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Frequency Domain Studies on Sampled-Data Systems Using FR-Operators

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Abstract

The FR-operator is an operator that can completely describe the frequency domain nature of sampled-data systems. This thesis deals with three topics involving the use of the FR-operator for understanding the frequency-domain characteristics of sampled-data systems: (i) computation of the frequency response gain of sampled-data systems; (ii) sensitivity and complementary sensitivity reduction problems of sampled-data systems; and (iii) frequency response of sampled-data systems using a time-sharing multirate sample-hold scheme.

First, we review the notion of the FR-operator and provide a definition of the frequency response of sampled-data systems. We also introduce some useful properties of FR-operators. Based on these fundamental results, we provide an exact computation method for the frequency response gain under the assumption that $P_{11}(s) = 0$ in the generalized plant setting.

In approaching topic (i), we derive a bisection method for computing the frequency response gain of sampled-data systems that do not necessarily satisfy the condition $P_{11}(s) = 0$. We first give the notation used and provide some useful results for the infinite-dimensional congruent transformation of operators. Next, by using these results, we consider the decision of whether the frequency response gain at a given angular frequency is smaller than a given positive number γ. We show that this decision is reduced to the problem of counting the number of negative eigenvalues of a certain block-diagonal, self-adjoint infinite-dimensional matrix. Related issues are also presented, including a lifting-based algorithm for the bisection method, a one-dimensional search algorithm, relations to the several methods of solving the H^∞ problem of sampled-data systems, and possible applications of the congruent transformation to systems and control theory.

Moving on to topic (ii), we clarify several relationships between the SR/CSR problems of a given sampled-data system and those of its 'hold equivalent' discretized system (called the naively discretized system), where SR and CSR stand for sensitivity reduction and complementary sensitivity reduction. First, after summarizing the results for inner-outer factorization as well as the Nevanlinna problem and its solution, we consider the SR/CSR problems of naively discretized systems

and show several interesting results for the best achievable performances in these problems by using the coprime factorization approach. Next, we introduce the doubly sensitivity-preserving (DSP) discretized system and show that the SR/CSR problems of sampled-data systems can be equivalently reduced to those of the DSP discretized system. Based on this result, we show the relationships between the SR/CSR problems of sampled-data systems (which are equivalent to those of DSP discretized systems) and the corresponding problems of the naively discretized systems. A quantitative relationship between the best achievable performances of the SR and CSR problems of sampled-data systems is also clarified.

Finally, for topic (iii), we give the FR-operator-based representation for the frequency-domain characteristics of sampled-data systems using a time-sharing multirate sample-hold scheme. First, we introduce the time-sharing multirate sample-hold scheme together with our motivation for introducing such a control scheme. Next, using the notion of FR-operators, we provide a frequency domain representation of a sampled-data system using the time-sharing multirate sample-hold scheme and, moreover, derive some methods for computing the frequency response gain of the sampled-data system. Finally, we show that such a sample-hold scheme can be applied to solving the reliable stabilization problem, which is known to be a hard problem under the use of LTI controllers.

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Notations and Definitions

$\mathbf{R}$	Set of real numbers.
$\mathbf{R}^n$	Set of real vectors of size n.
$\mathbf{R}^{n \times m}$	Set of real matrices of size $n \times m$.
$L^2[0, \tau)$	Set of square Lebesgue-integrable functions on $[0, \tau)$.
l^2	Set of square summable sequences.
0	Zero matrix of appropriate size.
I_n	Identity matrix of size n. The subscript n is omitted when the size is not relevant or can be determined from the context.
$\underline{I}$	Identity on l^2.
A^T	Transpose of the matrix A.
A^*	Complex conjugate transpose of the matrix A.
A^{-1}	Inverse of the matrix A.
$\sigma_i(A)$	The ith largest singular value of the matrix A.
$\text{trace }(A)$	Trace of the matrix A.
$X^{\sim}(s)$	$X^T(-s)$.
$X^{\sim}(z)$	$X^T(z^{-1})$.
$\mathcal{Z}[X(s)]$	z-transform of $X(s)$.
$\mathbf{D}$	Open unit disc in the complex plane.
$\bar{\mathbf{D}}$	Closed unit disc in the complex plane.
$\mathbf{R}_{\_}$	Set of rational functions analytic on $\bar{\mathbf{D}}$, or set of matrices whose every entry belongs to $\mathbf{R}_{\_}$.
$\left[\begin{array}{c\|c} A & B \\ \hline C & D \end{array} \right]$	Transfer matrix $C(sI - A)^{-1} + D$ (continuous-time system), or transfer matrix $C(zI - A)^{-1} + D$ (discrete-time system).
$\mathcal{F}_l \left(\begin{bmatrix} P_{11}(\cdot) & P_{12}(\cdot) \\ P_{21}(\cdot) & P_{22}(\cdot) \end{bmatrix}, Q(\cdot) \right)$	The lower linear fractional transformation given by $P_{11}(\cdot) + P_{12}(\cdot)Q(\cdot)\big(I - P_{22}(\cdot)Q(\cdot)\big)^{-1}P_{21}(\cdot)$.

Contents

Chapter 1

Introduction

A control system using a digital controller is called a digital control system. In most cases, the plant to be controlled is a continuous-time system, whereas a digital controller is a discrete-time system. Therefore, a digital control system is a hybrid system that involves both continuous-time and discrete-time signals. In dealing with such a digital control system, the system obtained by the following procedure has often been introduced. The continuous-time outputs are replaced with discrete-time outputs produced by sampling the original continuous-time outputs; at the same time, the continuous-time inputs are replaced with signals produced by holding discrete-time inputs. If we observe the behavior of the system only at the sampling instants and do not take into account the behavior between the sampling instants (i.e., the "intersample behavior"), we can regard the system as a discrete-time system. Such a discrete-time system has been exploited for analysis and synthesis of digital control systems. However, such an approach to digital control systems causes serious problems, as explained below.

In Chen and Francis [16], the optimal tracking problem for step input was considered. It was shown that the above approach to a digital control system attains complete tracking at the sampling instants but sometimes causes considerable intersample ripple. Another example was given by Hara et al. [38], who showed that a robust stability margin against the plant uncertainty estimated by the above discrete-time system leads to a "too optimistic" result. These two problematic results can be attributed to an inappropriate perspective on digital control systems. In considering the optimal tracking problem, the original digital control system was handled only in an approximate manner by completely ignoring the intersample behavior of the digital control system. In estimating the robust stability margin against plant uncertainty, the above handling of the digital control system regards the uncertainty of a continuous-time plant as a cascade connection of a hold circuit, an uncertain discrete-time system, and a sampler due to the introduction of ficti-

tious hold and sampling devices. However, such an uncertainty does not reflect the actual uncertainty of the continuous-time plant. Therefore, in order to resolve the above problems, we have to deal with digital control systems as they are, that is, we should not use an approximative discrete-time system obtained by introducing fictitious sample and hold devices but we should take into account, directly and exactly, the intersample behavior of the original digital control system. When we deal with digital control systems from this standpoint, we refer to the digital control system as a sampled-data system. This thesis is devoted to studying sampled-data systems, particularly from the frequency-domain point of view.

The notion of the frequency response plays an important role in various areas of science and engineering such as physics, chemistry, signal processing, communications, and system control. In system and control engineering, the Bode plot obtained by plotting the frequency response gains against frequencies provides many insights to engineers for understanding the exact frequency-domain characteristics of a given system. Furthermore, the frequency response has close relationships to many control problems such as the sensitivity reduction problem, the robust stability problem, the filtering problem, and, more generally, H^2 and H^∞ problems [17]. Therefore, the frequency response is also utilized as a useful tool for formulating and solving practical problems in designing control systems.

For continuous-time systems and discrete-time systems, the notion of the frequency response is well-established, and it has been exploited for analysis and synthesis of control systems. On the other hand, for sampled-data systems, the notion of the frequency response taking into account the intersample behavior and influence of aliasing exactly was not well-established for a long time. The reason is as follows: In the classical theory of sampled-data control, e.g., Jury [54], it is well-known that the frequency-separation property is lost when the intersample behavior is taken into account. Specifically, a sampled-data system maps a single sinusoidal input $\exp(j\varphi t)$ to a continuous-time output consisting of many sinusoidal components $\exp(j(\varphi + m\omega_s)t); \ m = 0, \pm 1, \cdots$ in the steady-state, where $\omega_s := 2\pi/\tau$ is the sampling angular frequency under the given sampling period τ. Due to this "multi-frequency response" nature, it has been difficult to define a single function of φ that completely describes the frequency-domain characteristics of sampled-data systems. Another important reason making an exact understanding of sampled-data systems difficult is that they become periodically time-varying even if both the continuous-time plant and the discrete-time controller are time-invariant, and thus, we cannot define a transfer function as in the cases of continuous-time systems and discrete-time systems.

In addressing the problem of defining the frequency response of sampled-data

systems that completely describes the intersample behavior and the influence of aliasing, Yamamoto and Khargonekar [81] and Araki et al. [8] provided interesting methods based on the lifting approach and the FR-operator approach, respectively. In Yamamoto and Khargonekar [81], a function space model of a given sampled-data system via lifting is employed, that is, a continuous-time signal is regarded as a sequence of functions defined on $[0, \tau)$, and the sampled-data system is regarded as a mapping defined on the set of sequences of functions. The lifting technique enables us to view the sampled-data system as a time-invariant discrete-time system, and thus the notion of the transfer function $\mathcal{Q}(z)$, which is an operator defined on infinite-dimensional function space, can be introduced even for sampled-data systems as in the case of discrete-time systems. Furthermore, it was shown that $\mathcal{Q}(e^{j\varphi\tau})$ can be regarded as the frequency response operator of the sampled-data system and that the frequency response gain at angular frequency φ is defined as the norm of $\mathcal{Q}(e^{j\varphi\tau})$ induced on $L^2[0, \tau)$, i.e., $\|\mathcal{Q}(e^{j\varphi\tau})\|$. The lifting approach was first introduced by Yamamoto [79], and it has been applied to various problems of sampled-data systems, e.g., the tracking problem by Yamamoto [80], the H^∞ problem by Bamieh and Pearson [10], Toivonen [74] and Hayakawa et al. [40], and the H^2 problem by Bamieh and Pearson [9], to name a few.

An alternative approach for defining the frequency response gain of sampled-data systems was developed by Araki et al. [8]. To resolve the problem due to the "multi-frequency response" nature described above, Araki et al. [8] introduced the set $\mathcal{X}_\varphi$ of all signals that consist of sinusoidal components $\exp(j(\varphi + m\omega_s)t); m = 0, \pm 1, \cdots$ and that have finite power. They showed that a stable sampled-data system with a strictly proper pre-filter before the sampler can map $\mathcal{X}_\varphi$ into $\mathcal{X}_\varphi$ in the steady-state. This means that the frequency-separation property is recovered in a generalized sense, if all signals of $\mathcal{X}_\varphi$ are treated as a group. Based on this result, the sampled-data system is associated with an operator from $\mathcal{X}_\varphi$ to $\mathcal{X}_\varphi$. The operator is referred to as the FR-operator, and its matrix expression denoted by $\underline{Q}(j\varphi)$ is called the FR-matrix, where FR stands for Frequency Response. Since $\mathcal{X}_\varphi$ can be related with l^2 in an isometrically isomorphic fashion, $\underline{Q}(j\varphi)$ is regarded as a mapping on l^2. Accordingly, the frequency response gain at angular frequency φ is defined as the norm of $\underline{Q}(j\varphi)$ induced on l^2, i.e., $\|\underline{Q}(j\varphi)\|_{l^2/l^2}$. In Yamamoto and Araki [78], the relationships between the frequency response operator $\mathcal{Q}(e^{j\varphi\tau})$ and the FR-matrix $\underline{Q}(j\varphi)$ are clarified, and it is shown that the frequency response gain defined by $\mathcal{Q}(e^{j\varphi\tau})$ and that by $\underline{Q}(j\varphi)$ coincide with each other. The FR-operator was first introduced by Araki and Ito [7], and it has been applied to fundamental problems of sampled-data systems such as the H^2 problem by Hagiwara and Araki [31], a Nyquist-type stability condition by Hagiwara and Araki [33], and the absolute

stability by Hagiwara and Araki [30].

In addition to the above two approaches, there are some important approaches that can deal with sampled-data systems taking the intersample behavior into account. In Hara and Kabamba [37] and Kabamba and Hara [57], the hybrid state-space representation was introduced to describe sampled-data systems. The hybrid state involves the state of the continuous-time plant and that of the discrete-time controller, and thus it contains both the continuous-time signal and the discrete-time signal. The most important contribution of these works is that they were the first to provide a method for solving the H^∞ problem of sampled-data systems by showing that the H^∞ problem of sampled-data systems can be equivalently reduced to that of discrete-time systems. Sivashankar and Khargonekar [69] and Sun et al. [72] introduced a system whose state varies discontinuously at sampling instants. The system is referred to as a jump system, and it has been applied to the H^∞ problem of sampled-data systems. The jump system was also used to solve the H^2 problem of sampled-data systems by Ichikawa and Katayama [42]. An interesting feature of the jump-system-based approach is that it directly solves problems by using matrix Riccati equations obtained by the continuous-time plant data, which contrasts with other existing approaches in which the problems of sampled-data systems are reduced to those of equivalent discrete-time systems. More interestingly, Sun et al. [72] showed that this approach enables us to design not only a discrete-time controller but also a hold function.

All of the above approaches enable us to deal with sampled-data systems taking into account the intersample behavior and the influence of aliasing directly and exactly, and thus they have provided us with many fruitful results on analysis and synthesis of sampled-data systems. However, several important problems still remain unsolved for sampled-data systems. Among these problems, this thesis is devoted to investigating three issues:

  (i) Computation of the frequency response gain of sampled-data systems.

 (ii) Sensitivity and complementary sensitivity reduction problems of sampled-data systems.

(iii) Frequency response of sampled-data systems using a time-sharing multirate sample-hold scheme.

Since these three problems are closely related to the frequency-domain characteristics of sampled-data systems, it is appropriate to use a frequency-domain approach. In this thesis, we apply the FR-operator simply because some arguments presented here (particularly in Chapter 4) seem to belong to a class for which intuitive ideas

can work out relatively more easily within such an approach. Therefore, we have decided to unify the discussions in this thesis to use of the FR-operator approach, but this certainly does not imply that parallel arguments are difficult to develop for the lifting approach. Indeed, a lifting-based counterpart to the FR-operator-based bisection algorithm for computing the frequency response gain of sampled-data systems can be obtained in a similar manner as shown in Sec. 3.3.1 and Ito et al. [48].

In the following, we describe the above three issues in detail.

Computation of the frequency response gain of sampled-data systems.

As mentioned above, both the frequency response operator $\mathcal{Q}(e^{j\varphi\tau})$ given by Yamamoto and Khargonekar [81] and the FR-matrix $\underline{Q}(j\varphi)$ given by Araki et al. [8] are infinite-dimensional operators, and thus the computation of the frequency response gain of sampled-data systems is a nontrivial matter.

A general computation method for the frequency response gain of sampled-data systems was first given in Yamamoto [77] and in Yamamoto and Khargonekar [81]. This method computes the frequency response gain γ_φ at the angular frequency φ by searching for the maximum value γ such that a γ-dependent generalized eigenvalue problem has an eigenvalue $e^{j\varphi\tau}$. In spite of its theoretical significance, it is necessary to check whether $e^{j\varphi\tau}$ is exactly equal to an eigenvalue for each step of the search with respect to γ. However, there is no obvious way to carry out such a check in a numerically reliable fashion. A few other methods are able to compute the frequency response gain approximately (e.g., Araki et al. [8], Hara et al. [39], Yamamoto et al. [82], Yamamoto et al. [83]), but they require us to compute the norm of a huge matrix as we increase the degree of the approximation, and this is undesirable from the viewpoints of computational load and numerical reliability; to satisfy those requirements, a bisection algorithm for computing the frequency response gain of sampled-data systems would be highly desirable. However, the previous attempts to derive such an algorithm (e.g., Hagiwara et al. [29], Hara et al. [39]) have been only partially successful because their methods require certain conditions, which are often overly restrictive. Therefore, deriving a complete bisection algorithm without any restrictive conditions has been one of the most important open problems in modern sampled-data control theory. As one of the major contributions of this thesis, we propose for the first time a bisection algorithm for computing the frequency response gain of sampled-data systems without any restrictive conditions. This enables us to compute the frequency response gain of sampled-data systems with any desired accuracy, and thus we can completely grasp the frequency-domain characteristics of a given sampled-data system.

Essentially, deriving a bisection algorithm is reduced to deciding whether the frequency response gain at each frequency is smaller than a given positive number γ. In this thesis, we exploit useful properties of the infinite-dimensional congruent transformation (i.e., the Schur complement arguments and the Sylvester law of inertia) to reduce this binary decision to the task of counting up the number of negative eigenvalues of a certain block-diagonal self-adjoint operator. This operator consists of two γ-dependent blocks: a finite-dimensional matrix block and an infinite-dimensional matrix block. The former can be computed with the finite-dimensional state-space matrices of the sampled-data system by an exponentiation formula, and thus its negative eigenvalues are easy to count. On the other hand, regarding the infinite-dimensional matrix block, we can derive an exact and effective method for counting up the number of negative eigenvalues in the block. Consequently, we can readily obtain a bisection algorithm for the computation of the frequency response gain. We also provide possible applications of infinite-dimensional congruent transformation to various areas of system and control theory.

The above bisection method was first reported in [45]. Further studies on the bisection algorithm and the related issues were reported in [47]. These studies have been published in [48].

Sensitivity and complementary sensitivity reduction problems of sampled-data systems.

Our investigations into the sensitivity reduction (SR) and complementary sensitivity reduction (CSR) problems of sampled-data systems are motivated by the recent studies on the problem of what determines the best achievable performance of a control system. For example, the H^2 tracking performances of continuous-time systems, discrete-time systems, and sampled-data systems were studied in Chen and Toker [14], Okajima et al. [66], and Chen et al. [15], respectively, and these works derived elegant *analytic* solutions of the best achievable performance of control systems.

On the other hand, concerning the SR/CSR problems of sampled-data systems, analytic solutions of the best achievable performance have not yet been derived. Of course, we could obtain the best achievable performance of SR/CSR problems of sampled-data systems *numerically* by formulating the problem as the H^∞ problem of sampled-data systems and by applying a well-established solving method (e.g., Kabamba and Hara [57], Bamieh and Pearson [10], Hayakawa et al. [40], Mirkin and Tadmor [64]). However, such a numerical method would not provide us with a clear insight into what determines the best achievable performance of sampled-data systems.

For the SR/CSR problems of the discrete-time system obtained by the 'hold equivalent' discretization of the original sampled-data system, pioneering work was done by Sung and Hara [73], where several interesting results were derived, including integral constraints on the sensitivity and complementary sensitivity and lower bounds of the best achievable performance for the SR and CSR problems. The results of the integral constraint on the sensitivity and the lower bound of the best achievable performance for the SR problem were extended to multiple-input multiple-output cases by Hara and Sung [36]. Several results of these works were obtained by a "transfer matrix-based approach," in which the coprime factorization and problem formulation via the Nevanlinna problem, as well as its solution using the Pick matrix, play fundamental roles. For obtaining the analytic solutions of the SR/CSR problems of sampled-data systems, the transfer matrix-based approach would also be appropriate, but such an approach has not been developed because the infinite-dimensionality nature of sampled-data systems would make such a strategy extremely difficult.

In this thesis, in order to overcome the difficulty due to the infinite-dimensionality nature of sampled-data systems, we first show that, as far as the SR and CSR problems are concerned, we can have a discretization method to reduce them to equivalent discrete-time problems. This gives a discretized system that we call the doubly sensitivity-preserving (DSP) discretized system, which can be used for both the SR and CSR problems of sampled-data systems. For the derivation of the DSP discretized system, the FR-operator approach would be helpful. Through a comparison between the sensitivity transfer matrix of a discrete-time system and the sensitivity FR-operator of a sampled-data system, we can infer the condition that should be satisfied by a DSP discretized system. This is one of the advantages of the FR-operator approach.

On the other hand, we refer to the discretized system obtained by 'hold equivalent' discretization as the naively discretized system. In this thesis, we consider the SR/CSR problems of sampled-data systems by applying the transfer-matrix-based approach to the DSP discretized system and clarify important relationships between the DSP discretized system and the naively discretized system. The contribution of this part is not only clarifying the relationships between the SR/CSR problems of sampled-data systems and those of the naively discretized systems but also providing a fundamental basis for dealing with SR/CSR problems of sampled-data systems. The method presented in this part will be useful for further study on the best achievable performance of sampled-data systems.

The notion of DSP discretized systems and that of the aliasing factors were first introduced in [41]. The properties of aliasing factors and their relationships to the

best achievable performance were reported in [50] and [51]. The studies related to these topics have been published in [52] and [53].

Frequency response of sampled-data systems using time-sharing multirate sample-hold scheme.

It is well known that *non-standard* sample-hold schemes, such as generalized holds/samplers and multirate holds/samplers, make it possible for us to solve various problems that are difficult or impossible to solve by the *standard* sample-hold scheme using the zero-order hold and the ideal sampler. For example, the following problems have been shown to be easily solved under mild conditions by using non-standard sample-hold schemes: strong stabilization, simultaneous stabilization, exact model matching, exact linearization, and adaptive control (e.g., Araki [2], Araki and Hagiwara [3], Araki and Hagiwara [4], Araki et al. [5], Chammas and Leondes [13], Hagiwara and Araki [26], Hagiwara and Araki [27], Hagiwara et al. [32], Kabamba [55], Kabamba and Yang [56], Khargonekar et al. [58], Khargonekar and Poolla [59], Ortega and Kreisselmeyer [67]). However, the frequency-domain characteristics of sampled-data systems using such non-standard sample-hold schemes have not been fully studied due to the lack of a useful tool for completely describing the frequency domain nature of these systems.

One of the main contributions of this thesis is to provide a fundamental tool for studying frequency-domain characteristics of sampled-data systems using non-standard sample-hold schemes. In particular, by applying the FR-operator-based approach, we provide a frequency domain representation for the sampled-data system using a "time-sharing multirate sample-hold scheme." This scheme is a special kind of control scheme that uses a multirate sample-hold scheme, where "multirate" implies that the sampler and the hold operate with different periods and "time-sharing" implies that the manipulation of the plant input and the detection of the plant output work at separate time intervals. We also provide a bisection method for computing the frequency response gain of sampled-data systems using a time-sharing multirate sample-hold scheme. The derivation is based on the infinite-dimensional congruent transformation approach used in the study of issue (i). A computation method based on the norm-equivalent discrete-time system is also provided for a class of sampled-data systems.

Since the time-sharing multirate sample-hold scheme can be regarded as an extension of existing multirate control schemes, the technique used for deriving the frequency-domain representation of the sampled-data system using the time-sharing multirate sample-hold scheme can also be applied in deriving the frequency-domain representation of sampled-data systems using existing multirate control schemes.

Through pursuing our study of issue (iii) in this way, we provide a fundamental basis to consider the frequency-domain characteristics of sampled-data systems using non-standard sample-hold schemes.

As an application of the time-sharing multirate sample-hold scheme, we consider the reliable stabilization problem, in which we are to find a set of controllers that stabilize a given plant when they all act together, as well as when any one of them fails. It was shown in Vidyasagar and Viswanadham [76] and Minto and Ravi [62] that when linear time-invariant (LTI) controllers are employed, strong stabilizability (e.g., Vidyasagar [75]) of the plant is necessary for reliable stabilization under a two-controller configuration. Concerning the reliable stabilization using LTI controllers under a configuration using more than two controllers, neither solvability conditions nor design methods have been obtained so far. We show that the reliable stabilization problem under an N-controller configuration with $N \geq 2$ is solvable and that controllers achieving reliable stabilization can be designed provided that the time-sharing multirate sample-hold scheme is employed.

The notion of the time-sharing sample-hold scheme was first introduced in [43], and its application to reliable stabilization was reported in [44]. The studies related to this topic have been published in [46] and [49].

This thesis is organized as follows:

In Chapter 2, following the arguments in Araki et al. [8], we review the notion of the FR-operator, and provide a definition of the frequency response of sampled-data systems. We also introduce useful properties of FR-operators. Based on these fundamental results, we provide an exact computation method for the frequency response gain under the assumption that $P_{11}(s) = 0$ in the generalized plant setting. The technique used for the derivation of the method is fundamental to deriving related results in the later chapters.

In Chapter 3, we derive a bisection method for computing the frequency response gain of sampled-data systems that do not necessarily satisfy the condition $P_{11}(s) = 0$. We first give the notation used and provide some useful results for the infinite-dimensional congruent transformation of operators. Next, by using these results, we show that the decision of whether the frequency response gain at a given angular frequency is smaller than a given positive number γ is reduced to the problem of counting the number of negative eigenvalues of a certain block-diagonal self-adjoint infinite-dimensional matrix. Related issues are also presented, including a lifting-based algorithm for the bisection method, a one-dimensional search algorithm, the relationship to the several methods for solving the H^∞ problem of sampled-data systems, and possible applications of the congruent transformation to systems and control theory. Finally, we evaluate the proposed method through a comparison

with existing methods.

In Chapter 4, we consider the sensitivity reduction (SR) and complementary sensitivity reduction (CSR) problems of sampled-data systems by using the notion of FR-operators. First, after summarizing the results on inner-outer factorization and the Nevanlinna problem and its solution, we consider the SR/CSR problems of naively discretized systems and show several interesting results for the best achievable performance in these problems by using the coprime factorization approach. Next, we introduce the doubly sensitivity-preserving discretized system, abbreviated as DSP discretized system, and show that the SR/CSR problems of sampled-data systems can be equivalently reduced to those of the DSP discretized system. Based on this result, we show the relationship between the SR/CSR problems of sampled-data systems (equivalent to those of DSP discretized systems) and the corresponding problems of the naively discretized systems. The relationship between the best achievable performances of the SR and CSR problems of sampled-data systems is also clarified.

In Chapter 5, we give the FR-operator-based representation for the frequency-domain characteristics of sampled-data systems using the time-sharing multirate sample-hold scheme. First, we introduce the time-sharing multirate sample-hold scheme together with the motivation for introducing such a control scheme. Next, using the notion of FR-operators, we provide a frequency-domain representation of the sampled-data system using the time-sharing multirate sample-hold scheme and derive methods for computing the frequency response gain of such a system. Finally, we show that this sample-hold scheme can be applied to solving the reliable stabilization problem, where the result for decomposition of a matrix into the sum of stability matrices plays a key role in deriving the solvability condition of the reliable stabilization problem.

