}[b, \tilde{b}, c, \tilde{c}] \, \exp\left[-S_{\mathrm{gh}}(b, \tilde{b}, c, \tilde{c})\right] = \tau_2 \, |\eta(\tau)|^4 \; . \tag{2.19}$$

$U(1)_V$-sector : This sector is described by a single compact boson Y $(Y \sim Y + 2\pi)$. Recalling (2.1), we can explicitly compute the relevant world-sheet action as

$$\begin{aligned} S_Y(Y; u, z) &\equiv -\kappa S^{(A)}_{\mathrm{gWZW}}\left(e^{iY\sigma_2},\, h[u+\frac{2}{k}z],\, h[u+\frac{2}{k}z]^{\dagger\,-1}\right) \\ &\quad +2S^{(A)}_{\mathrm{gWZW}}\left(e^{iY\sigma_2},\, h[u+\frac{k+2}{k}z],\, h[u+\frac{k+2}{k}z]^{\dagger\,-1}\right) \\ &= \frac{k+2}{\pi}\int d^2w \left|\partial_{\bar{w}}\left(Y+\Phi[u+\frac{2}{k}z]\right)\right|^2 \\ &\quad -\frac{2}{\pi}\int d^2w \left|\partial_{\bar{w}}\left(Y+\Phi[u+\frac{k+2}{k}z]\right)\right|^2 \\ &= \frac{k}{\pi}\int d^2w \, |\partial_{\bar{w}}Y^u|^2 + \frac{k+2}{\pi}\int d^2w \left|\partial_{\bar{w}}\Phi[\frac{2}{k}z]\right|^2 - \frac{2}{\pi}\int d^2w \left|\partial_{\bar{w}}\Phi[\frac{k+2}{k}z]\right|^2 \\ &= \frac{k}{\pi}\int d^2w \, |\partial_{\bar{w}}Y^u|^2 - \frac{2\pi}{\tau_2}\hat{c}|z|^2, \end{aligned} \tag{2.20}$$

where $Y^u \equiv Y + \Phi[u]$ and satisfies the twisted boundary conditions;

$$\begin{aligned} &Y^u(w+2\pi, \bar{w}+2\pi) = Y^u(w, \bar{w}) - 2\pi(m_1+s_1), \\ &Y^u(w+2\pi\tau, \bar{w}+2\pi\bar{\tau}) = Y^u(w, \bar{w}) + 2\pi(m_2+s_2), \quad (m_1, m_2 \in \mathbb{Z}). \end{aligned} \tag{2.21}$$

Rescaling canonically the twisted boson Y^u as $Y^u \rightarrow Y^u/\sqrt{\alpha' k}$, we arrive at the theory of a twisted compact boson with radius $R = \sqrt{\alpha' k}$. Therefore, the relevant path-integration yields

$$
\begin{aligned}
Z_Y(\tau,u,z) &\equiv \int \mathcal{D}Y^u \, \exp\left[-\frac{1}{\pi\alpha'}\int d^2w \, |\partial_{\bar{w}} Y^u|^2 + \frac{2\pi}{\tau_2}\hat{c}\,|z|^2\right] \\
&= e^{\frac{2\pi}{\tau_2}\hat{c}|z|^2} \frac{\sqrt{k}}{\sqrt{\tau_2}\,|\eta(\tau)|^2} \sum_{m_1,m_2\in\mathbb{Z}} \exp\left(-\frac{\pi k}{\tau_2}\left|(m_1+s_1)\tau+(m_2+s_2)\right|^2\right). \qquad (2.22)
\end{aligned}
$$

Note that the linear couplings with the modulus parameters z, $\bar{z}$ with the currents $i\partial_w Y^u$, $i\partial_{\bar{w}} Y^u$ cancel out in (2.20). This fact justifies the identification of parameters z, $\bar{z}$ with the marginal deformation by $e^{2\pi i\left(zJ_0-\bar{z}\tilde{J}_0\right)}$ mentioned before.

Each sector (2.17), (2.18), (2.19), and (2.22) of the partition function is separately invariant under modular transformation

$$
\tau \mapsto \frac{a\tau+b}{c\tau+d}, \quad u \mapsto \frac{u}{c\tau+d}, \quad z \mapsto \frac{z}{c\tau+d}, \qquad \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}). \qquad (2.23)
$$

Combining all of them, we finally obtain the desired partition function;

$$
\begin{aligned}
Z(\tau,z) &= \int_\Sigma \frac{d^2u}{\tau_2} Z_g\left(\tau, u+\frac{2}{k}z\right) Z_\psi\left(\tau, u+\frac{k+2}{k}z\right) Z_Y(\tau,u,z)\, Z_{\mathrm{gh}}(\tau) \\
&= \mathcal{N}\, e^{\frac{2\pi}{\tau_2}\left(\hat{c}|z|^2-\frac{k+4}{k}z_2^2\right)} \sum_{m_1,m_2\in\mathbb{Z}} \int_\Sigma \frac{d^2u}{\tau_2} \left|\frac{\theta_1\left(\tau,u+\frac{k+2}{k}z\right)}{\theta_1\left(\tau,u+\frac{2}{k}z\right)}\right|^2 e^{-4\pi\frac{u_2z_2}{\tau_2}}\, e^{-\frac{\pi k}{\tau_2}|m_1\tau+m_2+u|^2} \\
&= \mathcal{N}\, e^{\frac{2\pi}{\tau_2}\left(\hat{c}|z|^2-\frac{k+4}{k}z_2^2\right)} \int_{\mathbb{C}} \frac{d^2u}{\tau_2} \left|\frac{\theta_1\left(\tau,u+\frac{k+2}{k}z\right)}{\theta_1\left(\tau,u+\frac{2}{k}z\right)}\right|^2 e^{-4\pi\frac{u_2z_2}{\tau_2}}\, e^{-\frac{\pi k}{\tau_2}|u|^2}, \qquad (2.24)
\end{aligned}
$$

where $\mathcal{N}$ is a normalization constant to be fixed later. In deriving the last line we made use of the identity; $\theta_1(\tau,z+m\tau+n) = (-1)^{m+n}\, q^{-\frac{m^2}{2}}\, e^{-2\pi imz}\,\theta_1(\tau,z)$.

This result is a natural generalization of the formula given in [3]. By construction, we expect that the obtained partition function (2.24) is modular invariant. However, there exists a subtlety here since the u-integration diverges logarithmically. This is due to the $\sim \left|u+\frac{2}{k}z\right|^{-2}$ behavior around $u+\frac{2}{k}z=0$, $(\bmod\ \mathbb{Z}\tau+\mathbb{Z})$ of the integrand, and such an IR divergence is inevitable for non-compact models like $SL(2;\mathbb{R})/U(1)$. It is thus important to introduce a suitable regularization not violating good modular behaviors, and we will discuss this issue in the next subsection.

2.3 Regularized Partition Function

Our next task is to extract the content of $\mathcal{N}=2$ representations of the partition function (2.24). Actually the partition function is infra-red divergent because the theta function in the denominator acquires zeros under u-integration. This is due to the non-compactness of the target manifold $SL(2,\mathbb{R})/U(1)$. Thus we have to introduce a suitable cut-off which preserves the modular invariance of the theory. In the following we take $k=\frac{N}{K}$ (N,K are positive co-prime integers) and consider models with the central charge $\hat{c}(\equiv\frac{c}{3})=1+\frac{2K}{N}=1+\frac{2}{k}$.

For convenience of analysis, we introduce minor modifications of the formula (2.24);

- We fix the normalization constant $\mathcal{N}$ as $\mathcal{N}=k$ so that

$$\lim_{z\to 0} Z(\tau,z)=1, \tag{2.25}$$

  which we find reasonable after performing the character decomposition.

- We redefine the integration variable $u \to -u$.

Namely, we shall start with the formula;

$$Z(\tau,z) \;=\; k e^{\frac{2\pi}{\tau_2}\left(\hat{c}|z|^2-\frac{k+4}{k}z_2^2\right)} \int_{\mathbb{C}} \frac{d^2u}{\tau_2}\left|\frac{\theta_1\left(\tau,-u+\left(1+\frac{2}{k}\right)z\right)}{\theta_1\left(\tau,-u+\frac{2}{k}z\right)}\right|^2 e^{4\pi\frac{u_2z_2}{\tau_2}} e^{-\frac{\pi k}{\tau_2}|u|^2}, \tag{2.26}$$

where we use the notations as $\mathrm{Re}\,z=z_1$, $\mathrm{Im}\,z=z_2$ etc. for the complex numbers z, τ, u.

We first discuss a suitable regularization of (2.26) which preserves modular invariance. By shifting the integration variable u as $u\to u+\frac{2}{k}z$ and introducing winding numbers w, m by $u=(s_1+w)\tau+(s_2+m)$, we can rewrite (2.26) as

$$\begin{aligned} Z(\tau,z) \;=\; & k e^{\frac{2\pi}{\tau_2}\left(\hat{c}|z|^2-z_2^2\right)} \sum_{w,m\in\mathbb{Z}} \int_0^1 ds_1 \int_0^1 ds_2 \left|\frac{\theta_1\left(\tau,-s_1\tau-s_2+z\right)}{\theta_1\left(\tau,-s_1\tau-s_2\right)}\right|^2 \\ & \times e^{4\pi s_1 z_2}\, e^{-\frac{\pi k}{\tau_2}\left|(s_1+w)\tau+(s_2+m)+\frac{2}{k}z\right|^2}. \end{aligned} \tag{2.27}$$

Now, we define the regularized partition function simply by replacing the integration region $D\equiv(0,1)\times(0,1)$ by $D(\epsilon):=(\epsilon,1-\epsilon)\times(0,1)$ where ϵ is a small positive constant;

$$\begin{aligned} Z_{\mathrm{reg}}(\tau,z;\epsilon) \;=\; & k e^{\frac{2\pi}{\tau_2}\left(\hat{c}|z|^2-z_2^2\right)} \sum_{w,m\in\mathbb{Z}} \int_\epsilon^{1-\epsilon} ds_1 \int_0^1 ds_2 \left|\frac{\theta_1\left(\tau,-s_1\tau-s_2+z\right)}{\theta_1\left(\tau,-s_1\tau-s_2\right)}\right|^2 \\ & \times e^{4\pi s_1 z_2}\, e^{-\frac{\pi k}{\tau_2}\left|(s_1+w)\tau+(s_2+m)+\frac{2}{k}z\right|^2}. \end{aligned} \tag{2.28}$$

We expect (2.28) to be convergent since all poles in the integrand are removed, and to exhibit a logarithmic divergence in $\epsilon\to 0$ limit. The integrand of (2.28) has a $\mathbb{Z}\times\mathbb{Z}$-periodicity with respect to s_1, s_2, and behaves like a modular form under the change of integration variables (s_1, s_2) as

$$T\ :\ (s_1, s_2)\ \longrightarrow\ (s_1, s_2+s_1), \qquad S\ :\ (s_1, s_2)\ \longrightarrow\ (s_2, -s_1).$$

This regularization does not strictly preserve the modular invariance, however, its violatation is under a good control;

$$Z_{\rm reg}(\tau+1, z;\epsilon) = Z_{\rm reg}(\tau, z;\epsilon), \qquad Z_{\rm reg}\left(-\frac{1}{\tau}, \frac{z}{\tau};\epsilon\right) - Z_{\rm reg}(\tau, z) = \mathcal{O}(\epsilon\log\epsilon, \epsilon). \tag{2.29}$$

The violation of S-invariance comes from the small change of the integration region $D(\epsilon)$ due to the S-transformation, while the T-transformation preserves it (up to the periodicity of s_1, s_2).

We shall now analyze $Z_{\rm reg}(\tau, z;\epsilon)$ in detail. By dualizing the temporal winding number m into the KK momentum n by means of the Poisson resummation formula (D.1), we can rewrite $Z_{\rm reg}(\tau, z;\epsilon)$ as follows;

$$\begin{aligned}
Z_{\rm reg}(\tau, z;\epsilon) &= \sqrt{k\tau_2}\, e^{\frac{2\pi}{\tau_2}\left(\hat{c}|z|^2 - z_2^2\right)} \sum_{w,n\in\mathbb{Z}} \int_\epsilon^{1-\epsilon} ds_1 \int_0^1 ds_2 \left|\frac{\theta_1(\tau, -s_1\tau - s_2 + z)}{\theta_1(\tau, -s_1\tau - s_2)}\right|^2 \\
&\quad \times e^{4\pi s_1 z_2}\, e^{-\pi\tau_2\left\{\frac{n^2}{k} + k\left(s_1 + w + \frac{2z_2}{k\tau_2}\right)^2\right\} + 2\pi i n\left\{(s_1+w)\tau_1 + s_2 + \frac{2z_1}{k}\right\}} \\
&= \sqrt{k\tau_2}\, e^{2\pi\frac{\hat{c}}{\tau_2} z_1^2} \sum_{w,n\in\mathbb{Z}} \int_\epsilon^{1-\epsilon} ds_1 \int_0^1 ds_2 \left|\frac{\theta_1(\tau, -s_1\tau - s_2 + z)}{\theta_1(\tau, -s_1\tau - s_2)}\right|^2 \\
&\quad \times e^{-\pi\tau_2\left\{\frac{n^2}{k} + k(s_1+w)^2\right\} + 2\pi i n\left\{(s_1+w)\tau_1 + s_2 + \frac{2z_1}{k}\right\} - 4\pi w z_2} \\
&= \sqrt{k\tau_2}\, e^{2\pi\frac{\hat{c}}{\tau_2} z_1^2} \sum_{w,n\in\mathbb{Z}} \int_\epsilon^{1-\epsilon} ds_1 \int_0^1 ds_2 \left|\frac{\theta_1(\tau, -s_1\tau - s_2 + z)}{\theta_1(\tau, -s_1\tau - s_2)}\right|^2 \\
&\quad \times e^{-\pi\tau_2\left\{\frac{n^2}{k} + k(s_1+w)^2\right\} + 2\pi i n\{(s_1+w)\tau_1 + s_2\}}\, y^{w+\frac{n}{k}}\, \bar{y}^{w-\frac{n}{k}}.
\end{aligned} \tag{2.30}$$

We next expand the ratio of θ_1 functions by using the identity (D.8);

$$\begin{aligned}
\left|\frac{\theta_1(\tau, -s_1\tau - s_2 + z)}{\theta_1(\tau, -s_1\tau - s_2)}\right|^2 &= \left|\frac{-i\theta_1(\tau, z)}{\eta(\tau)^3}\right|^2 \sum_{\ell,\tilde{\ell}\in\mathbb{Z}} \frac{(yq^\ell)}{(1-yq^\ell)} \cdot \left[\frac{(yq^{\tilde{\ell}})}{(1-yq^{\tilde{\ell}})}\right]^* \\
&\quad \times e^{-2\pi i(s_1\tau_1 + s_2)(\ell - \tilde{\ell}) + 2\pi s_1\tau_2(\ell + \tilde{\ell})}.
\end{aligned} \tag{2.31}$$

