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ON THE SPEED OF APPROACH TO EQUILIBRIUM
FOR A COLLISIONLESS GAS

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ABSTRACT. We investigate the speed of approach to Maxwellian equilibrium
for a collisionless gas enclosed in a vessel whose wall are kept at a uniform,
constant temperature, assuming diffuse reflection of gas molecules on the vessel
wall. We establish lower bounds for potential decay rates assuming uniform L^p
bounds on the initial distribution function. We also obtain a decay estimate in
the spherically symmetric case. We discuss with particular care the influence
of low-speed particles on thermalization by the wall.

In memory of Carlo Cercignani (1939-2010)

1. **Introduction.** Relaxation to equilibrium is a fundamental issue in statistical
mechanics. In the kinetic theory of gases, convergence to equilibrium is formulated
in terms of the asymptotic behavior of solutions of the Boltzmann equation in the
long time limit. In [4], Cercignani formulated a quite remarkable conjecture, assert-
ing the domination of Boltzmann's relative H functional by a constant multiple of
Boltzmann's entropy production functional. If true, Cercignani's conjecture would
imply exponential relaxation to equilibrium for space homogeneous solutions of the
Boltzmann equation, and would have important consequences even for space in-
homogeneous solutions. This conjecture was subsequently disproved in a series of
papers by Bobylev, Cercignani and Wennberg [2, 3, 14]. And yet, in the words of
Villani, "Cercignani's conjecture is sometimes true, and almost always true" [12].
Slightly later, Desvillettes and Villani [7] studied the speed of convergence to equi-
librium for space inhomogeneous solutions of the Boltzmann equation, under rather
stringent conditions of regularity, decay at large velocities and positivity, but with-
out assuming that the initial data is close to equilibrium. They found that the
distance between the distribution function at time t and the final equilibrium state
decays like $O(t^{-m})$ for each $m > 0$ as $t \to +\infty$. Their result makes critical use of

the Boltzmann collision integral and entropy production, and of their combination
with the free transport operator (the left-hand side of the Boltzmann equation.)
In an earlier paper [5], Desvillettes had proved that Boltzmann's entropy produc-
tion can be regarded as a distance between the distribution function and the set of
Maxwellian states. On the other hand, Guo proved in [9] that, for initial data close
enough to a uniform Maxwellian, solutions of the Boltzmann equation converge to
equilibrium at exponential speed.

In [7], Desvillettes and Villani treated the cases where the spatial domain is the
flat torus (equivalently, the Euclidian space with periodic initial data), or a smooth
bounded domain with either the specular or bounce-back reflection conditions at the
boundary. Therefore, convergence to equilibrium is driven by the collision process
alone, and the role of the boundary condition is rather indirect. In the long time
limit, the solution of the Boltzmann equation converges to a uniform Maxwellian
state whose density and temperature are given in terms of the initial data by the
conservation of total mass and total energy. The boundary condition is involved
only through its compatibility with the conservation laws of total mass and energy.
In a more recent work [13], Villani extended their result to the case where the gas is
enclosed in a vessel whose walls are maintained at a constant temperature, modeled
with an accommodation or diffuse reflection condition. In that case, mass is the
only quantity that is conserved under the dynamics of the Boltzmann equation, so
that the final equilibrium state is the centered, uniform Maxwellian with the wall
temperature and same total mass as the initial data. In that case, the boundary
condition has a direct influence on the relaxation to equilibrium. Nevertheless, the
combined effect of entropy production by the collision process and of free streaming,
an example of what is termed "hypocoercivity", is the driving mechanism used by
Villani in his proof of relaxation to equilibrium [13].

However, it seems obvious from physical consideration that the accommodation
or diffuse reflection condition on a wall maintained at some constant temperature
is enough to drive the gas to an equilibrium state, and that the role of the collision
process is not as important as in the periodic, or specular or bounce-back reflection
cases — which are in any case less realistic.

In order to establish the role of Boltzmann's collision integral in the rate of
relaxation to equilibrium, we consider the case of a collisionless gas enclosed in a
vessel whose surface is maintained at a constant temperature, and investigate the
speed of approach to equilibrium for such a system. The present paper provides
a theoretical confirmation of the numerical results obtained by the authors in [10]
in collaboration with Tsuji. Basically, while the collision process is not necessary
in the convergence to equilibrium, it greatly influences the speed of relaxation to
equilibrium. In the same spirit, Desvillettes and Salvarani [6] have investigated
the speed of relaxation to equilibrium in the case of linear collisional models where
the collision frequency is not uniformly bounded away from 0, by adapting the
hypocoercivity method.

We dedicate this piece of work to the memory of our teacher, friend and colleague
Carlo Cercignani, in recognition of his leading role in the mathematical theory of
the Boltzmann equation.

2. **Statement of the problem.** Consider a collisionless gas enclosed in a container
materialized by a bounded domain Ω in $\mathbf{R}^N$ whose boundary $\partial\Omega$ is maintained at

a constant temperature $\theta_w > 0$. Its distribution function $F \equiv F(t, x, v)$ satisfies

$$\begin{cases} \partial_t F + v \cdot \nabla_x F = 0, & (x, v) \in \Omega \times \mathbf{R}^N, \\ F(t, x, v)|v \cdot n_x| = \displaystyle\int_{v' \cdot n_x > 0} F(t, x, v')v' \cdot n_x K(x, v, dv'), & (x, v) \in \Gamma_-, \quad (1) \\ F\big|_{t=0} = F^{in}. \end{cases}$$

The notation n_x designates the unit outward normal at the point $x \in \partial\Omega$, and we have used the notation

$$\Gamma_+ := \{(x, v) \in \partial\Omega \times \mathbf{R}^N \,|\, v \cdot n_x > 0\},$$
$$\Gamma_- := \{(x, v) \in \partial\Omega \times \mathbf{R}^N \,|\, v \cdot n_x < 0\}.$$

The boundary condition is given by the measure-valued kernel $K(x, v, dv') \geq 0$ satisfying the assumptions

(i) for each $x \in \partial\Omega$, one has

$$\int_{v \cdot n_x < 0} K(x, v, dv')dv = dv'$$

(ii) for each $x \in \partial\Omega$, one has

$$\mathcal{M}_{(1,0,\theta)}(v)|v \cdot n_x| = \int_{v' \cdot n_x > 0} \mathcal{M}_{(1,0,\theta)}(v')v' \cdot n_x K(x, v, dv'), \quad v \cdot n_x < 0,$$

iff $\theta = \theta_w$. Throughout this paper, we use the following notation for the Maxwellian distribution:

$$\mathcal{M}_{(\rho,u,\theta)}(v) = \frac{\rho}{(2\pi\theta)^{N/2}} e^{-\frac{|v-u|^2}{2\theta}}$$

(Henceforth, we denote for simplicity $M_w = \mathcal{M}_{(1,0,\theta_w)}$.) With these notations, the case of diffuse reflection corresponds with

$$K(x, v, dv') := \frac{\mathcal{M}_{(1,0,\theta_w)}(v)|v \cdot n_x|dv'}{\displaystyle\int_{u \cdot n_x < 0} \mathcal{M}_{(1,0,\theta_w)}(u)|u \cdot n_x|du}.$$

The case of an accommodation boundary condition (i.e. the so-called Maxwell-type condition) corresponds with

$$K(x, v, dv') := (1 - \alpha(x))\delta(v' - v + 2(v \cdot n_x)n_x)$$
$$+ \alpha(x)\frac{\mathcal{M}_{(1,0,\theta_w)}(v)|v \cdot n_x|dv'}{\displaystyle\int_{u \cdot n_x < 0} \mathcal{M}_{(1,0,\theta_w)}(u)|u \cdot n_x|du}.$$

In the case of diffuse reflection, Arkeryd and Nouri [1] have proved that the solution of (1) satisfies

$$F(t + \cdot, \cdot, \cdot) \to \frac{\mathcal{M}_{(1,0,\theta_w)}}{|\Omega|} \iint_{\Omega \times \mathbf{R}^N} F^{in}dxdv \text{ in } L^1([0, T] \times \Omega \times \mathbf{R}^N)$$

as $t \to +\infty$.

In the present paper, we study in detail the following

Problem: Is there a decay rate $E(t) \to 0$ as $t \to +\infty$, for which an asymptotic estimate of the form

$$\left\| F(t + \cdot, \cdot, \cdot) - \frac{\mathcal{M}_{(1,0,\theta_w)}}{|\Omega|} \iint_{\Omega \times \mathbf{R}^N} F^{in}dxdv \right\|_{L^1([0,T] \times \Omega \times \mathbf{R}^N)} = O(E(t)) \quad (2)$$

holds for each solution F of (1) uniformly as F^{in} runs through certain classes of bounded subsets of $L^1(\mathbf{R}^N \times \mathbf{R}^N)$?

3. **Negative results and lower bounds on** E. We first consider the problem of finding a decay rate that is uniform as F^{in} runs through all bounded subsets of $L^1(\mathbf{R}^N \times \mathbf{R}^N)$.