Chapter 6 offers our conclusions, where we summarize the results obtained in this thesis and discuss future topics.

Chapter 2

Frequency Response and FR-Operator of Sampled-data Systems

The purpose of this chapter is to provide several fundamental notions for developing frequency domain theory of sampled-data systems that enables us to deal with intersample behavior and the influence of aliasing directly and exactly. The notions introduced in this chapter play significant roles, and are frequently used throughout this dissertation. Among them, the frequency response of sampled-data system is an important notion as well as useful tool, since it has close relationships, e.g., to the H^2 and H^∞ problems of sampled-data systems, the robust stability problem against LTI perturbations, and the digital re-design of continuous-time controllers. Such a notion has been introduced with the lifting approach by Yamamoto and Khargonekar [81], and independently with an FR-operator approach by Araki et al. [8]. In both approaches, an operator that completely characterizes the frequency-domain nature of sampled-data systems is introduced, and hence, by using the operator, we can develop a frequency domain theory that completely reflects the intersample behavior and the influence of aliasing. In this thesis, we apply FR-operator approach.

Following the arguments in [8], this chapter introduces the notion of the FR-operator, and provides a definition of the frequency response of sampled-data systems. We also clarify the useful properties of FR-operators. Here, we only state these results without proofs; for the detailed proofs and derivations, see the earlier study [8]. This chapter also provides a method for computing the frequency response gain of a class of sampled-data systems proposed by Hagiwara et al. [29]. The technique used for the derivation of the method is fundamental for deriving other related results appearing in the later chapters, and thus we provide the detailed derivation for the computation method. Numerical examples are also studied.

2.1 Basic Concept behind Defining the Frequency Response of Sampled-data Systems

In this section, we state the basic idea for defining the frequency response gain of sampled-data systems. Before considering sampled-data systems, however, we briefly summarize the frequency response of continuous-time systems. It is well known that when we input the sinusoidal signal $\exp(j\varphi t)$ to a stable finite-dimensional linear time-invariant (FDLTI) continuous-time system with the transfer matrix $G(s)$, we obtain the sinusoidal output $G(j\varphi)\exp(j\varphi t)$ in the steady state. This implies that a stable FDLTI continuous-time system maps a sinusoidal input with frequency φ to a sinusoidal output with the *same* single frequency φ, and no other sinusoidal component appears in the output. This property of a sinusoidal component of the input affecting only the sinusoidal component of the output with the same frequency, while not affecting other sinusoidal components of the output, is referred to as the *frequency-separation property*. Based on this property, we can define the frequency response of a stable FDLTI continuous-time system by the way in which a single sinusoidal component of the input is transferred to a sinusoidal component of the output having the same frequency. The frequency-separation property is also satisfied by stable FDLTI discrete-time systems, and we can define the frequency response of such systems in a similar manner.

Now, let us begin our consideration of the frequency response of sampled-data systems. In the classical theory of sampled-data control [54], it is well known that the frequency-separation property is lost when the intersample behavior is taken into account, i.e., a sampled-data system maps a single sinusoidal input $\exp(j\varphi t)$ to a continuous-time output consisting of an infinite number of sinusoidal components $\exp(j\varphi t + jm\omega_s t)$ $(m = 0, \pm 1, \cdots)$ in the steady-state (Figure 2.1), where $\omega_s := 2\pi/\tau$
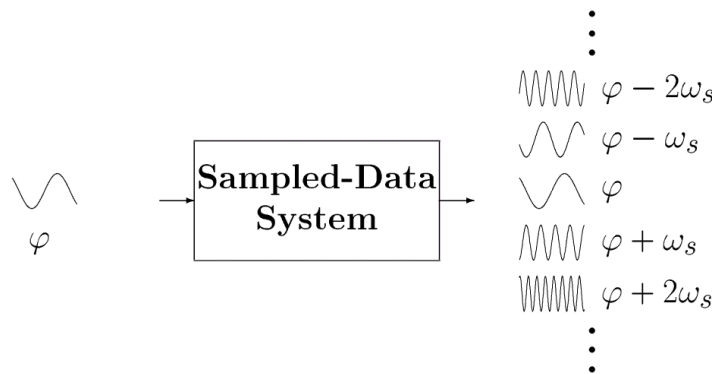
Figure 2.1: Output of a sampled-data system for single sinusoidal input.

is the sampling angular frequency and τ is the sampling period. Due to this "multi-frequency response" nature, it has been difficult to define a single function of φ that completely describes the frequency-domain characteristics of sampled-data systems.

To solve the above difficulty, we consider the set $\mathcal{X}_\varphi$ of all signals that consist of an infinite number of sinusoidal components $\exp(j\varphi_m t)$ $(m = 0, \pm 1, \cdots)$, where $\varphi_m = \varphi + m\omega_s$, and that have finite power, i.e.,

$$\mathcal{X}_\varphi := \left\{ \; x(t) \;\; | \;\; x(t) = \sum_{m=-\infty}^{\infty} x_m \exp(j\varphi_m t), \quad \sum_{m=-\infty}^{\infty} \|x_m\|^2 < \infty \right\}. \tag{2.1}$$

The range of φ is restricted to $\varphi \in \mathcal{I}_0 := (\,-\omega_s/2, \omega_s/2\,]$. It was previously shown [8] that a stable sampled-data system with a strictly proper pre-filter before the sampler maps $\mathcal{X}_\varphi$ into $\mathcal{X}_\varphi$ in the steady-state (Figure 2.2). This means that the frequency-separation property is recovered, in a generalized sense, if all of the signals in $\mathcal{X}_\varphi$ are treated as a group. Based on this result, we associate an operator $\mathcal{Q}(j\varphi)$ from $\mathcal{X}_\varphi$ to $\mathcal{X}_\varphi$ with the sampled-data system, and then the operator describes all of the characteristics of the sampled-data system in the frequency domain. We refer to this operator as an *FR-operator*, where FR stands for "frequency response".

The fact that a sampled-data system maps $\mathcal{X}_\varphi$ into $\mathcal{X}_\varphi$ implies that the infinite-dimensional vector $\underline{x} = \left[\cdots, x_{-1}^T, x_0^T, x_1^T, \cdots\right]^T \in l^2$, which consists of the coefficient vectors of the input signal $x(t) \in \mathcal{X}_\varphi$, is mapped to $\underline{y} = \left[\cdots, y_{-1}^T, y_0^T, y_1^T, \cdots\right]^T \in l^2$, which consists of the coefficient vectors of the output signal $y(t) \in \mathcal{X}_\varphi$. The mapping from $\underline{x} \in l^2$ to $\underline{y} \in l^2$ is expressed by using an infinite-dimensional matrix $\underline{Q}(j\varphi)$ as $\underline{y} = \underline{Q}(j\varphi)\underline{x}$. We refer to the infinite-dimensional matrix $\underline{Q}(j\varphi)$ as an *FR-matrix* of the system or the *matrix expression* of the FR-operator $\mathcal{Q}(j\varphi)$. Using the FR-matrix $\underline{Q}(j\varphi)$, the frequency response gain of the sampled-data system at angular
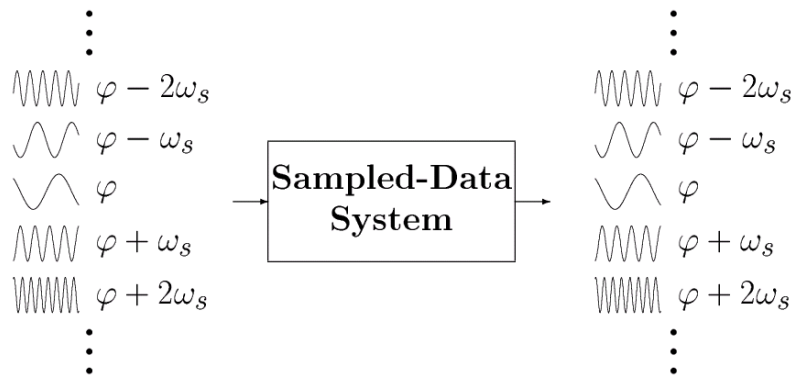
Figure 2.2: Output of a sampled-data system for multi sinusoidal input $\mathcal{X}_\varphi$.

frequency φ is defined as the l^2-induced norm of $\underline{Q}(j\varphi)$, i.e., $\|\underline{Q}(j\varphi)\|_{l^2/l^2}$ [8].

By using the notion of FR-operators, we can develop a frequency domain theory for sampled-data systems that takes into account the intersample behavior. The specific form of the FR-matrix $\underline{Q}(j\varphi)$ under the generalized plant setting and the related issues involving the definitions of the frequency response gain and the H^∞ norm of the sampled-data system are presented in Sec. 2.2 and Sec. 2.3, respectively.

2.2 FR-Operator Representation of Sampled-data Systems

In this section, we introduce the FR-operator representation of the sampled-data system expressed by a *generalized plant*. It is well known that an expression using a generalized plant is very useful for dealing with various problems of modern control theory, such as H^2 and H^∞ design problems, multi-objective control design problems, and robust stability analysis.

Consider the sampled-data system shown in Figure 2.3, where solid lines stand for continuous-time signals and dashed lines represent discrete-time signals. Here, P is an FDLTI continuous-time system called the generalized plant, which expresses not only actual systems but also virtual systems representing weighting functions. The continuous-time signals z, y, w, and u are referred to as the controlled output, measured output, exogenous input, and control input, respectively, and we denote their Laplace transforms by $Z(s)$, $Y(s)$, $W(s)$, and $U(s)$, respectively. These signals are related by

$$\begin{bmatrix} Z(s) \\ Y(s) \end{bmatrix} = P(s) \begin{bmatrix} W(s) \\ U(s) \end{bmatrix}, \quad P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix}, \tag{2.2}$$

where $P(s)$ is the transfer matrix of the generalized plant P, and $P_{ij}(s)$'s are component blocks of $P(s)$ divided according to the sizes of the signals z, y, w and u. C_d is an FDLTI discrete-time controller whose pulse transfer matrix is given by $C_d(z)$. $\mathcal{S}$ is the ideal sampler with sampling period τ, which works according to $y_d[k] = y(k\tau)$. $\mathcal{H}$ is a generalized hold, which works according to $u(k\tau + t) = h(t)u_d[k]$ $(t \in [0, \tau))$, where $h(t)$ is referred to as the hold function. The transfer matrix $H(s)$ of the generalized hold is defined as $H(s) = \int_0^\tau h(t)e^{-st}dt$.

The FR-matrix $\underline{Q}(j\varphi)$ from w to z is represented as follows [8, 29]:

$$\underline{Q}(j\varphi) = \underline{P_{11}}(j\varphi) + \frac{1}{\tau}\underline{P_{12}}H(j\varphi)\Lambda(e^{j\varphi\tau})\underline{P_{21}}(j\varphi), \tag{2.3}$$

where

$$\underline{P_{11}}(j\varphi) = \text{block diag}[\cdots, P_{11}(j\varphi_{-1}), P_{11}(j\varphi_0), P_{11}(j\varphi_1), \cdots], \tag{2.4}$$
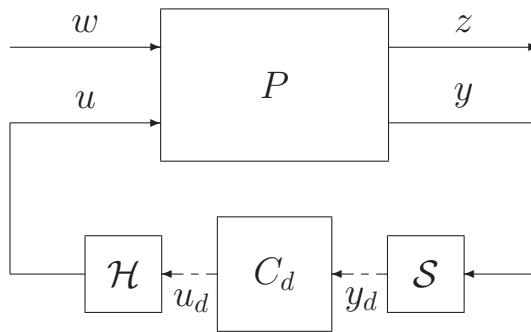
Figure 2.3: Sampled-data control system under generalized plant setting.

$$\underline{P_{12}H}(j\varphi) = \begin{bmatrix} \vdots \\ P_{12}(j\varphi_{-1})H(j\varphi_{-1}) \\ P_{12}(j\varphi_0)H(j\varphi_0) \\ P_{12}(j\varphi_1)H(j\varphi_1) \\ \vdots \end{bmatrix}, \tag{2.5}$$

$$\underline{P_{21}}(j\varphi) = \begin{bmatrix} \cdots & P_{21}(j\varphi_{-1}) & P_{21}(j\varphi_0) & P_{21}(j\varphi_1) & \cdots \end{bmatrix}, \tag{2.6}$$

$$\Lambda(e^{j\varphi\tau}) = C_d(e^{j\varphi\tau})\big(I - \Pi_{22}(e^{j\varphi\tau})C_d(e^{j\varphi\tau})\big)^{-1}, \tag{2.7}$$

$$\Pi_{22}(e^{j\varphi\tau}) = \frac{1}{\tau}\sum_{m=-\infty}^{\infty} P_{22}(j\varphi_m)H(j\varphi_m) = \mathcal{Z}\Big[P_{22}(s)H(s)\Big]_{z=e^{j\varphi\tau}}, \tag{2.8}$$

$$\varphi_m = \varphi + m\omega_s \ (m = 0, \pm 1, \cdots), \quad \omega_s = \frac{2\pi}{\tau}, \tag{2.9}$$

and $\mathcal{Z}$ stands for the z-transform.

Since the FR-matrix $\underline{Q}(j\varphi)$ is an infinite-dimensional matrix, the computation of $\|\underline{Q}(j\varphi)\|_{l^2/l^2}$ would seem very difficult. For the computation of the l^2-induced norm of FR-matrices, we give an approximation method via finite-dimensional truncation in Sec. 2.3. The exact method for a special case ($P_{11}(s) = 0$) and a bisection method for the fully generalized case are discussed in Sec. 2.4 and in Chapter 3, respectively. In Sec. 4.1, the sensitivity FR-operator and the complementary sensitivity FR-operator are defined for closed-loop sampled-data systems, and the exact methods for computing their l^2-induced norms are derived in Sec. 4.3.

2.3   Fundamental Properties of FR-Operators

In this section, we present several fundamental properties of FR-operators and give the definitions of the frequency response gain and the H^∞ norm of the sampled-data system shown in Figure 2.3. The derivations of the properties are essentially the same as the ones presented in the earlier work [8], and so they are omitted.

First, we begin with the following property by which the FR-matrix $\underline{Q}(j\varphi)$ is bounded on l^2.

Theorem 2.1 Suppose that $P_{11}(s)$ and $P_{12}(s)$ are proper, $P_{21}(s)$ and $P_{22}(s)$ are strictly proper, and $P_{11}(s)$, $P_{12}(s)$ and $P_{21}(s)$ have no poles at $s = j(\varphi + m\omega_s)$ ($m = 0, \pm 1, \cdots$). Assume that $\Lambda(z)$ is proper and has no poles at $z = e^{j\varphi h}$. Let $\underline{Q}(j\varphi)$ be given by (2.3), and $\underline{z} = \underline{Q}(j\varphi)\underline{w}$. Then, the inequality

$$\|\underline{z}\|_{l^2} \le K\|\underline{w}\|_{l^2} \tag{2.10}$$

holds for some nonnegative constant K independent of $\underline{w}$.

From Theorem 2.1, we can regard the sampled-data system of Figure 2.3 as a linear bounded operator on l^2. Thus, using the FR-matrix $\underline{Q}(j\varphi)$, we define the frequency response gain of the sampled-data system of Figure 2.3 at angular frequency φ as $\|\underline{Q}(j\varphi)\|_{l^2/l^2}$. The H^∞ norm of the sampled-data system is defined by $\|\underline{Q}(j\varphi)\|_\infty := \max_\varphi \|\underline{Q}(j\varphi)\|_{l^2/l^2}$.

Remark 2.1 An alternative way to define the frequency response gain of sampled-data systems was proposed by Yamamoto and Khargonekar [81], where the frequency response gain is defined as the induced norm of the $L^2[0, \tau)$-bounded operator obtained by a lifting technique [80]. Later work [78] showed that the frequency response gain defined by Yamamoto and Khargonekar [81] and that defined by the FR-operator are equivalent.

Next, we introduce a property that validates the approximation method for frequency response gain computation by the FR-matrix via finite-dimensional truncation.

Theorem 2.2 Suppose that $P_{11}(s)$ and $P_{12}(s)$ are proper, $P_{21}(s)$ and $P_{22}(s)$ are strictly proper, and $P_{11}(s)$, $P_{12}(s)$ and $P_{21}(s)$ have no poles at $s = j(\varphi + m\omega_s)$ ($m = 0, \pm 1, \cdots$). Assume that $\Lambda(z)$ is proper and has no poles at $z = e^{j\varphi h}$. Let $\underline{Q}(j\varphi)$ be given by (2.3). Then, for any $\epsilon > 0$, there exists an integer N_0 such that $N \ge N_0$ implies that

$$\left| \|\underline{Q}(j\varphi)\|_{l^2/l^2} - \|Q_{[N]}(j\varphi)\| \right| < \epsilon, \tag{2.11}$$

where $Q_{[N]}(j\varphi)$ is given by

$$Q_{[N]}(j\varphi) = P_{11[N]}(j\varphi)$$
$$+ \frac{1}{\tau}\{P_{12}H\}_{[N]}(j\varphi)\left(I - C_d(e^{j\varphi\tau})\Pi_{22}(e^{j\varphi\tau})\right)^{-1}C_d(e^{j\varphi\tau})P_{21[N]}(j\varphi), \quad (2.12)$$

$$P_{11[N]}(j\varphi) = \text{block diag}[P_{11}(j\varphi_{-N}), \cdots, P_{11}(j\varphi_0), \cdots, P_{11}(j\varphi_N)], \quad (2.13)$$

$$\{P_{12}H\}_{[N]}(j\varphi) = \begin{bmatrix} P_{12}(j\varphi_{-1})H(j\varphi_{-N}) \\ \vdots \\ P_{12}(j\varphi_0)H(j\varphi_0) \\ \vdots \\ P_{12}(j\varphi_1)H(j\varphi_N) \end{bmatrix}, \quad (2.14)$$

$$P_{21[N]}(j\varphi) = \begin{bmatrix} P_{21}(j\varphi_{-N}) & \cdots & P_{21}(j\varphi_0) & \cdots & P_{21}(j\varphi_N) \end{bmatrix}, \quad (2.15)$$

and $\|Q_{[N]}(j\varphi)\|$ is the matrix norm (the maximum singular value) of the finite-dimensional matrix $Q_{[N]}(j\varphi)$. Furthermore, when $P_{11}(s)$, $P_{12}(s)$ and $P_{21}(s)$ do not have poles on the imaginary axis and $(I - C_d(z)\Pi_{22}(z))^{-1}C_d(z)$ does not have poles on the unit circle, the above N_0 can be chosen independent of φ, that is, $\|\underline{Q}(j\varphi)\|_{l^2/l^2}$ is approximated by $\|Q_{[N]}(j\varphi)\|$ *uniformly* with respect to φ.

Theorem 2.2 implies that $\|\underline{Q}(j\varphi)\|_{l^2/l^2} = \lim_{N\to\infty} \bar{\sigma}(Q_{[N]}(j\varphi))$, where $\bar{\sigma}$ denotes the maximum singular value.

Finally, we present the property of the H^∞ norm of sampled-data systems defined by the FR-matrix. This is related to the bounded operators on L^2 associated with internally stable sampled-data systems. Before presenting the results, we introduce the definitions of the internal stability and the L^2-stability of the sampled-data system shown in Figure 2.3.

Let the continuous-time vector $x_{\text{sd}}(t)$ be given by

$$x_{\text{sd}}(t) = \begin{bmatrix} x(t) \\ \xi[k] \end{bmatrix} \quad (k\tau \le t < (k+1)\tau), \quad (2.16)$$

where $x(t)$ and $\xi[k]$ are the state vector of the continuous-time generalized plant P and that of the discrete-time controller C_d, respectively. Then, the sampled-data system is said to be *internally stable* if for every initial time t_0 ($0 \le t_0 < \tau$) and initial state $x_{\text{sd}}(t_0)$ we have $x_{\text{sd}}(t) \to 0$ as $t \to \infty$ when $w(t) \equiv 0$ [16].

The sampled-data system is said to be L^2-*stable* if the sampled-data system maps every input $w(t) \in L^2$ to the output $z(t) \in L^2$. It was also previously shown [16] that if the sampled-data system is internally stable, it is also L^2-stable. (This can be proved by using the lifting technique [80].) Accordingly, a linear bounded operator $\mathcal{L} : L^2 \in w(t) \mapsto z(t) \in L^2$ can be associated with the sampled-data system of Figure 2.3 when the sampled-data system is internally stable.

Now, we turn back to the relationship between the L^2-stability and the H^∞ norm of sampled-data systems.

Theorem 2.3 Consider the sampled-data system of Figure 2.3. Suppose that $P_{11}(s)$ and $P_{12}(s)$ are proper, $P_{21}(s)$ and $P_{22}(s)$ are strictly proper, and $P_{11}(s)$, $P_{12}(s)$ and $P_{21}(s)$ have no poles on the imaginary axis. Assume also that the sampled-data system is internally stable. Then, the induced norm of the operator $\mathcal{L} : L^2 \in w(t) \mapsto z(t) \in L^2$ associated with the sampled-data system of Figure 2.3 is given by

$$\|\mathcal{L}\|_{L^2/L^2} = \|\underline{Q}(j\varphi)\|_\infty. \tag{2.17}$$

Theorem 2.3 can be interpreted as the counterpart to the well-known fact that, for an FDLTI continuous-time system, its L^2-induced norm and H_∞ norm coincide.

2.4 Direct Norm Computation of Finite-rank FR-Operators

As seen from (2.3) to (2.6), FR-matrix $\underline{Q}(j\varphi)$ is an infinite-dimensional matrix, and thus the computation of the frequency response gain, i.e., $\|\underline{Q}(j\varphi)\|_{l^2/l^2}$, would seem very difficult. In Sec. 2.3, we suggested a computation method for $\|\underline{Q}(j\varphi)\|_{l^2/l^2}$ that utilizes Theorem 2.2, but that method is based on the finite-dimensional truncation of the FR-matrix, and thus we only obtain approximated values.

In this section, following the earlier work [29], we show that it is possible to compute frequency response gains exactly when the generalized plant shown in Figure 2.3 satisfies the condition that $P_{11}(s) = 0$. It should be noted that when $P_{11}(s) = 0$, the FR-matrix becomes a *finite-rank* matrix as seen from (2.3), although the matrix itself is *infinite-dimensional*. This fact is frequently used for solving various problems of sampled-data systems such as frequency response gain computation for a special class of sampled-data systems and the sensitivity reduction problems of sampled-data systems.

2.4.1 Norm equivalent discrete-time system

The exact computation method is obtained by showing that there is an equivalent discrete-time system whose frequency response gain coincides with that of the sampled-data system *for each angular frequency* φ. The derivation of the equivalent system uses only such elementary frequency-domain notions as (conventional) z-transform, pulse transfer function, and impulse modulation formula, while numerical computation can be carried out using state-space equations.

Here, we deal with the internally stable sampled-data system shown in Figure 2.3 and suppose that the state-space representation of the generalized plant $P(s)$ and the hold function $h(t)$ of the generalized hold $\mathcal{H}$ are given, respectively, by the following equations[1]:

$$P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & 0 & 0 \end{array} \right], \quad h(t) = C_H \exp(A_H t) B_H. \quad (2.18)$$

Here, we have assumed that $D_{21} = 0$ and $D_{22} = 0$ so as to satisfy the conditions of Theorems 2.1-2.3, in which the strictly properness of $P_{21}(s)$ and $P_{22}(s)$ is assumed. These assumptions are naturally satisfied because a strictly proper pre-filter is usually placed before the sampler.

Under the assumption that $P_{11}(s) = 0$, the condition

$$D_{11} = 0, \quad C_1 A^k B_1 = 0 \ (k = 0, 1, \cdots) \tag{2.19}$$

holds, and the FR-matrix $\underline{Q}(j\varphi)$ shown in (2.3) can be rewritten as

$$\underline{Q}(j\varphi) = \frac{1}{\tau} \underline{P_{12}H}(j\varphi) \Lambda(e^{j\varphi\tau}) \underline{P_{21}}(j\varphi), \tag{2.20}$$

where $\underline{P_{12}H}(j\varphi)$, $\underline{P_{21}}(j\varphi)$, and $\Lambda(e^{j\varphi\tau})$ are given, respectively, by (2.5), (2.6), and (2.7) with $\Pi_{22} := \mathcal{S}P_{22}\mathcal{H}$, i.e.,

$$\Pi_{22}(z) = \mathcal{Z}\left[P_{22}(s)H(s)\right] = \left[\begin{array}{c|c} \widehat{A} & B_{2d} \\ \hline C_2 & 0 \end{array} \right], \tag{2.21}$$

where

$$\widehat{A} = \exp(A\tau), \quad B_{2d} = \int_0^\tau \exp\left\{A(\tau - t)\right\} B_2 C_H \exp(A_H t) B_H dt. \tag{2.22}$$

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[1]Here, $\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$ implies that the transfer matrix $C(sI - A)^{-1}B + D$ for continuous-time systems, or transfer matrix $C(zI - A)^{-1}B + D$ for discrete-time systems.