After substituting (2.31) into (2.30), one can easily integrate s_2 out, which just yields the constraint

$$n = \ell - \tilde{\ell}. \tag{2.32}$$

We next evaluate the s_1-integral. Picking up relevant terms, we obtain

$$e^{-\pi\tau_2\frac{N}{K}s_1^2-2\pi s_1\left\{\tau_2\frac{N}{K}w-i\tau_1 n+i\tau_1(\ell-\tilde{\ell})-\tau_2(\ell+\tilde{\ell})\right\}}=e^{-\pi\tau_2\frac{N}{K}s_1^2-2\pi s_1\tau_2\frac{v}{K}}, \tag{2.33}$$

where we set

$$v:=Nw-K(\ell+\tilde{\ell}), \tag{2.34}$$

and used the condition (2.32). With the help of a simple Gaussian integral, we further obtain

$$\begin{aligned}\int_{\epsilon}^{1-\epsilon}ds_1\, e^{-\pi\tau_2\frac{N}{K}s_1^2-2\pi s_1\tau_2\frac{v}{K}} &= \sqrt{\frac{\tau_2}{NK}}\int_{\epsilon}^{1-\epsilon}ds_1\int_{\mathbb{R}-i0}dp\, e^{-\frac{\pi}{NK}\tau_2 p^2-2\pi i\tau_2\frac{s_1}{K}(p-iv)}\\ &= \sqrt{\frac{K}{N\tau_2}}\frac{1}{2\pi i}\int_{\mathbb{R}-i0}dp\,\frac{e^{-\frac{\pi}{NK}\tau_2p^2}}{p-iv}\,\left\{e^{-\varepsilon(v+ip)}-e^{\varepsilon(v+ip)}e^{-2\pi i\tau_2\frac{1}{K}(p-iv)}\right\},\end{aligned} \tag{2.35}$$

where we set $\varepsilon\equiv 2\pi\frac{\tau_2}{K}\epsilon\,(>0)$. Using

$$2K\ell+v=K(kw+n)\,(\equiv Nw+Kn)\,,\quad 2K\tilde{\ell}+v=K(kw-n)\,(\equiv Nw-Kn)\,. \tag{2.36}$$

we obtain

$$\begin{aligned}e^{-\pi\tau_2\left(\frac{n^2}{k}+kw^2\right)+2\pi i\tau_1 nw} &= q^{\frac{(n+kw)^2}{4k}}\,\bar{q}^{\frac{(n-kw)^2}{4k}}=q^{\frac{(2K\ell+v)^2}{4NK}}\,\bar{q}^{\frac{(2K\tilde{\ell}+v)^2}{4NK}}\\ &= q^{\frac{1}{N}\left(K\ell^2+\ell v\right)}\bar{q}^{\frac{1}{N}\left(K\tilde{\ell}^2+\tilde{\ell}v\right)}e^{-\pi\tau_2\frac{v^2}{NK}},\end{aligned} \tag{2.37}$$

We also note that

$$y^{w+\frac{n}{k}}\bar{y}^{w-\frac{n}{k}}=y^{\frac{Nw+Kn}{N}}\,\bar{y}^{\frac{Nw-Kn}{N}}=y^{\frac{2K}{N}\left(\ell+\frac{v}{2K}\right)}\bar{y}^{\frac{2K}{N}\left(\tilde{\ell}+\frac{v}{2K}\right)}. \tag{2.38}$$

Combining all the pieces, we obtain the following expression of $Z_{\text{reg}}(\tau,z;\epsilon)$;

$$\begin{aligned}Z_{\text{reg}}(\tau,z;\epsilon) &= e^{2\pi\frac{\hat{c}}{\tau_2}z_1^2}\left|\frac{\theta_1(\tau,z)}{\eta(\tau)^3}\right|^2\sum_{\substack{v,\ell,\tilde{\ell}\in\mathbb{Z}\\ v+K(\ell+\tilde{\ell})\in N\mathbb{Z}}}\frac{1}{2\pi i}\int_{\mathbb{R}-i0}dp\,\frac{e^{-\pi\tau_2\frac{p^2+v^2}{NK}}}{p-iv}\,\left\{e^{-\varepsilon(v+ip)}-e^{\varepsilon(v+ip)}\,e^{-2\pi i\tau_2\frac{1}{K}(p-iv)}\right\}\\ &\qquad\times\frac{(yq^{\ell})^{1+\frac{v}{N}}}{1-yq^{\ell}}\left[\frac{(yq^{\tilde{\ell}})^{1+\frac{v}{N}}}{1-yq^{\tilde{\ell}}}\right]^*y^{\frac{2K}{N}\ell}q^{\frac{K}{N}\ell^2}\,\left[y^{\frac{2K}{N}\tilde{\ell}}q^{\frac{K}{N}\tilde{\ell}^2}\right]^*\\ &= e^{2\pi\frac{\hat{c}}{\tau_2}z_1^2}\left|\frac{\theta_1(\tau,z)}{\eta(\tau)^3}\right|^2\sum_{\substack{v,\ell,\tilde{\ell}\in\mathbb{Z}\\ v+K(\ell+\tilde{\ell})\in N\mathbb{Z}}}\frac{1}{2\pi i}\left[\int_{\mathbb{R}-i0}dp\,e^{-\varepsilon(v+ip)}\,yq^{\ell}\,\left[yq^{\tilde{\ell}}\right]^*-\int_{\mathbb{R}+i(N-0)}dp\,e^{\varepsilon(v+ip)}\right]\\ &\qquad\times\frac{e^{-\pi\tau_2\frac{p^2+v^2}{NK}}}{p-iv}\,\frac{(yq^{\ell})^{\frac{v}{N}}}{1-yq^{\ell}}\left[\frac{(yq^{\tilde{\ell}})^{\frac{v}{N}}}{1-yq^{\tilde{\ell}}}\right]^*y^{\frac{2K}{N}\ell}q^{\frac{K}{N}\ell^2}\,\left[y^{\frac{2K}{N}\tilde{\ell}}q^{\frac{K}{N}\tilde{\ell}^2}\right]^*\end{aligned} \tag{2.39}$$

Here we have absorbed the factor

$$yq^{\ell}\cdot\left[yq^{\tilde{\ell}}\right]^{*}\cdot e^{-2\pi i\tau_2\frac{1}{K}(p-iv)}$$

by the change of variables $p=:p'-iN$, $v=:v'-N$ with

$$\frac{p^2+v^2}{NK}=\frac{p'^2+v'^2}{NK}-\frac{2i}{K}(p'-iv'),\qquad p-iv=p'-iv',$$
$$\left(yq^{\ell}\right)^{\frac{v}{N}}\left[\left(yq^{\tilde{\ell}}\right)^{\frac{v}{N}}\right]^{*}=\left(yq^{\ell}\right)^{\frac{v'}{N}-1}\left[\left(yq^{\tilde{\ell}}\right)^{\frac{v'}{N}-1}\right]^{*}.$$

(2.39) is further rewritten as

$$Z_{\rm reg}(\tau,z;\epsilon)=e^{2\pi\frac{\hat{c}}{\tau_2}z_1^2}\left|\frac{\theta_1(\tau,z)}{\eta(\tau)^3}\right|^2\sum_{\substack{v,\ell,\tilde{\ell}\in\mathbb{Z}\\ v+K(\ell+\tilde{\ell})\in N\mathbb{Z}}}\frac{1}{2\pi i}\left[\int_{\mathbb{R}-i0}dp\,e^{-\varepsilon(v+ip)}\,yq^{\ell}\,\left[yq^{\tilde{\ell}}\right]^{*}-\int_{\mathbb{R}+i(N-0)}dp\,e^{\varepsilon(v+ip)}\right]$$
$$\times\frac{e^{-\pi\tau_2\frac{p^2}{NK}}}{p-iv}\,\frac{1}{1-yq^{\ell}}\left[\frac{1}{1-yq^{\tilde{\ell}}}\right]^{*}y^{\frac{2K}{N}\left(\ell+\frac{v}{2K}\right)}q^{\frac{K}{N}\left(\ell+\frac{v}{2K}\right)^2}\left[y^{\frac{2K}{N}\left(\tilde{\ell}+\frac{v}{2K}\right)}q^{\frac{K}{N}\left(\tilde{\ell}+\frac{v}{2K}\right)^2}\right]^{*}.\qquad(2.40)$$

We note that the power series expansion in $v,\ell,\tilde{\ell}$ (2.40) converges as is expected. In fact, as is obvious from the last line of (2.40), a potential divergence could happen when taking the limit $\ell,\tilde{\ell}\rightarrow\pm\infty$, $v\rightarrow\mp\infty$ with keeping $\ell+\frac{v}{2K}$, $\tilde{\ell}+\frac{v}{2K}$ finite. However, it is easy to confirm that the factors $e^{-\varepsilon(v+ip)}q^{\ell}/(1-yq^{\ell})$, $e^{\varepsilon(v+ip)}/(1-yq^{\tilde{\ell}})$ suitably produce damping effects to make the power series convergent in both sides of $v\rightarrow\pm\infty$.

3 Character Decomposition of Partition Function

Now, we would like to discuss the decomposition of partition function (2.40) in terms of $\mathcal{N}=2$ characters, which will be the main part of our analysis in this paper.

First of all, by shifting the integration contour in the second term as $\mathbb{R}+i(N-0)\rightarrow\mathbb{R}-i0$, we can decompose (2.40) into the pole contributions $Z_{\rm dis}(\epsilon)$ and the 'remainder part' $Z_{\rm rem}(\epsilon)$

as follows;

$$
\begin{aligned}
&Z_{\rm reg}(\tau,z;\epsilon) = Z_{\rm dis}(\tau,z;\epsilon) + Z_{\rm rem}(\tau,z;\epsilon),\\
&Z_{\rm dis}(\tau,z;\epsilon) \equiv e^{2\pi\frac{\hat{c}}{\tau_2}z_1^2}\left|\frac{\theta_1(\tau,z)}{\eta(\tau)^3}\right|^2 \sum_{v=0}^{N-1} \sum_{\substack{\ell,\tilde{\ell}\in\mathbb{Z}\\ v+K(\ell+\tilde{\ell})\in N\mathbb{Z}}} \frac{(yq^{\ell})^{\frac{v}{N}}}{1-yq^{\ell}} \left[\frac{(yq^{\tilde{\ell}})^{\frac{v}{N}}}{1-yq^{\tilde{\ell}}}\right]^* y^{\frac{2K}{N}\ell} q^{\frac{K}{N}\ell^2} \left[y^{\frac{2K}{N}\tilde{\ell}} q^{\frac{K}{N}\tilde{\ell}^2}\right]^*,
\end{aligned} \tag{3.1}
$$

$$
\begin{aligned}
Z_{\rm rem}(\tau,z;\epsilon) \equiv\ & e^{2\pi\frac{\hat{c}}{\tau_2}z_1^2}\left|\frac{\theta_1(\tau,z)}{\eta(\tau)^3}\right|^2 \sum_{\substack{v,\ell,\tilde{\ell}\in\mathbb{Z}\\ v+K(\ell+\tilde{\ell})\in N\mathbb{Z}}} \frac{1}{2\pi i}\int_{\mathbb{R}-i0} dp\, \frac{e^{-\pi\tau_2\frac{p^2+v^2}{NK}}}{p-iv}\\
&\times\left\{yq^{\ell}\cdot\left[yq^{\tilde{\ell}}\right]^* e^{-\varepsilon(v+ip)} - e^{\varepsilon(v+ip)}\right\} \frac{(yq^{\ell})^{\frac{v}{N}}}{1-yq^{\ell}} \left[\frac{(yq^{\tilde{\ell}})^{\frac{v}{N}}}{1-yq^{\tilde{\ell}}}\right] y^{\frac{2K}{N}\ell} q^{\frac{K}{N}\ell^2} \left[y^{\frac{2K}{N}\tilde{\ell}} q^{\frac{K}{N}\tilde{\ell}^2}\right]^*
\end{aligned} \tag{3.2}
$$

Note that the ϵ-dependence disappears in the discrete part $Z_{\rm dis}$, because the pole occurs at $p = iv$, $v = 0, 1, \ldots, N-1$.

We next elaborate on each of these contributions.

3.1 Discrete Part

The discrete part (3.1) is rewritten in terms of the (extended) discrete characters [2, 3, 6]. To make notations simple, we here adopt a slightly different notation for extended characters (see Appendix C);

$$
\chi_{\rm dis}(v,a;\tau,z) \equiv \frac{i\theta_1(\tau,z)}{\eta(\tau)^3} \sum_{n\in\mathbb{Z}} \frac{\left(yq^{Nn+a}\right)^{\frac{v}{N}}}{1-yq^{Nn+a}}\, y^{2K\left(n+\frac{a}{N}\right)} q^{NK\left(n+\frac{a}{N}\right)^2}. \tag{3.3}
$$

This is the sum over spectral flows of the discrete (BPS) representation generated by Ramond vacuum with the $U(1)$-charge $Q = \frac{v}{N} - \frac{1}{2}$, whose flow momenta are taken to be $n \in a + N\mathbb{Z}$, ($a \in \mathbb{Z}_N$). Setting $\ell = Nn_L + a_L$, $\tilde{\ell} = Nn_R + a_R$ ($n_L, n_R \in \mathbb{Z}$, $a_L, a_R \in \mathbb{Z}_N$) in (3.1), we obtain

$$
Z_{\rm dis}(\tau,z;\epsilon) = e^{2\pi\frac{\hat{c}}{\tau_2}z_1^2} \sum_{v=0}^{N-1} \sum_{\substack{a_L,a_R\in\mathbb{Z}_N\\ v+K(a_L+a_R)\in N\mathbb{Z}}} \chi_{\rm dis}(v,a_L;\tau,z)\chi_{\rm dis}(v,a_R;\tau,z)^*. \tag{3.4}
$$

This gives essentially the same result as in [3].

3.2 Remainder Part

To evaluate the remainder part $Z_{\text{rem}}(\tau, z; \epsilon)$, it is convenient to decompose (3.2) into three pieces;

$$\begin{aligned} Z_{\text{rem}}(\tau, z; \epsilon) &= -e^{2\pi \frac{\hat{c}}{\tau_2} z_1^2} \left| \frac{\theta_1(\tau, z)}{\eta(\tau)^3} \right|^2 \sum_{\substack{v, \ell, \tilde{\ell} \in \mathbb{Z} \\ v + K(\ell + \tilde{\ell}) \in N\mathbb{Z}}} \frac{1}{2\pi i} \int_{\mathbb{R} - i0} dp \, \frac{e^{-\pi \tau_2 \frac{p^2 + v^2}{NK}}}{p - iv} \\ & \qquad \times y^{\frac{1}{N}(2K\ell + v)} q^{\frac{1}{N}\left(K\ell^2 + \ell v\right)} \left[y^{\frac{1}{N}\left(2K\tilde{\ell} + v\right)} q^{\frac{1}{N}\left(K\tilde{\ell}^2 + \tilde{\ell} v\right)} \right]^* \\ & \qquad \times \left[\frac{yq^\ell e^{-\varepsilon(v+ip)}}{1 - yq^\ell} + \frac{e^{\varepsilon(v+ip)}}{1 - \bar{y}\bar{q}^{\tilde{\ell}}} + \frac{yq^\ell \left\{ e^{\varepsilon(v+ip)} - e^{-\varepsilon(v+ip)} \right\}}{(1 - yq^\ell)(1 - \bar{y}\bar{q}^{\tilde{\ell}})} \right] \\ &=: \; Z_{(1)} + Z_{(2)} + Z_{(3)}. \end{aligned} \tag{3.5}$$

Here $Z_{(1)}$, $Z_{(2)}$, $Z_{(3)}$ correspond to the choice of three terms within the square bracket [ ]. We here made use of a formula ;

$$\frac{XY\alpha^{-1} - \alpha}{(1 - X)(1 - Y)} = -\left[\frac{X\alpha^{-1}}{1 - X} + \frac{\alpha}{1 - Y} + \frac{X(\alpha - \alpha^{-1})}{(1 - X)(1 - Y)} \right].$$

(1) $Z_{(1)}$ **:**

Let us first consider the contribution $Z_{(1)}$. To carry out the v-summation, we decompose it as $v = v_0 + Nr$, $(r \in \mathbb{Z},\ v_0 = 0, 1, \ldots, N-1)$, and make use of the identities;

$$\begin{aligned} \frac{(yq^\ell)^{\frac{v_0}{N} + r + 1}}{1 - yq^\ell} &= \frac{(yq^\ell)^{\frac{v_0}{N}}}{1 - yq^\ell} - \sum_{j=0}^{r} (yq^\ell)^{\frac{v_0}{N} + j}, & (r \geq 0), \\ \frac{(yq^\ell)^{\frac{v_0}{N} + r + 1}}{1 - yq^\ell} &= \frac{(yq^\ell)^{\frac{v_0}{N}}}{1 - yq^\ell} + \sum_{j=-1}^{r+1} (yq^\ell)^{\frac{v_0}{N} + j}, & (r \leq -2). \end{aligned} \tag{3.6}$$

Corresponding to various terms in the above expansion, we can further decompose $Z_{(1)}$ as follows;

- **(i)** We consider the contribution to $Z_{(1)}$ which is multiplied by the term $\dfrac{(yq^\ell)^{\frac{v_0}{N}}}{1 - yq^\ell}$ in the expansion (3.6) and denote it as $Z_{(1),(\text{i})}$.
- **(ii)** We consider the contribution to $Z_{(1)}$ which is multiplied by the term $-(yq^\ell)^{\frac{v_0}{N} + r}$ in (3.6). This contribution is considered to be dominant as compared with those multiplied by the terms $(yq^\ell)^{\frac{v_0}{N} + j}$ and will be denoted as $Z_{(1),\,(\text{ii})}$. Note that the 'saturation' $|j| = |r|$ is possible only for $r \geq 0$. This fact is important in the following analysis.