Proposition 1. *There does not exist any decay rate function* $E : \mathbf{R}_+ \to \mathbf{R}_+$ *such that* $E(t) \to 0$ *as* $t \to +\infty$ *and*

$$\left\| F(t + \cdot, \cdot, \cdot) - \frac{\mathcal{M}_{(1,0,\theta_w)}}{|\Omega|} \iint_{\Omega \times \mathbf{R}^N} F^{in} dx dv \right\|_{L^1([0,T] \times \Omega \times \mathbf{R}^N)} \tag{3}$$
$$\leq E(t) \| F^{in} \|_{L^1(\mathbf{R}^N \times \mathbf{R}^N)}$$

for each $t \geq 0$.

Since the initial boundary value problem (1) is linear, if there is a long time decay as in (2) uniformly as the initial data F^{in} runs through bounded subsets of L^1, an inequality of the form (3) must hold.

Proof. Assume that the origin $0 \in \Omega$ — if not, one can apply a translation in the x-variable without changing the nature of the problem — and set $R = \frac{1}{2}\text{dist}(0, \partial\Omega)$. We henceforth denote by $\mathbf{B}$ the closed unit ball of $\mathbf{R}^N$, and by $\mathbf{1}_A$ the indicator function of the set A.

For $0 < \varepsilon \ll 1$, set

$$F^{in}(x, v) = \frac{1}{\varepsilon^{2N}} \mathbf{1}_{\varepsilon\mathbf{B}}(x) \mathbf{1}_{\varepsilon\mathbf{B}}(v)$$

(with $\varepsilon\mathbf{B} := \{\varepsilon z \,|\, z \in \mathbf{B}\}$) so that

$$\iint F^{in}(x, v) dx dv = |\mathbf{B}|^2.$$

Call $\tau(x, v) := \inf\{t \geq 0 \,|\, x - tv \notin \Omega\}$; obviously the solution F of (1) satisfies $F \geq \Phi$, where Φ is the solution of the same free transport equation, with the same initial data, but with absorbing boundary condition:

$$\begin{cases} \partial_t \Phi + v \cdot \nabla_x \Phi = 0, & (x, v) \in \Omega \times \mathbf{R}^N, \\ \Phi(t, x, v) = 0, & (x, v) \in \Gamma_-, \\ \Phi|_{t=0} = F^{in}. \end{cases} \tag{4}$$

Solving for Φ along characteristics, one finds that

$$\Phi(t, x, v) = F^{in}(x - tv, v) \mathbf{1}_{0 \leq t \leq \tau(x,v)}.$$

Now

$$\left\| F(t + \cdot, \cdot, \cdot) - \frac{\mathcal{M}_{(1,0,\theta_w)}}{|\Omega|} \iint_{\Omega \times \mathbf{R}^N} F^{in} dx dv \right\|_{L^1([0,T] \times \Omega \times \mathbf{R}^N)}$$
$$= \left\| F(t + \cdot, \cdot, \cdot) - \frac{|\mathbf{B}|^2}{|\Omega|} \mathcal{M}_{(1,0,\theta_w)} \right\|_{L^1([0,T] \times \Omega \times \mathbf{R}^N)}$$
$$\geq \int_0^T \int_\Omega \int_{\mathbf{R}^N} \left(F(t + s, x, v) - \frac{|\mathbf{B}|^2}{|\Omega|} \mathcal{M}_{(1,0,\theta_w)}(v) \right)^+ dv dx ds.$$

Henceforth, if $A \subset \mathbf{R}^3$ and $u \in \mathbf{R}^3$, we denote $A + u := \{z + u \mid z \in A\}$. Since the function $z \mapsto z^+ = \max(z, 0)$ is nondecreasing, one has

$$\left\| F(t + \cdot, \cdot, \cdot) - \frac{\mathcal{M}_{(1,0,\theta_w)}}{|\Omega|} \iint_{\Omega \times \mathbf{R}^N} F^{in} dx dv \right\|_{L^1([0,T] \times \Omega \times \mathbf{R}^N)}$$

$$\geq \int_0^T \int_\Omega \int_{\mathbf{R}^N} \left(\Phi(t + s, x, v) - \frac{|\mathbf{B}|^2}{|\Omega|} \mathcal{M}_{(1,0,\theta_w)}(v) \right)^+ dv dx ds$$

$$= \int_0^T \int_\Omega \int_{\mathbf{R}^N} \left(\frac{1}{\varepsilon^{2N}} \mathbf{1}_{\varepsilon \mathbf{B} + (t+s)v}(x) \mathbf{1}_{\varepsilon \mathbf{B}}(v) \mathbf{1}_{0 \leq t+s \leq \tau(x,v)} \right.$$
$$\left. - \frac{|\mathbf{B}|^2}{|\Omega|} \mathcal{M}_{(1,0,\theta_w)}(v) \right)^+ dv dx ds$$

$$\geq \int_0^T \int_\Omega \int_{\mathbf{R}^N} \left(\frac{1}{\varepsilon^{2N}} \mathbf{1}_{\varepsilon \mathbf{B} + (t+s)v}(x) \mathbf{1}_{\varepsilon \mathbf{B}}(v) \mathbf{1}_{0 \leq (t+s)|v| \leq R-\varepsilon} \right.$$
$$\left. - \frac{|\mathbf{B}|^2}{|\Omega|} \mathcal{M}_{(1,0,\theta_w)}(v) \right)^+ dv dx ds$$

since $\tau(x,v)|v| \geq 2R - \varepsilon$ for each $x \in \varepsilon \mathbf{B}$. Now

$$\left(\frac{1}{\varepsilon^{2N}} \mathbf{1}_{\varepsilon \mathbf{B} + (t+s)v}(x) \mathbf{1}_{\varepsilon \mathbf{B}}(v) \mathbf{1}_{0 \leq (t+s)|v| \leq R-\varepsilon} - \frac{|\mathbf{B}|^2}{|\Omega|} \mathcal{M}_{(1,0,\theta_w)}(v) \right)^+$$

$$= \frac{1}{\varepsilon^{2N}} \mathbf{1}_{\varepsilon \mathbf{B} + (t+s)v}(x) \mathbf{1}_{\varepsilon \mathbf{B}}(v) \mathbf{1}_{0 \leq (t+s)|v| \leq R-\varepsilon} \left(1 - \frac{\varepsilon^{2N} |\mathbf{B}|^2}{|\Omega|} \mathcal{M}_{(1,0,\theta_w)}(v) \right)^+$$

$$\geq \frac{1}{\varepsilon^{2N}} \mathbf{1}_{\varepsilon \mathbf{B} + (t+s)v}(x) \mathbf{1}_{\varepsilon \mathbf{B}}(v) \mathbf{1}_{0 \leq (t+s)|v| \leq R-\varepsilon} \left(1 - \frac{\varepsilon^{2N} |\mathbf{B}|^2}{|\Omega|(2\pi\theta_w)^{N/2}} \right)^+.$$

Then

$$\int_0^T \int_\Omega \int_{\mathbf{R}^N} \frac{1}{\varepsilon^{2N}} \mathbf{1}_{\varepsilon \mathbf{B} + (t+s)v}(x) \mathbf{1}_{\varepsilon \mathbf{B}}(v) \mathbf{1}_{0 \leq (t+s)|v| \leq R-\varepsilon} dv dx ds$$

$$= |\mathbf{B}| \int_0^T \int_{\mathbf{R}^N} \frac{1}{\varepsilon^N} \mathbf{1}_{\varepsilon \mathbf{B}}(v) \mathbf{1}_{0 \leq (t+s)|v| \leq R-\varepsilon} dv ds$$

$$= |\mathbf{B}| \int_0^T \int_{\mathbf{R}^N} \frac{1}{\varepsilon^N} \mathbf{1}_{0 \leq |v| \leq \min(\varepsilon, (R-\varepsilon)/t+s)} dv ds$$

$$= |\mathbf{B}| \int_0^T \frac{1}{\varepsilon^N} \min \left(\varepsilon^N, \frac{(R-\varepsilon)^N}{(t+s)^N} \right) ds$$

$$\geq |\mathbf{B}| \frac{T}{\varepsilon^N} \min \left(\varepsilon^N, \frac{(R-\varepsilon)^N}{(t+T)^N} \right).$$

Therefore, if there is a uniform decay rate E as in (3), it must satisfy

$$T|\mathbf{B}| \left(1 - \frac{\varepsilon^{2N} |\mathbf{B}|^2}{|\Omega|(2\pi\theta_w)^{N/2}} \right)^+ \left(\min \left(1, \frac{(R-\varepsilon)}{\varepsilon(t+T)} \right) \right)^N \leq |\mathbf{B}|^2 E(t).$$

Here ε only has to satisfy $0 < \varepsilon < R$. If one lets $\varepsilon \to 0^+$, this contradicts the existence of a profile E in (3) such that $E(t) \to 0$ as $t \to +\infty$. $\qquad \square$

The negative result in Proposition 1 could have been easily anticipated: as $\varepsilon \to 0$, our choice of F^{in} converges to $|\mathbf{B}|^2 \delta_{x=0} \delta_{v=0}$ in the weak topology of measures. Then particles distributed under $\delta_{x=0} \delta_{v=0}$ do not move, hence cannot reach the boundary

and therefore cannot be thermalized. Thus, in some sense, the problem of finding a decay rate as in (3) is trivial if stated in this way.