Now, let us focus on the relation $\|Q(j\varphi)\|_{l^2/l^2} = \lim_{N\to\infty} \bar{\sigma}(Q_{[N]}(j\varphi))$, where $Q_{[N]}(j\varphi)$ is given by

$$Q_{[N]}(j\varphi) = \frac{1}{\tau}\{P_{12}H\}_{[N]}(j\varphi)\Lambda(e^{j\varphi\tau})P_{21[N]}(j\varphi), \tag{2.23}$$

when $P_{11}(s) = 0$. Since $Q_{[N]}(j\varphi)$ is a finite-dimensional matrix, we have

$$\bar{\sigma}\left(Q_{[N]}(j\varphi)\right) = \lambda_{\max}^{1/2}\left(Q_{[N]}^*(j\varphi)Q_{[N]}(j\varphi)\right)$$
$$= \lambda_{\max}^{1/2}\left(\frac{1}{\tau}P_{21[N]}^*(j\varphi)\Lambda^*(e^{j\varphi\tau})\{P_{12}H\}_{[N]}^*(j\varphi) \cdot \frac{1}{\tau}\{P_{12}H\}_{[N]}(j\varphi)\Lambda(e^{j\varphi\tau})P_{21[N]}(j\varphi)\right), \tag{2.24}$$

where $*$ denotes complex conjugate transposition of a complex matrix, and $\lambda_{\max}$ is the maximum eigenvalue of a Hermite matrix. Since the (nonzero) eigenvalues of the product of two matrices are invariant to a change in the order of their multiplication, we obtain

$$\bar{\sigma}\left(Q_{[N]}(j\varphi)\right)$$
$$= \lambda_{\max}^{1/2}\left(P_{21[N]}(j\varphi)\frac{1}{\tau}P_{21[N]}^*(j\varphi)\Lambda^*(e^{j\varphi\tau})\{P_{12}H\}_{[N]}^*(j\varphi)\frac{1}{\tau}\{P_{12}H\}_{[N]}(j\varphi)\Lambda(e^{j\varphi\tau})\right). \tag{2.25}$$

Here, note that the eigenvalues of a matrix are continuous with respect to its entries, since the coefficients of the characteristic polynomial are continuous with respect to the entries and the roots of the polynomial are continuous with respect to its coefficients. From this, together with (2.25), we obtain

$$\|Q(j\varphi)\|_{l^2/l^2} = \lim_{N\to\infty} \bar{\sigma}\left(Q_{[N]}(j\varphi)\right)$$
$$= \lim_{N\to\infty} \lambda_{\max}^{1/2}\left(P_{21[N]}(j\varphi)\frac{1}{\tau}P_{21[N]}^*(j\varphi)\Lambda^*(e^{j\varphi\tau})\{P_{12}H\}_{[N]}^*(j\varphi)\frac{1}{\tau}\{P_{12}H\}_{[N]}(j\varphi)\Lambda(e^{j\varphi\tau})\right)$$
$$= \lambda_{\max}^{1/2}\left(\lim_{N\to\infty} P_{21[N]}(j\varphi)\frac{1}{\tau}P_{21[N]}^*(j\varphi)\Lambda^*(e^{j\varphi\tau})\{P_{12}H\}_{[N]}^*(j\varphi)\frac{1}{\tau}\{P_{12}H\}_{[N]}(j\varphi)\Lambda(e^{j\varphi\tau})\right)$$
$$= \lambda_{\max}^{1/2}\left(\frac{1}{\tau}\underline{P_{21}}(j\varphi)\underline{P_{21}}^*(j\varphi)\Lambda^*(e^{j\varphi\tau})\frac{1}{\tau}\underline{P_{12}H}^*(j\varphi)\underline{P_{12}H}(j\varphi)\Lambda(e^{j\varphi\tau})\right). \tag{2.26}$$

Here, note that $\frac{1}{\tau}\underline{P_{21}}(j\varphi)\underline{P_{21}}^*(j\varphi)$ and $\frac{1}{\tau}\underline{P_{12}H}^*(j\varphi)\underline{P_{12}H}(j\varphi)$ are finite-dimensional matrices, which can be rewritten, respectively, as

$$\frac{1}{\tau}\underline{P_{21}}(j\varphi)\underline{P_{21}}^*(j\varphi) = \frac{1}{\tau}\sum_{m=-\infty}^{\infty} P_{21}(j\varphi_m)P_{21}(j\varphi_m)^*, \tag{2.27}$$

$$\frac{1}{\tau}\underline{P_{12}H}^*(j\varphi)\underline{P_{12}H}(j\varphi) = \frac{1}{\tau}\sum_{m=-\infty}^{\infty} (P_{12}(j\varphi_m)H(j\varphi_m))^* (P_{12}(j\varphi_m)H(j\varphi_m)). \tag{2.28}$$

Now, from the impulse modulation formula [54], we have

$$\frac{1}{\tau}\underline{P_{21}}(j\varphi)\underline{P_{21}}^*(j\varphi) = \mathcal{Z}\Big[P_{21}(s)P_{21}\tilde{}(s)\Big]_{z=e^{j\varphi\tau}}, \tag{2.29}$$

$$\frac{1}{\tau}\underline{P_{12}H}^*(j\varphi)\underline{P_{12}H}(j\varphi) = \mathcal{Z}\Big[H\tilde{}(s)P_{12}\tilde{}(s)P_{12}(s)H(s)\Big]_{z=e^{j\varphi\tau}}, \tag{2.30}$$

where, for continuous-time systems, $X\tilde{}(s)$ is defined as $X\tilde{}(s) := X^T(-s)$. Here, note that if there exist discrete-time time transfer matrices $\Pi_{21}(z)$ and $\Pi_{12}(z)$ satisfying

$$\Pi_{21}(z)\Pi_{21}\tilde{}(z) = \mathcal{Z}\Big[P_{21}(s)P_{21}\tilde{}(s)\Big], \tag{2.31}$$

$$\Pi_{12}\tilde{}(z)\Pi_{12}(z) = \mathcal{Z}\Big[H\tilde{}(s)P_{12}\tilde{}(s)P_{12}(s)H(s)\Big], \tag{2.32}$$

where $X\tilde{}(z)$ is defined as $X\tilde{}(z) := X^T(z^{-1})$ for discrete-time systems, (2.26) can be rewritten as

$$\begin{aligned}\|\underline{Q}(j\varphi)\|_{l^2/l^2} &= \lambda_{\max}^{1/2}\left(\Pi_{21}(j\varphi)\Pi_{21}^*(j\varphi)\Lambda^*(e^{j\varphi\tau})\Pi_{12}^*(j\varphi)\Pi_{12}(j\varphi)\Lambda(e^{j\varphi\tau})\right)\\ &= \lambda_{\max}^{1/2}\left(\Pi_{21}^*(e^{j\varphi\tau})\Lambda^*(e^{j\varphi\tau})\Pi_{12}^*(e^{j\varphi\tau})\Pi_{12}(e^{j\varphi\tau})\Lambda(e^{j\varphi\tau})\Pi_{21}(e^{j\varphi\tau})\right)\\ &= \bar{\sigma}\left(\widehat{Q}(e^{j\varphi\tau})\right), \end{aligned} \tag{2.33}$$

where $\widehat{Q}(e^{j\varphi\tau}) = \Pi_{12}(e^{j\varphi\tau})\Lambda(e^{j\varphi\tau})\Pi_{21}(e^{j\varphi\tau})$. Consequently, the frequency response gain $\|\underline{Q}(j\varphi)\|_{l^2/l^2}$ can be computed as the maximum singular value of the pulse transfer matrix $\widehat{Q}(z)$ evaluated at $z = e^{j\varphi\tau}$, provided that there exist discrete-time transfer matrices $\Pi_{21}(z)$ and $\Pi_{12}(z)$ satisfying the conditions (2.31) and (2.32), respectively. As we shall see in Sec. 2.4.2 and Sec. 2.4.3, the state space representations of $\Pi_{21}(z)$ and $\Pi_{12}(z)$ are given, respectively, by the following equations,

$$\Pi_{21}(z) := \left[\begin{array}{c|c}\widehat{A} & W \\ \hline C_2 & 0\end{array}\right], \tag{2.34}$$

$$\Pi_{12}(z) := \left[\begin{array}{c|c}\widehat{A} & B_{2d} \\ \hline V_1 & V_2 B_H\end{array}\right], \tag{2.35}$$

where W, V_1, and V_2 are any matrices such that

$$WW^T = \int_0^\tau \exp(At)B_1 B_1^T \exp(A^T t)\, dt, \tag{2.36}$$

$$\begin{bmatrix}V_1^T \\ V_2^T\end{bmatrix}\begin{bmatrix}V_1 & V_2\end{bmatrix}$$
$$= \int_0^\tau \exp\left(\begin{bmatrix}A & B_2 C_H \\ 0 & A_H\end{bmatrix}t\right)^T \begin{bmatrix}C_1^T \\ C_H^T D_{12}^T\end{bmatrix}\begin{bmatrix}C_1 & D_{12}C_H\end{bmatrix}\exp\left(\begin{bmatrix}A & B_2 C_H \\ 0 & A_H\end{bmatrix}t\right)dt. \tag{2.37}$$
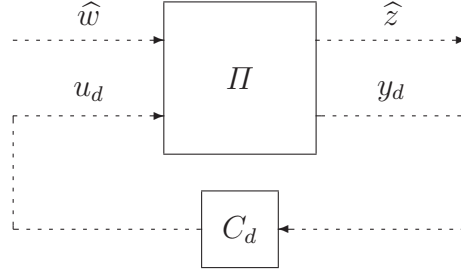
Figure 2.4: Discrete-time system $\widehat{Q}(z)$ with the same frequency response gain

Now, let us introduce

$$
\Pi(z) := \left[\begin{array}{c|cc} \widehat{A} & W & B_{2d} \\ \hline V_1 & 0 & V_2 B_H \\ C_2 & 0 & 0 \end{array}\right]. \tag{2.38}
$$

Then, from (2.19), (2.21), (2.34), (2.35), (2.36), and (2.37), it is easy to verify that

$$
\Pi(z) = \left[\begin{array}{cc} 0 & \Pi_{12}(z) \\ \Pi_{21}(z) & \Pi_{22}(z) \end{array}\right]. \tag{2.39}
$$

The reason for $\Pi_{11}(z) = V_1(zI - \widehat{A})^{-1}W = 0$ is due the fact that $V_1\widehat{A}^k W = 0$ ($k = 0, 1, \cdots$), which is obtained by (2.19), (2.36), and (2.37). From (2.39), together with (2.7), $\widehat{Q}(z)$ is nothing but the pulse transfer matrix from $\widehat{w}$ to $\widehat{z}$ of the discrete-time system shown in Figure 2.4 (i.e., $\widehat{Q}(z) = \mathcal{F}_l(\Pi(z), C_d(z))$, where $\mathcal{F}_l$ denotes the lower linear fractional transformation. For the definition, see Notations and Definitions). In this way, we finally obtain the following theorem, which gives a method to compute the frequency response gains.

Theorem 2.4 Consider the sampled-data system shown in Figure 2.3. Suppose that $P_{11}(s) = 0$. Then, the frequency response gain $\|Q(j\varphi)\|_{l^2/l^2}$ coincides with $\bar{\sigma}(\widehat{Q}(e^{j\varphi\tau}))$ for every φ, where $\widehat{Q}(z)$ is the pulse transfer matrix from $\widehat{w}$ to $\widehat{z}$ of the discrete-time system of Figure 2.4.

2.4.2   Derivation of the state-space representation of Π_{21}

In this subsection, we show that a state space representation of $\Pi_{21}(z)$ that satisfies (2.31) is given by (2.34) and (2.36). We begin with the following results, which are very useful for deriving state-space representations of equivalent discrete-time systems. These results are frequently used in the later sections.

Lemma 2.1 Let the state-space representations of the continuous-time transfer matrices $G(s)$, $G_1(s)$, and $G_2(s)$ be given, respectively, by

$$G(s) = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right], \quad G_1(s) = \left[\begin{array}{c|c} A_1 & B_1 \\ \hline C_1 & D_1 \end{array}\right], \quad G_2(s) = \left[\begin{array}{c|c} A_2 & B_2 \\ \hline C_2 & D_2 \end{array}\right]. \qquad (2.40)$$

Then,

$$G^{\sim}(s) = \left[\begin{array}{c|c} -A^T & -C^T \\ \hline B^T & D^T \end{array}\right] = \left[\begin{array}{c|c} -A^T & C^T \\ \hline -B^T & D^T \end{array}\right], \qquad (2.41)$$

$$G_1(s)G_2(s) = \left[\begin{array}{cc|c} A_1 & B_1C_2 & B_1D_2 \\ 0 & A_2 & B_2 \\ \hline C_1 & D_1C_2 & D_1D_2 \end{array}\right] = \left[\begin{array}{cc|c} A_2 & 0 & B_2 \\ B_1C_2 & A_1 & B_1D_2 \\ \hline D_1C_2 & C_1 & D_1D_2 \end{array}\right]. \qquad (2.42)$$

Lemma 2.2 Let A and C be square matrices. Then,

$$\exp\left(\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}\tau\right) = \begin{bmatrix} \exp(A\tau) & \int_0^\tau \exp\{A(\tau-t)\}B\exp(Ct)dt \\ 0 & \exp(C\tau) \end{bmatrix}, \qquad (2.43)$$

$$= \begin{bmatrix} \exp(A\tau) & \int_0^\tau \exp(At)B\exp\{C(\tau-t)\}dt \\ 0 & \exp(C\tau) \end{bmatrix}, \qquad (2.44)$$

$$\exp\left(\begin{bmatrix} A & 0 \\ B & C \end{bmatrix}\tau\right) = \begin{bmatrix} \exp(A\tau) & 0 \\ \int_0^\tau \exp\{C(\tau-t)\}B\exp(At)dt & \exp(C\tau) \end{bmatrix}, \qquad (2.45)$$

$$= \begin{bmatrix} \exp(A\tau) & 0 \\ \int_0^\tau \exp(Ct)B\exp\{A(\tau-t)\}dt & \exp(C\tau) \end{bmatrix}. \qquad (2.46)$$

Lemma 2.3 Let A and C be invertible matrices. Then,

$$\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}BC^{-1} \\ 0 & C^{-1} \end{bmatrix}, \quad \begin{bmatrix} A & 0 \\ B & C \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & 0 \\ -C^{-1}BA^{-1} & C^{-1} \end{bmatrix}. \qquad (2.47)$$

Lemma 2.4

$$\mathcal{Z}\left[C(sI-A)^{-1}B\right] = C\left\{I - \exp(A\tau)z^{-1}\right\}^{-1}B. \qquad (2.48)$$

$$\mathcal{Z}\left[C(sI-A)^{-1}Be^{-\tau s}\right] = C\left\{I - \exp(A\tau)z^{-1}\right\}^{-1}Bz^{-1}. \qquad (2.49)$$

$$\mathcal{Z}\left[C(sI-A)^{-1}Be^{\tau s}\right] = C\left\{I - \exp(A\tau)z^{-1}\right\}^{-1}Bz, \text{ when } CB = 0. \qquad (2.50)$$

Concerning Lemmas 2.1 and 2.2, see e.g., Chen and Francis [16]. The equalities in Lemma 2.3 are verified easily. Here, we only show Lemma 2.4.

Proof of Lemma 2.4: Note that the z-transform of $X(s)$ is defined by $\mathcal{Z}\Big[X(s)\Big] = \sum_{k=0}^{\infty} x(k\tau)z^{-k}$, where $x(t)$ is the inverse Laplace transform of $X(s)$. Also note that the inverse Laplace transform of $C(sI - A)^{-1}e^{as}$ is given by $Ce^{A(t+a)}B\mathbf{u}(t+a)$ where $\mathbf{u}(t)$ is the unit step function. By this, we obtain

$$
\mathcal{Z}\Big[C(sI-A)^{-1}B\Big] = \sum_{k=0}^{\infty} C\exp(Ak\tau)Bz^{-k}
$$

$$
= \sum_{k=0}^{\infty} C\left\{\exp(A\tau)z^{-1}\right\}^{k}B = C\left\{I - \exp(A\tau)z^{-1}\right\}^{-1}B, \tag{2.51}
$$

$$
\mathcal{Z}\Big[C(sI-A)^{-1}Be^{-\tau s}\Big] = \sum_{k=1}^{\infty} C\exp(A(k\tau - \tau))Bz^{-k}
$$

$$
= \sum_{k=1}^{\infty} C\left\{\exp(A\tau)z^{-1}\right\}^{k-1}Bz^{-1} = C\left\{I - \exp(A\tau)z^{-1}\right\}^{-1}Bz^{-1}. \tag{2.52}
$$

When $CB = 0$, we obtain

$$
\mathcal{Z}\Big[C(sI-A)^{-1}Be^{\tau s}\Big] = \sum_{k=0}^{\infty} C\exp(A(k\tau + \tau))Bz^{-k} = \sum_{k=-1}^{\infty} C\exp(A(k\tau + \tau))Bz^{-k}
$$

$$
= \sum_{k=-1}^{\infty} C\left\{\exp(A\tau)z^{-1}\right\}^{k+1}Bz = C\left\{I - \exp(A\tau)z^{-1}\right\}^{-1}Bz. \tag{2.53}
$$

This completes the proof.

Now, we show that $\Pi_{21}(z)$ given by (2.34) and (2.36) satisfies $\Pi_{21}(z)\Pi_{21}{}^{\sim}(z) = \mathcal{Z}\Big[P_{21}(s)P_{21}{}^{\sim}(s)\Big]$. From (2.41) and (2.42), we have

$$
P_{21}(s)P_{21}{}^{\sim}(s) = \left[\begin{array}{c|c} A & B_1 \\ \hline C_2 & 0 \end{array}\right] \cdot \left[\begin{array}{c|c} -A^T & C_2^T \\ \hline -B_1^T & 0 \end{array}\right] = \left[\begin{array}{cc|c} A & -B_1B_1^T & 0 \\ 0 & -A^T & C_2^T \\ \hline C_2 & 0 & 0 \end{array}\right]. \tag{2.54}
$$

From (2.44), we have

$$
\exp\left(\begin{bmatrix} A & -B_1B_1^T \\ 0 & -A^T \end{bmatrix}\tau\right) = \begin{bmatrix} \exp(A\tau) & -\int_0^\tau \exp(At)B_1B_1^T\left\{-A^T(\tau - t)\right\}dt \\ 0 & \exp(-A^T\tau) \end{bmatrix} \tag{2.55}
$$

$$
= \begin{bmatrix} \widehat{A} & -WW^T\widehat{A}^{-T} \\ 0 & \widehat{A}^{-T} \end{bmatrix}, \tag{2.56}
$$

where $\widehat{A}$ and W are given by (2.22) and (2.36), respectively. Thus, from Lemmas 2.3 and 2.4, we obtain

$$\mathcal{Z}\Big[P_{21}(s)P_{21}\tilde{\ }(s)\Big] = \begin{bmatrix} C_2 & 0 \end{bmatrix} \left\{ \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} - \begin{bmatrix} \widehat{A} & -WW^T\widehat{A}^{-T} \\ 0 & \widehat{A}^{-T} \end{bmatrix} z^{-1} \right\}^{-1} \begin{bmatrix} 0 \\ C_2^T \end{bmatrix} \quad (2.57)$$

$$= \begin{bmatrix} C_2 & 0 \end{bmatrix} \begin{bmatrix} I - \widehat{A}z^{-1} & WW^T\widehat{A}^{-T}z^{-1} \\ 0 & I - \widehat{A}^{-T}z^{-1} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ C_2^T \end{bmatrix} \quad (2.58)$$

$$= -C_2(I - \widehat{A}z^{-1})^{-1}WW^T\widehat{A}^{-T}z^{-1}(I - \widehat{A}^{-T}z^{-1})^{-1}C_2^T \quad (2.59)$$

$$= C_2(zI - \widehat{A})^{-1}WW^T(z^{-1}I - \widehat{A}^T)^{-1}C_2^T. \quad (2.60)$$

This implies that $\mathcal{Z}\Big[P_{21}(s)P_{21}\tilde{\ }(s)\Big] = \Pi_{21}(z)\Pi_{21}\tilde{\ }(z)$.

2.4.3 Derivation of the state-space representation of Π_{12}

In this subsection, we show that a state space representation of $\Pi_{12}(z)$ that satisfies (2.32) is given by (2.35) and (2.37). We begin with the following lemma.

Lemma 2.5 Suppose that $X(s) = C(sI - A)^{-1}B$ with $CB = 0$ and that $Y(e^{\tau s})$ is a matrix of the form $Y(e^{\tau s}) = Y_0 + Y_1/e^{\tau s} = Y_0 + e^{-\tau s}Y_1$. Then,

$$\mathcal{Z}\Big[Y\tilde{\ }(e^{\tau s})X(s)Y(e^{\tau s})\Big] = Y\tilde{\ }(z)\mathcal{Z}\Big[X(s)\Big]Y(z). \quad (2.61)$$

Proof of Lemma 2.5: By Lemma 2.4, the left-hand side of (2.61) is rewritten as

$$\mathcal{Z}\Big[Y\tilde{\ }(e^{\tau s})X(s)Y(e^{\tau s})\Big] = \mathcal{Z}\Big[(Y_0^T + e^{\tau s}Y_1^T)X(s)(Y_0 + e^{-\tau s}Y_1)\Big]$$

$$= \mathcal{Z}\Big[Y_0^T X(s)Y_0\Big] + \mathcal{Z}\Big[Y_0^T X(s)Y_1 e^{-\tau s}\Big] + \mathcal{Z}\Big[Y_1^T X(s)Y_0 e^{\tau s}\Big] + \mathcal{Z}\Big[Y_1^T X(s)Y_1\Big]$$

$$= Y_0^T \mathcal{Z}\Big[X(s)\Big]Y_0 + Y_0^T \mathcal{Z}\Big[X(s)\Big]Y_1 z^{-1} + Y_1^T \mathcal{Z}\Big[X(s)\Big]Y_0 z + Y_1^T \mathcal{Z}\Big[X(s)\Big]Y_1$$

$$= (Y_0^T + zY_1^T)\mathcal{Z}\Big[X(s)\Big](Y_0 + z^{-1}Y_1) = Y\tilde{\ }(z)\mathcal{Z}\Big[X(s)\Big]Y(z). \quad (2.62)$$

This completes the proof.

Now, let us consider the transfer matrix $H(s)$ of the generalized hold $\mathcal{H}$ with the hold function $h(t) = C_H \exp(A_H t)B_H$. This is given by

$$H(s) = \int_0^\tau C_H \exp(A_H t)B_H \exp(-st)dt = C_H(sI - A_H)^{-1}(I - \widehat{A}_H e^{-\tau s})B_H$$

$$= \left[\begin{array}{c|c} A_H & I \\ \hline C_H & 0 \end{array} \right] H_d(e^{\tau s}), \quad (2.63)$$

where

$$\widehat{A}_H = \exp(A_H \tau), \quad H_d(e^{\tau s}) = (I - \widehat{A}_H e^{-\tau s})B_H. \quad (2.64)$$

Therefore, from (2.42), $P_{12}(s)H(s)$ is rewritten as

$$P_{12}(s)H(s) = \left[\begin{array}{c|c} A & B_2 \\ \hline C_1 & D_{12} \end{array}\right] \cdot \left[\begin{array}{c|c} A_H & I \\ \hline C_H & 0 \end{array}\right] H_d(e^{\tau s}) = \left[\begin{array}{c|c} A_{12} & B_{12} \\ \hline C_{12} & 0 \end{array}\right] H_d(e^{\tau s}), \qquad (2.65)$$

where

$$A_{12} = \begin{bmatrix} A & B_2 C_H \\ 0 & A_H \end{bmatrix}, \quad B_{12} = \begin{bmatrix} 0 \\ I \end{bmatrix}, \quad C_{12} = \begin{bmatrix} C_1 & D_{12} C_H \end{bmatrix}. \qquad (2.66)$$

Thus, from (2.41), (2.42), and Lemma 2.5, $\mathcal{Z}\left[H^\sim(s)P_{12}^\sim(s)P_{12}(s)H(s)\right]$ becomes

$$\mathcal{Z}\left[H^\sim(s)P_{12}^\sim(s)P_{12}(s)H(s)\right] = \mathcal{Z}\left[H^\sim_d(e^{\tau s})\left[\begin{array}{c|c} -A_{12}^T & -C_{12}^T \\ \hline B_{12}^T & 0 \end{array}\right]\left[\begin{array}{c|c} A_{12} & B_{12} \\ \hline C_{12} & 0 \end{array}\right]H_d(e^{\tau s})\right]$$

$$= \mathcal{Z}\left[H^\sim_d(e^{\tau s})\left[\begin{array}{cc|c} -A_{12}^T & -C_{12}^T C_{12} & 0 \\ 0 & A_{12} & B_{12} \\ \hline B_{12}^T & 0 & 0 \end{array}\right]H_d(e^{\tau s})\right] = H^\sim_d(z)\mathcal{Z}\left[\begin{array}{cc|c} -A_{12}^T & -C_{12}^T C_{12} & 0 \\ 0 & A_{12} & B_{12} \\ \hline B_{12}^T & 0 & 0 \end{array}\right]H_d(z),$$

$$(2.67)$$

where

$$H_d(z) = (I - \widehat{A}_H z^{-1})B_H, \quad H^\sim_d(z) = H_d^T(z^{-1}) = B_H^T(I - \widehat{A}_H^T z). \qquad (2.68)$$

In a similar manner to the derivation of (2.60) from (2.54), we obtain

$$\mathcal{Z}\left[\begin{array}{cc|c} -A_{12}^T & -C_{12}^T C_{12} & 0 \\ 0 & A_{12} & B_{12} \\ \hline B_{12}^T & 0 & 0 \end{array}\right] = B_{12}^T(z^{-1}I - \widehat{A}_{12}^T)^{-1}V^T V(zI - \widehat{A}_{12})^{-1}B_{12}, \qquad (2.69)$$

where $\widehat{A}_{12} = \exp(A_{12}\tau)$, and $V = \begin{bmatrix} V_1 & V_2 \end{bmatrix}$ is a matrix satisfying the condition $V^T V = \int_0^\tau \exp(A_{12}^T t)C_{12}^T C_{12}\exp(A_{12}t)dt$. Note that the latter equality is nothing but (2.37).

Now, we evaluate $(zI - \widehat{A}_{12})^{-1}B_{12}H_d(z)$. From (2.66), together with Lemmas 2.2 and 2.3, we have

$$(zI - \widehat{A}_{12})^{-1}B_{12}H_d(z)$$

$$= \left\{zI - \begin{bmatrix} \widehat{A} & \int_0^\tau \exp\{A(\tau - t)\}B_2 C_H \exp(A_H t)dt \\ 0 & \widehat{A}_H \end{bmatrix}\right\}^{-1}\begin{bmatrix} 0 \\ (I - \widehat{A}_H z^{-1})B_H \end{bmatrix}$$

$$= \begin{bmatrix} * & (zI - \widehat{A})^{-1}\int_0^\tau \exp\{A(\tau - t)\}B_2 C_H \exp(A_H t)dt(zI - \widehat{A}_H)^{-1} \\ 0 & (zI - \widehat{A}_H)^{-1} \end{bmatrix}\begin{bmatrix} 0 \\ (I - \widehat{A}_H z^{-1})B_H \end{bmatrix}$$

$$= \begin{bmatrix} (zI - \widehat{A})^{-1}\int_0^\tau \exp\{A(\tau - t)\}B_2 C_H \exp(A_H t)dt z^{-1}B_H \\ z^{-1}B_H \end{bmatrix} = z^{-1}\begin{bmatrix} (zI - \widehat{A})^{-1}B_{2d} \\ B_H \end{bmatrix},$$

$$(2.70)$$

where B_{2d} is given by (2.22). Thus, from (2.67), (2.69), and (2.70), we obtain

$$\mathcal{Z}\Big[H^\sim(s)P_{12}^\sim(s)P_{12}(s)H(s)\Big]$$

$$= \Big[B_{2d}^T(z^{-1}I - \widehat{A}^T)^{-1} \quad B_H^T\Big] \begin{bmatrix} V_1^T \\ V_2^T \end{bmatrix} \begin{bmatrix} V_1 & V_2 \end{bmatrix} \begin{bmatrix} (zI - \widehat{A})^{-1}B_{2d} \\ B_H \end{bmatrix}, \qquad (2.71)$$

which implies $\mathcal{Z}\Big[H^\sim(s)P_{12}^\sim(s)P_{12}(s)H(s)\Big] = \Pi_{12}^\sim(z)\Pi_{12}(z)$.