(iii) We collect all the remaining terms, that is, $(yq^{\ell})^{\frac{v_0}{N}+j}$ with $|j| < |r|$ and denote it as $Z_{(1),(\mathrm{iii})}$.

We first evaluate $Z_{(1),(\mathrm{i})}$;

$$
\begin{aligned}
Z_{(1),(\mathrm{i})} &= -e^{2\pi\frac{\hat{c}}{\tau_2}z_1^2}\left|\frac{\theta_1(\tau,z)}{\eta(\tau)^3}\right|^2 \sum_{v_0=0}^{N-1}\sum_{r\in\mathbb{Z}}\sum_{\substack{a_L,a_R\in\mathbb{Z}_N\\ v_0+K(a_L+a_R)\in N\mathbb{Z}}}\sum_{n_L,n_R\in\mathbb{Z}}\frac{1}{2\pi i}\int_{\mathbb{R}-i0}dp\,\frac{e^{-\pi\tau_2\frac{p^2+(v_0+Nr)^2}{NK}}}{p-i(v_0+Nr)}\\
&\quad\times\frac{(yq^{Nn_L+a_L})^{\frac{v_0}{N}}}{1-yq^{Nn_L+a_L}}\,y^{2K\left(n_L+\frac{a_L}{N}\right)}\,q^{NK\left(n_L+\frac{a_L}{N}\right)^2}\\
&\quad\times\left[y^{2K\left(n_R+\frac{v_0+Nr+2Ka_R}{2NK}\right)}\,q^{NK\left(n_R+\frac{v_0+Nr+2Ka_R}{2NK}\right)^2}\,q^{-\frac{(v_0+Nr)^2}{4NK}}\right]^*\times e^{-\varepsilon(v_0+Nr+ip)}\\
&= -e^{2\pi\frac{\hat{c}}{\tau_2}z_1^2}\sum_{v_0=0}^{N-1}\sum_{\substack{a_L,a_R\in\mathbb{Z}_N\\ v_0+K(a_L+a_R)\in N\mathbb{Z}}}\chi_{\mathrm{dis}}(v_0,a_L;\tau,z)\\
&\quad\times\left[\frac{i\theta_1(\tau,z)}{\eta(\tau)^3}\cdot\frac{1}{2}\sum_{j\in\mathbb{Z}_{2K}}R^{(+)}_{v_0+Nj,NK}\,\Theta_{v_0+Nj+2Ka_R,NK}\left(\tau,\frac{2z}{N}\right)\right]^*+\mathcal{O}(\varepsilon). \qquad (3.7)
\end{aligned}
$$

In the last line we introduced the function $R^{(+)}_{m,k}$ (C.8), given explicitly as

$$
\begin{aligned}
R^{(+)}_{m,NK} &\equiv \frac{1}{i\pi}\sum_{\lambda\in m+2NK\mathbb{Z}}\int_{\mathbb{R}-i0}dp\,\frac{e^{-\pi\tau_2\frac{p^2+\lambda^2}{NK}}}{p-i\lambda}\,q^{-\frac{\lambda^2}{4NK}}\\
&\equiv \sum_{\lambda\in m+2NK\mathbb{Z}}\mathrm{sgn}(\lambda+0)\mathrm{Erfc}\left(\sqrt{\frac{\pi\tau_2}{NK}}\,|\lambda|\right)\,q^{-\frac{\lambda^2}{4NK}}, \qquad (3.8)
\end{aligned}
$$

where $\mathrm{Erfc}(x)$ is the error-function defined in (D.3). The power series in (3.7) converges even at $\varepsilon = 0$, and one can simply take the limit $\varepsilon \to 0$. The emergence of non-holomorphic function $R^{(+)}_{*,*}$ (C.8) is crucial in our analysis.

On the other hand, $Z_{(1),\mathrm{(ii)}}$ is evaluated as

$$
\begin{aligned}
Z_{(1),\mathrm{(ii)}} &= -e^{2\pi\frac{\hat{c}}{\tau_2}z_1^2}\left|\frac{\theta_1(\tau,z)}{\eta(\tau)^3}\right|^2\sum_{v_0=0}^{N-1}\sum_{r=0}^{\infty}\sum_{\substack{a_L,a_R\in\mathbb{Z}_N\\ v_0+K(a_L+a_R)\in N\mathbb{Z}}}\sum_{n_L,n_R\in\mathbb{Z}}\frac{1}{2\pi i}\int_{\mathbb{R}-i0}dp\,\frac{e^{-\pi\tau_2\frac{p^2+(v_0+Nr)^2}{NK}}}{p-i(v_0+Nr)}\\
&\quad\times(-1)(yq^{Nn_L+a_L})^{\frac{v_0+Nr}{N}}\,y^{2K\left(n_L+\frac{a_L}{N}\right)}\,q^{NK\left(n_L+\frac{a_L}{N}\right)^2}\\
&\quad\times\left[y^{2K\left(n_R+\frac{v_0+Nr+2Ka_R}{2NK}\right)}\,q^{NK\left(n_R+\frac{v_0+Nr+2Ka_R}{2NK}\right)^2}\,q^{-\frac{(v_0+Nr)^2}{4NK}}\right]^*\,e^{-\varepsilon(v_0+Nr+ip)}\\
&= e^{2\pi\frac{\hat{c}}{\tau_2}z_1^2}\left|\frac{\theta_1(\tau,z)}{\eta(\tau)^3}\right|^2\sum_{v_0=0}^{N-1}\sum_{r=0}^{\infty}\sum_{\substack{a_L,a_R\in\mathbb{Z}_N\\ v_0+K(a_L+a_R)\in N\mathbb{Z}}}\sum_{n_L,n_R\in\mathbb{Z}}\frac{1}{2\pi i}\int_{\mathbb{R}-i0}dp\,\frac{e^{-\pi\tau_2\frac{p^2}{NK}}}{p-i(v_0+Nr)}\\
&\quad\times y^{2K\left(n_L+\frac{v_0+Nr+2Ka_L}{2NK}\right)}\,q^{NK\left(n_L+\frac{v_0+Nr+2Ka_L}{2NK}\right)^2}\\
&\quad\times\left[y^{2K\left(n_R+\frac{v_0+Nr+2Ka_R}{2NK}\right)}\,q^{NK\left(n_R+\frac{v_0+Nr+2Ka_R}{2NK}\right)^2}\right]^*\,e^{-\varepsilon(v_0+Nr+ip)}. \qquad (3.9)
\end{aligned}
$$

The power series is convergent due to the damping factor $e^{-\varepsilon(v_0+Nr+ip)}$ and an expected logarithmic divergence emerges in the limit $\varepsilon\to 0$.

We can rewrite (3.9) in a simpler form by using the (extended) continuous characters (C.1). Introducing new quantum numbers $n_0\in\mathbb{Z}_N$, $w_0\in\mathbb{Z}_{2K}$ by the relation

$$
v_0+Nr+2Ka_L\equiv Nw_0+Kn_0\ (\bmod\ 2NK),\qquad v_0+Nr+2Ka_R\equiv Nw_0-Kn_0\ (\bmod\ 2NK),
$$

which solves the constraint $v_0+K(a_L+a_R)\in N\mathbb{Z}$, we obtain

$$
\begin{aligned}
Z_{(1),\mathrm{(ii)}} &= e^{2\pi\frac{\hat{c}}{\tau_2}z_1^2}\sum_{n_0\in\mathbb{Z}_N,\,w_0\in\mathbb{Z}_{2K}}\int_{\mathbb{R}-i0}dp\,\rho_{(1)}(p,n_0,w_0;\varepsilon)\\
&\qquad\times\chi_{\mathrm{con}}(p,Nw_0+Kn_0;\tau,z)\chi_{\mathrm{con}}(p,Nw_0-Kn_0;\tau,z)^*, \qquad (3.10)
\end{aligned}
$$

where $\rho_{(1)}$ denotes some spectral density. To evaluate it, it is convenient to introduce the symbol '$[m]_{2K}$' defined by

$$
[m]_{2K}\equiv m\ (\bmod\ 2K),\quad 0\le[m]_{2K}\le 2K-1,
$$

and set $m_L:=Nw_0+Kn_0$, $m_R:=Nw_0-Kn_0$. Note that

$$
v_0+Nr=[m_L]_{2K}+2Ks=[m_R]_{2K}+2K\tilde{s},\quad ({}^{\exists}\,s,\tilde{s}\in\mathbb{Z}_{\ge 0}). \qquad (3.11)
$$

With the helps of (D.11), we obtain[8]

$$\begin{aligned}\rho_{(1)}(p,n_0,w_0;\varepsilon) &= \frac{1}{4\pi i}\left[\sum_{s=0}^{\infty}\frac{e^{-\varepsilon([m_L]_{2K}+2Ks+ip)}}{p-i\left([m_L]_{2K}+2Ks\right)}+\sum_{\tilde{s}=0}^{\infty}\frac{e^{-\varepsilon([m_R]_{2K}+2K\tilde{s}+ip)}}{p-i\left([m_R]_{2K}+2K\tilde{s}\right)}\right] \\ &= C(\varepsilon)-\frac{1}{4\pi i}\frac{\partial}{\partial p}\log\left[\Gamma\left(\frac{[m_L]_{2K}}{2K}+\frac{ip}{2K}\right)\Gamma\left(\frac{[m_R]_{2K}}{2K}+\frac{ip}{2K}\right)\right]+\mathcal{O}(\varepsilon),\end{aligned} \tag{3.12}$$

where $C(\varepsilon)$ denotes some positive constant independent of n_0, w_0, which logarithmically diverges in the $\varepsilon\rightarrow 0$ limit.

Finally, $Z_{(1),(\mathrm{iii})}$ has a complicated form;

$$\begin{aligned}Z_{(1),(\mathrm{iii})} &= e^{2\pi\frac{\hat{c}}{\tau_2}z_1^2}\sum_{v_0=0}^{N-1}\sum_{\substack{a_L,a_R\in\mathbb{Z}_N \\ v_0+K(a_L+a_R)\in N\mathbb{Z}}}\left[\sum_{r=1}^{\infty}\sum_{j=0}^{r-1}-\sum_{r=-2}^{-\infty}\sum_{j=-1}^{r+1}\right]\frac{1}{2\pi i}\int_{\mathbb{R}-i0}dp\,\frac{1}{p-i(v_0+Nr)} \\ &\quad\times\chi_{\mathrm{con}}\left(p(r,j),v_0+Nj+2Ka_L;\tau,z\right)\,\chi_{\mathrm{con}}\left(p,v_0+Nj+2Ka_R;\tau,z\right)^*,\end{aligned} \tag{3.13}$$

where we set

$$p(r,j)^2 := p^2+(v_0+Nr)^2-(v_0+Nj)^2.$$

Note that we always have $p(r,j)^2>p^2$ for $0\leq j\leq r-1$ $(r\geq 1)$ or $-1\geq j\geq r+1$ $(r\leq -2)$.

A few remarks are in order;

- In the IR region $\tau_2\sim +\infty$, $Z_{(1),(\mathrm{iii})}$ is negligible in comparison with the "dominant" continuous part $Z_{(1),(\mathrm{ii})}$ for each fixed values of left and right $U(1)$ charge. In fact, by construction $p(r,j)^2>p^2$ and this leads to $|Z_{(1),(\mathrm{ii})}|\gg|Z_{(1),(\mathrm{iii})}|$, around $\tau_2\sim+\infty$.
- $Z_{(1),\,(\mathrm{iii})}$ is expanded into continuous characters $\chi_{\mathrm{con}}(p,m)$ which has left-right *asymmetric* momenta;

  $$\sim\chi_{\mathrm{con}}(p,m_L;\tau,z)\chi_{\mathrm{con}}(p',m_R;\tau,z)^*,\qquad p\neq p' \text{ in general.}$$

  It is, however, easy to check that the level matching condition $h_L-h_R\in\mathbb{Z}$, and invariance under T-transformation are satisfied.
- The q-expansion of $Z_{(1),(\mathrm{iii})}$ converges *even at* $\varepsilon=0$ (without the damping factor).

(2) $Z_{(2)}$ **:**

The evaluation of $Z_{(2)}$ is almost parallel to that of $Z_{(1)}$. We again make use of the expansion similar to (3.6) and decompose $Z_{(2)}$ as follows;

[8] We shall here adopt the 'left-right symmetric form' of spectral density just for convention.

(i) We extract the terms multiplied by $\left[\frac{(yq^{\tilde{\ell}})^{\frac{v_0}{N}}}{1-yq^{\ell}}\right]^*$ and denote them as $Z_{(2),\,\text{(i)}}$.

(ii) We extract the terms multiplied by $\left[(yq^{\ell})^{\frac{v_0}{N}+r}\right]^*$ for $v = v_0 + Nr$ and $r \leq -1$. This contribution is denoted as $Z_{(2),\,\text{(ii)}}$.

(iii) The remaining part is denoted as $Z_{(2),\,\text{(iii)}}$.