Before going further, let us comment on the kind of inequality considered in Proposition 1. In order to measure the speed of approach to equilibrium for the problem (1), it might be more natural to consider a pointwise in time inequality of the type

$$\left\| F(t, \cdot, \cdot) - \frac{\mathcal{M}_{(1,0,\theta_w)}}{|\Omega|} \iint_{\Omega \times \mathbf{R}^N} F^{in} dx dv \right\|_{L^1(\Omega \times \mathbf{R}^N)} \leq E(t) \|F^{in}\|_{L^1(\mathbf{R}^N \times \mathbf{R}^N)} \quad (5)$$

instead of (3). Observe that the l.h.s. of (3) is the time-average of the l.h.s. of (5), so that (5) is a stronger statement than (3) — as a matter of fact, (5) entails (3) with $E(t)$ changed into $TE(t)$. Therefore, negative results obtained on time-averaged estimates like (3) are stronger than their analogues for the corresponding pointwise estimate similar to (5). This is why we sought to confirm numerically pointwise in time bounds like (5) in our previous paper [10], and not time-averaged estimates as in the work of Arkeryd-Nouri [1]. Throughout the present section, we discuss negative results on the speed of approach to equilibrium and therefore consider time-averaged decay estimates.

A slightly less trivial issue is to investigate the existence of such a decay rate under the additional assumption ruling out the possibility of a complete concentration as above. This could be done by controlling not only the L^1 norm of the initial data, but (for instance) its entropy, or relative entropy with respect to the final state. For instance, one could look instead at a bound of the form

$$\left\| F(t + \cdot, \cdot, \cdot) - \frac{\mathcal{M}_{(1,0,\theta_w)}}{|\Omega|} \iint_{\Omega \times \mathbf{R}^N} F^{in} dx dv \right\|_{L^1([0,T] \times \Omega \times \mathbf{R}^N)}$$
$$\leq E(t) \iint_{\Omega \times \mathbf{R}^N} F^{in}(x, v) |\ln F^{in}(x, v)| dx dv . \quad (6)$$

Proposition 2. *If there is a time decay rate E such that the solution of (1) satisfies (6), then it must satisfy*

$$E(t) \geq \frac{C}{\ln t} \quad \text{as } t \to +\infty .$$

Proof. For the same initial data as before

$$\iint F^{in}(x, v) |\ln F^{in}(x, v)| dx dv = \iint \frac{1}{\varepsilon^{2N}} \mathbf{1}_{\varepsilon B}(x) \mathbf{1}_{\varepsilon B}(v) \ln \left(\frac{1}{\varepsilon^{2N}} \right) dx dv$$
$$= 2N |\mathbf{B}|^2 |\ln \varepsilon| .$$

Proceeding as before to obtain a lower bound for the l.h.s. of (6), one arrives at the inequality

$$T|\mathbf{B}| \left(1 - \frac{\varepsilon^{2N} |\mathbf{B}|^2}{|\Omega|(2\pi\theta_w)^{N/2}} \right)^+ \left(\min \left(1, \frac{(R - \varepsilon)}{\varepsilon(t + T)} \right) \right)^N \leq 2N |\mathbf{B}|^2 |\ln \varepsilon| E(t) .$$

Choosing $\varepsilon_0 > 0$ small enough, this inequality takes the form

$$C \left(\min \left(1, \frac{1}{\varepsilon t} \right) \right)^N \leq |\ln \varepsilon| E(t)$$

for all $t > T$ and all $\varepsilon \in (0, \varepsilon_0)$. Picking then $\varepsilon = 1/t$ with $t \gg 1$ so that in particular $t > \max(T, 1/\varepsilon_0)$ leads to

$$E(t) \geq \frac{C}{\ln t} \quad \text{as } t \to +\infty.$$

$\square$

The constraint on the decay rate E obtained in Proposition 2 remains the same if one replaces the r.h.s. in the bound (6) with either

$$E(t) \iint_{\Omega \times \mathbf{R}^N} F^{in}(x, v)(1 + |v|^2 + |\ln F^{in}(x, v)|) dx dv$$

or

$$E(t) \iint_{\Omega \times \mathbf{R}^N} \left(F^{in}(x, v) \ln \left(\frac{F^{in}}{\bar{\rho} \mathcal{M}_{(1,0,\theta_w)}} \right) - F^{in} + \bar{\rho} \mathcal{M}_{(1,0,\theta_w)} \right)(x, v) dx dv,$$

where

$$\bar{\rho} = \frac{1}{|\Omega|} \iint_{\Omega \times \mathbf{R}^N} F^{in} dx dv.$$

Indeed, compared to the r.h.s. in the bound (6), both these quantities involve in addition the total energy integral

$$\iint_{\Omega \times \mathbf{R}^N} |v|^2 F^{in}(x, v) dx dv.$$

Since the inequality (6) is tested with velocity distribution functions concentrated near $v = 0$, this additional term is negligible and its presence does not change the final result.

Of course, prescribing an entropy bound is not the only possibility for avoiding concentrations in the initial data F^{in}. For instance, one could ask for a control of F^{in} in some L^p-norm with $1 < p \leq \infty$. In other words, one could investigate the possibility of an inequality of the form

$$\left\| F(t + \cdot, \cdot, \cdot) - \frac{\mathcal{M}_{(1,0,\theta_w)}}{|\Omega|} \iint_{\Omega \times \mathbf{R}^N} F^{in} dx dv \right\|_{L^1([0,T] \times \Omega \times \mathbf{R}^N)} \tag{7}$$
$$\leq E(t) \|F^{in}\|_{L^p(\Omega \times \mathbf{R}^N)}.$$

Proposition 3. *If there is a time decay rate E such that the solution of (1) satisfies (7), then it must satisfy*

$$E(t) \geq \frac{C}{t^{N \min(1, 2/p')}} \quad \text{as } t \to +\infty.$$

Proof. With the same initial data as before

$$\|F^{in}\|_{L^p(\Omega \times \mathbf{R}^N)} = \frac{|\mathbf{B}|^{2/p}}{\varepsilon^{2N/p'}},$$

with $p' = \frac{p}{p-1} < \infty$ the dual exponent of p, so that, estimating the l.h.s. by the same argument as before, one arrives at the inequality

$$T|\mathbf{B}| \left(1 - \frac{\varepsilon^{2N} |\mathbf{B}|^2}{|\Omega|(2\pi\theta_w)^{N/2}} \right)^+ \left(\min \left(1, \frac{(R - \varepsilon)}{\varepsilon(t + T)} \right) \right)^N \leq \frac{|\mathbf{B}|^{2/p}}{\varepsilon^{2N/p'}} E(t).$$

Assume that $p \geq 2$. Since the growth of the l.h.s. in $1/\varepsilon$ is smaller than that of the r.h.s., there is no point in letting $\varepsilon \to 0$. One therefore chooses $\varepsilon > 0$ small enough but fixed, so that $\varepsilon < R/2$ and $\varepsilon^{2N}|\mathbf{B}|^2 < \frac{1}{2}|\Omega|(2\pi\theta_w)^{N/2}$, so that

$$\frac{T}{2}|\mathbf{B}|\left(\min\left(1, \frac{R}{2\varepsilon(t+T)}\right)\right)^N \leq \frac{|\mathbf{B}|^{2/p}}{\varepsilon^{2N/p'}}E(t)\,.$$

In that case, one sees that there exists $C = C(T, R, \varepsilon) > 0$ such that

$$E(t) \geq \frac{C}{t^N} \quad \text{as } t \gg 1\,.$$

If $1 < p < 2$, by picking $\varepsilon_0 > 0$ small enough, the inequality above takes the form

$$C\left(\min\left(1, \frac{1}{\varepsilon t}\right)\right)^N \leq \frac{|\mathbf{B}|^{2/p}}{\varepsilon^{2N/p'}}E(t)\,,$$

for all $t > 0$ and all $\varepsilon \in (0, \varepsilon_0)$. Picking $\varepsilon = 1/t$ with $t \gg 1$ so that in particular $t > \max(T, 1/\varepsilon_0)$, one concludes from this last estimate that

$$E(t) \geq \frac{C}{t^{2N/p'}} \quad \text{as } t \to +\infty\,.$$

$\square$

At this point, it is worth comparing the result in Proposition 3 for the collisionless gas driven to equilibrium by the boundary, and the following important result, obtained by Ukai, Point and Ghidouche [11], for the Boltzmann equation linearized at a uniform Maxwellian equilibrium.