2.5 Numerical Examples

In this section, we compute the frequency response gain of a sampled-data system by using the exact method presented in Sec. 2.4 and the approximation method presented in Sec. 2.3.

Let us consider the sampled-data system of Figure 2.5, where $\mathcal{S}$ is the ideal sampler with sampling period τ and $\mathcal{H}$ is the zero-order hold. The system is expressed by the generalized plant setting by taking $P_{11}(s) = 0$, $P_{12}(s) = 1$, $P_{21}(s) = 1/(s+1)$, $P_{22}(s) = 0$, and $C_d(z) = 1$. Since $P_{11}(s) = 0$, we can apply the exact computation method presented in Sec. 2.4. The generalized plant $P(s)$ can be expressed by the following state-space representation:

$$P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & 0 & 0 \end{array}\right] = \left[\begin{array}{c|cc} -1 & 1 & 0 \\ \hline 0 & 0 & 1 \\ 1 & 0 & 0 \end{array}\right]. \qquad (2.72)$$

First, by applying the exact computation method using the norm-equivalent discrete-time system, we compute the frequency response gain for $\tau = 0.5, 1, 2$ (up to the Nyquist frequencies), together with the frequency response gain of the continuous-time system $1/(s+1)$. The result is shown in Figure 2.6. From this figure, we can observe that the frequency response gain approaches that of the continuous-time system $1/(s+1)$ as the sampling period decreases, although the frequency response gains at around the Nyquist frequencies do not decrease by 20dB/decade due to the aliasing effect.

Next, we compute the frequency response gain of the sampled-data system of Figure 2.5 for $\tau = 2$ by the approximation method using the finite-dimensional
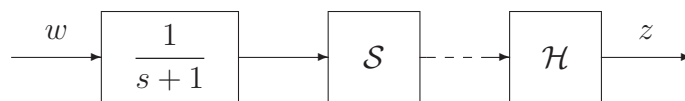
Figure 2.5: Sampled-data system for numerical examples.

truncation of $N = 1$, 5, 25. The results are shown in Figure 2.7. From this figure, as stated in Theorem 2.2, we can observe that the approximated frequency response gains obtained by truncating the FR-matrix approach the exact value as the size of the finite-dimensional matrix increases.
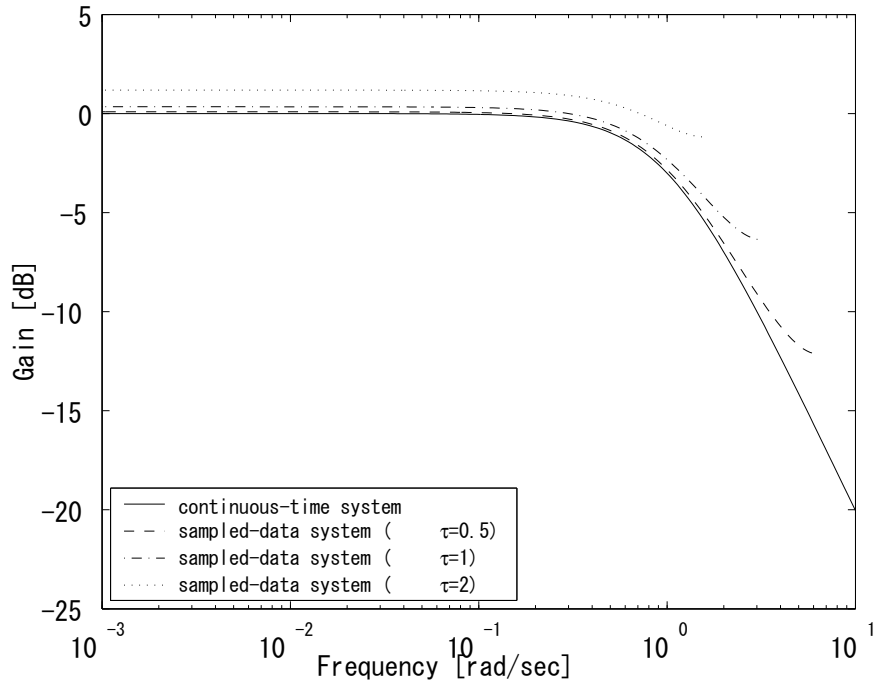
Figure 2.6: Frequency response gain of the sampled-data system of Figure 2.5 and that of continuous-time system $1/(s + 1)$.

2.6 Summary

In this chapter, we introduced the notion of an FR-operator, which enables us to develop the frequency domain theory of sampled-data systems while taking into account intersample behavior and the influence of aliasing. After introducing the FR-operator, we provided definitions of the frequency response gain and the H^∞ norm of sampled-data systems. Furthermore, we presented several properties of FR-operators, which involve the l^2-boundedness of the FR-operator, an approximative method for frequency response gain computation via finite-dimensional truncation, and the relationship between the L^2-induced norm and the H^∞ norm of sampled-data systems. The last property implies that the FR-operator satisfies the basic relation between the time-domain and the frequency-domain characterizations and can be regarded as the exact sampled-data counterpart of the frequency transfer
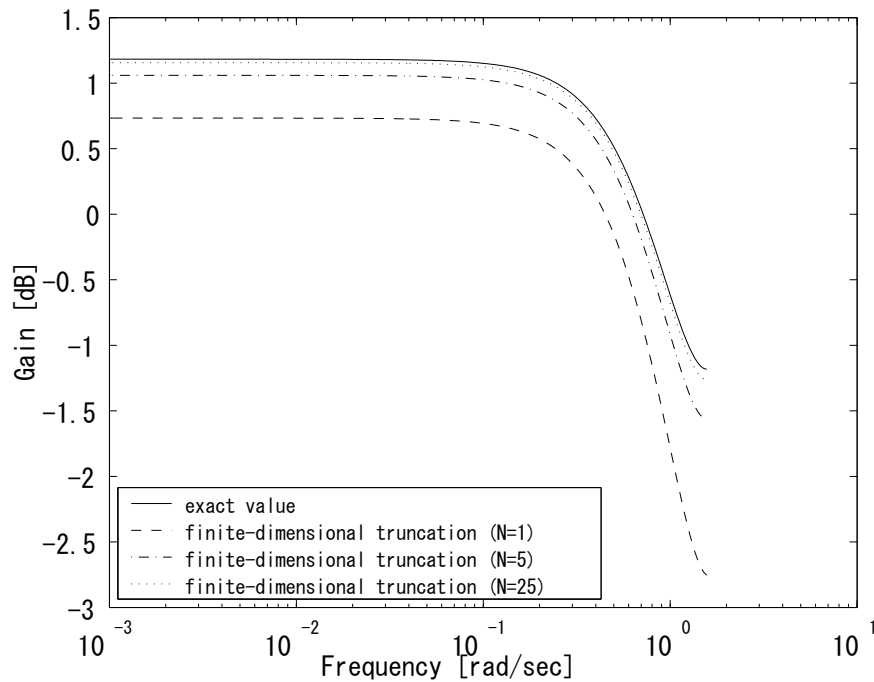
Figure 2.7: Frequency response gain and its approximations via finite-dimensional truncations.

matrix of a continuous-time system. Other related issues, such as the extension of the FR-operator to the Laplace domain and the relation to Goodwin and Salgado's method of defining the frequency response of sampled-data systems [24], can be found in the earlier work mentioned above [8].

This chapter has also provided an exact and direct method for computing the frequency response gain of a class of sampled-data systems satisfying the condition that $P_{11}(s) = 0$ under the generalized plant setting. The key result we obtained is that, under the condition $P_{11}(s) = 0$, there exists an 'equivalent discrete-time system' whose *frequency response gain* coincides with that of the given sampled-data system *for every frequency*. The technique used for the derivation of the method is fundamental for deriving other related results appearing in the later chapters, such as a bisection algorithm for computing the frequency response gain of general sampled-data systems (without the condition $P_{11}(s) = 0$), which is presented in Chapter 3, and an analysis of the sensitivity and complementary sensitivity reduction problems of sampled-data systems, which are presented in Chapter 4.

Chapter 3

Norm Computation of FR-Operators via Infinite-Dimensional Congruent Transformation

In Chapter 2, we defined the frequency response of sampled-data systems as the l^2-induced norm of the infinite-dimensional matrix called FR-matrix, and we presented an exact computation method under the assumption that $P_{11}(s) = 0$ in the generalized plant setting. In this chapter, we derive fully generalized method for computing the frequency response gain of sampled-data systems. The method is based on the bisection algorithm, that is, the algorithm decides whether the frequency response gain at each angular frequency is smaller than a given positive number γ. The bisection method is derived from the useful properties of infinite-dimensional congruent transformation, i.e., Schur complement arguments and Sylvester's Law of Inertia. The techniques used for the derivation of the state-space representation of the norm-equivalent discrete-time systems, which are used in Secs. 2.4.2 and 2.4.3, are also used to derive explicit formulae for the matrices required for the bisection method.

In this chapter, we first give the notation used and provide some useful fundamental results for the infinite-dimensional congruent transformation of operators. Next, by using these results, we show that the decision of whether the frequency response gain is smaller than a given positive number γ is reduced to the problem of counting the number of negative eigenvalues of a certain block-diagonal self-adjoint matrix consisting of two γ-dependent blocks: a finite-dimensional matrix block and an infinite-dimensional matrix block. As shown in the later sections, the numbers of negative eigenvalues of these two matrices are easy to count, and thus the frequency response gain of sampled-data systems can be easily computed by this algorithm.

Related issues are also presented, including a lifting-based algorithm for the bisection method, one-dimensional search algorithm, the relation to the several methods of solving the H^∞ problem of sampled-data systems, and possible applications of the congruent transformation to systems and control theory. Finally, we evaluate the proposed method through a comparison with existing methods.

3.1 Properties of Infinite-Dimensional Congruent Transformation

In this section, we give the notation used in this chapter and prepare some useful results for computing the frequency response gain of sampled-data systems.

Let $\mathcal{M}$ be a separable Hilbert space. The class of linear self-adjoint compact operators on $\mathcal{M}$ is denoted by $\mathcal{W}$. For $\underline{X} \in \mathcal{W}$, it is well-known [65] that (i) every λ in the spectrum is a real number, and is in fact an eigenvalue provided that $\lambda \neq 0$, (ii) the eigenvalues are at most countably infinite[2], and (iii) 0 is the only possible point of their accumulation. For a linear compact operator $\underline{X}$ on $\mathcal{M}$, $\sigma_i(\underline{X})$ denotes the i-th largest singular value of $\underline{X}$, which is given by the square root of the i-th largest eigenvalue of $\underline{X}^*\underline{X}$. The class of operators of the form $\gamma I - \underline{X}$ ($\underline{X} \in \mathcal{W}$, γ is a positive number) is denoted by $\mathcal{N}$. Note that every linear self-adjoint operator on a finite-dimensional space $\mathcal{M}$ belongs to $\mathcal{N}$, even with a prescribed value of γ, and this fact will serve us throughout the chapter. If the underlying space $\mathcal{M}$ is infinite-dimensional, on the other hand, every $\underline{Y} \in \mathcal{N}$ can be expressed uniquely as $\underline{Y} = \gamma I - \underline{X}$, and thus it is noncompact but has the property that (iv) every λ in the spectrum is a real number, and is in fact an eigenvalue provided that $\lambda \neq \gamma$, which corresponds to the property (i). It also has the above property (ii). Furthermore, it has the property that (v) the number of its negative eigenvalues, as well as the multiplicity of the zero eigenvalue, is finite, which corresponds to the property (iii). For $\underline{Y} \in \mathcal{N}$, $\mathrm{N}(\underline{Y})$ denotes the pair of integers (z, n), where z and n are the multiplicity of the zero eigenvalue and the number of the negative eigenvalues of $\underline{Y}$, respectively. We define $(z_1, n_1) + (z_2, n_2) := (z_1 + z_2, n_1 + n_2)$, and it readily follows that $\mathrm{N}\left(\begin{bmatrix} \underline{Y}_1 & 0 \\ 0 & \underline{Y}_2 \end{bmatrix} \right) = \mathrm{N}(\underline{Y}_1) + \mathrm{N}(\underline{Y}_2)$ if $\begin{bmatrix} \underline{Y}_1 & 0 \\ 0 & \underline{Y}_2 \end{bmatrix} \in \mathcal{N}$. For $\underline{Y} \in \mathcal{N}$, $\underline{Y}$ is said to be positive definite [65] (and we denote it $\underline{Y} > 0$) if $\langle \underline{Y}x, x \rangle > 0$ for every nonzero $x \in \mathcal{M}$, where $\langle x, y \rangle$ denotes the inner product of x, $y \in \mathcal{M}$.

Now, we begin with the following lemma which is an operator version of Sylvester's Law of Inertia under congruent transformations.

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[2]Throughout this chapter, the eigenvalues are counted according to their multiplicities.

Lemma 3.1 Suppose that $\underline{Y} \in \mathcal{N}$ and that $\underline{X}$ is an invertible operator on $\mathcal{M}$. If $\underline{X}^*\underline{Y}\,\underline{X} \in \mathcal{N}$, then $\mathrm{N}\,(\underline{Y}) = \mathrm{N}\,(\underline{X}^*\underline{Y}\,\underline{X})$.

From this lemma, the following lemma is immediate, which is an operator version of the Schur complement arguments.

Lemma 3.2 Suppose that $\begin{bmatrix} \underline{Q} & \underline{S} \\ \underline{S}^* & \underline{R} \end{bmatrix} \in \mathcal{N}$ and suppose that $\underline{Q}$ and $\underline{R}$ are invertible when their inverses are referred to. Then,

$$\mathrm{N}\left(\begin{bmatrix} \underline{Q} & \underline{S} \\ \underline{S}^* & \underline{R} \end{bmatrix}\right) = \mathrm{N}\left(\begin{bmatrix} \underline{Q} - \underline{S}\,\underline{R}^{-1}\underline{S}^* & 0 \\ 0 & \underline{R} \end{bmatrix}\right) = \mathrm{N}\left(\begin{bmatrix} \underline{Q} & 0 \\ 0 & \underline{R} - \underline{S}^*\underline{Q}^{-1}\underline{S} \end{bmatrix}\right).$$
(3.1)

We also have the following lemmas.

Lemma 3.3 Suppose $\underline{Y} \in \mathcal{N}$. Then, $\underline{Y} > 0$ if and only if all eigenvalues of $\underline{Y}$ are positive (i.e., $\mathrm{N}(\underline{Y}) = (0,0)$).

Lemma 3.4 Let $X = \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}$, where A is an n-dimensional invertible matrix. Then, $\mathrm{N}(X) = (0, n)$.

The proofs of the above lemmas are given below.

Proof of Lemma 3.1: Let $\underline{Y} = \gamma I - \underline{Z}$ where $\gamma > 0$ and $\underline{Z} \in \mathcal{W}$. The spectral decomposition theorem ensures that there exists a complete orthonormal set $\{e_1, e_2, \cdots\}$ consisting of the eigenvectors of $\underline{Z}$, such that for any $x \in \mathcal{M}$, $\underline{Z}x$ can be expressed as

$$\underline{Z}x = \sum_{i=1}^{\infty} \lambda_i \langle x, e_i \rangle e_i,$$
(3.2)

where λ_i is the eigenvalue of $\underline{Z}$ corresponding to e_i (Corollary 8.16 of [85]). We first show that the above spectral decomposition can be extended to $\underline{Y} \in \mathcal{N}$. Since $\{e_i\}$ is a complete orthonormal set, we have

$$x = \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i.$$
(3.3)

From (3.2) and (3.3), we obtain

$$\underline{Y}x = (\gamma I - \underline{Z})x = \gamma \sum_{i=1}^{\infty} \langle x, e_i \rangle e_i - \sum_{i=1}^{\infty} \lambda_i \langle x, e_i \rangle e_i$$

$$= \sum_{i=1}^{\infty} (\gamma - \lambda_i) \langle x, e_i \rangle e_i,$$
(3.4)

which implies that the spectral decomposition is also possible for $\underline{Y} \in \mathcal{N}$.

Now, we return to the proof of Lemma 3.1. Since $\underline{Y} \in \mathcal{N}$ and $\underline{X}^*\underline{Y}\,\underline{X} \in \mathcal{N}$ by assumption, it follows from the above arguments that there exist complete orthonormal sets $\{e_1, e_2, \cdots\}$ and $\{f_1, f_2, \cdots\}$ consisting of the eigenvectors of $\underline{Y}$ and $\underline{X}^*\underline{Y}\,\underline{X}$, respectively, such that for any $x \in \mathcal{M}$, $\underline{Y}x$ and $\underline{X}^*\underline{Y}\,\underline{X}x$ can be expressed respectively as

$$\underline{Y}x = \sum_{i=1}^{\infty} \mu_i \langle x, e_i \rangle e_i, \tag{3.5}$$

$$\underline{X}^*\underline{Y}\,\underline{X}x = \sum_{i=1}^{\infty} \nu_i \langle x, f_i \rangle f_i, \tag{3.6}$$

where μ_i is the eigenvalue of $\underline{Y}$ corresponding to e_i, and ν_i is the eigenvalue of $\underline{X}^*\underline{Y}\,\underline{X}$ corresponding to f_i. Let s and t be the numbers of negative eigenvalues of $\underline{Y}$ and $\underline{X}^*\underline{Y}\,\underline{X}$, respectively. Here, we aim to show $s = t$. To this end, suppose that $s < t$.

Note that the convergences of (3.5) and (3.6) are unconditional, i.e., these limits are invariant under reordering of the terms. (See Corollary 5.17.11 of [65].) Therefore, we can assume without loss of generality that $\mu_1, \cdots, \mu_s < 0$ while $\mu_{s+i} \geq 0$ $(i \geq 1)$, and $\nu_1, \cdots, \nu_t < 0$ while $\nu_{t+i} \geq 0$ $(i \geq 1)$. Also, consider the following linear equation for $[a_1, \cdots a_t]^T$.

$$\begin{bmatrix} \langle \underline{X}f_1, e_1 \rangle & \cdots & \langle \underline{X}f_t, e_1 \rangle \\ \vdots & & \vdots \\ \langle \underline{X}f_1, e_s \rangle & \cdots & \langle \underline{X}f_t, e_s \rangle \end{bmatrix} \begin{bmatrix} a_1 \\ \vdots \\ a_t \end{bmatrix} = 0. \tag{3.7}$$

Then, as an under-determined equation, there exists a nontrivial solution $[a_1, \cdots, a_t]^T$ to (3.7). Let $z = \sum_{j=1}^{t} a_j f_j$. Since

$$\langle z, f_i \rangle = \langle \sum_{j=1}^{t} a_j f_j, f_i \rangle = \begin{cases} a_i & 1 \leq i \leq t \\ 0 & t < i, \end{cases} \tag{3.8}$$

it follows from (3.6) that

$$\underline{X}^*\underline{Y}\,\underline{X}z = \sum_{i=1}^{\infty} \nu_i \langle z, f_i \rangle f_i = \sum_{i=1}^{t} \nu_i a_i f_i.$$

Therefore, again from (3.8), we have

$$\langle \underline{X}^*\underline{Y}\,\underline{X}z, z \rangle = \sum_{i=1}^{t} \nu_i a_i \langle f_i, z \rangle = \sum_{i=1}^{t} \nu_i |a_i|^2 < 0. \tag{3.9}$$

On the other hand, for $1 \le i \le s$, $\langle \underline{X}z, e_i \rangle = \sum_{j=1}^{t} a_j \langle \underline{X}f_j, e_i \rangle = 0$ by (3.7). This together with (3.5) implies

$$\underline{Y}\,\underline{X}z = \sum_{i=1}^{\infty} \mu_i \langle \underline{X}z, e_i \rangle e_i = \sum_{i=s+1}^{\infty} \mu_i \langle \underline{X}z, e_i \rangle e_i. \tag{3.10}$$

Therefore, we have

$$\langle \underline{Y}\,\underline{X}z, \underline{X}z \rangle = \sum_{i=s+1}^{\infty} \mu_i \langle \underline{X}z, e_i \rangle \langle e_i, \underline{X}z \rangle = \sum_{i=s+1}^{\infty} \mu_i |\langle \underline{X}z, e_i \rangle|^2 \ge 0. \tag{3.11}$$

This together with (3.9) contradicts $\langle \underline{X}^*\underline{Y}\,\underline{X}z, z \rangle = \langle \underline{Y}\,\underline{X}z, \underline{X}z \rangle$. Therefore, $s \ge t$. We can establish $s \le t$ in a similar manner, and thus we obtain $s = t$.

Finally, we have to show that the multiplicity of the zero eigenvalue is invariant under infinite-dimensional congruent transformations. However, this is immediate since we can readily see that the dimension of the eigenspace corresponding to the zero eigenvalue is invariant. This completes the proof.

Proof of Lemma 3.2: Suppose that $\underline{Y} = \begin{bmatrix} \underline{Q} & \underline{S} \\ \underline{S}^* & \underline{R} \end{bmatrix} = \gamma I - \begin{bmatrix} \check{Q} & \check{S} \\ \check{S}^* & \check{R} \end{bmatrix}$ and $\underline{X} =$

$\begin{bmatrix} I & 0 \\ -\underline{R}^{-1}\underline{S}^* & I \end{bmatrix}$ where $\begin{bmatrix} \check{Q} & \check{S} \\ \check{S}^* & \check{R} \end{bmatrix} \in \mathcal{W}$. Then,

$$\underline{X}^*\underline{Y}\,\underline{X} = \begin{bmatrix} \underline{Q} - \underline{S}\,\underline{R}^{-1}\underline{S}^* & 0 \\ 0 & \underline{R} \end{bmatrix} = \gamma I - \begin{bmatrix} \check{Q} + \check{S}\underline{R}^{-1}\check{S}^* & 0 \\ 0 & \check{R} \end{bmatrix}.$$

Similarly, by letting $\underline{X} = \begin{bmatrix} I & -\underline{Q}^{-1}\underline{S} \\ 0 & I \end{bmatrix}$, we obtain

$$\underline{X}^*\underline{Y}\,\underline{X} = \begin{bmatrix} \underline{Q} & 0 \\ 0 & \underline{R} - \underline{S}^*\underline{Q}^{-1}\underline{S} \end{bmatrix} = \gamma I - \begin{bmatrix} \check{Q} & 0 \\ 0 & \check{R} + \check{S}^*\underline{Q}^{-1}\check{S} \end{bmatrix}.$$

Therefore, from Lemma 3.1, the proof becomes complete if we show that $\check{Q} \in \mathcal{W}$, $\check{R} \in \mathcal{W}$, $\check{S}\underline{R}^{-1}\check{S}^* \in \mathcal{W}$, and $\check{S}^*\underline{Q}^{-1}\check{S} \in \mathcal{W}$. However, this is immediate since $\begin{bmatrix} \check{Q} & \check{S} \\ \check{S}^* & \check{R} \end{bmatrix} \in \mathcal{W}$ if and only if $\check{Q} \in \mathcal{W}$, $\check{R} \in \mathcal{W}$, and $\check{S}$ is compact.

Proof of Lemma 3.3: By definition, $\underline{Y} > 0$ if and only if $\langle \underline{Y}x, x \rangle > 0$ for every non-zero $x \in \mathcal{M}$. However, since $\langle \underline{Y}x, x \rangle = \sum_{i=1}^{\infty} \mu_i |\langle x, e_i \rangle|^2$ by (3.5), we can see that $\underline{Y} > 0$ if and only if $\mu_i > 0$ $(\forall i)$. This completes the proof.

Proof of Lemma 3.4: $X = \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}$ is similar to $-X$ since $-X = J^{-1}XJ$ where

$J = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$. Therefore, the set of eigenvalues of X and that of $-X$ coincide, including multiplicities. By this, together with the assumption that A is invertible, we obtain $\mathrm{N}(X) = (0, n)$.

3.2 Bisection-Based Norm Computation of FR-Operators

In this section, by exploiting the useful lemmas shown in Sec. 3.1, we provide a bisection-based method for computing the frequency response gain of the sampled-data system in Figure 2.3. Here, we are interested in checking whether

$$\|\underline{Q}(j\varphi)\|_{l^2/l^2} < \gamma \tag{3.12}$$

for a given γ. Regarding the problem of checking whether (3.12) is true, we introduce the following assumption without loss of generality.

$$D_{11} = 0. \tag{3.13}$$

This is because an appropriate J-unitary transformation can always reduce the problem of checking (3.12) to that of checking an inequality of the same form but with $D_{11} = 0$. Concerning the existence of such a J-unitary transformation, see Hagiwara et al. [29] and Zhou et al. [86] for more details. The condition (3.13) guarantees the compactness of the FR-operator $\underline{P}_{11}(j\varphi)$ and, consequently, that of $\underline{Q}(j\varphi)$. This enables us to employ the lemmas of Sec. 3.1. Note that $\|\underline{Q}(j\varphi)\|_{l^2/l^2}$ is equal to the maximum singular value $\sigma_1\big(\underline{Q}(j\varphi)\big)$.

3.2.1 Criterion based on the numbers of negative eigenvalues of operators

Now, we proceed to the topic of deriving a bisection algorithm for computing the frequency response gain at given angular frequency φ. In the following, we assume that $\gamma > 0$ does not coincide with a singular value of $\underline{P}_{11}(j\varphi)$. Since only 0 is the accumulation point of the singular values of $\underline{P}_{11}(j\varphi)$, this assumption will be satisfied for almost every γ.