$Z_{(2),\text{(i)}}$, $Z_{(2),\text{(ii)}}$ are calculated as follows;

$$
\begin{aligned}
Z_{(2),\text{(i)}} &= -e^{2\pi\frac{\hat{c}}{\tau_2}z_1^2}\sum_{v_0=0}^{N-1}\sum_{\substack{a_L,a_R\in\mathbb{Z}_N\\ v_0+K(a_L+a_R)\in N\mathbb{Z}}}\chi_{\text{dis}}(v_0,a_R;\tau,z)^* \\
&\quad\times\frac{i\theta_1(\tau,z)}{\eta(\tau)^3}\cdot\frac{1}{2}\sum_{j\in\mathbb{Z}_{2K}}R^{(+)}_{v_0+Nj,NK}\,\Theta_{v_0+Nj+2Ka_L,NK}\left(\tau,\frac{2z}{N}\right)+\mathcal{O}(\varepsilon), \qquad (3.14)\\
Z_{(2),\text{(ii)}} &= e^{2\pi\frac{\hat{c}}{\tau_2}z_1^2}\sum_{n_0\in\mathbb{Z}_N,\,w_0\in\mathbb{Z}_{2K}}\int_{\mathbb{R}-i0}dp\,\rho_{(2)}(p,n_0,w_0;\varepsilon)\\
&\quad\times\chi_{\text{con}}(p,Nw_0+Kn_0;\tau,z)\chi_{\text{con}}(p,Nw_0-Kn_0;\tau,z)^*, \qquad (3.15)\\
\rho_{(2)}(p,n_0,w_0;\varepsilon) &= -\frac{1}{4\pi i}\left[\sum_{s=0}^{\infty}\frac{e^{-\varepsilon([-m_L]_{2K}+2Ks-ip)}}{p+i\,([-m_L]_{2K}+2Ks)}+\sum_{\tilde{s}=0}^{\infty}\frac{e^{-\varepsilon([-m_R]_{2K}+2K\tilde{s}-ip)}}{p+i\,([-m_R]_{2K}+2K\tilde{s})}\right]\\
&= C'(\varepsilon)+\frac{1}{4\pi i}\frac{\partial}{\partial p}\log\left[\Gamma\left(\frac{[-m_L]_{2K}}{2K}-\frac{ip}{2K}\right)\Gamma\left(\frac{[-m_R]_{2K}}{2K}-\frac{ip}{2K}\right)\right]+\mathcal{O}(\varepsilon),\\
&\qquad (3.16)
\end{aligned}
$$

Again $C'(\varepsilon)$ is a positive logarithmically divergent constant independent of p, n_0, w_0.

The subleading part $Z_{(2),\text{(iii)}}$ has a similar form to (3.13), and we omit it here.

(3) $Z_{(3)}$:

Finally, let us consider $Z_{(3)}$. We note that the power series including the factor

$$
\frac{yq^{\ell}}{(1-yq^{\ell})(1-\bar{y}\bar{q}^{\tilde{\ell}})}
$$

is converging. Then it is easy to see

$$
\lim_{\varepsilon\to+0}Z_{(3)}=0.
$$

3.3 Summary of Decomposition

Now, let us collect all the pieces of information on partition function. A crucial fact is that contributions of discrete (BPS) representations to $Z_{\rm dis}$, $Z_{(1),{\rm (i)}}$ and $Z_{(2),{\rm (i)}}$ are precisely combined into the form of the *"modular completion"* (see Appendix C),

$$\widehat{\chi}_{\rm dis}(v,a;\tau,z) \equiv \chi_{\rm dis}(v,a;\tau,z) - \frac{1}{2}\sum_{j\in\mathbb{Z}_{2K}} R^{(+)}_{v+Nj,NK}(\tau)\Theta_{v+Nj+2Ka,NK}\left(\tau,\frac{2z}{N}\right)\frac{i\theta_1(\tau,z)}{\eta(\tau)^3} \tag{3.17}$$

and we can safely take the $\epsilon \to 0$ limit. In the modular completion (3.17) the anomalous transformation property of the discrete character $\chi_{\rm dis}(v,a;\tau,z)$ is compensated by the transformation law of the auxiliary function $R^{(+)}_{v+Nj,NK}(\tau)$ and the combination transforms like a Jacobi form.

Thus, we have found that the true discrete part of the partition function is given by the bilinear form of modular completions;

$$Z_{\rm dis}(\tau,z) := e^{2\pi\frac{\hat{c}}{\tau_2}z_1^2}\sum_{v=0}^{N-1}\sum_{\substack{a_L,a_R\in\mathbb{Z}_N\\ v+K(a_L+a_R)\in N\mathbb{Z}}}\widehat{\chi}_{\rm dis}(v,a_L;\tau,z)\widehat{\chi}_{\rm dis}(v,a_R;\tau,z)^*. \tag{3.18}$$

This is indeed modular invariant as is directly confirmed by using the modular transformation formula (C.14). Process of completion cures the modular property of the BPS characters, however, we have to pay the price of the non-holomorphic dependence of the function $R^{(+)}_{v+Nj,NK}(\tau)$ in this construction. To be precise we need quadratic terms of $R^{(+)}_{m,NK}$ in order to obtain the expression (3.18). However, they can be naturally regarded as a part of $Z_{\rm subleading}$ defined below.

The rest of partition function can be expanded only in terms of the continuous (non-BPS) characters $\chi_{\rm con}(p,m)$. We shall first combine $Z_{(1),{\rm (ii)}}$ and $Z_{(2),{\rm (ii)}}$

$$\begin{aligned} Z_{\rm con}(\tau,z;\epsilon) &:= Z_{(1),{\rm (ii)}} + Z_{(2),{\rm (ii)}} \\ &\equiv \sum_{n_0\in\mathbb{Z}_N,\, w_0\in\mathbb{Z}_{2K}}\int_0^\infty dp\,\rho(p,n_0,w_0;\epsilon) \\ &\qquad \times\chi_{\rm con}(p,Nw_0+Kn_0;\tau,z)\chi_{\rm con}(p,Nw_0-Kn_0;\tau,z)^*, \end{aligned} \tag{3.19}$$

where the spectral density $\rho(p,n_0,w_0;\epsilon)$ is given by

$$\begin{aligned} \rho(p,n_0,w_0;\epsilon) &:= \rho_{(1)}(p,n_0,w_0;\epsilon)+\rho_{(1)}(-p,n_0,w_0;\epsilon)+\rho_{(2)}(p,n_0,w_0;\epsilon)+\rho_{(2)}(-p,n_0,w_0;\epsilon) \\ &= \mathcal{C}(\epsilon)+\frac{1}{4\pi i}\frac{\partial}{\partial p}\log\frac{\prod_{\alpha,\beta=\pm1}\Gamma\left(\frac{1}{2K}[\alpha Nw_0+\beta Kn_0]_{2K}-\frac{ip}{2K}\right)}{\prod_{\alpha,\beta=\pm1}\Gamma\left(\frac{1}{2K}[\alpha Nw_0+\beta Kn_0]_{2K}+\frac{ip}{2K}\right)}+\mathcal{O}(\epsilon). \end{aligned} \tag{3.20}$$

$\mathcal{C}(\epsilon)$ is a positive logarithmically divergent constant independent of p, n_0, w_0. Note here that the density function function $\rho(p, n_0, w_0; \epsilon)$ has no singularity at $p = 0$, and thus the integral $\int_0^\infty dp$ is well-defined[9].

We also have the 'subleading part' of partition function $Z_{\text{subleading}}$, which consists of contributions from $Z_{(1),(\text{iii})}$, $Z_{(2),\,(\text{iii})}$ as well as the quadratic term of $R^{(+)}_{*,*}$ appearing in the modular completion $\widehat{\chi}_{\text{dis}}$ (3.17). This is expressible as a convergent series of the terms such as

$$\int dp_L \int dp_R \sum_{m_L, m_R} \sigma(p_L, p_R, m_L, m_R)\, \chi_{\text{con}}(p_L, m_L; \tau, z) \chi_{\text{con}}(p_R, m_R; \tau, z)^*,$$

with some density $\sigma(p_L, p_R, m_L, m_R)$ (including delta-functions in general) not specified here. $Z_{\text{subleading}}$ damps more rapidly than other parts of partition function Z_{dis}, Z_{con}, when taking the IR limit $\tau_2 \to +\infty$.

The asymmetry of radial momenta p_L, p_R appearing in $Z_{\text{subleading}}$ may be an interesting feature. We also note that such an asymmetry is not observed in Z_{con}, especially, in the divergent term proportional to the volume factor $\mathcal{C}(\epsilon)$. These facts suggest a non-compact and curved geometry, which is asymptotic to a flat space-time. In the context of string compactification, the divergent term in Z_{con} corresponds to the strings freely propagating in the asymptotic region of space-time. On the other hand, the sectors of Z_{dis} and $Z_{\text{subleading}}$ could be contributed from strings localized in the strongly curved region around the tip of cigar. The former corresponds to massless sectors, whereas the latter would be regarded as 'too heavy' string modes to propagate in the asymptotic region, and the asymmetry of radial momenta mentioned above would originate from curvature of cigar geometry. Note that neither p_L nor p_R are good quantum numbers in such a curved background, while conformal weights still make sense. It can be checked that the level-matching condition is satisfied and the T-invariance is maintained.

Here we notice a general phenomenon: when one tries to describe the string theory on some non-compact target manifold, there occurs in general a clash between the holomorphy and modular property of the theory. We have to either give up holomorphy and keep modular invariance or weaken the condition of modular invariance and keep holomorphy. If one starts from the path-integral formulation of the theory, modular invariance is automatically enforced and one ends up with a non-holomorphic dependence in the character decomposition.

[9] It may be worthwhile to note that it is *not* the case for each of $\rho_{(1)}$, $\rho_{(2)}$. In fact, each of $\rho_{(1)}(p, 0, 0)$ and $\rho_{(2)}(p, 0, 0)$ shows a singularity when approaching to $p = 0$. After taking the sum, the singularity at $p = 0$ is canceled, and one may simply replace the integration contour $\mathbb{R} - i0$ with $\mathbb{R}$.

3.4 Orbifolding

In the last part of this section, we discuss some variants of the modular invariant[10]. Among others, we focus on the $\mathbb{Z}_N$-orbifold of $SL(2;\mathbb{R})/U(1)$ supercoset. Geometrically it amounts to reducing the size of asymptotic circle of the cigar to $1/N$, and we obtain

$$Z_{\rm orb}(\tau,z) = e^{\frac{2\pi}{\tau_2}\left(\hat{c}|z|^2-\frac{k+4}{k}z_2^2\right)}\frac{k}{N}\sum_{a,b\in\mathbb{Z}_N}\int_{\mathbb{C}}\frac{d^2u}{\tau_2}\,e^{4\pi\frac{u_2z_2}{\tau_2}}\left|\frac{\theta_1\left(\tau,-u+\left(1+\frac{2}{k}\right)z\right)}{\theta_1\left(\tau,-u+\frac{2}{k}z\right)}\right|^2 e^{-\frac{\pi k}{\tau_2}\left|u+\frac{a\tau+b}{N}\right|^2}. \tag{3.21}$$

It is obvious that this is also modular invariant.

More generally, one may consider the '$\mathbb{Z}_N$-twisted partition function' ($\alpha,\beta\in\mathbb{Z}_N$ denotes the parameters of twisting);

$$\begin{aligned} Z_{[\alpha,\beta]}(\tau,z) &= e^{\frac{2\pi}{\tau_2}\left(\hat{c}|z|^2-\frac{k+4}{k}z_2^2\right)}\frac{k}{N}\sum_{a,b\in\mathbb{Z}_N}e^{-2\pi i\frac{1}{N}(\alpha b-\beta a)}\int_{\mathbb{C}}\frac{d^2u}{\tau_2}\,e^{4\pi\frac{u_2z_2}{\tau_2}} \\ &\quad\times\left|\frac{\theta_1\left(\tau,-u+\left(1+\frac{2}{k}\right)z\right)}{\theta_1\left(\tau,-u+\frac{2}{k}z\right)}\right|^2 e^{-\frac{\pi k}{\tau_2}\left|u+\frac{a\tau+b}{N}\right|^2}. \end{aligned} \tag{3.22}$$

It is not modular invariant, but rather behaves modular covariantly;

$$Z_{[\alpha,\beta]}(\tau+1,z) = Z_{[\alpha,\alpha+\beta]}(\tau), \qquad Z_{[\alpha,\beta]}\left(-\frac{1}{\tau},\frac{z}{\tau}\right) = Z_{[\beta,-\alpha]}(\tau,z). \tag{3.23}$$

We also note that

$$Z_{[0,0]}(\tau,z) = Z_{\rm orb}(\tau,z), \qquad \frac{1}{N}\sum_{\alpha,\beta\in\mathbb{Z}_N}Z_{[\alpha,\beta]}(\tau,z) = Z(\tau,z). \tag{3.24}$$

The partition function (3.22) can be evaluated in almost the same way, though the analysis gets a bit more complicated. Relevant changes are summarized as follows;

- The winding number w is replaced with $\frac{w}{N}$, while the KK momentum n is changed into $Nn+\alpha$. Thus (2.32) is replaced with

  $$Nn+\alpha = \ell-\tilde{\ell}, \tag{3.25}$$

  and (2.34) becomes

  $$v = w-K(\ell+\tilde{\ell}). \tag{3.26}$$

[10]Precisely speaking, one should understand the partition functions given here as the ones defined with the regularization scheme presented above.

- By the same reason,

$$
\begin{aligned}
w+\frac{n}{k} &\longrightarrow \frac{w}{N}+\frac{Nn+\alpha}{k}=\frac{w+K(Nn+\alpha)}{N}=\frac{2K}{N}\left(\ell+\frac{v}{2K}\right),\\
w-\frac{n}{k} &\longrightarrow \frac{w}{N}-\frac{Nn+\alpha}{k}=\frac{w-K(Nn+\alpha)}{N}=\frac{2K}{N}\left(\tilde{\ell}+\frac{v}{2K}\right).
\end{aligned}
$$

Thus the combinations $\frac{2K}{N}\left(\ell+\frac{v}{2K}\right)$, $\frac{2K}{N}\left(\tilde{\ell}+\frac{v}{2K}\right)$ are unchanged.

The final expression is found to be

$$
\begin{aligned}
Z_{[\alpha,\beta]}(\tau,z) &= e^{2\pi\frac{\hat{c}}{\tau_2}z_1^2}\left|\frac{\theta_1(\tau,z)}{\eta(\tau)^3}\right|^2 \sum_{\substack{v,\ell,\tilde{\ell}\in\mathbb{Z}\\ \ell-\tilde{\ell}\in\alpha+N\mathbb{Z}}} \frac{1}{2\pi i}\left[\int_{\mathbb{R}-i0} dp\,(yq^{\ell})\,\overline{(yq^{\tilde{\ell}})}-\int_{\mathbb{R}+i(N-0)} dp\right]\\
&\times e^{2\pi i\frac{\beta}{N}\left\{v+K(\ell+\tilde{\ell})\right\}}\,\frac{e^{-\pi\tau_2\frac{p^2+v^2}{NK}}}{p-iv}\,\frac{(yq^{\ell})^{\frac{v}{N}}}{1-yq^{\ell}}\left[\frac{(yq^{\tilde{\ell}})^{\frac{v}{N}}}{1-yq^{\tilde{\ell}}}\right]^* y^{\frac{2K}{N}\ell}q^{\frac{K}{N}\ell^2}\left[y^{\frac{2K}{N}\tilde{\ell}}q^{\frac{K}{N}\tilde{\ell}^2}\right]^*
\end{aligned}
\tag{3.27}
$$

Especially, the discrete part is determined as follows;

$$
Z_{[\alpha,\beta]\,\mathrm{dis}}(\tau,z)=e^{2\pi\frac{\hat{c}}{\tau_2}z_1^2}\sum_{v=0}^{N-1}\sum_{a\in\mathbb{Z}_N} e^{2\pi i\frac{\beta}{N}\{v+K(\alpha+2a)\}}\,\widehat{\chi}_{\mathrm{dis}}(v,a+\alpha;\tau,z)\widehat{\chi}_{\mathrm{dis}}(v,a;\tau,z)^*. \tag{3.28}
$$

One can easily check that (3.18) and (3.28) are consistent with the relation (3.24).