Consider the Boltzmann equation linearized at the uniform Maxwellian state $M = \mathcal{M}_{(1,0,1)}$, assuming hard sphere collisions, in the case where the spatial domain is the periodic box with unit side $\mathbf{T}^3$. In other words, consider the Cauchy problem

$$\begin{cases} (\partial_t + v \cdot \nabla_x)g + \mathcal{L}_M g = 0\,, & (t,x,v) \in \mathbf{R}_+^* \times \mathbf{T}^3 \times \mathbf{R}^3\,, \\ g\big|_{t=0} = g^{in} \end{cases} \tag{8}$$

The linearized collision operator is given by

$$\mathcal{L}_M\phi(v) := \iint_{\mathbf{R}^3 \times \mathbf{S}^2} (\phi(v') + \phi(v_*') - \phi(v) - \phi(v_*))|(v - v_*) \cdot \omega|M(v_*)dv_* d\omega$$

with

$$\begin{cases} v' = v - (v - v_*) \cdot \omega\omega \\ v_*' = v_* + (v - v_*) \cdot \omega\omega \end{cases}$$

where ω runs through the unit sphere $\mathbf{S}^2$. It is understood that the notation $\mathcal{L}_M\phi(t, x, v)$ designates $(\mathcal{L}_M\phi(t, x, \cdot))(v)$. Define

$$\Pi g^{in} = \langle g^{in}\rangle + \langle vg^{in}\rangle \cdot v + \langle(\tfrac{1}{3}|v|^2 - 1)g^{in}\rangle\tfrac{1}{2}(|v|^2 - 3)$$

with the notation

$$\langle\phi\rangle = \int_{\mathbf{R}^3} \phi(v)M(v)dv\,.$$

Theorem 3.1. [11] *There exist $\gamma_* > 0$, and, for each $\gamma \in]0, \gamma_*[$, a positive constant C_γ such that, for each $g^{in} \in L^2(\mathbf{T}^3 \times \mathbf{R}^3; Mdvdx)$, the solution g of (8) satisfies, for each $t \geq 0$, the inequality*

$$\left\|g(t, \cdot, \cdot) - \int_{\mathbf{T}^3} \Pi g^{in}dx\right\|_{L^2(\mathbf{T}^3 \times \mathbf{R}^3; Mdvdx)} \leq C_\gamma e^{-\gamma t}\|g^{in}\|_{L^2(\mathbf{T}^3 \times \mathbf{R}^3; Mdvdx)}\,.$$

This result expresses that there is a spectral gap for the linearized Boltzmann (unbounded) operator

$$\mathbf{L}g(x,v) = v \cdot \nabla_x g(x,v) + \mathcal{L}_M g(x,v)$$

on $L^2(\mathbf{T}^3 \times \mathbf{R}^3; Mdvdx)$ with domain

$$D(\mathbf{L}) = \{\phi \in L^2(\mathbf{T}^3 \times \mathbf{R}^3; Mdvdx) \,|\, v \cdot \nabla_x \phi \text{ and } (1+|v|)\phi \in L^2(\mathbf{T}^3 \times \mathbf{R}^3; Mdvdx)\}.$$

In other words, the spectrum of $\mathbf{L}$ is included in a domain of the form

$$\mathrm{spec}(\mathbf{L}) \subset \{0\} \cup \{z \in \mathbf{C} \,|\, \mathrm{Re}(z) \geq \theta\}$$

for some $\theta > 0$ (the spectral gap). The eigenvalue 0 corresponds to the five-dimensional nullspace of $\mathbf{L}$, given by

$$\mathrm{Ker}\mathbf{L} = \{a + b \cdot v + c\tfrac{1}{2}(|v|^2 - 3) \,|\, a, c \in \mathbf{R} \text{ and } b \in \mathbf{R}^3\}.$$

One reason explaining the spectral gap as above is the fact that $\mathcal{L}_M$ is an (unbounded) Fredholm operator in the v variable, of the form

$$\mathcal{L}_M \phi(v) = \nu(|v|)\phi(v) - \mathcal{K}\phi$$

where $\nu(|v|)$ is a multiplication operator by the collision frequency satisfying the inequality $\nu_*(1 + |v|) \leq \nu(|v|) \leq \nu^*(1 + |v|)$ for some positive constants ν_* and ν^*.

Proposition 3 for $p = 2$ rules out the possibility of a similar spectral gap property for the problem (1).

Indeed, assume for simplicity that $\theta_w = 1$ in the problem (1). Then by the Cauchy-Schwarz inequality

$$|\Omega|^{1/2} \left\| \frac{F(t,\cdot,\cdot)}{M} - \frac{1}{|\Omega|} \iint_{\Omega \times \mathbf{R}^3} F^{in} dxdv \right\|_{L^2(\Omega \times \mathbf{R}^3; Mdvdx)}$$
$$\geq \left\| F(t,\cdot,\cdot) - \frac{M}{|\Omega|} \iint_{\Omega \times \mathbf{R}^3} F^{in} dxdv \right\|_{L^1(\Omega \times \mathbf{R}^3; dvdx)}$$

so that a spectral gap inequality of the form

$$\left\| \frac{F(t,\cdot,\cdot)}{M} - \frac{1}{|\Omega|} \iint_{\Omega \times \mathbf{R}^3} F^{in} dxdv \right\|_{L^2(\Omega \times \mathbf{R}^3; Mdvdx)} \tag{9}$$
$$\leq C_\gamma e^{-\gamma t} \left\| \frac{F^{in}}{M} \right\|_{L^2(\Omega \times \mathbf{R}^3; Mdvdx)}$$

would entail

$$\left\| F(t + \cdot, \cdot, \cdot) - \frac{M}{|\Omega|} \iint_{\Omega \times \mathbf{R}^3} F^{in} dxdv \right\|_{L^1([0,T] \times \Omega \times \mathbf{R}^3; dvdxds)}$$
$$\leq T|\Omega|^{1/2} C_\gamma e^{-\gamma t} \left\| \frac{F^{in}}{M} \right\|_{L^2(\Omega \times \mathbf{R}^3; Mdvdx)} \cdot$$

Now, for

$$F^{in}(x,v) = \frac{1}{\varepsilon^6} \mathbf{1}_{\varepsilon \mathbf{B}}(x) \mathbf{1}_{\varepsilon \mathbf{B}}(v),$$

one has, assuming that $0 < \varepsilon < 1$,

$$\left\| \frac{F^{in}}{M} \right\|_{L^2(\Omega \times \mathbf{R}^3; Mdvdx)} \leq ((2\pi)^3 e)^{1/4} \left\| F^{in} \right\|_{L^2(\Omega \times \mathbf{R}^3; dvdx)} \cdot$$

In other words, the existence of a spectral gap for the problem (1) in $\Omega \times \mathbf{R}^3$ — i.e. the inequality (9) would entail the inequality

$$\left\| F(t + \cdot, \cdot, \cdot) - \frac{\mathcal{M}_{(1,0,\theta_w)}}{|\Omega|} \iint_{\Omega \times \mathbf{R}^3} F^{in} dx dv \right\|_{L^1([0,T] \times \Omega \times \mathbf{R}^3; dv dx ds)}$$
$$\leq T |\Omega|^{1/2}((2\pi)^3 e)^{1/4} C_\gamma e^{-\gamma t} \left\| F^{in} \right\|_{L^2(\Omega \times \mathbf{R}^3; dv dx)},$$

which stands in contradiction with Proposition 3.

4. **Algebraic decay.** After the negative results obtained in the previous section, we now consider the problem of establishing decay estimates compatible with the results of section 3.

We shall do so on a special class of boundary value problems, satisfying the following assumptions:

(a) the spatial domain Ω is a ball of the Euclidian space $\mathbf{R}^3$;
(b) the distribution function of the collisionless gas at $t = 0$ is a radial function of both the space and velocity variables;
(c) the boundary condition is a diffuse reflection (complete accommodation), with a constant boundary temperature (no temperature gradient at the boundary of the vessel.)

None of these assumptions excludes the mechanism producing the negative results in section 3. Indeed, all these results are based on the choice of an initial distribution function that is concentrated, in the velocity variable, near $v = 0$, and in the space variable, near a point that is away from the boundary. This is obviously possible with the symmetries above, by choosing the initial data concentrated, in the space variable, near the center of the ball, and in the velocity variable, near $v = 0$ as in section 3. Particles so distributed at time $t = 0$ will not reach the boundary of the spatial domain until after very late, and, meanwhile, the exact nature of the boundary condition is immaterial for such particles.

Therefore, we set the spatial domain to be $\Omega = B(0,1)$, the unit ball of $\mathbf{R}^3$. The temperature on $\partial\Omega$ is set to be $\theta_w = 1$. Consider the initial boundary value problem

$$\begin{cases} (\partial_t + v \cdot \nabla_x) f = 0, & (t, x, v) \in \mathbf{R}_+ \times \Omega \times \mathbf{R}^3, \\ f(t, x, v) = \dfrac{\langle f \rangle_+}{\langle M \rangle_+} M(v), & v \cdot n_x < 0, \ x \in \partial\Omega, \\ f\big|_{t=0} = f^{in}(|x|, |v|), \end{cases} \tag{10}$$

with the notation

$$\langle \phi \rangle_+ = \int_{v \cdot n_x > 0} \phi(v) v \cdot n_x dv,$$

and we recall the notation $M = \mathcal{M}_{(1,0,1)}$ for the Maxwellian distribution with density 1, zero bulk velocity and temperature 1.