Regarding (3.12), we can readily show that this condition is equivalent to the following condition by using the Schur complement arguments (Lemma 3.2).

$$\begin{bmatrix} \gamma I & -\underline{Q}^*(j\varphi) \\ -\underline{Q}(j\varphi) & \gamma I \end{bmatrix} > 0 \tag{3.14}$$

By substituting (2.3) into (3.14), the above condition can be restated as $\underline{K}(\varphi, \gamma) > 0$,

where

$$
\underline{K}(\varphi,\gamma) := \begin{bmatrix} \gamma I & -\underline{P_{11}}^*(j\varphi) \\ -\underline{P_{11}}(j\varphi) & \gamma I \end{bmatrix} - \frac{1}{\tau} \begin{bmatrix} \underline{P_{21}}^*(j\varphi) & 0 \\ 0 & \underline{P_{12}H}(j\varphi)C_d(e^{j\varphi\tau}) \end{bmatrix}
$$
$$
\times \begin{bmatrix} 0 & I - \Pi_{22}(e^{j\varphi\tau})C_d(e^{j\varphi\tau}) \\ I - C_d^*(e^{j\varphi\tau})\Pi_{22}^*(e^{j\varphi\tau}) & 0 \end{bmatrix}^{-1} \begin{bmatrix} \underline{P_{21}}(j\varphi) & 0 \\ 0 & C_d^*(e^{j\varphi\tau})\underline{P_{12}H}^*(j\varphi) \end{bmatrix}.
$$
$$(3.15)$$

Here, note that $I - \Pi_{22}(e^{j\varphi\tau})C_d(e^{j\varphi\tau})$ is invertible because the feedback system is assumed to be internally stable. By this, together with Lemmas 3.3 and 3.4, we can readily see that the condition $\underline{K}(\varphi,\gamma) > 0$ is equivalent to the condition

$$
\mathrm{N}\left(\begin{bmatrix} \underline{K}(\varphi,\gamma) & 0 \\ 0 & \begin{bmatrix} 0 & I - \Pi_{22}(e^{j\varphi\tau})C_d(e^{j\varphi\tau}) \\ I - \Pi_{22}^*(e^{j\varphi\tau})C_d^*(e^{j\varphi\tau}) & 0 \end{bmatrix} \end{bmatrix}\right) = (0, l),
$$
$$(3.16)$$

where l denotes the number of the output y of $P_{22}(s)$, i.e., $y \in \mathbf{R}^l$. This condition in turn is equivalent to the condition

$$
\mathrm{N}\left(\begin{bmatrix} \begin{bmatrix} \gamma I & -\underline{P_{11}}^*(j\varphi) \\ -\underline{P_{11}}(j\varphi) & \gamma I \end{bmatrix} & 0 \\ 0 & \underline{F}(\varphi,\gamma) \end{bmatrix}\right) = (0, l)
$$
$$(3.17)$$

by (3.15) and Lemma 3.2, where the finite-dimensional matrix $F(\varphi,\gamma)$ is given by

$$
F(\varphi,\gamma) := \begin{bmatrix} 0 & I - \Pi_{22}(e^{j\varphi\tau})C_d(e^{j\varphi\tau}) \\ I - C_d^*(e^{j\varphi\tau})\Pi_{22}^*(e^{j\varphi\tau}) & 0 \end{bmatrix}
$$
$$
- \frac{1}{\tau} \begin{bmatrix} \underline{P_{21}}(j\varphi) & 0 \\ 0 & C_d^*(e^{j\varphi\tau})\underline{P_{12}H}^*(j\varphi) \end{bmatrix} \begin{bmatrix} \gamma I & -\underline{P_{11}}^*(j\varphi) \\ -\underline{P_{11}}(j\varphi) & \gamma I \end{bmatrix}^{-1}
$$
$$
\times \begin{bmatrix} \underline{P_{21}}^*(j\varphi) & 0 \\ 0 & \underline{P_{12}H}(j\varphi)C_d(e^{j\varphi\tau}) \end{bmatrix}.
$$
$$(3.18)$$

Summarizing the above, we can obtain the following theorem.

Theorem 3.1 Suppose that γ is not a singular value of $\underline{P_{11}}(j\varphi)$. Then, the following four statements are equivalent, where l is the number of outputs of $P_{22}(s)$, and ν is the number of singular values of $\underline{P_{11}}(j\varphi)$ larger than γ.

(i) $\|\underline{Q}(j\varphi)\|_{l^2/l^2} < \gamma$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ (3.19)

(ii) $\mathrm{N}\left(\underline{K}(\varphi,\gamma)\right) + \mathrm{N}\left(\begin{bmatrix} 0 & I - \Pi_{22}(e^{j\varphi\tau})C_d(e^{j\varphi\tau}) \\ I - C_d^*(e^{j\varphi\tau})\Pi_{22}^*(e^{j\varphi\tau}) & 0 \end{bmatrix}\right)$

$$= (0, l) \qquad (3.20)$$

(iii) $\mathrm{N}\left(\begin{bmatrix} \gamma I & -\underline{P_{11}}^*(j\varphi) \\ -\underline{P_{11}}(j\varphi) & \gamma I \end{bmatrix}\right) + \mathrm{N}\left(F(\varphi,\gamma)\right) = (0, l) \qquad (3.21)$

(iv) $\mathrm{N}\left(F(\varphi,\gamma)\right) = (0, l - \nu) \qquad (3.22)$

By Theorem 3.1, whether the condition (3.12) holds can be checked by counting the number of singular values of $\underline{P_{11}}(j\varphi)$ larger than γ, the number of negative eigenvalues of the finite-dimensional matrix $F(\varphi,\gamma)$, and the multiplicity of the zero eigenvalue of $F(\varphi,\gamma)$. For the computation of the first number, we can apply the bisection algorithm that will be developed in Sec. 3.2.2; for the second and third numbers, the computation of $F(\varphi,\gamma)$ will be described in Sec. 3.2.3. Accordingly, we can readily obtain a complete bisection algorithm.

Remark 3.1 In Theorem 3.1, the size of the square matrix $F(\varphi,\gamma)$ is twice the number of the outputs of $P_{22}(s)$. The dual form to Theorem 3.1, in which the size of $F(\varphi,\gamma)$ is twice the number of the inputs of $P_{22}(s)$, can be obtained in a similar way.

Before closing this subsection, let us note that Theorem 3.1 can be extended to a result that can be used to compute other singular values (not only the maximum singular value) of sampled-data systems by a bisection search. To do this, it is sufficient to know the open interval $\left(\sigma_{i+1}\left(\underline{Q}(j\varphi)\right), \sigma_i\left(\underline{Q}(j\varphi)\right)\right)$ in which a given γ is included. Now, suppose that $\sigma_{i+1}\left(\underline{Q}(j\varphi)\right) < \gamma < \sigma_i\left(\underline{Q}(j\varphi)\right)$. We can see that this condition is equivalent to the condition $\mathrm{N}(\underline{K}(\varphi,\gamma)) = (0, i)$. Therefore, we can readily obtain the following result in a similar way to Theorem 3.1.

Theorem 3.2 Suppose that γ is not a singular value of $\underline{P_{11}}(j\varphi)$. Then, the following four statements are equivalent for all non-negative integers i, where $\sigma_0(\cdot)$ is defined to be infinity, l is the number of outputs of $P_{22}(s)$, and ν is the number of singular values of $\underline{P_{11}}(j\varphi)$ larger than γ.

(i) $\sigma_{i+1}\left(\underline{Q}(j\varphi)\right) < \gamma < \sigma_i\left(\underline{Q}(j\varphi)\right) \qquad (3.23)$

(ii) $\mathrm{N}\left(\underline{K}(\varphi,\gamma)\right) + \mathrm{N}\left(\begin{bmatrix} 0 & I - \Pi_{22}(e^{j\varphi\tau})C_d(e^{j\varphi\tau}) \\ I - C_d^*(e^{j\varphi\tau})\Pi_{22}^*(e^{j\varphi\tau}) & 0 \end{bmatrix}\right)$

$$= (0, l + i) \qquad (3.24)$$

$$\text{(iii) N} \left(\begin{bmatrix} \gamma I & -\underline{P_{11}}^*(j\varphi) \\ -\underline{P_{11}}(j\varphi) & \gamma I \end{bmatrix} \right) + \text{N}\left(F(\varphi,\gamma)\right) = (0, l+i) \tag{3.25}$$

$$\text{(iv) N}\left(F(\varphi,\gamma)\right) = (0, l+i-\nu) \tag{3.26}$$

3.2.2 Computation of negative eigenvalues of the infinite-rank block-diagonal matrix

In Sec. 3.2.1, we showed that it is possible to check whether the condition $\|\underline{Q}(j\varphi)\|_{l^2/l^2} < \gamma$ holds by counting the number of singular values of the block-diagonal matrix $\underline{P_{11}}(j\varphi)$ larger than γ, the number of negative eigenvalues of the finite-dimensional matrix $F(\varphi,\gamma)$, and the multiplicity of the zero eigenvalue of $F(\varphi,\gamma)$. In this subsection, we provide a method for counting the number of singular values of $\underline{P_{11}}(j\varphi)$ larger than γ.

Note that $P_{11}(s)$ is strictly proper by assumption (3.13). This implies that the elements $P_{11}(j\varphi_n)$ tend to zero as $|n|$ goes to infinity. Therefore, we can compute the *exact values* of all singular values of $\underline{P_{11}}(j\varphi)$ larger than γ by computing the singular values of the *finite-dimensional* matrix

$$P_{11[N]}(j\varphi) = \text{blockdiag}[P_{11}(j\varphi_{-N}), \cdots, P_{11}(j\varphi), \cdots, P_{11}(j\varphi_N)] \tag{3.27}$$

larger than γ, if N is sufficiently large. One such N is given as follows, as discussed in Dullerud [18]. Let

$$A_p = \begin{bmatrix} -A^T & -\frac{1}{\gamma}C_1^T C_1 \\ \frac{1}{\gamma}B_1 B_1^T & A \end{bmatrix} \tag{3.28}$$

and let

$$\mu = \max\{|\lambda| : \lambda \text{ is a purely imaginary eigenvalue of } A_p\}. \tag{3.29}$$

Then, it is enough to take N such that $\varphi_N > \mu$ and $\varphi_{-N} < -\mu$, which follows from the well-known results of Boyd et al. [11]. Therefore, it follows that we can obtain the number of singular values of $\underline{P_{11}}(j\varphi)$ larger than γ by computing the singular values of the finite-dimensional matrix of (3.27).

3.2.3 Computation of finite-dimensional matrix for norm computation

In this subsection, we show how to compute the finite-dimensional matrix $F(\varphi,\gamma)$ used in the bisection algorithm. Observe that (3.18) can be rewritten as

$$F(\varphi,\gamma) = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} - \begin{bmatrix} I & 0 \\ 0 & C_d^*(e^{j\varphi\tau}) \end{bmatrix} \Phi_\gamma(e^{j\varphi\tau}) \begin{bmatrix} I & 0 \\ 0 & C_d(e^{j\varphi\tau}) \end{bmatrix}, \tag{3.30}$$

where

$$\Phi_\gamma(e^{j\varphi\tau}) := \begin{bmatrix} 0 & \Pi_{22}(e^{j\varphi\tau}) \\ \Pi_{22}^*(e^{j\varphi\tau}) & 0 \end{bmatrix} + \frac{1}{\tau} \begin{bmatrix} \underline{P_{21}}(j\varphi) & 0 \\ 0 & \underline{P_{12}H}^*(j\varphi) \end{bmatrix}$$

$$\times \begin{bmatrix} \gamma I & -\underline{P_{11}}^*(j\varphi) \\ -\underline{P_{11}}(j\varphi) & \gamma I \end{bmatrix}^{-1} \begin{bmatrix} \underline{P_{21}}^*(j\varphi) & 0 \\ 0 & \underline{P_{12}H}(j\varphi) \end{bmatrix}. \quad (3.31)$$

For the computation of $\Phi_\gamma(e^{j\varphi\tau})$, we have the following theorem.

Theorem 3.3 $\Phi_\gamma(e^{j\varphi\tau})$ is the frequency pulse-transfer function of the discrete-time system given by

$$\Phi_\gamma(z) := \left[\begin{array}{c|cc} \Phi_{\gamma 11} & -\Phi_{\gamma 11} \begin{bmatrix} C_2^T \\ 0 \end{bmatrix} & \Phi_{\gamma 12} B_H \\ \hline \begin{bmatrix} 0 & C_2 \end{bmatrix} & 0 & 0 \\ B_H^T \Gamma_{33}^T \Phi_{\gamma 21} & -B_H^T \Gamma_{33}^T \Phi_{\gamma 21} \begin{bmatrix} C_2^T \\ 0 \end{bmatrix} & B_H^T \Gamma_{33}^T \Phi_{\gamma 22} B_H \end{array} \right], \quad (3.32)$$

where

$$\left[\begin{array}{c|c} \Phi_{\gamma 11} & \Phi_{\gamma 12} \\ \hline \Phi_{\gamma 21} & \Phi_{\gamma 22} \end{array} \right] := \left[\begin{array}{cc|c} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} \\ \Gamma_{21} & \Gamma_{22} & \Gamma_{23} \\ \hline \Gamma_{41} & \Gamma_{42} & \Gamma_{43} \end{array} \right], \quad (3.33)$$

and Γ_{ij} are given by

$$\begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & 0 \\ \Gamma_{21} & \Gamma_{22} & \Gamma_{23} & 0 \\ 0 & 0 & \Gamma_{33} & 0 \\ \Gamma_{41} & \Gamma_{42} & \Gamma_{43} & \Gamma_{44} \end{bmatrix} := \exp\left(\begin{bmatrix} -A^T & -\frac{1}{\gamma}C_1^T C_1 & -\frac{1}{\gamma}C_1^T D_{12} C_H & 0 \\ \frac{1}{\gamma}B_1 B_1^T & A & B_2 C_H & 0 \\ 0 & 0 & A_H & 0 \\ C_H^T B_2^T & \frac{1}{\gamma}C_H^T D_{12}^T C_1 & \frac{1}{\gamma}C_H^T D_{12}^T D_{12} C_H & -A_H^T \end{bmatrix} \tau \right). \quad (3.34)$$

The above equations give a formula for the computation of $F(\varphi, \gamma)$ via matrix exponentiation, and thus we can readily count the number of negative eigenvalues of the finite-dimensional matrix $F(\varphi, \gamma)$ and the multiplicity of the zero eigenvalue of $F(\varphi, \gamma)$.

Remark 3.2 The matrix exponentiation (3.34) can be regarded as a generalization of (A.3) in Hayakawa et al. [40], where (A.3) is obtained under the zero-order hold settings with $\gamma = 1$ by using a lifting-based method, whereas (3.34) is derived under the generalized hold settings by using the impulse modulation formula.

Proof of Theorem 3.3: From the impulse modulation formula, the right-hand side of (3.31) can be rewritten as follows:

$$\Phi_\gamma(e^{j\varphi\tau}) = \frac{1}{\tau} \sum_{m=-\infty}^{\infty} \begin{bmatrix} I & 0 \\ 0 & H^*(j\varphi_m) \end{bmatrix} T(j\varphi_m) \begin{bmatrix} I & 0 \\ 0 & H(j\varphi_m) \end{bmatrix}$$

$$= \mathcal{Z}\left[\begin{bmatrix} I & 0 \\ 0 & H^\sim(s) \end{bmatrix} T(s) \begin{bmatrix} I & 0 \\ 0 & H(s) \end{bmatrix}\right]_{z=e^{j\varphi\tau}}, \tag{3.35}$$

where

$$T(s) := \begin{bmatrix} 0 & P_{22}(s) \\ P_{22}^\sim(s) & 0 \end{bmatrix} + \begin{bmatrix} P_{21}(s) & 0 \\ 0 & P_{12}^\sim(s) \end{bmatrix} \begin{bmatrix} \gamma I & -P_{11}^\sim(s) \\ -P_{11}(s) & \gamma I \end{bmatrix}^{-1} \begin{bmatrix} P_{21}^\sim(s) & 0 \\ 0 & P_{12}(s) \end{bmatrix}. \tag{3.36}$$

As seen from (3.36), $T(s)$ can be expressed by the lower linear fractional representation[1] given by the following equation:

$$T(s) = \mathcal{F}_l\left(\begin{bmatrix} 0 & P_{22}(s) & P_{21}(s) & 0 \\ P_{22}^\sim(s) & 0 & 0 & P^\sim_{12}(s) \\ \hline P_{21}^\sim(s) & 0 & 0 & P_{11}^\sim(s) \\ 0 & P_{12}(s) & P_{11}(s) & 0 \end{bmatrix}, \frac{1}{\gamma}I\right)$$

$$= \mathcal{F}_l\left(\begin{bmatrix} \check{A} & \check{B}_1 & \check{B}_2 \\ \hline \check{C}_1 & 0 & \check{D}^T \\ \check{C}_2 & \check{D} & 0 \end{bmatrix}, \frac{1}{\gamma}I\right),$$

where

$$\check{A} = \begin{bmatrix} -A^T & 0 \\ 0 & A \end{bmatrix}, \quad \check{B}_1 = \begin{bmatrix} -C_2^T & 0 \\ 0 & B_2 \end{bmatrix}, \quad \check{B}_2 = \begin{bmatrix} 0 & -C_1^T \\ B_1 & 0 \end{bmatrix},$$

$$\check{C}_1 = \begin{bmatrix} 0 & C_2 \\ B_2^T & 0 \end{bmatrix}, \quad \check{C}_2 = \begin{bmatrix} B_1^T & 0 \\ 0 & C_1 \end{bmatrix}, \quad \check{D} = \begin{bmatrix} 0 & 0 \\ 0 & D_{12} \end{bmatrix}.$$

The above state space representation is obtained by Lemma 2.1. Using the fact that

$$\mathcal{F}_l\left(\begin{bmatrix} \check{A} & \check{B}_1 & \check{B}_2 \\ \hline \check{C}_1 & 0 & \check{D}^T \\ \check{C}_2 & \check{D} & 0 \end{bmatrix}, \frac{1}{\gamma}I\right) = \begin{bmatrix} \check{A} + \frac{1}{\gamma}\check{B}_2\check{C}_2 & \check{B}_1 + \frac{1}{\gamma}\check{B}_2\check{D} \\ \hline \check{C}_1 + \frac{1}{\gamma}\check{D}^T\check{C}_2 & \frac{1}{\gamma}\check{D}^T\check{D} \end{bmatrix},$$

---

[1]For the definition of the lower linear fractional representation, see Notations and Definitions.

and rearranging the righthand side leads to

$$T(s) = \left[\begin{array}{c|cc} E & F_1 & F_2 \\ \hline G_1 & 0 & 0 \\ G_2 & 0 & D \end{array}\right],$$

where

$$E = \begin{bmatrix} -A^T & -\frac{1}{\gamma}C_1^T C_1 \\ \frac{1}{\gamma}B_1 B_1^T & A \end{bmatrix}, \quad F_1 = \begin{bmatrix} -C_2^T \\ 0 \end{bmatrix}, \quad F_2 = \begin{bmatrix} -\frac{1}{\gamma}C_1^T D_{12} \\ B_2 \end{bmatrix},$$

$$G_1 = \begin{bmatrix} 0 & C_2 \end{bmatrix}, \quad G_2 = \begin{bmatrix} B_2^T & \frac{1}{\gamma}D_{12}^T C_1 \end{bmatrix}, \quad D = \frac{1}{\gamma}D_{12}^T D_{12}.$$

Note also that

$$\begin{bmatrix} I & 0 \\ 0 & H(s) \end{bmatrix} = \left[\begin{array}{c|cc} A_H & 0 & I \\ \hline 0 & I & 0 \\ C_H & 0 & 0 \end{array}\right] \begin{bmatrix} I & 0 \\ 0 & H_d(e^{\tau s}) \end{bmatrix}, \tag{3.37}$$

and

$$\begin{bmatrix} I & 0 \\ 0 & H^\sim(s) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & H_d^T(e^{-\tau s}) \end{bmatrix} \left[\begin{array}{c|cc} -A_H^T & 0 & C_H^T \\ \hline 0 & I & 0 \\ -I & 0 & 0 \end{array}\right], \tag{3.38}$$

where $H_d(e^{\tau s})$ is given by (2.64). Therefore, we obtain

$$\Phi_\gamma(z) = \mathcal{Z}\left[\begin{bmatrix} I & 0 \\ 0 & H^\sim(s) \end{bmatrix} T(s) \begin{bmatrix} I & 0 \\ 0 & H(s) \end{bmatrix}\right]$$

$$= \mathcal{Z}\left[\begin{bmatrix} I & 0 \\ 0 & H_d^\sim(e^{\tau s}) \end{bmatrix} \left[\begin{array}{c|cc} -A_H^T & 0 & C_H^T \\ \hline 0 & I & 0 \\ -I & 0 & 0 \end{array}\right] \left[\begin{array}{c|cc} E & F_1 & F_2 \\ \hline G_1 & 0 & 0 \\ G_2 & 0 & D \end{array}\right] \left[\begin{array}{c|cc} A_H & 0 & I \\ \hline 0 & I & 0 \\ C_H & 0 & 0 \end{array}\right] \begin{bmatrix} I & 0 \\ 0 & H_d(e^{\tau s}) \end{bmatrix}\right]$$

$$= \mathcal{Z}\left[\begin{bmatrix} I & 0 \\ 0 & H_d^\sim(e^{\tau s}) \end{bmatrix} \left[\begin{array}{cccc|cc} E & F_2 C_H & 0 & F_1 & 0 \\ 0 & A_H & 0 & 0 & I \\ C_H^T G_2 & C_H^T D C_H & -A_H^T & 0 & 0 \\ \hline G_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -I & 0 & 0 \end{array}\right] \begin{bmatrix} I & 0 \\ 0 & H_d(e^{\tau s}) \end{bmatrix}\right]$$

$$= \begin{bmatrix} I & 0 \\ 0 & H_d^\sim(z) \end{bmatrix} \mathcal{Z}\left[\begin{array}{cccc|cc} E & F_2 C_H & 0 & F_1 & 0 \\ 0 & A_H & 0 & 0 & I \\ C_H^T G_2 & C_H^T D C_H & -A_H^T & 0 & 0 \\ \hline G_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -I & 0 & 0 \end{array}\right] \begin{bmatrix} I & 0 \\ 0 & H_d(z) \end{bmatrix}, \tag{3.39}$$

where we used Lemmas 2.1 and 2.5 together with the fact that $G_1 F_1 = 0$.

Now, we apply Lemma 2.4 to the $\mathcal{Z}$-transform in (3.39). Let $\Phi_{\gamma 11}$, $\Phi_{\gamma 11}$, $\Phi_{\gamma 21}$, $\Phi_{\gamma 22}$, and $\widehat{A}_H$ be given by

$$\begin{bmatrix} \Phi_{\gamma 11} & \Phi_{\gamma 12} & 0 \\ 0 & \widehat{A}_H & 0 \\ \Phi_{\gamma 21} & \Phi_{\gamma 22} & \widehat{A}_H^{-T} \end{bmatrix} = \exp\left(\begin{bmatrix} E & F_2 C_H & 0 \\ 0 & A_H & 0 \\ C_H^T G_2 & C_H^T D C_H & -A_H^T \end{bmatrix} \tau \right). \tag{3.40}$$

Note that (3.40) is nothing but (3.34).

From (3.39) and (3.40), together with Lemma 2.4, we obtain

$$\Phi_\gamma(z) = \begin{bmatrix} G_1 & 0 & 0 \\ 0 & 0 & -H_{\tilde{d}}(z) \end{bmatrix} \left\{ I - \begin{bmatrix} \Phi_{\gamma 11} & \Phi_{\gamma 12} & 0 \\ 0 & \widehat{A}_H & 0 \\ \Phi_{\gamma 21} & \Phi_{\gamma 22} & \widehat{A}_H^{-T} \end{bmatrix} z^{-1} \right\}^{-1} \begin{bmatrix} F_1 & 0 \\ 0 & H_d(z) \\ 0 & 0 \end{bmatrix}. \tag{3.41}$$

Here, note that $H_d(z) = (I - \widehat{A}_H z^{-1}) B_H$ and $-H_{\tilde{d}}(z) = B_H^T \widehat{A}_H^T z(I - \widehat{A}_H^{-T} z^{-1})$ by (2.68). Also note that (1,1), (1,2), (3,1), and (3,2)-components of the matrix inverse appearing in (3.41) are given by

$$(1,1)\text{-component} = (I - \Phi_{\gamma 11} z^{-1})^{-1} \tag{3.42}$$

$$(1,2)\text{-component} = (I - \Phi_{\gamma 11} z^{-1})^{-1} \Phi_{\gamma 12} z^{-1} (I - \widehat{A}_H z^{-1})^{-1} \tag{3.43}$$

$$(3,1)\text{-component} = (I - \widehat{A}_H^{-T} z^{-1})^{-1} z^{-1} \Phi_{\gamma 21} (I - \Phi_{\gamma 11} z^{-1})^{-1} \tag{3.44}$$

$$(3,2)\text{-component} = (I - \widehat{A}_H^{-T} z^{-1})^{-1} z^{-1} \left\{ \Phi_{\gamma 21}(I - \Phi_{\gamma 11} z^{-1})^{-1} z^{-1} \Phi_{\gamma 12} \right.$$
$$\left. + \Phi_{\gamma 22} \right\} (I - \widehat{A}_H z^{-1})^{-1} \tag{3.45}$$

From these equations, we obtain

$$\Phi_\gamma(z)$$
$$= \begin{bmatrix} G_1 & 0 \\ 0 & B_H^T \widehat{A}_H^T \end{bmatrix} \begin{bmatrix} (I - \Phi_{\gamma 11} z^{-1})^{-1} & (I - \Phi_{\gamma 11} z^{-1})^{-1} \Phi_{\gamma 12} z^{-1} \\ \Phi_{\gamma 21}(I - \Phi_{\gamma 11} z^{-1})^{-1} & \Phi_{\gamma 21}(I - \Phi_{\gamma 11} z^{-1})^{-1} z^{-1} \Phi_{\gamma 12} + \Phi_{\gamma 22} \end{bmatrix} \begin{bmatrix} F_1 & 0 \\ 0 & B_H \end{bmatrix}$$
$$= \begin{bmatrix} G_1 & 0 \\ 0 & B_H^T \widehat{A}_H^T \end{bmatrix} \begin{bmatrix} (zI - \Phi_{\gamma 11})^{-1} \Phi_{\gamma 11} + I & (zI - \Phi_{\gamma 11})^{-1} \Phi_{\gamma 12} \\ \Phi_{\gamma 21}(zI - \Phi_{\gamma 11})^{-1} \Phi_{\gamma 11} + \Phi_{\gamma 21} & \Phi_{\gamma 21}(zI - \Phi_{\gamma 11})^{-1} \Phi_{\gamma 12} + \Phi_{\gamma 22} \end{bmatrix} \begin{bmatrix} F_1 & 0 \\ 0 & B_H \end{bmatrix}$$
$$= \begin{bmatrix} G_1 & 0 \\ 0 & B_H^T \widehat{A}_H^T \end{bmatrix} \left\{ \begin{bmatrix} I \\ \Phi_{\gamma 21} \end{bmatrix} (zI - \Phi_{\gamma 11})^{-1} \begin{bmatrix} \Phi_{\gamma 11} & \Phi_{\gamma 12} \end{bmatrix} + \begin{bmatrix} I & 0 \\ \Phi_{\gamma 21} & \Phi_{\gamma 22} \end{bmatrix} \right\} \begin{bmatrix} F_1 & 0 \\ 0 & B_H \end{bmatrix}$$
$$= \begin{bmatrix} G_1 \\ B_H^T \widehat{A}_H^T \Phi_{\gamma 21} \end{bmatrix} (zI - \Phi_{\gamma 11})^{-1} \begin{bmatrix} \Phi_{\gamma 11} F_1 & \Phi_{\gamma 12} B_H \end{bmatrix} + \begin{bmatrix} G_1 F_1 & 0 \\ B_H^T \widehat{A}_H^T \Phi_{\gamma 21} F_1 & B_H^T \widehat{A}_H^T \Phi_{\gamma 22} B_H \end{bmatrix}. \tag{3.46}$$

The last equation of (3.46) coincides with (3.32) because $\widehat{A}_H = \Gamma_{33}$, $G_1 = \begin{bmatrix} 0 & C_2 \end{bmatrix}$, and $F_1 = \begin{bmatrix} -C_2^T \\ 0 \end{bmatrix}$. This completes the proof of Theorem 3.3.