4 Elliptic Genus

4.1 Analysis Based on the Character Decomposition

The elliptic genus [17] is given by formally setting $\bar{z}=0$ in the partition function of $\widetilde{\mathrm{R}}$-sector Z, while leaving z at a generic value. Since we have already obtained the character decomposition of the partition function, it is straightforward to calculate the elliptic genus. It is obvious that only BPS representations Z_{dis} contributes in the right-moving ($\bar{z}$-dependent) sector. Using the result of (3.18) and the formulas of Witten index (C.15);

$$
\widehat{\chi}_{\mathrm{dis}}(v,a;\tau,0)=-\delta^{(N)}_{a,0}, \tag{4.1}
$$

we obtain the formula for the elliptic genus;

$$
\mathcal{Z}(\tau,z)=-\sum_{v=0}^{N-1}\sum_{\substack{a\in\mathbb{Z}_N\\ v+Ka\in N\mathbb{Z}}}\widehat{\chi}_{\mathrm{dis}}(v,a;\tau,z). \tag{4.2}
$$

Also, (3.28) yields

$$\mathcal{Z}_{[\alpha,\beta]}(\tau,z) = -\sum_{v=0}^{N-1} e^{2\pi i \frac{\beta}{N}(v+K\alpha)}\, \widehat{\chi}_{\mathrm{dis}}(v,\alpha;\tau,z), \tag{4.3}$$

$$\equiv -e^{2\pi i \frac{K}{N}\alpha\beta} q^{\frac{K}{N}\alpha^2} y^{\frac{2K}{N}\alpha}\, \widehat{\mathcal{K}}^{(2NK)}\left(\tau, \frac{z+\alpha\tau+\beta}{N}\right) \frac{i\theta_1(\tau,z)}{\eta(\tau)^3}. \tag{4.4}$$

Here we have introduced the notation of Appell function $\widehat{\mathcal{K}}$, see Appendix C.

It is obvious that the following relations hold;

$$\mathcal{Z}_{[0,0]}(\tau,z) = \mathcal{Z}_{\mathrm{orb}}(\tau,z), \qquad \frac{1}{N}\sum_{\alpha,\beta\in\mathbb{Z}_N} \mathcal{Z}_{[\alpha,\beta]}(\tau,z) = \mathcal{Z}(\tau,z), \tag{4.5}$$

corresponding to (3.24), where $\mathcal{Z}_{\mathrm{orb}}(\tau,z)$ is the elliptic genus of $\mathbb{Z}_N$-orbifold (3.21). Especially,

$$\mathcal{Z}_{\mathrm{orb}}(\tau,z) = -\sum_{v=0}^{N-1} \widehat{\chi}_{\mathrm{dis}}(v,0;\tau,z) \equiv -\widehat{\mathcal{K}}^{(2NK)}\left(\tau,\frac{z}{N}\right)\frac{i\theta_1(\tau,z)}{\eta(\tau)^3}, \tag{4.6}$$

which is the one proposed in [10]. We also note

$$\mathcal{Z}(\tau,z) = -\frac{1}{N}\sum_{\alpha,\beta\in\mathbb{Z}_N} e^{2\pi i \frac{K}{N}\alpha\beta} q^{\frac{K}{N}\alpha^2} y^{\frac{2K}{N}\alpha}\, \widehat{\mathcal{K}}^{(2NK)}\left(\tau, \frac{z+\alpha\tau+\beta}{N}\right) \frac{i\theta_1(\tau,z)}{\eta(\tau)^3}. \tag{4.7}$$

Thanks to the good modular property of $\widehat{\mathcal{K}}^{(2k)}(\tau,z)$ (C.11), the expressions (4.4), (4.6) and (4.7) all behave modular covariantly. For instance, $\mathcal{Z}_{[\alpha,\beta]}(\tau,z)$ (4.4) satisfies the following modular transformation formulas;

$$\mathcal{Z}_{[\alpha,\beta]}(\tau+1,z) = \mathcal{Z}_{[\alpha,\alpha+\beta]}(\tau,z), \qquad \mathcal{Z}_{[\alpha,\beta]}\left(-\frac{1}{\tau},\frac{z}{\tau}\right) = e^{i\pi\frac{\hat{c}}{\tau}z^2}\, \mathcal{Z}_{[\beta,-\alpha]}(\tau,z). \tag{4.8}$$

4.2 Path-integral Representation of Elliptic Genus

It would be interesting to clarify the 'path-integral representation' of elliptic genus in a way analogous to [10]. We begin with generalizing the partition function (2.26) so as to include two *independent* angular variables z_L, z_R that couple with the left/right $U(1)$-charges. To this aim, it is enough to formally replace z, $\bar{z}$ with z_L, $\overline{z_R}$ in the expression (2.26). Note also that $z_2 \equiv \frac{z-\bar{z}}{2i}$ should be replaced with $\frac{z_L-\overline{z_R}}{2i}$. We thus obtain the partition function

$$\begin{aligned} Z(\tau,z_L,z_R) &= k e^{\frac{2\pi}{\tau_2}\left\{\hat{c} z_L \bar{z}_R - \frac{k+4}{k}\left(\frac{z_L-\overline{z_R}}{2i}\right)^2\right\}} \int_{\mathbb{C}} \frac{d^2u}{\tau_2}\, e^{4\pi\frac{u_2}{\tau_2}\left(\frac{z_L-\overline{z_R}}{2i}\right)} \\ &\quad \times \frac{\theta_1\left(\tau,-u+\left(1+\frac{2}{k}\right)z_L\right)}{\theta_1\left(\tau,-u+\frac{2}{k}z_L\right)} \overline{\frac{\theta_1\left(\tau,-u+\left(1+\frac{2}{k}\right)z_R\right)}{\theta_1\left(\tau,-u+\frac{2}{k}z_R\right)}}\, e^{-\frac{\pi k}{\tau_2}|u|^2}. \end{aligned} \tag{4.9}$$

This partition function is complex, but still modular invariant, as can be directly confirmed.

Then, the desired elliptic genus should be given as

$$\mathcal{Z}(\tau, z) = \lim_{z_R \to 0} e^{-2\pi \frac{\hat{c}}{\tau_2}\left(\frac{z+\bar{z}_R}{2}\right)^2} Z(\tau, z_L = z, z_R). \tag{4.10}$$

(Recall that the partition function $Z(\tau, z)$ includes the anomaly factor $e^{2\pi \frac{\hat{c}}{\tau_2} z_1^2}$, which is absent in the definition of elliptic genus.) When setting $z_R = 0$, the right-moving θ_1-factors drop off, as is expected from supersymmetry. We thus obtain

$$\mathcal{Z}(\tau, z) \;=\; k e^{\pi \frac{z^2}{k\tau_2}} \int_{\mathbb{C}} \frac{d^2 u}{\tau_2} \frac{\theta_1\left(\tau, -u + \left(1+\frac{2}{k}\right) z\right)}{\theta_1\left(\tau, -u + \frac{2}{k} z\right)} e^{-2\pi i z \frac{u_2}{\tau_2}} e^{-\frac{\pi k}{\tau_2}|u|^2}. \tag{4.11}$$

This possesses the expected modular properties;

$$\mathcal{Z}(\tau+1, z) = \mathcal{Z}(\tau, z), \qquad \mathcal{Z}\left(-\frac{1}{\tau}, \frac{z}{\tau}\right) = e^{i\pi \frac{\hat{c}}{\tau} z^2} \mathcal{Z}(\tau, z). \tag{4.12}$$

If we started with the twisted partition function $Z_{[\alpha,\beta]}$ (3.22), we would similarly obtain

$$\begin{aligned} \mathcal{Z}_{[\alpha,\beta]}(\tau, z) \;=\; & \frac{1}{K} e^{\pi \frac{z^2}{k\tau_2}} \sum_{a,b \in \mathbb{Z}_N} e^{-2\pi i \frac{1}{N}(\alpha b - \beta a)} \int_{\mathbb{C}} \frac{d^2 u}{\tau_2} \frac{\theta_1\left(\tau, -u + \left(1+\frac{2}{k}\right) z\right)}{\theta_1\left(\tau, -u + \frac{2}{k} z\right)} \\ & \times e^{-2\pi i z \frac{u_2}{\tau_2}} e^{-\frac{\pi k}{\tau_2}\left|u + \frac{a\tau + b}{N}\right|^2} \qquad (\alpha, \beta \in \mathbb{Z}_N). \end{aligned} \tag{4.13}$$

This expression shows that $\mathcal{Z}_{[\alpha,\beta]}(\tau, z)$ has the expected modular properties (4.8).

4.2.1 Direct Evaluation of (4.13)

Let us try to rederive the formula (4.4) from the path-integral representation (4.13). This is parallel to the analysis presented in the previous section. However, contrary to the case of partition function, *we need not introduce any regularization.* This is because the integrand of (4.13) possesses at most simple poles, and thus the u-integral already converges.

Set $u =: \mathbf{s}_1 \tau + \mathbf{s}_2$. Since the integrand includes Gaussian factors which decrease rapidly at infinity, we can safely shift the contours of $\mathbf{s}_1$, $\mathbf{s}_2$-integrals as

$$\mathbb{R} \longrightarrow \mathbb{R} + i\xi_1, \qquad \mathbb{R} \longrightarrow \mathbb{R} + i\xi_2,$$

with arbitrary $\xi_1, \xi_2 \in \mathbb{R}$, without changing the value of the integral. Namely, we can rewrite (4.13) as

$$\begin{aligned} \mathcal{Z}_{[\alpha,\beta]}(\tau, z) \;=\; & \frac{1}{K} e^{\pi \frac{z^2}{k\tau_2}} \sum_{a,b \in \mathbb{Z}_N} e^{-2\pi i \frac{1}{N}(\alpha b - \beta a)} \int_{\mathbb{R}+i\xi_1} d\mathbf{s}_1 \int_{\mathbb{R}+i\xi_2} d\mathbf{s}_2 \frac{\theta_1\left(\tau, -\mathbf{s}_1\tau - \mathbf{s}_2 + \left(1+\frac{2}{k}\right) z\right)}{\theta_1\left(\tau, -\mathbf{s}_1\tau - \mathbf{s}_2 + \frac{2}{k} z\right)} \\ & \times e^{-2\pi i z \mathbf{s}_1} e^{-\frac{\pi k}{\tau_2}\left[\left\{\left(\mathbf{s}_1 + \frac{a}{N}\right)\tau_1 + \mathbf{s}_2 + \frac{b}{N}\right\}^2 + \left(\mathbf{s}_1 + \frac{a}{N}\right)^2 \tau_2^2\right]}. \end{aligned} \tag{4.14}$$

Moreover, we introduce the 'winding numbers' $w, m \in \mathbb{Z}$ and real parameters ζ_i as

$$\mathbf{s}_1 = (\zeta_1 + i\xi_1) + s_1 + w, \qquad \mathbf{s}_2 = (\zeta_2 + i\xi_2) + s_2 + m, \qquad (0 < s_1, s_2 < 1) \tag{4.15}$$

We shall choose suitable parameters ξ_i, ζ_i to simplify the relevant integral. A good choice is given by

$$\zeta_1 + i\xi_1 = -i\frac{z}{k\tau_2}, \qquad \zeta_2 + i\xi_2 = i\frac{\bar{\tau}z}{k\tau_2}. \tag{4.16}$$

Relevant calculations are as follows;

$$\begin{aligned}
\frac{\theta_1\left(\tau, -\mathbf{s}_1\tau - \mathbf{s}_2 + \left(1+\frac{2}{k}\right)z\right)}{\theta_1\left(\tau, -\mathbf{s}_1\tau - \mathbf{s}_2 + \frac{2}{k}z\right)} &= \frac{\theta_1\left(\tau, -(s_1+w)\tau - (s_2+m) + z\right)}{\theta_1\left(\tau, -(s_1+w)\tau - (s_2+m)\right)} \\
&= e^{2\pi i z w}\,\frac{\theta_1\left(\tau, -s_1\tau - s_2 + z\right)}{\theta_1\left(\tau, -s_1\tau - s_2\right)},
\end{aligned} \tag{4.17}$$

$$\begin{aligned}
& e^{-\frac{\pi k}{\tau_2}\left[\left\{\left(\mathbf{s}_1+\frac{a}{N}\right)\tau_1+\mathbf{s}_2+\frac{b}{N}\right\}^2+\left(\mathbf{s}_1+\frac{a}{N}\right)^2\tau_2^2\right]} \\
&= e^{-\frac{\pi k}{\tau_2}\left[\left\{\left(s_1+w+\frac{a}{N}\right)\tau_1+s_2+m+\frac{b}{N}+\frac{z}{k}\right\}^2+\left(s_1+w+\frac{a}{N}\right)^2\tau_2^2\right]}\cdot e^{\frac{\pi z^2}{k\tau_2}}\cdot e^{2\pi i z\left(s_1+w+\frac{a}{N}\right)},
\end{aligned} \tag{4.18}$$

$$e^{-2\pi i z \mathbf{s}_1} = e^{-2\pi i z (s_1+w)}\cdot e^{-2\pi\frac{z^2}{k\tau_2}}. \tag{4.19}$$

Substituting (4.17), (4.18), and (4.19) into (4.14), we obtain

$$\begin{aligned}
\mathcal{Z}_{[\alpha,\beta]}(\tau,z) &= \frac{1}{K}\sum_{w,m\in\mathbb{Z}}\sum_{a,b\in\mathbb{Z}_N} e^{-2\pi i\frac{1}{N}(\alpha b-\beta a)}\int_0^1 ds_1\int_0^1 ds_2\,\frac{\theta_1\left(\tau,-s_1\tau-s_2+z\right)}{\theta_1\left(\tau,-s_1\tau-s_2\right)} \\
&\qquad\times y^{w+\frac{a}{N}}\, e^{-\frac{\pi k}{\tau_2}\left[\left\{\left(s_1+w+\frac{a}{N}\right)\tau_1+s_2+m+\frac{b}{N}+\frac{z}{k}\right\}^2+\left(s_1+w+\frac{a}{N}\right)^2\tau_2^2\right]} \\
&= \frac{1}{K}\sum_{w,m\in\mathbb{Z}} e^{-2\pi i\frac{1}{N}(\alpha m-\beta w)}\int_0^1 ds_1\int_0^1 ds_2\,\frac{\theta_1\left(\tau,-s_1\tau-s_2+z\right)}{\theta_1\left(\tau,-s_1\tau-s_2\right)} \\
&\qquad\times y^{\frac{w}{N}}\, e^{-\frac{\pi}{NK\tau_2}\left[\left\{(Ns_1+w)\tau_1+Ns_2+m+Kz\right\}^2+(Ns_1+w)^2\tau_2^2\right]}.
\end{aligned} \tag{4.20}$$

In the 2nd line we rewrote $Nw+a$, $Nm+b$ as w, m.