The main result in this section is the following.

Theorem 4.1. *Assume that $0 \leq f^{in}(x, v) \leq CM(v)$ a.e. in $(x, v) \in \Omega \times \mathbf{R}^3$. Then the solution of (10) satisfies*

$$\iint_{\Omega \times \mathbf{R}^3} \left| \frac{f(t, x, v)}{M(v)} - \frac{1}{|\Omega|} \iint_{\Omega \times \mathbf{R}^3} f^{in}(|x|, |v|) dx dv \right|^p M(v) dv dx = O\left(\frac{1}{t^3}\right) + O\left(\frac{1}{t^p}\right)$$

so that

$$\left\| \frac{f(t,\cdot,\cdot)}{M} - \frac{1}{|\Omega|} \iint_{\Omega \times \mathbf{R}^3} f^{in}(|x|,|v|)dxdv \right\|_{L^p(\Omega \times \mathbf{R}^3; Mdvdx)} = O\left(\frac{1}{t^{\min(1,3/p)}} \right)$$

for all $1 \le p < \infty$.

The reader might want to compare the upper bound in this theorem with the lower bound in Proposition 3: this is done by taking $p = \infty$ in the estimate (7) and in Proposition 3, and $p = 1$ in the theorem above. The difference in the exponents observed suggests that one of these results at least is not sharp. The numerical investigations in [10] suggests that it is the lower estimate in Proposition 3 that is sharp, while the upper bound in the theorem above is not.

Our main task in the proof of this theorem is to use the symmetries in the problem (10) to reduce it to a renewal integral equation for a scalar unknown quantity, and to use classical results for such equations.

The role of symmetries in this problem is summarized in the following statement.

Lemma 4.2. *If f is the solution of the initial boundary value problem (10), its outgoing flux at the boundary*

$$\mu := \int_{v \cdot n_x > 0} f(t, x, v) v \cdot n_x dv$$

is independent of the position variable x as x runs through $\partial \Omega$, and a function of $t \ge 0$ only.

Proof. If $R \in O_3(\mathbf{R})$ is any orthogonal matrix, the function $f_R(t, x, v) := f(t, Rx, Rv)$ obviously satisfies $f_R\big|_{t=0} = f\big|_{t=0}$, and $\langle f_R \rangle_+(t, x) = \langle f \rangle_+(t, Rx)$, while

$$(\partial_t + v \cdot \nabla_x) f_R(t, x, v) = \partial_t f(t, Rx, Rv) + v \cdot (R^T \nabla_x f)(t, Rx, Rv)$$
$$= \partial_t f(t, Rx, Rv) + Rv \cdot \nabla_x f(t, Rx, Rv) = 0 \, .$$

By uniqueness of the solution of (10), we conclude that $f_R = f$ for each $R \in O_3(\mathbf{R})$. Thus

$$\langle f \rangle_+(t, x) = \langle f_R \rangle_+(t, x) = \langle f \rangle_+(t, Rx)$$

for each $(t, x) \in \mathbf{R}_+ \times \partial \Omega$ and $R \in O_3(\mathbf{R})$: hence $\mu := \langle f \rangle_+$ is independent of x as x runs through $\partial \Omega$. $\square$

Proof of Theorem 4.1. With Lemma 4.2, we easily arrive at an integral equation satisfied by μ. Pick $x \in \partial \Omega$, and set $\theta = \widehat{(n_x, v)}$; the backward exit time

$$\tau_{x,v} := \inf\{t > 0 \,|\, x - tv \notin \overline{\Omega}\}$$

is given by the explicit formula

$$\tau_{x,v} = \frac{2 \cos \theta}{|v|} \, ,$$

as can be seen on Figure 1.

By the method of characteristics, for each $x \in \partial \Omega$ and $v \in \mathbf{R}^3$ such that $v \cdot n_x > 0$, one has

$$f(t, x, v) = \mathbf{1}_{t > \tau_{x,v}} \frac{M(v)}{\langle M \rangle_+} \mu(t - \tau_{x,v}) + \mathbf{1}_{t < \tau_{x,v}} f^{in}(|x - tv|, |v|) \, .$$
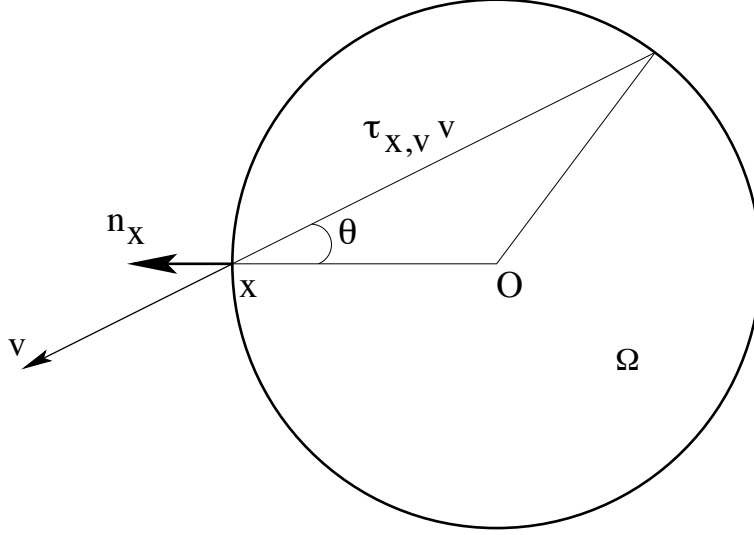
FIGURE 1. The exit time for a spherical domain

Hence

$$\mu(t) = \int_{v \cdot n_x > 0} f(t, x, v) v \cdot n_x dv$$

$$= \frac{1}{\langle M \rangle_+} \int_{v \cdot n_x > 0} \mu(t - \tau_{x,v}) \mathbf{1}_{t > \tau_{x,v}} M(v) v \cdot n_x dv$$

$$+ \int_{v \cdot n_x > 0} f^{in}(|x - tv|, |v|) \mathbf{1}_{t < \tau_{x,v}} v \cdot n_x dv$$

Introducing spherical coordinates in v, with n_x being the polar direction and θ the colatitude, we see on Figure 1 that, for each $x \in \partial \Omega$, one has $\tau_{x,v} |v| = 2 \cos \theta$, so that

$$\int_{v \cdot n_x > 0} \mu(t - \tau_{x,v}) \mathbf{1}_{t > \tau_{x,v}} M(v) v \cdot n_x dv$$

$$= 2\pi \int_0^\infty \int_0^{\pi/2} \mu \left(t - \frac{2 \cos \theta}{|v|} \right) \mathbf{1}_{t|v| > 2 \cos \theta} M(|v|) |v| \cos \theta |v|^2 \sin \theta d\theta d|v| \,,$$

abusing the notation $M(|v|)$ to designate the radial function $v \mapsto M(v)$.

Substituting $y = \cos \theta$, this integral is expressed as

$$\int_0^\infty \int_0^1 \mu \left(t - \frac{2y}{|v|} \right) \mathbf{1}_{t|v| > 2y} M(|v|) |v|^3 y dy d|v|$$

$$= \tfrac{1}{4} \int_0^\infty \int_0^{2/|v|} \mu(t - \tau) \mathbf{1}_{t > \tau} M(|v|) |v|^5 \tau d\tau d|v|$$

$$= \tfrac{1}{4} \int_0^\infty \int_0^t \mu(t - \tau) \mathbf{1}_{\tau < 2/|v|} M(|v|) |v|^5 \tau d\tau d|v|$$

$$= \tfrac{1}{4} \int_0^t \mu(t - \tau) \left(\int_0^{2/\tau} M(|v|) |v|^5 d|v| \right) \tau d\tau \,.$$

We henceforth define

$$K(\tau) := \frac{\pi\tau}{2\langle M\rangle_+} \int_0^{2/\tau} M(|v|)|v|^5 d|v| \,.$$

The function K is continuous on $\mathbf{R}_+^*$, satisfies $K(\tau) \geq 0$ for each $\tau \geq 0$, and

$$K(\tau) \to 0 \text{ as } \tau \to 0^+ \,, \quad K(\tau) \sim \frac{8}{3\tau^5} \text{ as } \tau \to +\infty \,.$$

Therefore μ satisfies

$$\mu(t) = \int_0^t K(\tau)\mu(t - \tau)d\tau + S(t) \tag{11}$$

where the source term is

$$S(t) := \int_{v\cdot n_x > 0} f^{in}(|x - tv|, |v|)\mathbf{1}_{t < \tau_{x,v}} v \cdot n_x dv \,.$$

This is a renewal equation, to which we shall apply the results of [8].