3.3 Related Issues and Further Developments

In the preceding sections, we derived a bisection method for computing the frequency response gain of general sampled-data systems by using the infinite-dimensional congruent transformation. In this section, we consider the following issues related to our bisection algorithm. In Sec. 3.3.1, we give a bisection algorithm based on a lifting technique. In Sec. 3.3.2, we derive a one-dimensional search algorithm. We also give a lifting-based counterpart to the one-dimensional search algorithm and clarify its relationship to the computation method of Yamamoto and Khargonekar [81]. In Sec. 3.3.3, we consider the relationship between the solutions of the H^∞ problem of sampled-data systems and the lifting-based bisection algorithm presented in Sec. 3.3.1. Possible applications of the infinite-dimensional congruent transformation are given in Sec. 3.3.4.

3.3.1 Bisection algorithm based on lifting technique

In this subsection, we give an alternative bisection algorithm, based on lifting, for computing the frequency response gain of sampled-data systems. Let us consider the sampled-data system of Figure 2.3, where the state space representation of the discrete-time controller C_d is given by

$$C_d(z) = \left[\begin{array}{c|c} A_{C_d} & B_{C_d} \\ \hline C_{C_d} & D_{C_d} \end{array}\right]. \tag{3.47}$$

According to the lifting theory [81], the counterpart to the FR-operator $\underline{Q}(j\varphi)$ is the frequency response operator $\mathcal{Q}(e^{j\varphi\tau})$ given by

$$\mathcal{Q}(e^{j\varphi\tau}) := \mathcal{C}(e^{j\varphi\tau}I - \mathcal{A})^{-1}\mathcal{B} + \mathcal{D}, \tag{3.48}$$

where

$$\mathcal{A} := \begin{bmatrix} \widehat{A} + B_{2d}D_{C_d}C_{2d} & B_{2d}C_{C_d} \\ B_{C_d}C_{2d} & A_{C_d} \end{bmatrix} \tag{3.49}$$

$$= \begin{bmatrix} I \\ 0 \end{bmatrix} \begin{bmatrix} \widehat{A} & B_{2d} \end{bmatrix} \begin{bmatrix} I & 0 \\ D_{C_d}C_{2d} & C_{C_d} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ B_{C_d}C_{2d} & A_{C_d} \end{bmatrix} \tag{3.50}$$

$$\mathcal{B} := \begin{bmatrix} \mathbf{B}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix} \mathbf{B}_1 \tag{3.51}$$

$$\mathcal{C} := \begin{bmatrix} \mathbf{C}_1 + \mathbf{D}_{12}D_{C_d}C_{2d} & \mathbf{D}_{12}C_{C_d} \end{bmatrix} = \begin{bmatrix} \mathbf{C}_1 & \mathbf{D}_{12} \end{bmatrix} \begin{bmatrix} I & 0 \\ D_{C_d}C_{2d} & C_{C_d} \end{bmatrix} \tag{3.52}$$

$$\mathcal{D} := \mathbf{D}_{11}, \tag{3.53}$$

with

$$\widehat{A} := e^{A\tau}, \quad B_{2d} := \int_0^\tau e^{A(\tau-\sigma)}h(\sigma)d\sigma, \quad C_{2d} := C_2 \tag{3.54}$$

$$\mathbf{B}_1 : \ w(\cdot) \mapsto \int_0^\tau e^{A(\tau-\sigma)}B_1 w(\sigma)d\sigma : \ L^2[0,\tau) \to \mathbf{R}^{n_x} \tag{3.55}$$

$$\mathbf{C}_1 : \ x \mapsto C_1 e^{A\theta}x : \ \mathbf{R}^{n_x} \to L^2[0,\tau) \tag{3.56}$$

$$\mathbf{D}_{11} : \ w(\cdot) \mapsto C_1 \int_0^\theta e^{A(\theta-\sigma)}B_1 w(\sigma)d\sigma + D_{11}w(\theta) : \ L^2[0,\tau) \to L^2[0,\tau) \tag{3.57}$$

$$\mathbf{D}_{12} : u_d \mapsto C_1 \int_0^\theta e^{A(\theta-\sigma)}B_2 h(\sigma)d\sigma u_d + D_{12}h(\theta)u_d \ : \ \mathbf{R}^m \to L^2[0,\tau). \tag{3.58}$$

and, $x \in \mathbf{R}^{n_x}$ and $u_d \in \mathbf{R}^m$.

The frequency response gain at angular frequency φ is defined as the norm of $\mathcal{Q}(e^{j\varphi\tau})$ induced on $L^2[0,\tau)$, i.e., $\|\mathcal{Q}(e^{j\varphi\tau})\|$ [81]. Note that the way an inverse appears in (3.48) and the way one appears in (2.3) are parallel. Also note that the condition $D_{11} = 0$ guarantees that $\mathcal{D}$ is compact, as is $\mathcal{Q}(e^{j\varphi\tau})$. Therefore, it is possible to derive a lifting counterpart to Theorem 3.1 in a similar way as described in the preceding section.

Theorem 3.4 Suppose that γ is not a singular value of $\mathcal{D}$. Then, the following four statements are equivalent, where n is the size of the square matrix $\mathcal{A}$ and $\widehat{\nu}$ is the number of singular values of $\mathcal{D}$ larger than γ.

(i) $\ \|\mathcal{Q}(e^{j\varphi\tau})\| < \gamma;$ $\tag{3.59}$

(ii) $\ \mathrm{N}\left(\mathbf{K}(e^{j\varphi\tau},\gamma)\right) + \mathrm{N}\left(\begin{bmatrix} 0 & e^{j\varphi\tau}I - \mathcal{A} \\ e^{-j\varphi\tau}I - \mathcal{A}^* & 0 \end{bmatrix}\right) = (0,n);$ $\tag{3.60}$

(iii) $\ \mathrm{N}\left(\begin{bmatrix} \gamma I & -\mathcal{D}^* \\ -\mathcal{D} & \gamma I \end{bmatrix}\right) + \mathrm{N}\left(\widehat{F}(e^{j\varphi\tau},\gamma)\right) = (0,n);$ $\tag{3.61}$

(iv) $\mathrm{N}\left(\widehat{F}(e^{j\varphi\tau},\gamma)\right) = (0, n - \widehat{\nu});$ $\tag{3.62}$

where

$$\mathbf{K}(e^{j\varphi\tau},\gamma) := \begin{bmatrix} \gamma I & -\mathcal{D}^* \\ -\mathcal{D} & \gamma I \end{bmatrix} - \begin{bmatrix} \mathcal{B}^* & 0 \\ 0 & \mathcal{C} \end{bmatrix} \begin{bmatrix} 0 & e^{j\varphi\tau}I - \mathcal{A} \\ e^{-j\varphi\tau}I - \mathcal{A}^* & 0 \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{B} & 0 \\ 0 & \mathcal{C}^* \end{bmatrix}, \tag{3.63}$$

$$\widehat{F}(e^{j\varphi\tau},\gamma) := \begin{bmatrix} 0 & e^{j\varphi\tau}I - \mathcal{A} \\ e^{-j\varphi\tau}I - \mathcal{A}^* & 0 \end{bmatrix} - \begin{bmatrix} \mathcal{B} & 0 \\ 0 & \mathcal{C}^* \end{bmatrix} \begin{bmatrix} \gamma I & -\mathcal{D}^* \\ -\mathcal{D} & \gamma I \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{B}^* & 0 \\ 0 & \mathcal{C} \end{bmatrix}. \tag{3.64}$$

From Theorem 3.4, we can check the condition (3.59) by (3.61) so that we can readily obtain a bisection algorithm for the computation of $\|\mathcal{Q}(e^{j\varphi\tau})\|(=\|\underline{Q}(j\varphi)\|_{l^2/l^2})$.

For the computation of $\widehat{F}(e^{j\varphi\tau}, \gamma)$, we can exploit the following result:

$$\widehat{F}(e^{j\varphi\tau}, \gamma) = \begin{bmatrix} 0 & e^{j\varphi\tau}I - \widetilde{E} \\ e^{-j\varphi\tau}I - \widetilde{E}^T & 0 \end{bmatrix} - \begin{bmatrix} \widetilde{B} & 0 \\ 0 & \widetilde{C}^T \end{bmatrix} G(\gamma) \begin{bmatrix} \widetilde{B}^T & 0 \\ 0 & \widetilde{C} \end{bmatrix}, \quad (3.65)$$

where

$$G(\gamma) := \begin{bmatrix} 0 & \widehat{A} & B_{2d} \\ \widehat{A}^T & 0 & 0 \\ B_{2d}^T & 0 & 0 \end{bmatrix} + \begin{bmatrix} \mathbf{B}_1 & 0 \\ 0 & \mathbf{C}_1^* \\ 0 & \mathbf{D}_{12}^* \end{bmatrix} \begin{bmatrix} \gamma I & -\mathbf{D}_{11}^* \\ -\mathbf{D}_{11} & \gamma I \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{B}_1^* & 0 & 0 \\ 0 & \mathbf{C}_1 & \mathbf{D}_{12} \end{bmatrix}, \quad (3.66)$$

$$\widetilde{E} := \begin{bmatrix} 0 & 0 \\ B_{C_d}C_{2d} & A_{C_d} \end{bmatrix}, \quad \widetilde{B} := \begin{bmatrix} I \\ 0 \end{bmatrix}, \quad \widetilde{C} := \begin{bmatrix} I & 0 \\ D_{C_d}C_{2d} & C_{C_d} \end{bmatrix}.$$

The above $G(\gamma)$ can be computed by

$$G(\gamma) = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & B_H^T \Gamma_{44}^{-1} \end{bmatrix} \begin{bmatrix} \Gamma_{21} & \Gamma_{22} & \Gamma_{23} \\ I & 0 & 0 \\ \Gamma_{41} & \Gamma_{42} & \Gamma_{43} \end{bmatrix} \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}^{-1} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & B_H \end{bmatrix}, \quad (3.67)$$

where Γ_{ij} are given by (3.34). The derivation of the above formula is similar to the arguments in earlier works [10],[40], and so it is omitted.

For the computation of $\widehat{\nu}$, i.e., the number of singular values of $\mathcal{D}$ larger than γ, we have a bisection algorithm for computing the singular values of $\mathcal{D} = \mathbf{D}_{11}$. To show this, we employ a technique similar to that used in Theorems 3.4 and 3.2, together with the FR-operator $\underline{P_{11}}(j\varphi)$. The counterpart to $\underline{P_{11}}(j\varphi)$ in the lifting approach is given by

$$\mathbf{P}_{11}(e^{j\varphi\tau}) = \mathbf{C}_1(e^{j\varphi\tau}I - \widehat{A})^{-1}\mathbf{B}_1 + \mathbf{D}_{11}. \quad (3.68)$$

In Yamamoto and Araki [78], it was shown that the frequency response based on the lifting approach and that based on the FR-operator are isometrically isomorphic. This means that the singular values of $\mathbf{P}_{11}(e^{j\varphi\tau})$ and those of $\underline{P_{11}}(j\varphi)$ all coincide for each φ, that is, the following relation holds for each φ and every $\gamma > 0$.

$$\mathrm{N}\left(\begin{bmatrix} \gamma I & -\mathbf{P}_{11}^*(e^{j\varphi\tau}) \\ -\mathbf{P}_{11}(e^{j\varphi\tau}) & \gamma I \end{bmatrix} \right) = \mathrm{N}\left(\begin{bmatrix} \gamma I & -\underline{P_{11}}^*(j\varphi) \\ -\underline{P_{11}}(j\varphi) & \gamma I \end{bmatrix}. \right) \quad (3.69)$$

Note that a method for computing the right-hand side of (3.69) has been already given in Sec. 3.2.2, and thus we can compute the left-hand side of (3.69). On the

other hand, using Lemma 3.2, we can obtain

$$
\mathrm{N}\left(\left[\begin{array}{cc} \begin{bmatrix} \gamma I & -\mathbf{P}_{11}^*(e^{j\varphi\tau}) \\ -\mathbf{P}_{11}(e^{j\varphi\tau}) & \gamma I \end{bmatrix} & 0 \\ 0 & \begin{bmatrix} 0 & e^{j\varphi\tau}I - \widehat{A} \\ e^{-j\varphi\tau}I - \widehat{A}^T & 0 \end{bmatrix} \end{array}\right]\right) = \mathrm{N}\left(\left[\begin{array}{cc} \begin{bmatrix} \gamma I & -\mathbf{D}_{11}^* \\ -\mathbf{D}_{11} & \gamma I \end{bmatrix} & 0 \\ 0 & \widehat{F}_{11}(e^{j\varphi\tau},\gamma) \end{array}\right]\right)
$$

(3.70)

in a similar way to the arguments in Sec. 3.2.1, where the finite-dimensional matrix $\widehat{F}_{11}(e^{j\varphi\tau},\gamma)$ is given by

$$
\widehat{F}_{11}(e^{j\varphi\tau},\gamma) := \begin{bmatrix} 0 & e^{j\varphi\tau}I \\ e^{-j\varphi\tau}I & 0 \end{bmatrix} - G_{11}(\gamma)
$$

(3.71)

$$
G_{11}(\gamma) := \begin{bmatrix} 0 & \widehat{A} \\ \widehat{A}^T & 0 \end{bmatrix} + \begin{bmatrix} \mathbf{B}_1 & 0 \\ 0 & \mathbf{C}_1^* \end{bmatrix} \begin{bmatrix} \gamma I & -\mathbf{D}_{11}^* \\ -\mathbf{D}_{11} & \gamma I \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{B}_1^* & 0 \\ 0 & \mathbf{C}_1 \end{bmatrix}
$$

$$
= \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \end{bmatrix} G(\gamma) \begin{bmatrix} I & 0 \\ 0 & I \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \Gamma_{21} & \Gamma_{22} \\ I & 0 \end{bmatrix} \begin{bmatrix} \Gamma_{11} & \Gamma_{12} \\ 0 & I \end{bmatrix}^{-1}.
$$

(3.72)

Since we already know the value of the left-hand side of (3.69) as mentioned above, we can compute the value of the left-hand side of (3.70) by Lemma 3.4. Therefore, if we focus on the right-hand side of (3.70), by computing the number of negative eigenvalues of $\widehat{F}_{11}(e^{j\varphi\tau},\gamma)$, we can obtain the exact number of negative eigenvalues of the operator $\begin{bmatrix} \gamma I & -\mathbf{D}_{11}^* \\ -\mathbf{D}_{11} & \gamma I \end{bmatrix}$. Since this is true for each $\gamma > 0$, we can compute every singular value of $\mathbf{D}_{11}(=\mathcal{D})$ through a bisection search with respect to γ. It is sufficient to carry out this bisection search at one fixed angular frequency for which $e^{j\varphi\tau}I - \widehat{A}$ is invertible, since $\mathbf{D}_{11}$ is independent of φ.

3.3.2 One-dimensional search algorithm

In this section, we show that another procedure can be obtained from condition (iv) of Theorem 3.2 for the computation of the frequency response gain $\|\underline{Q}(j\varphi)\|_{l^2/l^2}$.

Suppose that γ is larger than $\|\underline{Q}(j\varphi)\|_{l^2/l^2}$, and thus (3.23) holds for $i = 0$. Also suppose that

$$
\sigma_{\nu+1}(\underline{P_{11}}(j\varphi)) < \gamma < \sigma_\nu(\underline{P_{11}}(j\varphi))
$$

(3.73)

so that there are ν singular values of $\underline{P_{11}}(j\varphi)$ larger than γ. Now, as γ decreases, the condition (3.23) fails for $i = 0$ eventually when γ crosses the value of $\|\underline{Q}(j\varphi)\|_{l^2/l^2}$,

and we will have $i > 0$ instead. On the other hand, the value of ν remains unchanged as long as γ remains within the range (3.73), and hence by (3.26) the number of negative eigenvalues of $F(\varphi, \gamma)$ increases as γ crosses $\|\underline{Q}(j\varphi)\|_{l^2/l^2}$. This implies that $\|\underline{Q}(j\varphi)\|_{l^2/l^2}$ can be obtained by searching for the largest value of γ across which the number of negative eigenvalues of $F(\varphi, \gamma)$ increases as γ decreases. (In other words, $\|\underline{Q}(j\varphi)\|_{l^2/l^2}$ can be obtained by searching for the largest value of γ that satisfies the condition $\det\left(F(\varphi, \gamma)\right) = 0$.) Fortunately, it readily follows from the structure of $F(\varphi, \gamma)$ as shown in (3.18) that the eigenvalues of this matrix are uniformly non-increasing as γ decreases, provided that γ is within the range of (3.73) (i.e., provided that γ does not cross a singular value of $\underline{P}_{11}(j\varphi)$). Therefore, it should be easy to find, in a numerically reliable fashion, the largest value of γ across which the number of negative eigenvalues of $F(\varphi, \gamma)$ increases as γ decreases. This gives an alternative method for the computation of the frequency response gain $\|\underline{Q}(j\varphi)\|_{l^2/l^2}$ without a bisection search, which can be carried out without prior knowledge of the singular values of $\underline{P}_{11}(j\varphi)$.

In a similar manner to the above arguments, we can also obtain the lifting-based one-dimensional search algorithm for the computation of the frequency response gain $\|\mathcal{Q}(e^{j\varphi\tau})\|(= \|\underline{Q}(j\varphi)\|_{l^2/l^2})$ from the lifting counterpart to condition (iv) of Theorem 3.2, which is equivalent to $\sigma_{i+1}\left(\mathcal{Q}(e^{j\varphi\tau})\right) < \gamma < \sigma_i\left(\mathcal{Q}(e^{j\varphi\tau})\right)$, given by

$$\mathrm{N}\left(\widehat{F}(e^{j\varphi\tau}, \gamma)\right) = (0, n + i - \widehat{\nu}), \tag{3.74}$$

where $\widehat{\nu}$ is the number of singular values of $\mathcal{D}$ larger than γ, or equivalently, $\widehat{\nu}$ is the number satisfying

$$\sigma_{\widehat{\nu}+1}(\mathcal{D}) < \gamma < \sigma_{\widehat{\nu}}(\mathcal{D}). \tag{3.75}$$

Following the above arguments, it is shown that $\|\mathcal{Q}(e^{j\varphi\tau})\|$ can be obtained by searching for the largest value of γ across which the number of negative eigenvalues of $\widehat{F}(e^{j\varphi\tau}, \gamma)$ increases as γ decreases. (In other words, $\|\mathcal{Q}(e^{j\varphi\tau})\|$ can be obtained by searching for the largest value of γ that satisfies the condition $\det\left(\widehat{F}(e^{j\varphi\tau}, \gamma)\right) = 0$.) Furthermore, as in the case of $F(\varphi, \gamma)$, $\widehat{F}(e^{j\varphi\tau}, \gamma)$ also has an advantageous property in which the eigenvalues of this matrix are uniformly non-increasing as γ decreases, provided that γ is within the range of (3.73). Therefore, we can find the largest value of γ across which the number of negative eigenvalues of $\widehat{F}(e^{j\varphi\tau}, \gamma)$ increases as γ decreases in a numerically reliable fashion.

This one-dimensional search algorithm is in fact closely related to the computation method given by Yamamoto and Khargonekar [81], which is based on the following proposition (adapted and rearranged to fit our notation).

Proposition 3.1 (Theorem 2 of [81]) Suppose that γ is not a singular value of $\mathcal{D}$. Then, the following two statements are equivalent.

(i) γ is a singular value of $\mathcal{Q}(e^{j\varphi\tau})$

(ii) $\det\left(e^{j\varphi\tau}\overline{E}(\gamma) - \overline{A}(\gamma)\right) = 0$

where

$$\overline{E}(\gamma) = \begin{bmatrix} I & -\mathcal{B}(\gamma I - \frac{1}{\gamma}\mathcal{D}^*\mathcal{D})^{-1}\mathcal{B}^* \\ O & \mathcal{A}^* + \mathcal{C}^*\mathcal{D}(\gamma^2 I - \mathcal{D}^*\mathcal{D})^{-1}\mathcal{B}^* \end{bmatrix}$$

$$\overline{A}(\gamma) = \begin{bmatrix} \mathcal{A} + \mathcal{B}(\gamma^2 I - \mathcal{D}^*\mathcal{D})^{-1}\mathcal{D}^*\mathcal{C} & O \\ -\mathcal{C}^*(\gamma I - \frac{1}{\gamma}\mathcal{D}\mathcal{D}^*)^{-1}\mathcal{C} & I \end{bmatrix}.$$

Now, letting $A_\gamma := \mathcal{A} + \mathcal{B}(\gamma^2 I - \mathcal{D}^*\mathcal{D})^{-1}\mathcal{D}^*\mathcal{C}$, $B_\gamma := \mathcal{B}(\gamma I - \frac{1}{\gamma}\mathcal{D}^*\mathcal{D})^{-1}\mathcal{B}^*$ and $C_\gamma := \mathcal{C}^*(\gamma I - \frac{1}{\gamma}\mathcal{D}\mathcal{D}^*)^{-1}\mathcal{C}$, the condition (ii) can be restated as follows.

$$\det\left(\begin{bmatrix} e^{j\varphi\tau}I - A_\gamma & -e^{j\varphi\tau}B_\gamma \\ C_\gamma & e^{j\varphi\tau}A_\gamma^* - I \end{bmatrix}\right) = 0. \tag{3.76}$$

On the other hand, our condition $\det\left(\widehat{F}(e^{j\varphi\tau}, \gamma)\right) = 0$ can be rewritten as

$$\det\left(\begin{bmatrix} -B_\gamma & e^{j\varphi\tau}I - A_\gamma \\ e^{-j\varphi\tau}I - A_\gamma^* & -C_\gamma \end{bmatrix}\right) = 0. \tag{3.77}$$

Naturally, it can easily be verified that (3.76) and (3.77) are equivalent. However, the latter has a numerically desirable feature that the involved matrix $\widehat{F}(e^{j\varphi\tau}, \gamma)$ is self-adjoint (i.e., Hermitian). Furthermore, the eigenvalues of the matrix involved have numerically amenable properties as stated above.

Before closing this subsection, we mention that a one-dimensional search algorithm is also obtained for the computation of singular values of $\mathcal{D}$. Suppose that γ crosses one of the singular values of $\mathcal{D}$ before it crosses $\|\mathcal{Q}(e^{j\varphi\tau})\|$ as γ decreases. In this case, the inequality (3.75) fails for the original $\widehat{\nu}$, and this inequality will be satisfied for a larger value of $\widehat{\nu}$. This implies that the right-hand side of (3.74) must become smaller as γ crosses a singular value of $\mathcal{D}$. However, since the eigenvalues of $\widehat{F}(e^{j\varphi\tau}, \gamma)$ do not increase as γ decreases, which is derived in a similar way to the above arguments for $F(\varphi, \gamma)$, this can happen only by a discontinuous jump of some of the eigenvalues of this matrix at a singular value of $\mathcal{D}$. To state this in reverse, the singular values of $\mathcal{D}$ can be found, including their multiplicities, by observing the place and the number of such kinds of discontinuous jumps of the eigenvalues of $\widehat{F}(e^{j\varphi\tau}, \gamma)$. This gives an alternative method for the computation of the singular values of $\mathcal{D}$. For this computation, it is obviously enough to take only one fixed value φ.

3.3.3 Relationship to the solution of sampled-data H^∞ problem

The exponentiation formula for computing the finite-dimensional matrix $\widehat{F}(e^{j\varphi\tau}, \gamma)$, or equivalently, that for $G(\gamma)$, appearing in the lifting-based methods for computing the frequency response gain of sampled-data systems, frequently also appears in the existing research on the H^∞ control problem of sampled-data systems, e.g., Bamieh and Pearson [10] and Hayakawa et al. [40]. This suggests a strong connection between our study and those existing studies, and we indeed have constructed the following theorem that establishes such a connection.