The Poisson resummation (D.1) yields

$$\begin{aligned}
\mathcal{Z}_{[\alpha,\beta]}(\tau,z) &= \sqrt{k\tau_2}\sum_{w,n\in\mathbb{Z}} e^{2\pi i\frac{\beta w}{N}}\int_0^1 ds_1\int_0^1 ds_2\,\frac{\theta_1\left(\tau,-s_1\tau-s_2+z\right)}{\theta_1\left(\tau,-s_1\tau-s_2\right)} \\
&\qquad\times y^{\frac{w}{N}}\; e^{-\pi\tau_2\left\{NK\left(n+\frac{\alpha}{N}\right)^2+\frac{1}{NK}(Ns_1+w)^2\right\}+2\pi i\left(n+\frac{\alpha}{N}\right)\left\{(Ns_1+w)\tau_1+Ns_2+Kz\right\}}.
\end{aligned} \tag{4.21}$$

Due to the identity (D.8), we find

$$\frac{\theta_1\left(\tau,-s_1\tau-s_2+z\right)}{\theta_1\left(\tau,-s_1\tau-s_2\right)} = \frac{-i\theta_1(\tau,z)}{\eta(\tau)^3}\sum_{\ell\in\mathbb{Z}}\frac{yq^\ell}{1-yq^\ell}\,e^{-2\pi i(s_1\tau_1+s_2)\ell+2\pi s_1\tau_2\ell}. \tag{4.22}$$

The s_2-integral simply gives

$$\ell = Nn+\alpha. \tag{4.23}$$

Relevant terms of the s_1-integral are now calculated as

$$e^{-\pi\tau_2\frac{N}{K}s_1^2-2\pi s_1\left\{\frac{1}{K}w\tau_2-i(Nn+\alpha)\tau_1+i\ell\tau_1-\ell\tau_2\right\}}=e^{-\pi\tau_2\frac{N}{K}s_1^2-2\pi s_1\tau_2\frac{v}{K}}, \tag{4.24}$$

where we set

$$v:=w-K\ell=w-K(Nn+\alpha), \tag{4.25}$$

under the constraint (4.23). The s_1-integral is performed in the same way;

$$\begin{aligned}\int_0^1 ds_1\, e^{-\pi\tau_2\frac{N}{K}s_1^2-2\pi\tau_2\frac{s_1}{K}v} &= \sqrt{\frac{\tau_2}{NK}}\int_0^1 ds_1\int_{\mathbb{R}-i0} dp\, e^{-\frac{\pi}{NK}\tau_2p^2-2\pi i\tau_2\frac{s_1}{K}(p-iv)}\\ &= \sqrt{\frac{K}{N\tau_2}}\frac{1}{2\pi i}\int_{\mathbb{R}-i0} dp\,\frac{e^{-\frac{\pi}{NK}\tau_2p^2}}{p-iv}\left\{1-e^{-2\pi i\tau_2\frac{1}{K}(p-iv)}\right\}. \end{aligned}\tag{4.26}$$

We also note

$$\begin{aligned} e^{-\pi\tau_2\left\{NK\left(n+\frac{\alpha}{N}\right)^2+\frac{w^2}{NK}\right\}}\, e^{2\pi i w\left(n+\frac{\alpha}{N}\right)} &= q^{\frac{1}{4NK}\{w+K(Nn+\alpha)\}^2}\left[q^{\frac{1}{4NK}\{w-K(Nn+\alpha)\}^2}\right]^*\\ &= e^{-\pi\tau_2\frac{v^2}{NK}}\, q^{NK\left(n+\frac{\alpha}{N}\right)^2+v\left(n+\frac{\alpha}{N}\right)}. \end{aligned}\tag{4.27}$$

In the 2nd line we used (4.23) and (4.25). On the other hand, the power of y is given by

$$y^{\frac{w}{N}+K\left(n+\frac{\alpha}{N}\right)}=y^{2K\left(n+\frac{\alpha}{N}\right)+\frac{v}{N}}.$$

Collecting all these factors, we obtain

$$\begin{aligned}\mathcal{Z}_{[\alpha,\beta]}(\tau,z) &= -\frac{i\theta_1(\tau,z)}{\eta(\tau)^3}\sum_{n,v\in\mathbb{Z}}\frac{1}{2\pi i}\int_{\mathbb{R}-i0} dp\,\frac{e^{-\pi\tau_2\frac{p^2+v^2}{NK}}}{p-iv}\left\{1-e^{-2\pi i\tau_2\frac{1}{K}(p-iv)}\right\}\\ &\qquad\times e^{2\pi i\frac{\beta}{N}(v+K\alpha)}\,\frac{(yq^{Nn+\alpha})^{1+\frac{v}{N}}}{1-yq^{Nn+\alpha}}\,y^{2K\left(n+\frac{\alpha}{N}\right)}q^{NK\left(n+\frac{\alpha}{N}\right)^2}\\ &= -\frac{i\theta_1(\tau,z)}{\eta(\tau)^3}\sum_{n,v\in\mathbb{Z}}\frac{1}{2\pi i}\left[\int_{\mathbb{R}-i0} dp\,\left(yq^{Nn+\alpha}\right)-\int_{\mathbb{R}+i(N-0)} dp\right]\frac{e^{-\pi\tau_2\frac{p^2+v^2}{NK}}}{p-iv}\\ &\qquad\times e^{2\pi i\frac{\beta}{N}(v+K\alpha)}\,\frac{(yq^{Nn+\alpha})^{\frac{v}{N}}}{1-yq^{Nn+\alpha}}\,y^{2K\left(n+\frac{\alpha}{N}\right)}q^{NK\left(n+\frac{\alpha}{N}\right)^2}. \end{aligned}\tag{4.28}$$

Finally, by shifting the integration contour;

$$\mathbb{R}+i(N-0)\longrightarrow\mathbb{R}-i0,$$

in the 2nd integral, we obtain the decomposition;

$$\mathcal{Z}_{[\alpha,\beta]}(\tau,z)=\mathcal{Z}_{[\alpha,\beta]\,\mathrm{dis}}(\tau,z)+\mathcal{Z}_{[\alpha,\beta]\,\mathrm{rem}}(\tau,z). \tag{4.29}$$

The discrete part (pole contribution) is easily calculated as

$$\begin{aligned}\mathcal{Z}_{[\alpha,\beta]\,\mathrm{dis}}(\tau,z) &= -\frac{i\theta_1(\tau,z)}{\eta(\tau)^3}\sum_{v=0}^{N-1}\sum_{n\in\mathbb{Z}} e^{2\pi i\frac{\beta}{N}(v+K\alpha)}\frac{(yq^{Nn+\alpha})^{\frac{v}{N}}}{1-yq^{Nn+\alpha}}\,y^{2K\left(n+\frac{\alpha}{N}\right)}q^{NK\left(n+\frac{\alpha}{N}\right)^2} \\ &= -e^{2\pi i\frac{K}{N}\alpha\beta}q^{\frac{K}{N}\alpha^2}y^{\frac{2K}{N}\alpha}\,\mathcal{K}^{(2NK)}\left(\tau,\frac{z+\alpha\tau+\beta}{N}\right)\frac{i\theta_1(\tau,z)}{\eta(\tau)^3}. \qquad (4.30)\end{aligned}$$

Here we used the identity (C.4). This part (4.30) is obviously holomorphic.

The remainder part is now computed as

$$\begin{aligned}\mathcal{Z}_{[\alpha,\beta],\mathrm{rem}}(\tau,z) &= -\frac{i\theta_1(\tau,z)}{\eta(\tau)^3}\sum_{n,v\in\mathbb{Z}}\frac{1}{2\pi i}\int_{\mathbb{R}-i0}dp\,\left(yq^{Nn+\alpha}-1\right)\frac{e^{-\pi\tau_2\frac{p^2+v^2}{NK}}}{p-iv} \\ &\qquad\times e^{2\pi i\frac{\beta}{N}(v+K\alpha)}\frac{(yq^{Nn+\alpha})^{\frac{v}{N}}}{1-yq^{Nn+\alpha}}\,y^{2K\left(n+\frac{\alpha}{N}\right)}q^{NK\left(n+\frac{\alpha}{N}\right)^2}, \\ &= \frac{i\theta_1(\tau,z)}{\eta(\tau)^3}\sum_{n,v\in\mathbb{Z}}\frac{1}{2\pi i}\int_{\mathbb{R}-i0}dp\,\frac{e^{-\pi\tau_2\frac{p^2+v^2}{NK}}}{p-iv} \\ &\qquad\times e^{2\pi i\frac{\beta}{N}(v+K\alpha)}\,y^{2K\left(n+\frac{v+2K\alpha}{2NK}\right)}q^{NK\left(n+\frac{v+2K\alpha}{2NK}\right)^2}q^{-\frac{v^2}{4NK}}, \\ &= \frac{i\theta_1(\tau,z)}{\eta(\tau)^3}\cdot\frac{1}{2}\sum_{v\in\mathbb{Z}_{2NK}} e^{2\pi i\frac{\beta}{N}(v+K\alpha)}\,R^{(+)}_{v,NK}\,\Theta_{v+2K\alpha,NK}\left(\tau,\frac{2z}{N}\right), \\ &= \frac{i\theta_1(\tau,z)}{\eta(\tau)^3}\cdot\frac{1}{2}\sum_{v\in\mathbb{Z}_{2NK}} e^{2\pi i\frac{K}{N}\alpha\beta}\,q^{\frac{K}{N}\alpha^2}y^{\frac{2K}{N}\alpha}\,R^{(+)}_{v,NK}\,\Theta_{v,NK}\left(\tau,\frac{2(z+\alpha\tau+\beta)}{N}\right). \qquad (4.31)\end{aligned}$$

Here we used the definition of function $R^{(+)}_{*,*}$ (C.8). Combining (4.30) and (4.31), and using the formula of modular completion (C.7), we finally obtain the expected result;

$$\begin{aligned}&\mathcal{Z}_{[\alpha,\beta]}(\tau,z) = \mathcal{Z}_{[\alpha,\beta]\,\mathrm{dis}}(\tau,z)+\mathcal{Z}_{[\alpha,\beta]\,\mathrm{rem}}(\tau,z) \\ &\quad= -e^{2\pi i\frac{K}{N}\alpha\beta}\,q^{\frac{K}{N}\alpha^2}y^{\frac{2K}{N}\alpha} \\ &\quad\times\left[\mathcal{K}^{(2NK)}\left(\tau,\frac{z+\alpha\tau+\beta}{N}\right)-\frac{1}{2}\sum_{v\in\mathbb{Z}_{2NK}}R^{(+)}_{v,NK}\,\Theta_{v,NK}\left(\tau,\frac{2(z+\alpha\tau+\beta)}{N}\right)\right]\frac{i\theta_1(\tau,z)}{\eta(\tau)^3} \\ &\quad= -e^{2\pi i\frac{K}{N}\alpha\beta}\,q^{\frac{K}{N}\alpha^2}y^{\frac{2K}{N}\alpha}\,\widehat{\mathcal{K}}^{(2NK)}\left(\tau,\frac{z+\alpha\tau+\beta}{N}\right)\frac{i\theta_1(\tau,z)}{\eta(\tau)^3}. \qquad (4.32)\end{aligned}$$

A few small remarks are in order;

- The decomposition into $\mathcal{Z}_{[\alpha,\beta]\mathrm{dis}}(\tau,z)$ and $\mathcal{Z}_{[\alpha,\beta]\mathrm{rem}}(\tau,z)$ depends on the choice of integration contour of the momentum p. However, $\mathcal{Z}_{[\alpha,\beta]}(\tau,z)$ itself is of course free from such an ambiguity. For example, suppose we instead takes the contour $\mathbb{R}+i0$ in (4.26), we reach a different decomposition;

$$\mathcal{Z}'_{[\alpha,\beta]\mathrm{dis}}(\tau,z) = e^{2\pi i \frac{K}{N}\alpha\beta} q^{\frac{K}{N}\alpha^2} y^{\frac{2K}{N}\alpha} \mathcal{K}^{(2NK)}\left(\tau, -\frac{z+\alpha\tau+\beta}{N}\right) \frac{i\theta_1(\tau,z)}{\eta(\tau)^3}, \tag{4.33}$$

$$\mathcal{Z}'_{[\alpha,\beta]\mathrm{rem}}(\tau,z) = \frac{i\theta_1(\tau,z)}{\eta(\tau)^3}\cdot\frac{1}{2}\sum_{v\in\mathbb{Z}_{2NK}} e^{2\pi i \frac{K}{N}\alpha\beta} q^{\frac{K}{N}\alpha^2} y^{\frac{2K}{N}\alpha} R^{(-)}_{v,NK}\,\Theta_{v,NK}\left(\tau, \frac{2(z+\alpha\tau+\beta)}{N}\right). \tag{4.34}$$

in place of (4.30) and (4.31). (Here $R^{(-)}_{m,k}$ is again defined by (C.8).) It is obvious that

$$\mathcal{Z}'_{[\alpha,\beta]\mathrm{dis}}(\tau,z) \neq \mathcal{Z}_{[\alpha,\beta]\mathrm{dis}}(\tau,z), \quad \mathcal{Z}'_{[\alpha,\beta]\mathrm{rem}}(\tau,z) \neq \mathcal{Z}_{[\alpha,\beta]\mathrm{rem}}(\tau,z),$$

but, the sum of them is unchanged, as should be;

$$\mathcal{Z}'_{[\alpha,\beta]\mathrm{dis}}(\tau,z) + \mathcal{Z}'_{[\alpha,\beta]\mathrm{rem}}(\tau,z) = \mathcal{Z}_{[\alpha,\beta]\mathrm{dis}}(\tau,z) + \mathcal{Z}_{[\alpha,\beta]\mathrm{rem}}(\tau,z).$$

See the formula (C.7) to check this equivalence directly.

- $\mathcal{Z}_{[\alpha,\beta]}(\tau,z)$ has the following simple parity property with respect to z;

$$\mathcal{Z}_{[\alpha,\beta]}(\tau,-z) = \mathcal{Z}_{[-\alpha,-\beta]}(\tau,z), \tag{4.35}$$

whereas the holomorphic part $\mathcal{Z}_{[\alpha,\beta]\mathrm{dis}}(\tau,z)$ is not. Especially, $\mathcal{Z}(\tau,z)$ and $\mathcal{Z}_{\mathrm{orb}}(\tau,z)\equiv\mathcal{Z}_{[0,0]}(\tau,z)$ are even functions, as is expected. See again the Appendix C.

4.2.2 Relation with [10]

In the case of $\alpha=\beta=0$, we obtain

$$\mathcal{Z}_{[0,0]}(\tau,z) = -\widehat{\mathcal{K}}^{(2NK)}\left(\tau,\frac{z}{N}\right)\frac{i\theta_1(\tau,z)}{\eta(\tau)^3}. \tag{4.36}$$

This is essentially the result given by Troost [10], as we already mentioned. At first glance, our path-integral representation (4.13) may look different from the one of [10], at $\alpha=\beta=0$. The latter reads as[11]

$$\mathcal{Z}^{\mathrm{Troost}}(\tau,z) \equiv \frac{1}{K}\sum_{w,m\in\mathbb{Z}}\int_0^1 ds_1\int_0^1 ds_2\,\frac{\theta_1\left(\tau,-s_1\tau-s_2+\left(1+\frac{1}{k}\right)z\right)}{\theta_1\left(\tau,-s_1\tau-s_2+\frac{1}{k}z\right)} \times y^{\frac{w}{N}}\, e^{-\frac{\pi k}{\tau_2}\left|\left(s_1+\frac{w}{N}\right)\tau+\left(s_2+\frac{m}{N}\right)\right|^2}. \tag{4.37}$$

[11]See eq. (18) in [10]. [10] deals only with the case of $k\in\mathbb{Z}_{>0}$, however, it is straightforward to generalize the analysis to cases with $k=\frac{N}{K}$.