Next we consider the source term. Define

$$\Phi(|v|) := \operatorname{supess}_{x\in\Omega} f^{in}(|x|, |v|) \,;$$

assuming that $0 \leq f^{in} \in L^\infty(\Omega \times \mathbf{R}^3)$ implies that $\Phi \in L^\infty(\mathbf{R}_+)$. Then,

$$\begin{aligned}
S(t) &\leq \int_{v\cdot n_x > 0} \Phi(|v|)\mathbf{1}_{t < \tau_{x,v}} v \cdot n_x dv \\
&= 2\pi \int_0^\infty \Phi(|v|)|v|^3 \left(\int_0^{\pi/2} \mathbf{1}_{t|v| < 2\cos\theta} \cos\theta \sin\theta d\theta \right) d|v| \\
&= 2\pi \int_0^\infty \Phi(|v|)|v|^3 \left(\int_0^1 \mathbf{1}_{t|v| < 2y} y dy \right) d|v| \\
&= \pi \int_0^\infty \Phi(|v|)|v|^3 (1 - \tfrac{1}{4}t^2|v|^2)_+ d|v| \\
&\leq \pi\|\Phi\|_{L^\infty} \int_0^{2/t} |v|^3 d|v| = \frac{4\pi\|\Phi\|_{L^\infty}}{t^4} \,.
\end{aligned}$$

Apply Theorem 4 of [8]: with $n = 3$, one has

$$\int_0^\infty t^m K(t)dt < +\infty \quad \text{for } m = 0, 1, 2, 3$$

while

$$S(t) = o\left(\frac{1}{t}\right) \,, \quad \int_t^\infty S(\tau)d\tau = o\left(\frac{1}{t}\right)$$

as $t \to +\infty$. Then

$$\mu(t) \to \mu_\infty = \frac{\displaystyle\int_0^\infty S(\tau)d\tau}{\displaystyle\int_0^\infty \tau K(\tau)d\tau}$$

as $t \to +\infty$, and

$$\mu(t) - \mu_\infty = o\left(\frac{1}{t}\right) \,.$$

Now let $g := f/M$, where f is the solution of (10); g satisfies

$$\begin{cases} (\partial_t + v \cdot \nabla_x)g = 0\,, & (t,x,v) \in \mathbf{R}_+ \times \Omega \times \mathbf{R}^3\,, \\ g(t,x,v) = \sqrt{2\pi}\mu(t)\,, & v \cdot n_x < 0\,,\ x \in \partial\Omega\,, \\ g\big|_{t=0} = f^{in}(|x|,|v|)/M(v)\,. \end{cases} \tag{12}$$

The function g is given by

$$g(t,x,v) = \sqrt{2\pi}\mu(t - \tau_{x,v})\mathbf{1}_{t>\tau_{x,v}} + \frac{f^{in}(|x|,|v|)}{M(v)}\mathbf{1}_{t<\tau_{x,v}}\,,$$

so that

$$\begin{aligned} g(t,x,v) - \sqrt{2\pi}\mu_\infty =& \sqrt{2\pi}\left(\mu(t-\tau_{x,v}) - \mu_\infty\right)\mathbf{1}_{t>\tau_{x,v}} \\ &+ \left(\frac{f^{in}(|x|,|v|)}{M(v)} - \sqrt{2\pi}\mu_\infty\right)\mathbf{1}_{t<\tau_{x,v}} = I + II\,. \end{aligned}$$

Now $\tau_{x,v} \le 2/|v|$, so that

$$\int_{\mathbf{R}^3} \mathbf{1}_{t<\tau_{x,v}}M(v)dv \le \int_{|v|\le 2/t} M(v)dv \sim \frac{2^3|\mathbf{B}^3|}{(2\pi)^{3/2}}\frac{1}{t^3}\,.$$

Hence

$$\int_{\mathbf{R}^3} |II|^p M(v)dv = O\left(\frac{1}{t^3}\right)\,. \tag{13}$$

On the other hand

$$\begin{aligned} \int_{\mathbf{R}^3} |I|^p M(v)dv \le& \int_{\mathbf{R}^3} \frac{C\mathbf{1}_{t>\tau_{x,v}}}{(t - \tau_{x,v} + 1)^p} M(v)dv \\ \le& \int_{|v|<\varepsilon} \frac{C}{((t - \frac{2}{|v|})_+ + 1)^p} M(v)dv \\ &+ \int_{|v|\ge\varepsilon} \frac{C}{((t - \frac{2}{|v|})_+ + 1)^p} M(v)dv \\ \le& C\int_{|v|<\varepsilon} M(v)dv + \frac{C}{((t - \frac{2}{\varepsilon})_+ + 1)^p}\int_{|v|\ge\varepsilon} M(v)dv \\ \le& C'\varepsilon^3 + \frac{C'}{(\frac{t}{2} + 1)^p} \end{aligned}$$

for some other constant $C' > 0$, provided that $t \ge 4/\varepsilon$. Choosing $\varepsilon = 4/t < 1$, we arrive at

$$\int_{\mathbf{R}^3} |I|^p M(v)dv \le \frac{4^3 C'}{t^3} + \frac{C'}{t^p}\,. \tag{14}$$

Putting together (13)-(14) entails the first estimate in the theorem. The second follows from the first and interpolation with the maximum principle for the solution g of (12).

Finally, the fact that

$$\sqrt{2\pi}\mu_\infty = \frac{1}{|\Omega|}\iint_{\Omega\times\mathbf{R}^3} f^{in}(|x|,|v|)dxdv$$

follows from the conservation of mass for the solution of (10). $\qquad\square$

Notice that the decay requirements on the renewal kernel in Feller's Theorem 4 [8] forbid extending our proof to space dimensions less than 3. In the one-dimensional case, Yu obtained in [15] an algebraic rate of convergence for the outgoing flux (specifically, of order $t^{-1/10}$), by probabilistic arguments significantly more involved than the above proof.

5. **The monokinetic case: exponential decay.** The discussion in section 3 shows the existence of too many slow particles is the reason for the slow return to equilibrium in the case of a collisionless gas in a container with constant wall temperature. In the present section, we confirm this effect by proving that monokinetic populations of particles in the same conditions converge to equilibrium exponentially fast.

Radiative transfer provides a natural example of a monokinetic transport process. Consider the following (academic) problem: at time 0, a source emits radiation in an empty container surrounded by infinitely opaque material. Photons propagate at the speed of light, and are instantaneously thermalized on the wall. Denoting $I \equiv I(t, x, \omega, \nu)$ the radiative intensity at time t, position x, in the direction ω and with frequency ν, one arrives at the following initial-boundary value problem:

$$\begin{cases} \frac{1}{c}\partial_t I + \omega \cdot \nabla_x I = 0, & x \in \Omega, \ |\omega| = 1, \ \nu > 0, \\[2mm] I(t, x, \omega, \nu) = B_\nu(\theta_I(t, x)), & x \in \partial\Omega, \ \omega \cdot n_x < 0, \ \nu > 0, \\[2mm] I\big|_{t=0} = I^{in}. \end{cases} \quad (15)$$

Here $c > 0$ is the speed of light in vacuum, Ω is a bounded open set in $\mathbf{R}^3$ with C^1 boundary, n_x is the outward unit normal at $x \in \partial\Omega$. The notation $B_\nu(\theta)$ designates Planck's function for the radiative intensity emitted by a black body at temperature θ, i.e.

$$B_\nu(\theta) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/k\theta} - 1},$$

where h and k are respectively the Planck and the Boltzmann constants. The temperature θ_I is defined by

$$\iint_{\mathbf{S}^2 \times \mathbf{R}_+} I(t, x, \omega, \nu)(\omega \cdot n_x)_+ d\omega d\nu$$

$$= \iint_{\mathbf{S}^2 \times \mathbf{R}_+} B_\nu(\theta_I(t, x))(\omega \cdot n_x)_+ d\omega d\nu =: \sigma\theta_I(t, x)^4$$

where σ is the Stefan-Boltzmann constant

$$\sigma := \frac{2\pi k^4}{c^2 h^3} \int_0^\infty \frac{x^3 dx}{e^x - 1} = \frac{2\pi^5 k^4}{15 c^2 h^3}.$$

By definition of θ_I, the net flux of energy at each point of the wall is

$$\iint_{\mathbf{S}^2 \times \mathbf{R}_+} I(t, x, \omega, \nu)\omega \cdot n_x d\omega d\nu = 0,$$

so that the total energy in the container is conserved: for each $t \geq 0$, one has

$$\iiint_{\Omega \times \mathbf{S}^2 \times \mathbf{R}_+} I(t, x, \omega, \nu) dx d\omega d\nu = \iiint_{\Omega \times \mathbf{S}^2 \times \mathbf{R}_+} I^{in}(x, \omega, \nu) dx d\omega d\nu.$$

Therefore, one expects that, as $t \to +\infty$, the radiation inside the container thermalizes so that

$$I(t, x, \omega, \nu) \to B_\nu(\theta_\infty) \quad \text{as } t \to +\infty \,,$$

with

$$4\sigma|\Omega|\theta_\infty^4 := 4\pi|\Omega| \int_0^\infty B_\nu(\theta_\infty)d\nu = \iiint_{\Omega \times \mathbf{S}^2 \times \mathbf{R}_+} I^{in}(x, \omega, \nu)dxd\omega d\nu \,.$$

In this section, we seek the rate of convergence to equilibrium for the problem above.