Theorem 3.5 Suppose that the discrete-time system $\widetilde{P}$ given by

$$
\begin{aligned}
\widetilde{x}[k+1] &= \widetilde{A}\widetilde{x}[k] + \widetilde{B}_1\widetilde{w}[k] + \widetilde{B}_2\rho[k] \\
\widetilde{z}[k] &= \widetilde{C}_1\widetilde{x}[k] + \widetilde{D}_{11}\widetilde{w}[k] + \widetilde{D}_{12}\rho[k] \\
\eta[k] &= \widetilde{C}_2\widetilde{x}[k]
\end{aligned}
\tag{3.78}
$$

satisfies the following three conditions.
(i) $G(\gamma)$ given by (3.66) has the following decomposition.

$$
\begin{aligned}
&\begin{bmatrix} O & \widehat{A} & B_{2d} \\ \widehat{A}^T & O & O \\ B_{2d}^T & O & O \end{bmatrix} + \begin{bmatrix} \mathbf{B}_1 & O \\ O & \mathbf{C}_1^* \\ O & \mathbf{D}_{12}^* \end{bmatrix} \begin{bmatrix} \gamma I & -\mathbf{D}_{11}^* \\ -\mathbf{D}_{11} & \gamma I \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{B}_1^* & O & O \\ O & \mathbf{C}_1 & \mathbf{D}_{12} \end{bmatrix} \\
&= \begin{bmatrix} O & \widetilde{A} & \widetilde{B}_2 \\ \widetilde{A}^T & O & O \\ \widetilde{B}_2^T & O & O \end{bmatrix} + \begin{bmatrix} \widetilde{B}_1 & O \\ O & \widetilde{C}_1^T \\ O & \widetilde{D}_{12}^T \end{bmatrix} \begin{bmatrix} \gamma I & -\widetilde{D}_{11}^T \\ -\widetilde{D}_{11} & \gamma I \end{bmatrix}^{-1} \begin{bmatrix} \widetilde{B}_1^T & O & O \\ O & \widetilde{C}_1 & \widetilde{D}_{12} \end{bmatrix}.
\end{aligned}
\tag{3.79}
$$

(ii) $\gamma > \sigma_1(\widetilde{D}_{11})$.
(iii) For all A_{C_d}, B_{C_d}, C_{C_d} and D_{C_d}, the matrix $\mathcal{A}$ is stable if and only if the matrix $\widetilde{\mathcal{A}}$ is, where

$$
\mathcal{A} = \begin{bmatrix} \widehat{A} + B_{2d}D_{C_d}C_{2d} & B_{2d}C_{C_d} \\ B_{C_d}C_{2d} & A_{C_d} \end{bmatrix}, \quad \widetilde{\mathcal{A}} = \begin{bmatrix} \widetilde{A} + \widetilde{B}_2 D_{C_d}\widetilde{C}_2 & \widetilde{B}_2 C_{C_d} \\ B_{C_d}\widetilde{C}_2 & A_{C_d} \end{bmatrix}.
\tag{3.80}
$$

Then, $\widetilde{P}$ can be used as an equivalent discrete-time system for solving the H^∞ control problem of the original sampled-data system.

Theorem 3.5 can be proved by replacing $G(\gamma)$ in (3.65) with the right-hand side of (3.79) and tracing the arguments in Sec. 3.2.1 in the reverse way *mutatis mutandis*. The importance of Theorem 3.5 lies in that it provides a general class of discrete-time systems that can be used for solving the H^∞ problem of sampled-data

systems. Indeed, we can verify that both the discrete-time system given by Bamieh and Pearson [10] (which we call the B-P type realization) and that by Hayakawa et al. [40] (H-H-Y type) are special examples of $\widetilde{P}$ satisfying the conditions of the above theorem; we can interpret the B-P type realization as the one obtained by solving the equation (3.79) for $\widehat{P}$ under the constraint $\widetilde{D}_{11} = 0$, while the H-H-Y type is that under the constraint $\widetilde{A} = \widehat{A}$ and $\widetilde{B}_2 = B_{2d}$.

3.3.4 Possible applications of the properties of the congruent transformation

As seen in the preceding sections, the properties of the congruent transformation are very useful for the computation of the frequency response gain of sampled-data systems, as well as for the solution of the sampled-data H^∞ control problem. As a matter of fact, the congruent transformation approach developed in this chapter can be applied to a wide class of problems on systems and control. For example, applying a similar technique, the well-known bisection algorithm for computing the H^∞ norm of a continuous-time system [11] can be derived in an elementary way (i.e., purely algebraically, without resorting to a system-theoretic notion such as the inverse systems in [11]). In this subsection, we give further possible applications of the infinite-dimensional congruent transformation approach.

Gain margin and nonlinear stability analysis of sampled-data systems

In Hagiwara [35], the computation of the smallest positive γ (denoted by $\gamma_{\min}$), such that $\underline{Q}(j\varphi) + \gamma\underline{I}$ is positive real, was studied for the sampled-data system of Figure 2.3 with w and z the same size, where the notion of positive real sampled-data systems was introduced in Sugimoto and Suzuki [70],[71]. The inverse of such γ is closely related to the gain margin $k_{\max}$, where $k_{\max}$ is the maximum of k that guarantees the internal stability of the closed-loop system whose FR-operator is given by $(\underline{I} + k\underline{Q}(j\varphi))^{-1}$, and also to the stability of the feedback system consisting of the sampled-data system $\underline{Q}(j\varphi)$ and a sector-bounded nonlinearity. Since the positive real condition can be converted into the bounded real condition by the Cayley transformation, and since the bounded real (i.e., the H^∞ or small-gain) type of problem has been studied much more extensively than the positive real type of problem, one might try to compute $\gamma_{\min}$ in the Cayley transform domain. However, this is not simple because γ is involved in a very complex fashion in the Cayley transform. As a matter of fact, the congruent transformation approach can be applied directly to such positive real problems, too, and it is very useful for the computation of $\gamma_{\min}$. To demonstrate the usefulness of the congruent transformation

approach in a wide class of problems, we give a brief sketch of how it can be applied to this problem. By definition, $\underline{Q}(j\varphi) + \gamma\underline{I}$ is positive real if and only if the sampled-data system is internally stable and

$$\underline{Q}(j\varphi) + \underline{Q}^*(j\varphi) + 2\gamma\underline{I} > 0 \quad (\forall\varphi \in \mathcal{I}_0) \tag{3.81}$$

or equivalently,

$$\left\{2\gamma\underline{I} + \underline{P}_{11}(j\varphi) + \underline{P}_{11}{}^*(j\varphi)\right\} - \frac{1}{\tau} \left[\underline{P}_{21}{}^*(j\varphi) \quad \underline{P}_{12}H(j\varphi)C_d(e^{j\varphi\tau})\right]$$

$$\times \begin{bmatrix} 0 & -I + \Pi_{22}(e^{j\varphi\tau})C_d(e^{j\varphi\tau}) \\ -I + C_d^*(e^{j\varphi\tau})\Pi_{22}^*(e^{j\varphi\tau}) & 0 \end{bmatrix}^{-1} \begin{bmatrix} \underline{P}_{21}(j\varphi) \\ C_d^*(e^{j\varphi\tau})\underline{P}_{12}H^*(j\varphi) \end{bmatrix}$$

$$> 0 \tag{3.82}$$

for all $\varphi \in \mathcal{I}_0$. Assuming that $D_{11} = 0$ as before, we readily have $2\gamma\underline{I} + \underline{P}_{11}(j\varphi) + \underline{P}_{11}{}^*(j\varphi) \in \mathcal{N}$. Thus, we can apply the infinite-dimensional congruent transformation techniques as in Sec. 3.2, and thus we can reduce the checking of (3.82) into counting the following two numbers: the number of eigenvalues of the self-adjoint operator $-(\underline{P}_{11}(j\varphi) + \underline{P}_{11}{}^*(j\varphi))$ larger than 2γ (which we denote by $\nu(\varphi)$) and the number of negative eigenvalues of the finite-dimensional matrix

$$\begin{bmatrix} 0 & -I + \Pi_{22}(e^{j\varphi\tau})C_d(e^{j\varphi\tau}) \\ -I + C_d^*(e^{j\varphi\tau})\Pi_{22}^*(e^{j\varphi\tau}) & 0 \end{bmatrix} - \frac{1}{\tau}\begin{bmatrix} \underline{P}_{21}(j\varphi) \\ C_d^*(e^{j\varphi\tau})\underline{P}_{12}H^*(j\varphi) \end{bmatrix}$$

$$\times \left\{2\gamma\underline{I} + \underline{P}_{11}(j\varphi) + \underline{P}_{11}{}^*(j\varphi)\right\}^{-1} \left[\underline{P}_{21}{}^*(j\varphi) \quad \underline{P}_{12}H(j\varphi)C_d(e^{j\varphi\tau})\right] \tag{3.83}$$

(which we denote by $\mu(\varphi)$). To be more precise, we must check whether the number $\mu(\varphi)$ is equal to $l - \nu(\varphi)$ for all $\varphi \in \mathcal{I}_0$ (this corresponds to condition (iv) of Theorem 3.1). Here, the number $\nu(\varphi)$ can be computed easily since the eigenvalues of $\underline{P}_{11}(j\varphi) + \underline{P}_{11}{}^*(j\varphi)$ can be computed by a finite-dimensional truncation method, which can be derived by slightly modifying the arguments in Sec. 3.2.2. Concerning the computation of $\mu(\varphi)$, we can also derive the matrix exponentiation formula for the finite-dimensional matrix (3.83) by modifying the arguments in Sec. 3.2.3. Consequently, it is easy to check whether $\mu(\varphi)$ is equal to $l - \nu(\varphi)$ for all $\varphi \in \mathcal{I}_0$, or equivalently if $\underline{Q}(j\varphi) + \gamma\underline{I}$ is positive real. Thus, by using the arguments in Sec. 3.2, we can easily construct a bisection algorithm for the computation of $\gamma_{\min}$.

Quadratic constraints for sampled-data systems

A powerful method for the analysis of the feedback connections of two systems was proposed by Megretski and Rantzer [60]. It is based on the idea of integral quadratic constraints (IQC) and includes the bounded real and positive real approaches as special cases. Therefore, to get more general results than those based

on the bounded/positive real conditions as discussed above, it will be important to extend their arguments to the setting of sampled-data systems. Here, we use lifting-based representation for the following reasons: (i) IQC can be reduced to the conditions described by state space representation in general; (ii) the lifting-based representation of sampled-data systems can be regarded as a generalization of the state space representation of continuous or discrete-time systems. Therefore, it is more natural to use lifting-based representation than FR-operator-based representation.

In IQC-based analysis, it is important to check whether the quadratic constraint of the form

$$\begin{bmatrix} (e^{j\varphi\tau}I - \mathcal{A})^{-1}\mathcal{B} \\ I \end{bmatrix}^* \begin{bmatrix} M_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{12}^* & \mathcal{M}_{22} \end{bmatrix} \begin{bmatrix} (e^{j\varphi\tau}I - \mathcal{A})^{-1}\mathcal{B} \\ I \end{bmatrix} < 0 \quad (\forall \; \varphi) \tag{3.84}$$

is satisfied for all $\varphi \in \mathcal{I}_0$, where $\mathcal{A}$ is a stability matrix and M_{11}, $\mathcal{M}_{12}$, and $-\mathcal{M}_{22}$ are, respectively, a Hermitian matrix, a compact operator, and an invertible operator belonging to $\mathcal{N}$; furthermore, they generally depend on the operators $\mathcal{C}$ and $\mathcal{D}$. Such a quadratic constraint corresponds to the one regarding the Kalman-Yakubovich-Popov lemma in the continuous or discrete-time case [68]. Now, we readily see by a direct computation that (3.84) can be restated as follows:

$$-\mathcal{M}_{22} - \begin{bmatrix} \mathcal{B}^* & \mathcal{M}_{12}^* \end{bmatrix} \begin{bmatrix} 0 & e^{j\varphi\tau}I - \mathcal{A} \\ e^{-j\varphi\tau}I - \mathcal{A}^* & -M_{11} \end{bmatrix}^{-1} \begin{bmatrix} \mathcal{B} \\ \mathcal{M}_{12} \end{bmatrix} > 0 \quad (\forall \; \varphi). \tag{3.85}$$

Note that the inverse in (3.85), which we denote here by R^{-1}, is a finite-dimensional matrix, and we can show $\mathrm{N}(R) = (0, n)$, where n is the size of the matrix $\mathcal{A}$. Also note that $-\mathcal{M}_{22} \in \mathcal{N}$ by assumption. Therefore, again we can apply a technique similar to that in the preceding sections for reducing the test of the condition (3.85) to a finite-dimensional test. Thus, the congruent transformation approach can be applied to various problems in systems and control through the IQC method [60].

3.4 Numerical Study

In this section, we apply the bisection algorithm given in the preceding sections to compute the frequency response gain and singular values of sampled-data systems. We also apply the one-dimensional search algorithm, which is a slightly modified version of the method given by Yamamoto and Khargonekar [81], and compare the results with those obtained by our bisection algorithm.

3.4.1 Frequency response gain and singular values of a sampled-data system

Consider the continuous-time plant $\acute{P}(s)$ and the continuous-time controller $C_r(s)$ given by

$$\acute{P}(s) = \frac{1}{4s^2} \cdot \frac{(s/a + 1) \prod_{i=0}^{1}\{(s/\omega_i)^2 + 2\zeta_i(s/\omega_i) + 1\}}{\prod_{i=2}^{4}\{(s/\omega_i)^2 + 2\zeta_i(s/\omega_i) + 1\}}, \tag{3.86}$$

$$C_r(s) = \frac{0.0513s^3 + 0.00424s^2 + 0.0296s + 0.00157}{s^4 + 0.693s^3 + 0.779s^2 + 0.293s + 0.0739}, \tag{3.87}$$

where $a = 4.84$, $\zeta_0 = 0.02$, $\zeta_1 = -0.4$, $\zeta_2 = \zeta_3 = \zeta_4 = 0.02$, $\omega_0 = 1$, $\omega_1 = 5.65$, $\omega_2 = 0.765$, $\omega_3 = 1.41$, and $\omega_4 = 1.85$. This example is found in Anderson and Moore [1]. We compute the singular value plot of the following sampled-data system.

$P(s)$: $P_{11}(s) = P_{12}(s) = \acute{P}(s), P_{21}(s) = P_{22}(s) = -\acute{P}(s)$.

$C_d(z)$: discretization of $C_r(s)$ by the Tustin (bilinear) transformation with sampling period $\tau = 8$.
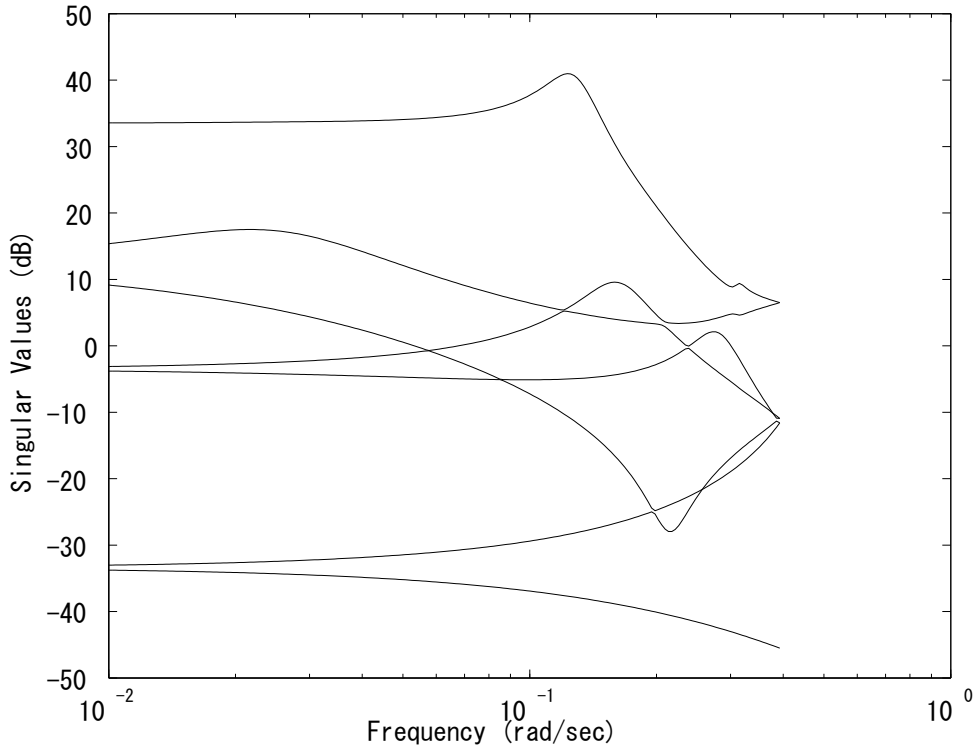
Figure 3.1: Singular value plot.

We suppose that P and C_d are connected with the zero-order hold and the ideal sampler.

Here, based on Theorem 3.2, we compute seven singular values from the largest for each angular frequency, where the number of angular frequencies is 200. To compute the initial upper and lower bounds of the singular values, we use the method given by Hagiwara et al. [34]. The CPU used for the computation is Intel Core$^{\text{TM}}$2 Duo P8700 2.53 GHz, and the amount of RAM is 4 GB. The bisection algorithm is implemented by MATLAB 6.0. The singular value plot of this example, which is accurate up to the tolerance of 0.01dB (can be guaranteed by employing a dB-based bisection method), is shown in Figure 3.1, and the computational load (CPU-time) is shown in Table 3.1. In this table, T_{inf} denotes the CPU-time for computing the singular values of the infinite-dimensional operator $\underline{P_{11}}(j\varphi)$, and T_i denotes that for computing, with the bisection algorithms, the i-th largest singular value for all frequencies.

Table 3.1: Computational load for bisection search (CPU-time [sec])

	CPU-time	number of γ-iterations
T_{inf}	0.109	
T_1	0	0^*
T_2	0	0^*
T_3	0.016	13
T_4	0.109	21
T_5	0.250	310
T_6	1.217	1579
T_7	1.450	1891

$*$ The initial upper and lower bounds for the first and second singular values were close enough, and so no bisection loop was run.

3.4.2 Comparison with the method of Yamamoto and Khargonekar

Here, we compute the frequency response gain of the sampled-data system given in Sec. 3.4.1 with the method by Yamamoto and Khargonekar [81]. More precisely, we use the condition (3.77) instead of (3.76) for better numerical properties. For simplicity, we decrease γ from 44dB to 4dB with a step of ΔdB, since we know
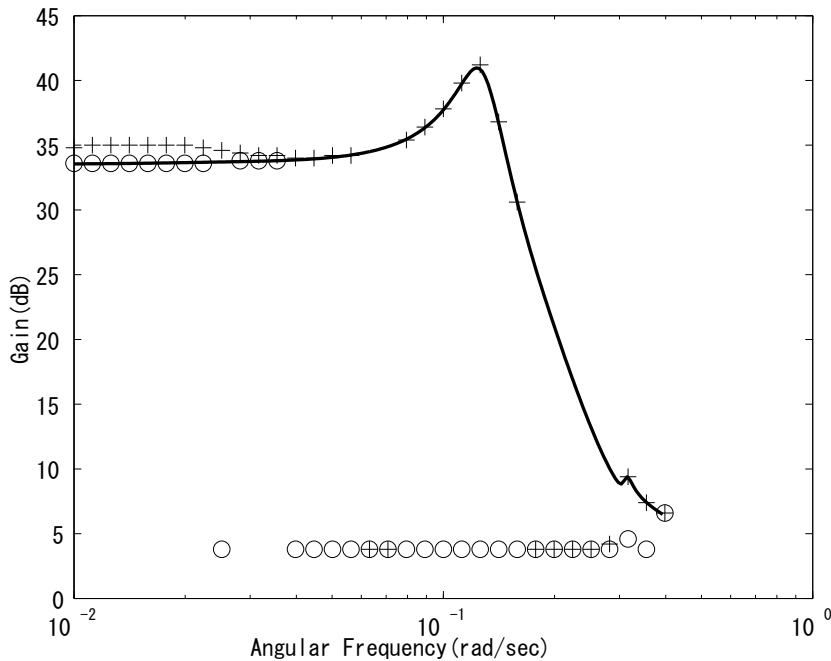
Figure 3.2: $\Delta = 0.2$, $e = 0.01(+)$, $0.001(\circ)$

from Figure 3.1 that the frequency response gain plot is actually within this range. The one-dimensional search is carried out at 33 angular frequencies, and, for each angular frequency, if the magnitude of the determinant (3.77) becomes less than e as we decrease γ, then the frequency response gain is regarded as equal to γ, where we take $e = 0.01$ or 0.001. Figure 3.2 and Figure 3.3 show the results for $\Delta = 0.2$ and for $\Delta = 0.002$, respectively, where "+" and "$\circ$" denote the plots for $e = 0.01$ and $e = 0.001$, respectively. Even though we refer to the computational load shown in Table 3.2, since it is surely an important factor that cannot be neglected, we stress that our primary intention here is to highlight the numerical reliability of our bisection algorithm.

From Figures 3.2 and 3.3, we can observe that a sufficiently small e must be taken to get accurate results, and we also need to take a correspondingly small Δ as we make e smaller. However, from Table 3.2, we can see that making both e and Δ small would require a long time. How much this computation time lengthens depends entirely on how good the initial upper and lower bounds are, but it is generally true that this one-dimensional search is time-consuming compared with the bisection algorithm. Indeed, for the same setting (i.e., the computation is carried out at 33 angular frequencies starting from the uniform upper bound of 44dB and lower bound of 4dB and is accurate up to a tolerance of 0.002dB), the bisection search takes only 0.375 sec with 495 iterations. Another important finding that
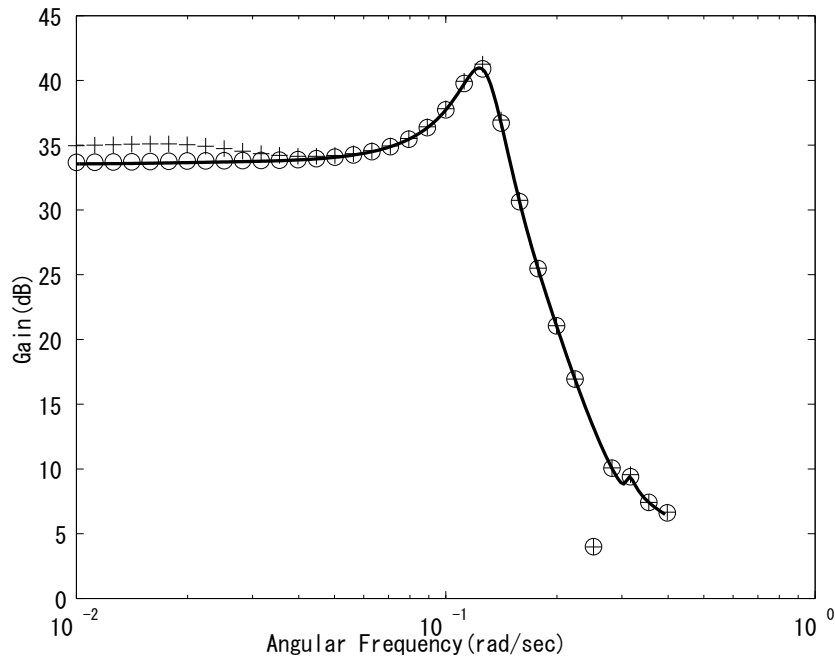
Figure 3.3: $\Delta = 0.002, e = 0.01(+), \ 0.001(\bigcirc)$

Table 3.2: Computational load for one-dimensional search (CPU-time [sec])

		Δ dB	
		0.2	0.002
e	0.01	1.34	98.8
	0.001	2.32	102

should be noted in Figure 3.3 is that the computation of the frequency response gain at $\varphi = 10^{-0.6} \approx 0.25$ rad/sec is very difficult with this one-dimensional search. The situation does not change so much by making e and Δ small. This can be seen from Figure 3.4, where the dependence of the determinant of (3.77) versus γ is plotted for this frequency. Since the slope of this curve at the zero-crossing point is very steep, it is difficult to detect this point by a discrete search of γ. Note that the slope here is steep because this point (i.e., the frequency response gain at this frequency) is close to the discontinuous point of the plot (namely, a singular value of $\mathcal{D}$ — recall the arguments of Sec. 3.3.2). In contrast, our bisection algorithm is free from such a problem since the algorithm can be applied even at those frequencies where the gain is close to a singular value of $\mathcal{D}$.

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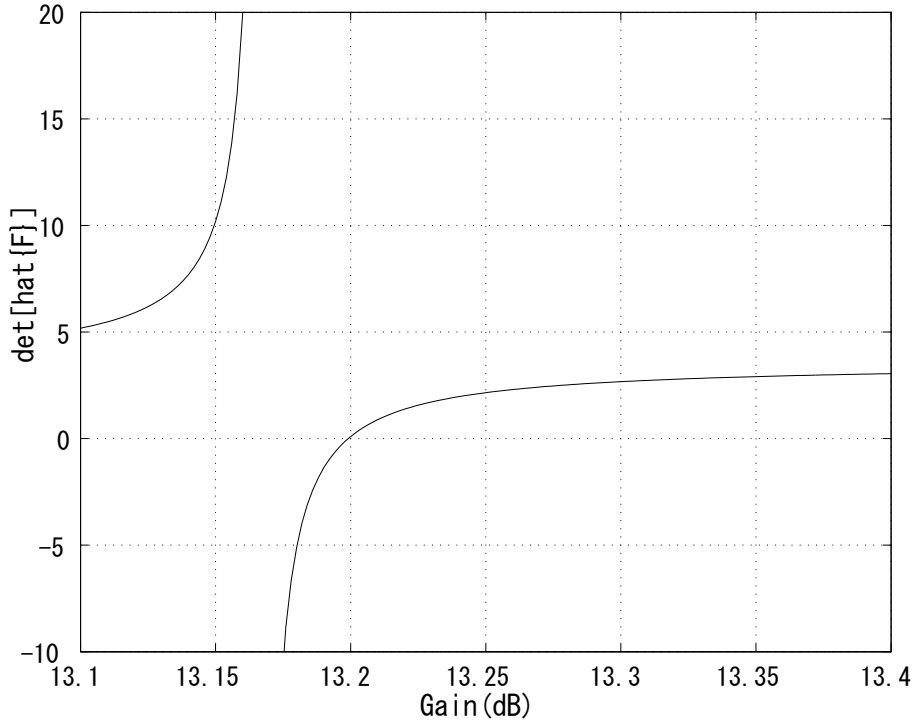
Figure 3.4: γ vs. $\det(\widehat{F}(e^{j\varphi h}, \gamma))$ plot at $\varphi = 10^{-0.6}$

3.5 Summary

In this chapter, we gave a complete bisection algorithm for computing the frequency response gain of sampled-data systems. The algorithm is based on the properties of the infinite-dimensional congruent transformation (i.e., the Schur complement arguments and Sylvester's law of inertia), and we focused on how to count the numbers of negative eigenvalues of self-adjoint matrices. The algorithm can be carried out easily with the state-space matrices of the sampled-data system, and the effectiveness of the derived bisection algorithm is demonstrated through a numerical example.