Now, we try to directly show the coincidence of $\mathcal{Z}_{[0,0]}(\tau,z)$ and $\mathcal{Z}^{\text{Troost}}(\tau,z)$ by using the contour deformation technique as above. In fact, rewriting the integration variable as;

$$\mathbf{s}_1=\widetilde{\mathbf{s}}_1-i\frac{z}{k\tau_2},\qquad \mathbf{s}_2=\widetilde{\mathbf{s}}_2+i\frac{\tau_1 z}{k\tau_2}, \tag{4.38}$$

and choosing the contours with $\xi_1=-\frac{z_1}{k\tau_2}$, $\xi_2=\frac{\tau_1 z_1}{k\tau_2}$, so as to make new integration variables $\widetilde{\mathbf{s}}_1$, $\widetilde{\mathbf{s}}_2$ real, we obtain from (4.14);

$$\begin{aligned}
\mathcal{Z}_{[0,0]}(\tau,z) &= \frac{1}{K}e^{\pi\frac{z^2}{k\tau_2}}\sum_{a,b\in\mathbb{Z}_N}\int_{\mathbb{R}+i\xi_1}d\mathbf{s}_1\int_{\mathbb{R}+i\xi_2}d\mathbf{s}_2\,\frac{\theta_1\left(\tau,-\mathbf{s}_1\tau-\mathbf{s}_2+\left(1+\frac{2}{k}\right)z\right)}{\theta_1\left(\tau,-\mathbf{s}_1\tau-\mathbf{s}_2+\frac{2}{k}z\right)}\,e^{-2\pi iz\mathbf{s}_1}\\
&\qquad\times e^{-\frac{\pi k}{\tau_2}\left[\left\{\left(\mathbf{s}_1+\frac{a}{N}\right)\tau_1+\mathbf{s}_2+\frac{b}{N}\right\}^2+\left(\mathbf{s}_1+\frac{a}{N}\right)^2\tau_2^2\right]},\\
&= \frac{1}{K}e^{\pi\frac{z^2}{k\tau_2}}\sum_{a,b\in\mathbb{Z}_N}\int_{\mathbb{R}}d\widetilde{\mathbf{s}}_1\int_{\mathbb{R}}d\widetilde{\mathbf{s}}_2\,\frac{\theta_1\left(\tau,-\widetilde{\mathbf{s}}_1\tau-\widetilde{\mathbf{s}}_2+\left(1+\frac{1}{k}\right)z\right)}{\theta_1\left(\tau,-\widetilde{\mathbf{s}}_1\tau-\widetilde{\mathbf{s}}_2+\frac{1}{k}z\right)}\,e^{-2\pi iz\widetilde{\mathbf{s}}_1}\,e^{-2\pi\frac{z^2}{k\tau_2}}\\
&\qquad\times e^{-\frac{\pi k}{\tau_2}\left[\left\{\left(\widetilde{\mathbf{s}}_1+\frac{a}{N}\right)\tau_1+\widetilde{\mathbf{s}}_2+\frac{b}{N}\right\}^2+\left(\widetilde{\mathbf{s}}_1+\frac{a}{N}\right)^2\tau_2^2\right]}\,e^{\pi\frac{z^2}{k\tau_2}}\,e^{2\pi iz\left(\widetilde{\mathbf{s}}_1+\frac{a}{N}\right)},\\
&= \frac{1}{K}\sum_{a,b\in\mathbb{Z}_N}\int_{\mathbb{R}}d\widetilde{\mathbf{s}}_1\int_{\mathbb{R}}d\widetilde{\mathbf{s}}_2\,\frac{\theta_1\left(\tau,-\widetilde{\mathbf{s}}_1\tau-\widetilde{\mathbf{s}}_2+\left(1+\frac{1}{k}\right)z\right)}{\theta_1\left(\tau,-\widetilde{\mathbf{s}}_1\tau-\widetilde{\mathbf{s}}_2+\frac{1}{k}z\right)}\,e^{2\pi iz\frac{a}{N}}\\
&\qquad\times e^{-\frac{\pi k}{\tau_2}\left[\left\{\left(\widetilde{\mathbf{s}}_1+\frac{a}{N}\right)\tau_1+\widetilde{\mathbf{s}}_2+\frac{b}{N}\right\}^2+\left(\widetilde{\mathbf{s}}_1+\frac{a}{N}\right)^2\tau_2^2\right]},\\
&= \mathcal{Z}^{\text{Troost}}(\tau,z).
\end{aligned} \tag{4.39}$$

Thus the agreement is shown.

5 Conclusions

In this paper we have studied the $SL(2,\mathbb{R})/U(1)$ SUSY gauged WZW model. After introducing a suitable IR regularization preserving good modular properties, we have found that the partition function (2.40) is decomposed as

$$Z(\tau,z;\epsilon)=Z_{\text{dis}}(\tau,z)+Z_{\text{con}}(\tau,z;\epsilon)+Z_{\text{subleading}}(\tau,z;\epsilon), \tag{5.1}$$

and main observations are summarized as follows;

- $Z_{\text{dis}}(\tau,z)$ (3.18) as a sum of a finite number of the modular completion $\widehat{\chi}_{\text{dis}}(\tau,z)$ (3.17) of discrete (BPS) characters and is by itself modular invariant.
- The remaining part $Z_{\text{con}}(\tau,z;\epsilon)+Z_{\text{subleading}}(\tau,z;\epsilon)$ contain only continuous (non-BPS) characters $\chi_{\text{con}}(\tau,z)$. The leading contribution shows a logarithmic divergence originating from the non-compactness of target space and is written in a modular invariant form;

$$\sim(-\log\epsilon)\sum_{n_0\in\mathbb{Z}_N,\,w_0\in\mathbb{Z}_{2K}}\int_0^\infty dp\,\chi_{\text{con}}(p,Nw_0+Kn_0;\tau,z)\chi_{\text{con}}(p,Nw_0-Kn_0;\tau,z)^*.$$

We should emphasize that partition function when formulated using path-integral must be modular invariant and thus expanded in terms of Jacobi forms. Since BPS characters do not have a good modular property, they must necessarily appear in modular completed forms in the partition function. Thus the partition function acquires non-holomorphic dependence.

In our previous attempt at describing elliptic genera for ALE spaces [18, 19], we have relaxed the condition of invariance under the full modular group and imposed invariance only under the $\Gamma(2)$ congruence subgroup. In this approach we did not encounter non-holomorphic dependence in the elliptic genera for ALE spaces.

We note that here is a basic crash between the holomorphy and modular invariance in string theory on non-compact background. Insistence on strict modular invariance would introduce non-holomorphy in the theory while the relaxed modular invariance may still be realized within holomorphic partition functions.

These alternatives will correspond to the choice of different boundary conditions at infinity of non-compact manifolds. From the physical point of view, however, we perhaps prefer invariance under the full modular group at the price of non-holomorphicity in the amplitudes. We, however, do not yet have a good physical feel on the effects of the non-holomorphic dependence in string theory amplitudes and detailed studies on non-holomorphic modular forms is just beginning to become initiated. We hope that this work sheds light on novel perspective in studies of non-compact superconformal fields theories.

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Appendix A: Conventions for Theta Functions

We assume $\tau \equiv \tau_1 + i\tau_2$, $\tau_2 > 0$ and set $q := e^{2\pi i\tau}$, $y := e^{2\pi iz}$;

$$\begin{aligned}
\theta_1(\tau,z) &= i\sum_{n=-\infty}^{\infty}(-1)^n q^{(n-1/2)^2/2}y^{n-1/2} \equiv 2\sin(\pi z)q^{1/8}\prod_{m=1}^{\infty}(1-q^m)(1-yq^m)(1-y^{-1}q^m),\\
\theta_2(\tau,z) &= \sum_{n=-\infty}^{\infty} q^{(n-1/2)^2/2}y^{n-1/2} \equiv 2\cos(\pi z)q^{1/8}\prod_{m=1}^{\infty}(1-q^m)(1+yq^m)(1+y^{-1}q^m),\\
\theta_3(\tau,z) &= \sum_{n=-\infty}^{\infty} q^{n^2/2}y^{n} \equiv \prod_{m=1}^{\infty}(1-q^m)(1+yq^{m-1/2})(1+y^{-1}q^{m-1/2}),\\
\theta_4(\tau,z) &= \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2/2}y^{n} \equiv \prod_{m=1}^{\infty}(1-q^m)(1-yq^{m-1/2})(1-y^{-1}q^{m-1/2}).
\end{aligned} \tag{A.1}$$

$$\Theta_{m,k}(\tau,z) = \sum_{n=-\infty}^{\infty} q^{k(n+\frac{m}{2k})^2}y^{k(n+\frac{m}{2k})}. \tag{A.2}$$

We also set

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n). \tag{A.3}$$

The spectral flow properties of theta functions are summarized as follows;

$$\begin{aligned}
&\theta_1(\tau,z+m\tau+n) = (-1)^{m+n}q^{-\frac{m^2}{2}}y^{-m}\theta_1(\tau,z)\ ,\\
&\theta_2(\tau,z+m\tau+n) = (-1)^{n}q^{-\frac{m^2}{2}}y^{-m}\theta_2(\tau,z)\ ,\\
&\theta_3(\tau,z+m\tau+n) = q^{-\frac{m^2}{2}}y^{-m}\theta_3(\tau,z)\ ,\\
&\theta_4(\tau,z+m\tau+n) = (-1)^{m}q^{-\frac{m^2}{2}}y^{-m}\theta_4(\tau,z)\ ,\\
&\Theta_{a,k}(\tau,2(z+m\tau+n)) = q^{-km^2}y^{-2km}\Theta_{a,k}(\tau,2z)\ .
\end{aligned} \tag{A.4}$$

Appendix B: Summary of H_3^+-Gauged WZW Model

In this appendix we summarize relevant formulas about the gauged WZW model associated with $H_3^+ \cong SL(2,\mathbb{C})/SU(2)$. We here denote the H_3^+-WZW action as $S_{\text{WZW}}(g)$, defined by

$$S_{\text{WZW}}(g) \equiv -\frac{1}{8\pi}\int_\Sigma d^2v\,\text{Tr}\left(\partial_\alpha g^{-1}\partial_\alpha g\right) + \frac{i}{12\pi}\int_B \text{Tr}\left((g^{-1}dg)^3\right), \quad (g(v,\bar{v}) \in H_3^+). \tag{B.1}$$

This is formally defined as an analytic continuation of $-S_{\text{WZW}}^{SU(2)}(g)$, and positive definite for ${}^\forall g(v,\bar{v}) \in H_3^+$.

The Polyakov-Wiegmann identity is written as

$$S_{\rm WZW}(gh) = S_{\rm WZW}(g) + S_{\rm WZW}(h) + \frac{1}{\pi}\int_\Sigma d^2v\,\mathrm{Tr}\left(g^{-1}\partial_{\bar v}g\,\partial_v h h^{-1}\right). \tag{B.2}$$

which plays a fundamental role in the analysis of (gauged) WZW model. It is convenient to define the vector and axial-type gauged WZW actions in the forms of

$$S^{(V)}_{\rm gWZW}(g,h,h^\dagger) \equiv S_{\rm WZW}(hgh^\dagger) - S_{\rm WZW}(hh^\dagger), \tag{B.3}$$
$$S^{(A)}_{\rm gWZW}(g,h,h^\dagger) \equiv S_{\rm WZW}(hgh^\dagger) - S_{\rm WZW}(hh^{\dagger\,-1}), \tag{B.4}$$

where we assume $h(v,\bar v) \in \exp(\mathbb{C}\sigma_2)$, and the gauge field (defined to be a hermitian 1-form) is identified as

$$A_{\bar v}\frac{\sigma_2}{2} = \partial_{\bar v} h h^{-1}, \qquad A_v\frac{\sigma_2}{2} = \partial_v h^\dagger h^{\dagger\,-1}.$$

The chiral gauge transformation is defined as

$$g \longmapsto \Omega g \Omega^\dagger, \quad h \longmapsto h\Omega^{-1}, \quad h^\dagger \longmapsto \Omega^{\dagger\,-1}h^\dagger, \quad ({}^\forall\Omega(v,\bar v) \in \exp(\mathbb{C}\sigma_2)), \tag{B.5}$$

and the vector-like (axial-like) action (B.3) ((B.4)) is anomaly free along the vector (axial) direction $\Omega(v,\bar v) \in \exp(i\mathbb{R}\sigma_2)$ $(\Omega(v,\bar v) \in \exp(\mathbb{R}\sigma_2))$.

The next path-integral formula has been given in [14, 20], and is useful for our analysis (up to some normalization constant C;

$$\int \mathcal{D}g\,\exp\left[-\kappa S_{\rm WZW}(h[u]gh[u]^\dagger)\right] \equiv \mathrm{Tr}\left(q^{L_0-\frac{c_g}{24}}\bar q^{\tilde L_0-\frac{c_g}{24}}\,e^{2\pi i(uj_0^3-\bar u\tilde j_0^3)}\right) = C\frac{e^{-(\kappa-2)\pi\frac{u_2^2}{\tau_2}}}{\sqrt{\tau_2}|\theta_1(\tau,u)|^2}, \tag{B.6}$$

where we used the notation $u_1 \equiv \mathrm{Re}(u) \equiv s_1\tau_1 + s_2$, $u_2 \equiv \mathrm{Im}(u) \equiv s_1\tau_2$, and $h[u]$ is defined in (2.10), (2.11), namely,

$$h[u] \equiv e^{i\Phi[u]\frac{\sigma_2}{2}}, \qquad \Phi[u](w,\bar w) \equiv \frac{1}{\tau_2}\mathrm{Im}(w\bar u).$$

One can easily find

$$S_{\rm WZW}(h[u]h[u]^\dagger) = \frac{\pi u_2^2}{\tau_2}, \qquad S_{\rm WZW}(h[u]h[u]^{\dagger\,-1}) = -\frac{\pi u_1^2}{\tau_2} \equiv \frac{\pi u_2^2}{\tau_2} - \frac{\pi|u|^2}{\tau_2}$$

by direct calculations, and thus we further obtain

$$\int \mathcal{D}g\,\exp\left[-\kappa S^{(V)}(g,\,h[u],\,h[u]^\dagger)\right] = C\frac{e^{2\pi\frac{u_2^2}{\tau_2}}}{\sqrt{\tau_2}|\theta_1(\tau,u)|^2}, \tag{B.7}$$

$$\int \mathcal{D}g\,\exp\left[-\kappa S^{(A)}(g,\,h[u],\,h[u]^\dagger)\right] = C\frac{e^{2\pi\frac{u_2^2}{\tau_2}-\pi\kappa\frac{|u|^2}{\tau_2}}}{\sqrt{\tau_2}|\theta_1(\tau,u)|^2}. \tag{B.8}$$

Note that the expressions (B.7) and (B.8) are modular invariant, whereas (B.6) is not.

Appendix C: Extended Characters and Modular Completion

We consider the case of $\hat{c} \equiv 1 + \frac{2K}{N}$, and focus only on the $\widetilde{\mathrm{R}}$-sector.

Extended Continuous (non-BPS) Characters [2, 3]:

$$\chi_{\mathrm{con}}(p, m; \tau, z) := q^{\frac{p^2}{4NK}} \Theta_{m,NK}\left(\tau, \frac{2z}{N}\right) \frac{i\theta_1(\tau, z)}{\eta(\tau)^3}. \tag{C.1}$$

This corresponds to the spectral flow sum of the non-degenerate representation with $h = \frac{p^2+m^2}{4NK} + \frac{\hat{c}}{8}$, $Q = \frac{m}{N} \pm \frac{1}{2}$ $(p \geq 0,\ m \in \mathbb{Z}_{2NK})$, whose flow momenta are taken to be $n \in N\mathbb{Z}$.