Theorem 5.1. *Assume that I^{in} is smooth and satisfies* $\operatorname{supp}(I^{in}) \subset \Omega \times \mathbf{R}_+$ *together with the bound $0 \le I^{in}(x, \nu) \le B_\nu(\Theta)$ with $\Theta > 0$, for all $x \in \Omega$ and $\nu \ge 0$. Then, the solution I of the initial boundary value problem* (15) *satisfies*

$$I(t, x, \omega, \nu) - B_\nu(\theta_\infty) = O(e^{-\alpha' t})$$

as $t \to +\infty$, uniformly on $\Omega \times \mathbf{S}^1 \times \mathbf{R}_+$, for each $\alpha' < \alpha$, where $\theta_\infty \ge 0$ is given by

$$\theta_\infty^4 := \frac{\pi}{\sigma|\Omega|} \iint_{\Omega \times \mathbf{S}^2} I^{in}(|x|^2, \nu)dxd\nu$$

while α is

$$\alpha = \inf \left\{ -\Re(z) \,|\, z \in \mathbf{C}^* \text{ such that } \tfrac{1}{2} \int_0^2 e^{-sz} sds = 1 \right\} \,.$$

Proof. The problem is simplified by considering the radiative intensity integrated over frequencies, the grey intensity u defined as

$$u(t, x, \omega) := \int_0^\infty I(t, x, \omega, \nu)d\nu \,.$$

The formula defining θ_I reads

$$\int_{\mathbf{S}^2} u(t, x, \omega)(\omega \cdot n_x)_+ d\omega = \sigma\theta_I(t, x)^4$$

while

$$\int_0^\infty B_\nu(\theta_I)d\nu = \frac{\sigma}{\pi}\theta_I^4 \,.$$

Therefore, choosing for simplicity units of time and space so that $c = 1$, the grey intensity u satisfies

$$\begin{cases} \partial_t u + \omega \cdot \nabla_x u = 0 \,, & x \in \Omega, \ |\omega| = 1 \,, \\[2mm] u(t, x, \omega) = \frac{1}{\pi} \int_{\mathbf{S}^2} u(t, x, \omega')(\omega' \cdot n_x)_+ d\omega' & x \in \partial\Omega, \ \omega \cdot n_x < 0 \,, \\[2mm] u\big|_{t=0} = u^{in} := \int_0^\infty I^{in} d\nu \,. \end{cases} \tag{16}$$

Henceforth, we assume the same geometric setting as in section 4: $\Omega = B(0, 1) \subset \mathbf{R}^3$, and I^{in} is radial in x and isotropic, meaning that $I^{in} \equiv I^{in}(|x|^2, \nu)$. In that case, reasoning as in Lemma 4.2 shows that

$$f(t) := \frac{1}{\pi} \int_{\mathbf{S}^2} u(t, x, \omega')(\omega' \cdot n_x)_+ d\omega'$$

is independent of the position $x \in \partial\Omega$.

Proceeding as in section 4, we derive a renewal integral equation for f. Indeed, integrating the transport equation in (16) by the method of characteristics, we see that

$$u(t,x,\omega) = \mathbf{1}_{t \le \tau_{x,\omega}} u^{in}(|x - t\omega|^2) + \mathbf{1}_{t > \tau_{x,\omega}} f(t - \tau_{x,\omega}) \tag{17}$$

whenever $x \in \partial\Omega$ with $\omega \cdot n_x > 0$, so that

$$f(t) = \tfrac{1}{\pi} \int_{\mathbf{S}^2} u(t,x,\omega)(\omega \cdot n_x)_+ d\omega$$

$$= \tfrac{1}{\pi} \int_{\mathbf{S}^2} \mathbf{1}_{t \le \tau_{x,\omega}} u^{in}(|x - t\omega|^2)(\omega \cdot n_x)_+ d\omega$$

$$+ \tfrac{1}{\pi} \int_{\mathbf{S}^2} \mathbf{1}_{t > \tau_{x,\omega}} f(t - \tau_{x,\omega})(\omega \cdot n_x)_+ d\omega$$

$$= S(t) + \int_0^t K(s) f(t - s) ds.$$

Here

$$\int_0^t K(s) f(t - s) ds = \tfrac{1}{\pi} \int_{\mathbf{S}^2} \mathbf{1}_{t > \tau_{x,\omega}} f(t - \tau_{x,\omega})(\omega \cdot n_x)_+ d\omega$$

$$= \tfrac{1}{\pi} \int_0^{\pi/2} \mathbf{1}_{t > 2\cos\theta} f(t - 2\cos\theta)\cos\theta \cdot 2\pi \sin\theta d\theta$$

$$= \int_0^2 \mathbf{1}_{t > s} f(t - s) \tfrac{1}{2} s ds$$

so that

$$K(s) = \tfrac{1}{2} s \mathbf{1}_{s < 2},$$

while

$$S(t) := \tfrac{1}{\pi} \int_{\mathbf{S}^2} \mathbf{1}_{t \le \tau_{x,\omega}} u^{in}(|x - t\omega|^2)(\omega \cdot n_x)_+ d\omega$$

$$= \tfrac{1}{\pi} \int_{\mathbf{S}^2} \mathbf{1}_{t \le \tau_{x,\omega}} u^{in}(1 + t^2 - 2tx \cdot \omega)(\omega \cdot n_x)_+ d\omega$$

$$= \tfrac{1}{\pi} \int_0^{\pi/2} \mathbf{1}_{t \le 2\cos\theta} u^{in}(1 + t^2 - 2t\cos\theta)\cos\theta \cdot 2\pi \sin\theta d\theta$$

$$= \tfrac{1}{2} \int_0^2 \mathbf{1}_{t \le s} u^{in}(1 + t^2 - ts) s ds$$

$$= \int_0^{+\infty} u^{in}(1 - t\tau) K(t + \tau) d\tau.$$

Since $u^{in} \in L^\infty(\Omega)$, and K is bounded with support in $[0, 2]$, so is S and one has

$$\int_0^\infty t^m K(t) dt < +\infty \quad \text{and} \quad S(t) = O(t^{-m}) \text{ as } t \to +\infty \text{ for each } m \ge 1.$$

Applying Theorem 4 of [8] already implies that

$$f(t) \to f_\infty := \frac{\displaystyle\int_0^\infty S(t) dt}{\displaystyle\int_0^\infty t K(t) dt} \quad \text{as } t \to +\infty,$$

and that $f(t) - f_\infty$ is rapidly decaying as $t \to +\infty$:

$$|f(t) - f_\infty| = O(t^{-m}) \quad \text{as } t \to +\infty \text{ for each } m \ge 1.$$

We improve this estimate and obtain exponential convergence, as follows. Given $\phi \in L^2(\mathbf{R}_+)$, we denote its Laplace transform by

$$\tilde{\phi}(z) := \int_0^\infty e^{-zt}\phi(t)dt \,.$$

We recall that the Laplace transform $\phi \mapsto \tilde{\phi}$ is an isomorphism from $L^2(\mathbf{R}_+)$ to the Hardy space $\mathcal{H}^2(\Pi_0)$, where $\Pi_a := \{z \in \mathbf{C} \,|\, \Re(z) > a\}$. We recall that an element of $\mathcal{H}^2(\Pi_a)$ is a holomorphic function $\psi \equiv \psi(z)$ on the half-plane Π_a that satisfies

$$\sup_{x>a} \int_{\mathbf{R}} |\psi(x+iy)|^2 dy < +\infty \,.$$

Applying the Laplace transform to both sides of the renewal equation

$$f(t) = S(t) + \int_0^t K(s)f(t-s)ds \,, \qquad t \ge 0 \tag{18}$$

satisfied by f, we see that

$$\tilde{f}(z)(1 - \tilde{K}(z)) = \tilde{S}(z) \,, \quad \Re(z) > 0 \,. \tag{19}$$

Since $K \in L^\infty(\mathbf{R})$ with support in $[0,2]$, the Laplace transform of K is entire holomorphic; it is given by

$$\tilde{K}(z) = \frac{1 - e^{-2z}(1+2z)}{2z^2} \,, \quad z \ne 0 \,, \qquad \tilde{K}(0) = 1 \,.$$

Since $\tilde{K}'(0) = -\frac{4}{3}$, the only zero of $1 - \tilde{K}$ near $z = 0$ is 0. Besides

$$|\tilde{K}(z)| = \int_0^\infty e^{-t\Re(z)} K(t)dt < 1 \qquad \text{whenever } \Re(z) > 0 \,,$$

since K is a probability distribution on $\mathbf{R}_+$. Furthermore

$$\Re(\tilde{K}(iy)) = \int_0^\infty \cos(ty)K(t)dt < \int_0^\infty K(t)dt = 1$$

so that $z = 0$ is the only zero of $1 - K$ on the imaginary axis. Finally, for each $a \in \mathbf{R}$, one has

$$|\tilde{K}(z)| \le \frac{1}{2|z|^2}(1 + (1+2|z|)e^{-2\Re(z)}) \,. \tag{20}$$

In particular $|\tilde{K}(z)| \to 0$ as $\Im(z) \to +\infty$ uniformly in $\Re(z) \ge a$. Therefore, there exists $\alpha > 0$ such that

$$1 - \tilde{K}(z) = 0 \text{ and } \Re(z) > -\alpha \quad \Leftrightarrow \quad z = 0 \,.$$

Hence $z \mapsto \frac{z}{1-\tilde{K}(z)}$ is holomorphic on the half-plane $\Pi_{-\alpha}$, and one recasts (19) as

$$z\tilde{f}(z) = \frac{z\tilde{S}(z)}{1 - \tilde{K}(z)} \,, \quad \Re(z) > -\alpha \,.$$

Indeed, since $S \in L^\infty(\mathbf{R})$ with support in $[0,2]$, its Laplace transform is an entire holomorphic function.