Extended Discrete (BPS) Characters [2, 3]:

$$\chi_{\mathrm{dis}}(v, a; \tau, z) := \sum_{n \in \mathbb{Z}} \frac{(yq^{Nn+a})^{\frac{v}{N}}}{1 - yq^{Nn+a}} y^{2K\left(n+\frac{a}{N}\right)} q^{NK\left(n+\frac{a}{N}\right)^2} \frac{i\theta_1(\tau, z)}{\eta(\tau)^3}. \tag{C.2}$$

This again corresponds to the sum of the Ramond vacuum representation with $h = \frac{\hat{c}}{8}$, $Q = \frac{v}{N} - \frac{1}{2}$ $(v = 0, 1, \ldots, N-1)$ over spectral flow with flow momentum m taken to be mod.N, as $m = a + N\mathbb{Z}$ $(a \in \mathbb{Z}_N)$. If one introduces the notation of Appell function or Lerch sum [21, 22, 9],

$$\mathcal{K}^{(2k)}(\tau, z) := \sum_{n \in \mathbb{Z}} \frac{q^{kn^2} y^{2kn}}{1 - yq^n} \tag{C.3}$$

one can write as

$$\chi_{\mathrm{dis}}(v, a; \tau, z) = \frac{1}{N} \sum_{b \in \mathbb{Z}_N} e^{-2\pi i \frac{vb}{N}} q^{\frac{K}{N}a^2} y^{\frac{2K}{N}a} \mathcal{K}^{(2NK)}\left(\tau, \frac{z + a\tau + b}{N}\right) \frac{i\theta_1(\tau, z)}{\eta(\tau)^3},$$

$$q^{\frac{K}{N}a^2} y^{\frac{2K}{N}a} \mathcal{K}^{(2NK)}\left(\tau, \frac{z + a\tau + b}{N}\right) \frac{i\theta_1(\tau, z)}{\eta(\tau)^3} = \sum_{v=0}^{N-1} e^{2\pi i \frac{vb}{N}} \chi_{\mathrm{dis}}(v, a; \tau, z). \tag{C.4}$$

The anomalous modular transformation formula of $\chi_{\mathrm{dis}}(v, a)$ and $\mathcal{K}^{(2k)}$ can be expressed as [2, 3, 9, 22];

$$\begin{aligned}\chi_{\mathrm{dis}}\left(v, a; -\frac{1}{\tau}, \frac{z}{\tau}\right) = e^{i\pi \frac{\hat{c}}{\tau} z^2} &\left[\sum_{v=0}^{N-1} \sum_{a \in \mathbb{Z}_N} \frac{1}{N} e^{2\pi i \frac{vv' - (v+2Ka)(v'+2Ka')}{2NK}} \chi_{\mathrm{dis}}(v', a'; \tau, z)\right. \\ &\left. - \frac{i}{2NK} \sum_{m' \in \mathbb{Z}_{2NK}} e^{-2\pi i \frac{(v+2Ka)m'}{2NK}} \int_{\mathbb{R}+i0} dp' \frac{e^{-2\pi \frac{vp'}{2NK}}}{1 - e^{-2\pi \frac{p'+im'}{2K}}} \chi_{\mathrm{con}}(p', m'; \tau, z)\right],\end{aligned} \tag{C.5}$$

$$\mathcal{K}^{(2k)}\left(-\frac{1}{\tau},\frac{z}{\tau}\right)=\tau e^{i\pi\frac{2kz^2}{\tau}}\left[\mathcal{K}^{(2k)}(\tau,z)-\frac{i}{\sqrt{2k}}\sum_{m\in\mathbb{Z}_{2k}}\int_{\mathbb{R}+i0}dp'\,\frac{q^{\frac{1}{2}p'^2}}{1-e^{-2\pi\left(\frac{p'}{\sqrt{2k}}+i\frac{m}{2k}\right)}}\,\Theta_{m,k}(\tau,2z)\right] \tag{C.6}$$

Integral over p' in the above formulas is called the Mordell's integral [23, 24].

Modular Completion :

Following [9][12], we define the modular completion of Appell function $\mathcal{K}^{(2k)}$ by the following combination;

$$\begin{aligned}\widehat{\mathcal{K}}^{(2k)}(\tau,z) &:= \mathcal{K}^{(2k)}(\tau,z)-\frac{1}{2}\sum_{m\in\mathbb{Z}_{2k}}R^{(+)}_{m,k}(\tau)\,\Theta_{m,k}(\tau,2z)\\ &\equiv -\mathcal{K}^{(2k)}(\tau,-z)-\frac{1}{2}\sum_{m\in\mathbb{Z}_{2k}}R^{(-)}_{m,k}(\tau)\,\Theta_{m,k}(\tau,2z).\end{aligned} \tag{C.7}$$

Here we have

$$R^{(\pm)}_{m,k}(\tau):=\sum_{\lambda\in m+2k\mathbb{Z}}\mathrm{sgn}(\lambda\pm 0)\mathrm{Erfc}\left(\sqrt{\frac{\pi\tau_2}{k}}\,|\lambda|\right)\,q^{-\frac{\lambda^2}{4k}}, \tag{C.8}$$

and Erfc$(*)$ denotes the error-function (D.3). In (C.8) it supplies a strong enough damping factor to make the power series convergent. Note that $\widehat{\mathcal{K}}^{(k)}(\tau,z)$ is holomorphic with respect to z, but *not* with respect to τ, since $R^{(\pm)}_{m,k}$ depends on τ_2. $R^{(+)}_{m,k}(\tau)$ is constructed in such a way that it generates the Mordell's integral under S-transformation [9] $(0<t<1)$;

$$R^{(+)}_{m,k}(\tau)+\frac{i}{\sqrt{-i\tau}}\frac{1}{\sqrt{2k}}\sum_{\ell\in\mathbb{Z}_{2k}}e^{-\frac{i\pi m\ell}{k}}R^{(+)}_{\ell,k}\left(-\frac{1}{\tau}\right)=2ie^{-\frac{i\pi m^2\tau}{2k}}\int_{\mathbb{R}-it}dp\,\frac{e^{2\pi ik\tau p^2-2\pi m\tau p}}{1-e^{2\pi p}}. \tag{C.9}$$

Mordell integrals then cancel in the combination $\widehat{\mathcal{K}}^{(2k)}(\tau,z)$ and the completed Appell function has a good transformation law [9];

$$\widehat{\mathcal{K}}^{(2k)}\left(-\frac{1}{\tau},\frac{z}{\tau}\right)=\tau e^{i\pi\frac{2kz^2}{\tau}}\,\widehat{\mathcal{K}}^{(2k)}(\tau,z). \tag{C.10}$$

To be precise, $\widehat{\mathcal{K}}^{(2k)}(\tau,z)$ is a *real-analytic* Jacobi form [25] of weight 1 and index k;

$$\widehat{\mathcal{K}}^{(2k)}\left(\frac{a\tau+b}{c\tau+d},\frac{z}{c\tau+d}\right)=(c\tau+d)\,e^{2\pi ik\frac{cz^2}{c\tau+d}}\,\widehat{\mathcal{K}}^{(2k)}(\tau,z)\ ,\quad\begin{pmatrix}a&b\\c&d\end{pmatrix}\in SL(2;\mathbb{Z}) \tag{C.11}$$

$$\widehat{\mathcal{K}}^{(2k)}(\tau,z+\ell\tau+m)=q^{-k\ell^2}y^{-2k\ell}\widehat{\mathcal{K}}^{(2k)}(\tau,z)\ ,\qquad \ell,m\in\mathbb{Z}. \tag{C.12}$$

[12]See Chapter 3 (especially, 'Definition 3.4' and related propositions) in [9].

The modular completion of the discrete character (C.2) is defined as

$$\begin{aligned}\widehat{\chi}_{\rm dis}(v,a;\tau,z) &:= \chi_{\rm dis}(v,a;\tau,z) - \frac{1}{2}\sum_{j\in\mathbb{Z}_{2K}} R^{(+)}_{v+Nj,NK}(\tau)\Theta_{v+Nj+2Ka,NK}\left(\tau,\frac{2z}{N}\right)\frac{i\theta_1(\tau,z)}{\eta(\tau)^3}, \\ &\equiv \frac{1}{N}\sum_{b\in\mathbb{Z}_N} e^{-2\pi i\frac{vb}{N}} q^{\frac{K}{N}a^2} y^{\frac{2K}{N}a}\,\widehat{\mathcal{K}}^{(2NK)}\left(\tau,\frac{z+a\tau+b}{N}\right)\frac{i\theta_1(\tau,z)}{\eta(\tau)^3}. \end{aligned} \quad \text{(C.13)}$$

The modular completion $\widehat{\chi}_{\rm dis}$ is again non-holomorphic with respect to τ, however, has a good modular property;

$$\widehat{\chi}_{\rm dis}\left(v,a;-\frac{1}{\tau},\frac{z}{\tau}\right) = e^{i\pi\frac{\hat{c}}{\tau}z^2}\sum_{v'=0}^{N-1}\sum_{a'\in\mathbb{Z}_N}\frac{1}{N}e^{2\pi i\frac{vv'-(v+2Ka)(v'+2Ka')}{2NK}}\,\widehat{\chi}_{\rm dis}(v',a';\tau,z). \quad \text{(C.14)}$$

This is easily proven by using the modular transformation formula of $\widehat{\mathcal{K}}^{(2k)}$ (C.10) and the second line of (C.13).

Witten Index :

$$\lim_{z\to 0}\chi_{\rm dis}(v,a;\tau,z) = \lim_{z\to 0}\widehat{\chi}_{\rm dis}(v,a;\tau,z) = -\delta^{(N)}_{a,0}. \quad \text{(C.15)}$$

Appendix D: Useful Formulas

Poisson Resummation Formula :

$$\sum_{n\in\mathbb{Z}}\exp\left(-\pi\alpha(n+a)^2+2\pi ib(n+a)\right) = \frac{1}{\sqrt{\alpha}}\sum_{m\in\mathbb{Z}}\exp\left(-\frac{\pi(m-b)^2}{\alpha}+2\pi ima\right), \quad ({\rm Re}\,\alpha>0) \quad \text{(D.1)}$$

Error-functions :

We define the error-functions as

$$\mathrm{Erf}(x) := \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt \quad \text{(D.2)}$$

$$\mathrm{Erfc}(x) := \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}\,dt \equiv 1-\mathrm{Erf}(x) \quad \text{(D.3)}$$

Here we normalize $\mathrm{Erf}(x)$ as $\mathrm{Erf}(\infty)=1$.

A relevant integration formula is as follows;

$$\begin{aligned}\int_0^\infty \frac{e^{-a^2x^2}}{x^2+b^2}\,dx &= \frac{\sqrt{\pi}}{b}e^{a^2b^2}\int_{ab}^\infty e^{-t^2}\,dt \\ &\equiv \frac{\pi}{2b}e^{a^2b^2}\,\mathrm{Erfc}(ab), \quad (a,b>0). \end{aligned} \quad \text{(D.4)}$$

From this formula we readily obtain

$$\int_{\mathbb{R}\mp i0} dp\, \frac{e^{-\alpha p^2}}{p-i\lambda} = i\pi e^{\alpha\lambda^2}\mathrm{sgn}(\lambda\pm 0)\mathrm{Erfc}(\sqrt{\alpha}|\lambda|), \qquad (\lambda\in\mathbb{R},\ \alpha>0), \tag{D.5}$$

which is useful for our calculation. Especially, the function $R^{(\pm)}_{m,k}(\tau)$ (C.8) is expressible as

$$R^{(\pm)}_{m,k}(\tau) = \frac{1}{i\pi}\sum_{\lambda\in m+2k\mathbb{Z}}\int_{\mathbb{R}\mp i0} dp\, \frac{e^{-\pi\tau_2\frac{p^2+\lambda^2}{k}}}{p-i\lambda}\, q^{-\frac{\lambda^2}{4k}}. \tag{D.6}$$

Useful Formulas for $\theta_1(\tau,z)$:

Following expansion is useful for our calculations; ($u\equiv s_1\tau+s_2$, $0<s_1<1$);

$$\begin{aligned}\frac{\theta_1(\tau,u+z)}{\theta_1(\tau,u)} &= \frac{-i\theta_1(\tau,z)}{\eta(\tau)^3}\sum_{n\in\mathbb{Z}}\frac{e^{2\pi inu}}{1-yq^n}, \qquad \left(y\equiv e^{2\pi iz}\right) && \text{(D.7)}\\ \frac{\theta_1(\tau,-u+z)}{\theta_1(\tau,-u)} &= \frac{-i\theta_1(\tau,z)}{\eta(\tau)^3}\sum_{n\in\mathbb{Z}}\frac{yq^n e^{-2\pi inu}}{1-yq^n}. && \text{(D.8)}\end{aligned}$$

Note that the convergence of power series in R.H.S of (D.7) and (D.8) requires the condition $0<s_1<1$. These identities are essentially the same as the branching relation of $SL(2;\mathbb{R})/U(1)$ supercoset theory analysed *e.g.* in [3] (see also [26]).

A proof of the first identity (D.7) is given by comparing the poles and their residues of both sides. In fact, the L.H.S has simples poles at $u=r\tau+s$, ($r,s\in\mathbb{Z}$), whose residues are found to be

$$\mathrm{Res}_{u=r\tau+s}[\text{L.H.S}] = \frac{y^{-r}}{2\pi}\frac{\theta_1(\tau,z)}{\eta(\tau)^3}. \tag{D.9}$$

On the other hand, the R.H.S of (D.7) is analytically continued into a meromorphic function of $u\in\mathbb{C}$ through a double power series expansion ($\xi\equiv e^{2\pi iu}$);

$$\begin{aligned}\text{R.H.S} &= \frac{-i\theta_1(\tau,z)}{\eta(\tau)^3}\left[\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\xi^n y^m q^{nm} - \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\xi^{-n}y^{-m}q^{nm}\right]\\ &= \frac{-i\theta_1(\tau,z)}{\eta(\tau)^3}\sum_{m\in\mathbb{Z}}\frac{y^m}{1-\xi q^m}. && \text{(D.10)}\end{aligned}$$

(The power series in the second line is convergent for $z=t_1\tau+t_2$, $0<t_1<1$.) It is easy to check that (D.10) also possesses simple poles at $u=r\tau+s$ ($r,s\in\mathbb{Z}$) with the residues equal to (D.9), which completes the proof.

(D.8) is readily derived from (D.7).

'Regularization formula' :

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{e^{-\varepsilon(n+z)}}{n+z} &= \int_{\varepsilon}^{\infty} \frac{e^{-t}}{t}\, dt - \frac{d}{dz} \log \Gamma(z) + \mathcal{O}(\varepsilon) \\ &= -\log(\varepsilon) + C - \frac{d}{dz} \log \Gamma(z) + \mathcal{O}(\varepsilon) \qquad (\varepsilon > 0,\ \operatorname{Re} z > 0), \quad \text{(D.11)}\end{aligned}$$

where C expresses some $\mathcal{O}(\varepsilon^0)$-constant independent of z.

This formula is derived from the identity of the digamma function;

$$\psi(z) \equiv \frac{d}{dz} \log \Gamma(z) = \int_0^{\infty} \left(\frac{e^{-t}}{t} - \frac{e^{-tz}}{1 - e^{-t}} \right) dt, \qquad (\operatorname{Re} z > 0).$$

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