Moreover, assuming that $u^{in} \in C_c^1((0,1))$, we deduce from the formula

$$S(t) = \int_t^\infty u^{in}(1 + t^2 - ts)K(s)ds$$

that $S \in C^1(\mathbf{R}_+)$, with

$$S'(t) = \int_t^\infty (u^{in})'(1 + t^2 - ts)(2t - s)K(s)ds$$

so that, for all $t \geq 0$, one has

$$|S'(t)| \leq 2t\|(u^{in})'\|_{L^\infty} \int_t^\infty K(s)ds \leq 4\|(u^{in})'\|_{L^\infty}.$$

Since $S(0) = u^{in}(1) = 0$, one has $z\tilde{S}(z) = \widetilde{S'}(z)$; and since S' is continuous with compact support on $\mathbf{R}$, its Laplace transform $\widetilde{S'} \in \mathcal{H}^2(\Pi_a)$ for each $a \in \mathbf{R}$.

In view of (20), for each $\alpha' < \alpha$, the function $z \mapsto (1 - \tilde{K}(z))^{-1}$ is bounded on $\Pi_{-\alpha'} \setminus B(0, \alpha'/2)$, so that the function

$$z \mapsto \frac{z\tilde{S}(z)}{1 - \tilde{K}(z)} \quad \text{belongs to } \mathcal{H}^2(\Pi_{-\alpha'})$$

for each $\alpha' < \alpha$.

Since $\widetilde{e^{at}\phi} = \tilde{\phi}(z - a)$, this implies the existence of a function g such that $e^{\alpha't}g \in L^2(\mathbf{R}_+)$ for each $\alpha' < \alpha$ and

$$\tilde{g}(z) = \frac{z\tilde{S}(z)}{1 - \tilde{K}(z)}, \quad \text{whenever } \Re(z) > -\alpha'.$$

Consider the function G defined by

$$G(t) = \int_0^t g(s)ds, \quad t \geq 0.$$

Then G is continuous on $\mathbf{R}$ with support in $\mathbf{R}_+$, satisfies

$$0 \leq |G(t)| \leq \frac{\|e^{\alpha't}g\|_{L^2}}{\sqrt{2\alpha'}}, \quad t \geq 0,$$

so that $\tilde{G}$ belongs to $\mathcal{H}^2(\Pi_\eta)$ for each $\eta > 0$.

Now, for each $z \in \Pi_\eta$

$$z\tilde{G}(z) = \int_0^\infty ze^{-zt}\int_0^t g(s)ds\,dt = \int_0^\infty g(s)\int_s^\infty ze^{-zt}dt\,ds$$

$$= \int_0^\infty g(s)e^{-zs}ds = \tilde{g}(z) = \frac{z\tilde{S}(z)}{1 - \tilde{K}(z)} = z\tilde{f}(z).$$

In other words, G and f are bounded continuous functions on $\mathbf{R}$ supported in $\mathbf{R}_+$, whose Laplace transforms coincide on Π_0. Therefore

$$f(t) = G(t) = \int_0^t g(s)ds, \quad \text{for each } t \geq 0,$$

and since $e^{\alpha't}g \in L^2(\mathbf{R}_+)$ for each $\alpha' < \alpha$, one conclude that

$$|f(t) - f_\infty| = \int_t^{+\infty} |g(s)|ds \leq \|e^{\alpha't}g\|_{L^2}\left(\int_t^{+\infty} e^{-\alpha's}ds\right)^{1/2}$$

$$= \frac{\|e^{\alpha't}g\|_{L^2}}{\sqrt{2\alpha'}}e^{-\alpha't} \leq C_{\alpha'}e^{-\alpha't}$$

for each $t > 0$.

Going back to the original problem (15) or (16), we deduce from the method of charateristics (17) that

$$u(t, x, \omega) = f(t), \quad \text{whenever } t > \text{diam}(\Omega),$$

or equivalently, in the frequency dependent case,

$$I(t, x, \omega, \nu) = B_\nu \left(\left(\tfrac{\pi}{\sigma} f(t) \right)^{1/4} \right), \quad \text{whenever } t > \frac{\text{diam}(\Omega)}{c},$$

since $\tau_{x,\omega} \leq \text{diam}(\Omega)$. □

Before concluding, it is worth discussing the role of the monokinetic assumption in the exponential decay estimate obtained here.

Intrinsically, the integral kernel $K(s)ds$ in the proof of Theorem 5.1 is the image under $\omega \mapsto \tau_{x,\omega}$ of the probability measure proportional to $(\omega \cdot n_x)_+ d\omega$ on $\mathbf{S}^2$. Since $0 \leq \tau_{x,\omega} \leq 2 = \text{diam}(\Omega)$, this kernel K is supported in $[0, 2]$. Therefore its Laplace transform is holomorphic entire, and since solving the renewal integral equation is equivalent to the possibility of inverting $1 - \tilde{K}$, the only constraint is the potential presence of zeros of $1 - \tilde{K}$; besides, since K has integral 1, the half-plane $\Re(z) \geq 0$ is zero-free, except for $z = 0$, and zeros of $1 - \tilde{K}$ cannot accumulate on the imaginary axis. Hence the solution of the renewal equation is obtained by inverting the Laplace transform, deforming the contour of integration to be $\Re(z) = -\alpha'$ for some $\alpha' > 0$.

In the case of the collisionless gas in Theorem 4.1, the integral kernel $K(s)ds$ is the image under $v \mapsto \tau_{x,v}$ of the probability measure proportional to $(v \cdot n_x)_+ M(v)dv$ on $\mathbf{R}^3$. Because of the low-speed gas molecules, its behavior near $s = 0$ is the same as that of the image measure of, say, $(v_1)_+ dv$ on the unit ball $B(0,1)$ under $v \mapsto \tau_{x,v}$ — the Maxwellian weight $M(v)$ obviously plays no role near $v = 0$. Thus, the decay of K for $s \to +\infty$ only depends on the Jacobian weight $|v|^{N-1}$ that appears when integrating radial functions with the Lebesgue measure in $\mathbf{R}^N$. This results in only algebraic decay for K in the limit $s \to +\infty$, so that the Laplace transform of K does not have a holomorphic extension in any half-plane of the form $\Re(z) > -\alpha$ with $\alpha > 0$. Therefore, the same deformation of contour as in the proof of Theorem 5.1 is impossible in that case. In order to be able to push the integration contour in the Laplace inversion formula to the left of the imaginary axis, K should decay exponentially fast as $s \to +\infty$. This would be the case if the Jacobian weight $|v|^{N-1}$ appearing in the integration of radial functions with the Lebesgue measure in space dimension N could be replaced with $e^{-a/|v|}$ for some $a > 0$. Thus, the difference between the speed of approach to equilibrium observed in a collisionless gas with diffuse reflection of gas molecules on vessel walls kept at constant temperature, and the spectral gap property for the linear Boltzmann equation on the other hand, can be measured in terms of the vanishing order of $|v|^{N-1}$ and $e^{-a/|v|}$ as $|v| \to 0$.

6. **Conclusion.** We have discussed in this paper the speed of approach to equilibrium for a collisionless gas enclosed in a vessel whose wall is kept at a uniform and constant temperature, assuming diffuse reflection on the vessel wall. We propose lower bounds for this convergence rate, depending on the possibility of initial distribution functions that are concentrated at low particle speeds. Not only do these lower bounds exclude exponential decay to equilibrium, they also constrain the power in any potential algebraic decay rate. On the other hand, assuming spherical symmetry of the initial velocity distribution function and of the vessel, we establish an algebraic decay rate by reducing the problem to a renewal equation for

the flux of outgoing gas particles. All these results are consistent with the asymptotic behavior obtained in [10] with numerical simulations. To clarify the role of low-speed particles, we have studied the case of radiative transfer in the vacuum, in a container with infinitely opaque boundary, so that the total incoming radiation flux on the boundary is reemitted as a Planck distribution. We establish exponential convergence of the radiative intensity to a Planck distribution in the long time limit, assuming the same spherical symmetries as in the case of a collisionless gas.

There remain several open problems in this direction, such as a) establishing decay rates without any symmetry assumptions, and b) obtaining the optimal decay rates, even in the spherically symmetric case. Both problems might involve mathematical techniques quite different from the ones used here. We hope to return to these questions in subsequent publications.

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