We also derived a bisection algorithm with the lifting approach by using the same technique. Furthermore, since the algorithm requires computation of the singular values of the operator $\mathcal{D}$, we proposed a bisection algorithm for their computation as well, again with the same technique. The algorithm can also be carried out with the state-space matrices of the sampled-data system.

In addition, some important related issues were discussed such as the relationship to the existing one-dimensional search method by Yamamoto and Khargonekar [81]. Furthermore, the link between computing the frequency response gain and solving

the sampled-data H^∞ problem was made clearer. This contrasts with the apparently prevailing belief (e.g., earlier work [39]) that the former is essentially much harder than the latter, which comes from some difficulties stemming from the fact that the frequency response gain can be smaller than $\|\mathcal{D}\|$, while the H^∞ norm is no smaller than $\|\mathcal{D}\|$.

Chapter 4

FR-Operator Approach to the Sensitivity Reduction Problems of Sampled-Data Systems

In the preceding chapters, we provided several methods for computing the frequency response gain of sampled-data systems using the notion of FR-operators. FR-operators can be used not only for computing the frequency response gain of sampled-data systems but also for analyzing their performance while taking into account intersample behavior and the influence of aliasing. This is possible because FR-operators completely describe the frequency-domain characteristics of sampled-data systems. In this chapter, we consider the sensitivity reduction (SR) and complementary sensitivity reduction (CSR) problems of sampled-data systems by using the notion of FR-operators.

We show that, as far as the SR and CSR problems are concerned, we can have a discretization method to reduce them to equivalent discrete-time problems. This provides a discretized system that we call the *doubly sensitivity-preserving* (DSP) discretized system, which can be used for both the SR and CSR problems of sampled-data systems. We also introduce the discretized system obtained by the 'hold equivalent' discretization, which is referred to as the *naively* discretized system. This chapter is mainly devoted to the study of the SR and CSR problems of sampled-data systems through the DSP discretized system, including an investigation into the important relationship between the DSP discretized plant and the naively discretized plant.

This chapter is organized as follows. First, we define the sensitivity and complementary sensitivity of sampled-data systems and summarize useful notions for studying the SR/CSR problems such as the coprime factorization approach, inner-outer factorization, Nevanlinna problem, and its solution. For details, see Vidyasagar [75]

and Francis [19]. Before going on to a consideration of the SR/CSR problems of sampled-data systems, we consider much easier problems, that is, the SR/CSR problems of naively discretized systems, and show several interesting results for the best achievable performances in these problems by using the coprime factorization approach. Next, we introduce the DSP discretized system and show that the SR/CSR problems of sampled-data systems can be equivalently reduced to those of the DSP discretized system. Based on this result, we show the relationships between the SR (CSR) problem of sampled-data systems (that of DSP discretized systems) and the corresponding problems of the naively discretized systems. Finally, we give results for the best achievable performances of the SR and CSR problems of sampled-data systems through the coprime factorization of the DSP discretized plant and the solution of the Nevanlinna problem with the Pick matrix. A numerical example is also presented to show the relationship between the best achievable performances in the SR/CSR problems of sampled-data systems and those obtained by naively discretized systems.

4.1   Preliminary

4.1.1   Sensitivity and Complementary Sensitivity of Sampled-Data Systems

Consider the sampled-data system Σ shown in Figure 4.1, consisting of the continuous time plant P, the discrete-time controller C_d, the ideal sampler $\mathcal{S}$ with sampling period τ, which works according to $y_d[k] = y(k\tau)$, and the generalized hold $\mathcal{H}$, which works according to $u(k\tau + t) = h(t)u_d[k]$ $(t \in [0, \tau))$, where u and u_d are hold output and input, respectively, and $h(t)$ is called the hold function. The transfer matrices of P, $\mathcal{H}$, and C_d are denoted by $P(s)$, $H(s)$ $(= \int_0^\tau h(t)e^{-st}dt)$, and $C_d(z)$, respectively. In Figure 4.1, solid lines stand for continuous-time signals and dashed lines represent discrete-time signals.

The sensitivity $\underline{S}(j\varphi)$ and the complementary sensitivity $\underline{T}(j\varphi)$ of the sampled-data system are defined as the FR-operators from r to e and u, respectively. According to Araki et al. [8], $\underline{S}(j\varphi)$ and $\underline{T}(j\varphi)$ are operators on l^2 represented as infinite-dimensional matrices given by

$$\underline{S}(j\varphi) := \left(\underline{I} + \frac{1}{\tau}\underline{H}(j\varphi)C_d(e^{j\varphi\tau})\underline{P}(j\varphi) \right)^{-1} = \underline{I} - \frac{1}{\tau}\underline{H}(j\varphi)K_d(e^{j\varphi\tau})\underline{P}(j\varphi), \quad (4.1)$$

$$\underline{T}(j\varphi) := \frac{1}{\tau}\underline{H}(j\varphi)K_d(e^{j\varphi\tau})\underline{P}(j\varphi), \quad (4.2)$$
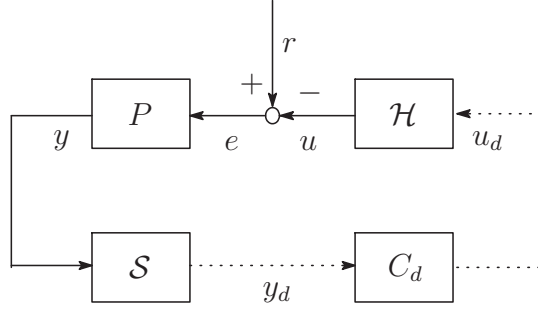
Figure 4.1: Sampled-data system Σ.

where and $\underline{I}$ is the identity on l^2, $\varphi \in \mathcal{I}_0 = (-\pi/\tau, \pi/\tau]$, and

$$
\underline{H}(j\varphi) = \begin{bmatrix} \vdots \\ H(j\varphi_{-1}) \\ H(j\varphi_0) \\ H(j\varphi_1) \\ \vdots \end{bmatrix}, \tag{4.3}
$$

$$
\underline{P}(j\varphi) = \begin{bmatrix} \cdots & P(j\varphi_{-1}) & P(j\varphi_0) & P(j\varphi_1) & \cdots \end{bmatrix}, \tag{4.4}
$$

$$
\varphi_m = \varphi + \frac{2\pi}{\tau}m \quad (m = 0, \pm 1, \cdots), \tag{4.5}
$$

$$
K_d(e^{j\varphi\tau}) = C_d(e^{j\varphi\tau})(I + P_d(e^{j\varphi\tau})C_d(e^{j\varphi\tau}))^{-1}, \tag{4.6}
$$

$$
P_d(e^{j\varphi\tau}) = \frac{1}{\tau}\underline{P}(j\varphi)\underline{H}(j\varphi) = \frac{1}{\tau}\sum_{m=-\infty}^{\infty} P(j\varphi_m)H(j\varphi_m), \tag{4.7}
$$

$$
= \mathcal{Z}[P(s)H(s)]_{z=e^{j\varphi\tau}}. \tag{4.8}
$$

In earlier work [8], it was shown that $\underline{S}(j\varphi)$ gives the improvement of the sensitivity due to the feedback control and also represents the ability to reject disturbances; it was also shown that $\underline{T}(j\varphi)$ represents the degree of robust stability and, at the same time, provides the effect of detection noise. Thus, $\underline{S}(j\varphi)$ and $\underline{T}(j\varphi)$ can be regarded as the natural counterparts to the sensitivity and the complementary sensitivity, respectively, of a continuous-time system. For details, see the earlier work [8]. In the following, we omit $j\varphi$ and $e^{j\varphi\tau}$ when no confusion occurs.

4.1.2 Mathematical Preliminaries

In this subsection, we summarize the useful notions for considering SR and CSR problems, such as the coprime factorization approach, inner-outer factorization, and Nevanlinna problem. See Vidyasagar [75] and Francis [19] for the details.

In this chapter, we often use $\lambda := 1/z$ instead of z to describe transfer matrices of discrete-time systems, and we consider the SR and CSR problems using the coprime factorization on $\mathbf{R}_{-}$, which denotes the set of rational functions analytic on $\bar{\mathbf{D}}$, the closed unit disc. We also use the notation $\mathbf{R}_{-}$ for the set of matrices whose every entry belongs to $\mathbf{R}_{-}$.

First, we summarize the coprime factorization approach. The importance of the coprime factorization approach is that it provides a parameterization of all controllers that stabilize a given plant. For scalar functions a, $b \in \mathbf{R}_{-}$, a and b are said to be coprime if they have no common zeros in $\bar{\mathbf{D}}$. It is well-known that, for a, $b \in \mathbf{R}_{-}$, a and b are coprime if and only if there exist x, $y \in \mathbf{R}_{-}$ satisfying $ax + by = 1$. The equation $ax + by = 1$ is referred to as the Bezout identity. Now, suppose that P_d is the pulse transfer matrix of a scalar discrete-time plant and let $P_d = n/d$ where n, $d \in \mathbf{R}_{-}$ are coprime. We refer to the pair (n, d) as a coprime factorization of P_d. In this case, the set of all controllers C_d that stabilize P_d, denoted by $\mathcal{C}$, is given by

$$\mathcal{C} = \left\{ C_d = \frac{x + dq}{y - nq} \,\middle|\, q \in \mathbf{R}_{-} \text{ and } y - nq \neq 0 \right\}, \tag{4.9}$$

where $(x, y) \in \mathbf{R}_{-}$ is a pair of the solutions of the Bezout identity $nx + dy = 1$. Thus, we can completely parameterize all stabilizing controllers by using free parameter $q \in \mathbf{R}_{-}$. Another importance of the coprime factorization approach is that the discrete-time sensitivity $S_d = 1/(1 + C_d P_d)$ and complementary sensitivity $T_d = C_d P_d/(1 + C_d P_d)$ are expressed by *affine* functions of the free parameter q as $S_d = d(y - nq)$ and $T_d = n(x + dq)$. This significantly reduces the difficulty of handling the SR/CSR problems.

Next, we briefly summarize the definitions of inner and outer on $\mathbf{R}_{-}$ and related issues. A matrix $G \in \mathbf{R}_{-}$ is inner if $G^{\sim}G = I$. A matrix $G \in \mathbf{R}_{-}$ is outer if G has full-row rank for every $\lambda \in \mathbf{D}$, the open unit disc. A matrix $G \in \mathbf{R}_{-}$ is a unit if it has an inverse in $\mathbf{R}_{-}$. It is well known that every $G \in \mathbf{R}_{-}$ has an inner-outer factorization $G = G^i G^o$ where G^i and G^o are inner and outer, respectively. In this case, all zeros of G^i in $\mathbf{D}$ consist of all zeros of G in $\mathbf{D}$. When G has no zeros on the unit circle, G^o turns out to be a unit. For a given matrix G, the set $G\mathbf{R}_{-}$ is defined as $G\mathbf{R}_{-} = \{GQ \mid Q \in \mathbf{R}_{-}\}$. A matrix G is said to be co-inner or co-outer if G^T is inner or outer, respectively. A co-inner-outer factorization has the form $G = G^{co}G^{ci}$, where G^{co} and G^{ci} are co-outer and co-inner, respectively. An inner-outer factorization of G^T yields a co-inner-outer factorization of G.

Finally, we review the Nevanlinna problem, which is closely related to the SR/CSR problems of discrete-time systems. Conforming to the forms that we encounter in the arguments in this chapter, we state the following "canonical" form of the Nevanlinna problem, which will be used to describe the explicit problems dealt with

later.

Problem 4.1 Suppose that α_i $(i = 1, \cdots, m)$ and β_j $(j = 1, \cdots, l)$ are distinct complex numbers with modulus less than 1. Determine whether there exists $F \in \mathbf{R}_\_$ satisfying the conditions $\|F\|_\infty < 1$, $F(\alpha_i) = \Xi_i$ $(i = 1, \ldots, m)$, and $F(\beta_j) = \Upsilon_j$ $(j = 1, \ldots, l)$, where Ξ_i and Υ_j are given complex matrices having a norm less than 1.

Obviously, the optimization problem that finds the infimum of γ such that there exists $F \in \mathbf{R}_\_$ satisfying the conditions $\|F\|_\infty < \gamma$, $F(\alpha_i) = \Xi_i$ $(i = 1, \ldots, m)$, and $F(\beta_j) = \Upsilon_j$ $(j = 1, \ldots, l)$ can be reduced to Problem 4.1 by scaling. It is well known that the following proposition is useful for studying the above problem.

Proposition 4.1 Suppose $\lambda_1, \cdots, \lambda_n$ are distinct complex numbers with modulus less than 1 and $\Lambda_1, \cdots, \Lambda_n$ are complex matrices with a norm less than 1. Define the Pick matrix Q as

$$Q = \begin{bmatrix} Q_{11} & \cdots & Q_{1n} \\ \vdots & \ddots & \vdots \\ Q_{n1} & \cdots & Q_{nn} \end{bmatrix}, \quad Q_{ij} = \frac{1}{1 - \bar{\lambda}_i \lambda_j} \cdot (I - \Lambda_i^* \Lambda_j). \tag{4.10}$$

Then, there exists $F \in \mathbf{R}_\_$ such that $\|F\|_\infty < 1$ and $F(\lambda_i) = \Lambda_i$ $(i = 1, \cdots, n)$ if and only if the matrix Q is positive definite.

4.2　SR and CSR Problems of Naively Discretized Systems

Before going on to consideration of the SR/CSR problems of sampled-data systems, we consider much easier problems, that is, the SR/CSR problems of discrete-time systems obtained by the 'hold equivalent' discretization. We call the discrete-time systems naively discretized systems. The precise definition of the naively discretized system will be given later.

Concerning the SR/CSR problems of naively discretized systems, pioneering work was done by Sung and Hara [73], where several interesting results were derived such as some integral constraints for the sensitivity and complementary sensitivity, and lower bounds of the best achievable performances for SR and CSR problems. The result of the integral constraint for the sensitivity and the result of the lower bound for the SR problem were extended to multiple-input multiple-output case by Hara and Sung [36]. In this section, we further study the SR/CSR problems of naively discretized systems and show that the best achievable performances in

the SR and CSR problems of naively discretized systems coincide. In addition, we show that the two lower bounds of the best achievable performances for these two problems derived by Sung and Hara [73] also coincide with each other.

Let us consider the discrete-time system Σ_d shown in Figure 4.2 consisting of the discrete-time plant P_d and the discrete-time controller C_d, where P_d is obtained by discretizing the continuous-time plant P as in (4.8). This system can be regarded as a discrete-time counterpart of the sampled-data system Σ shown in Figure 4.1 provided that r and r_d are zero. The use of this discrete-time system Σ_d is standard in the treatment of the sampled-data system Σ when we are not concerned about the intersample behavior, e.g., in stability analysis. That is, the discrete-time system Σ_d only describes the behavior at sampling instants, and there is no direct and obvious way to describe the intersample behavior of Σ with Σ_d. Throughout this chapter, we refer to Σ_d as the *naively discretized system*.

The sensitivity $S_d(e^{j\varphi\tau})$ and the complementary sensitivity $T_d(e^{j\varphi\tau})$ of the naively discretized system Σ_d are defined as the frequency transfer matrices from r_d to e_d and u_d, respectively:

$$S_d(e^{j\varphi\tau}) := \left\{ I + C_d(e^{j\varphi\tau})P_d(e^{j\varphi\tau}) \right\}^{-1}, \tag{4.11}$$

$$T_d(e^{j\varphi\tau}) := C_d(e^{j\varphi\tau})P_d(e^{j\varphi\tau}) \left\{ I + C_d(e^{j\varphi\tau})P_d(e^{j\varphi\tau}) \right\}^{-1}. \tag{4.12}$$

The best achievable performance for the SR problem of the naively discretized system Σ_d and that for the CSR problem are defined as $\inf_{C_d \in \mathcal{C}} \|S_d\|_\infty$ and $\inf_{C_d \in \mathcal{C}} \|T_d\|_\infty$, respectively, where $\mathcal{C}$ is the set of all stabilizing controllers C_d. In the rest of this section, we use $\lambda := 1/z$ instead of z to describe transfer matrices of discrete-time systems.

For the naively discretized system Σ_d, we set the following assumptions:

Assumption 4.1

(i) P and $\mathcal{H}$ are nonzero scalar systems.
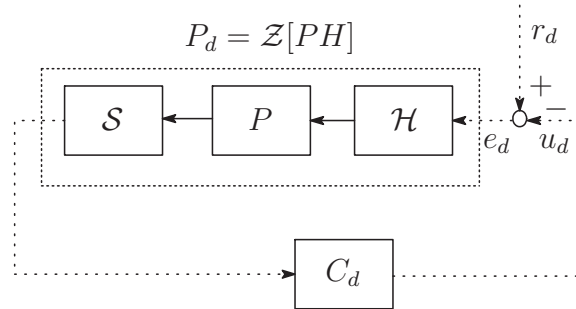
Figure 4.2: Naively discretized system Σ_d.

(ii) P is controllable and observable.

(iii) $P(s)$ is strictly proper and does not have poles on the imaginary axis.

(iv) $P(s)$ has $m(\neq 0)$ distinct poles of order one in $\Re(s) > 0$. They are referred to as unstable poles of $P(s)$, and are denoted by a_i $(i = 1, \cdots, m)$.

(v) The sampling period τ is selected such that the discrete-time plant P_d is controllable and observable.

(vi) P_d does not have zeros on the unit circle.

Remark 4.1 When $m = 0$, we can show that $C_d = 0$ attains the best achievable performances for both SR and CSR problems. It is to avoid such a trivial case that $m \neq 0$ is assumed in condition (iv) of Assumption 4.1.

From conditions (i) and (v) of Assumption 4.1, P_d is also a nonzero scalar system. In this case, according to the coprime factorization approach given in Sec. 4.1.2, the set of all stabilizing controllers is given by[1]

$$
\mathcal{C} = \left\{ C_d = \frac{x + dq}{y - nq} \,\middle|\, q \in \mathbf{R}_- \text{ and } y - nq \neq 0 \right\}, \tag{4.13}
$$

where the pair (n, d) is a coprime factorization of P_d on $\mathbf{R}_-$ satisfying

$$
P_d = n/d, \quad n, d \in \mathbf{R}_- \tag{4.14}
$$

and x, $y \in \mathbf{R}_-$ are the solutions of the Bezout identity

$$
nx + dy = 1. \tag{4.15}
$$

In the following, we arbitrarily fix the solutions x and y. Note that, from (4.11), (4.12), (4.13), (4.14), and (4.15), S_d and T_d are expressed, respectively, as follows:

$$
S_d = d(y - nq), \tag{4.16}
$$
$$
T_d = n(x + dq). \tag{4.17}
$$

Remark 4.2 From Assumption 4.1, together with the results obtained by Middleton and Freudenberg [61], it is shown that, for almost every sampling period τ, the relative degree of $P_d(z)$ is one, and $P_d(z)$ has m unstable poles. In other words, $P_d(1/\lambda)$ has zero of order one at $\lambda = 0$, m distinct unstable poles $e^{-a_i \tau}$ $(i = 1, \cdots, m)$ of order one, and no unstable poles and zeros on the unit circle. Therefore, n^i is expressed as $n^i = \lambda \widehat{n}^i$, where $\widehat{n}^i$ is inner consisting of unstable zeros of $P_d(1/\lambda)$ except for the origin, d^i has m distinct poles $e^{-a_i \tau}$ $(i = 1, \cdots, m)$ in the open unit disc $\mathbf{D}$, and both n^o and d^o become units, where $n^i n^o = n$ and $d^i d^o = d$ are inner-outer factorizations of n and d, respectively.

---

[1]Small letters imply scalar functions.

4.2.1 Relationship Between Best Achievable Performance in SR Problem and That in CSR Problem for Discrete-Time Systems

In this subsection, we show the following theorem, which implies that the best achievable performances in the SR and CSR problems of the naively discretized system Σ_d coincide.

Theorem 4.1 $\inf_{C_d \in \mathcal{C}} \|S_d\|_\infty = \inf_{C_d \in \mathcal{C}} \|T_d\|_\infty$

Proof of Theorem 4.1: Let $\alpha_i := e^{-a_i \tau}$ $(i = 1, \cdots, m)$ and β_j $(i = 1, \cdots, l)$ be unstable poles and zeros of $P_d(1/\lambda)$, respectively, and inner-outer factorizations of d and n be $d = d^i d^o$ and $n = n^i n^o$, respectively. Then, we have $|\alpha_i| < 1$, $d^i(\alpha_1) = \cdots = d^i(\alpha_m) = 0$ and $|\beta_j| < 1$, $n^i(\beta_1) = \cdots = n^i(\beta_l) = 0$. From (4.16) and (4.17), we obtain

$$\inf_{C_d \in \mathcal{C}} \|S_d\|_\infty = \inf_{q \in \mathbf{R}_-} \|d(y - nq)\|_\infty = \inf_{q' \in n^o d^o \mathbf{R}_-} \|dy - n^i d^i q'\|_\infty = \inf_{q' \in \mathbf{R}_-} \|dy - n^i d^i q'\|_\infty, \tag{4.18}$$

$$\inf_{C_d \in \mathcal{C}} \|T_d\|_\infty = \inf_{q \in \mathbf{R}_-} \|n(x + dq)\|_\infty = \inf_{q' \in n^o d^o \mathbf{R}_-} \|nx + n^i d^i q'\|_\infty = \inf_{q' \in \mathbf{R}_-} \|nx + n^i d^i q'\|_\infty. \tag{4.19}$$

The last equalities of (4.18) and (4.19) are due to the fact that $n^o d^o \mathbf{R}_- = \mathbf{R}_-$ because n^o and d^o are units from Assumption 4.1 (See Remark 4.2). Furthermore, from (4.18) and (4.19), we obtain

$$\inf_{C_d \in \mathcal{C}} \|S_d\|_\infty = \inf_{q' \in \mathbf{R}_-} \left\{ \gamma > 0 \ \middle| \ \left\| \frac{dy}{\gamma} - n^i d^i q' \right\|_\infty < 1 \right\}, \tag{4.20}$$

$$\inf_{C_d \in \mathcal{C}} \|T_d\|_\infty = \inf_{q' \in \mathbf{R}_-} \left\{ \gamma > 0 \ \middle| \ \left\| \frac{nx}{\gamma} + n^i d^i q' \right\|_\infty < 1 \right\}. \tag{4.21}$$

Here, note from (4.15) that $dy = 0$ and $nx = 1$ for λ such that $d(\lambda) = 0$, while $dy = 1$ and $nx = 0$ for λ such that $n(\lambda) = 0$. Hence, it follows that the computation of the best achievable performance $\inf_{C_d \in \mathcal{C}} \|S_d\|_\infty$ reduces to Problem 4.1 with $\Xi_i = 0$ $(i = 1, \cdots, m)$, and $\Upsilon_j = 1/\gamma$ $(j = 1, \cdots, l)$. We refer to this problem as Problem-S_d. Similarly, the problem of finding the best achievable performance $\inf_{C_d \in \mathcal{C}} \|T_d\|_\infty$ reduces to Problem 4.1 with $\Xi_i = 1/\gamma$ $(i = 1, \cdots, m)$, and $\Upsilon_j = 0$ $(j = 1, \cdots, l)$. We refer to this problem as Problem-T_d. From these interpolation conditions, we have $\gamma > 1$ for both problems (i.e., Problem-S_d and Problem-T_d) because γ (> 0) should satisfy the constraint $1/\gamma < 1$ in both cases.

From Proposition 4.1, the computation of the infimum γ for Problem-S_d reduces to the problem of finding the infimum of $\gamma > 1$ such that the Pick matrix

$$Q_{S_d} = \begin{bmatrix} X & Y \\ Y^* & (1-\gamma^{-2})Z \end{bmatrix} \tag{4.22}$$

is positive definite, where the (i,j)-elements of X, Y, and Z, which we denote by X_{ij}, Y_{ij}, and Z_{ij}, respectively, are given by $X_{ij} = \frac{1}{1-\bar{\alpha}_i\alpha_j}$ $(i,j = 1,\cdots,m)$, $Y_{ij} = \frac{1}{1-\bar{\alpha}_i\beta_j}$ $(i = 1,\cdots,m; j = 1,\cdots,l)$, and $Z_{ij} = \frac{1}{1-\bar{\beta}_i\beta_j}$ $(i,j = 1,\cdots,l)$. In a similar manner, the computation of the infimum γ for Problem-T_d reduces to the problem of finding the infimum of $\gamma > 1$ such that the Pick matrix

$$Q_{T_d} = \begin{bmatrix} (1-\gamma^{-2})X & Y \\ Y^* & Z \end{bmatrix} \tag{4.23}$$

is positive definite. Here, note that the positive definiteness of Q_{S_d} and that of Q_{T_d} are equivalent because these matrices are mutually related by a congruence transformation. Hence, Theorem 4.1 is established.

4.2.2 Relationship Between Lower Bound of Best Achievable Performance in SR Problem and That in CSR Problem Derived by Sung and Hara

In the preceding subsection, we showed that the best achievable performances in the SR and CSR problems of the naively discretized system Σ_d coincide. In this subsection, we show that the two lower bounds of these two best achievable performances obtained by Sung and Hara [73] also coincide. Their lower bounds are given by the following proposition.

Proposition 4.2

$$\inf_{C_d \in \mathcal{C}} \|S_d\|_\infty \geq e^{\alpha\tau}, \tag{4.24}$$

$$\inf_{C_d \in \mathcal{C}} \|T_d\|_\infty \geq \lambda_{\max}^{1/2}(A_\tau^{-1}B_\tau), \tag{4.25}$$

where $\alpha := \sum_{i=1}^m a_i$. A_τ and B_τ are square matrices of size m whose (i,j)-element $(A_\tau)_{ij}$ and $(B_\tau)_{ij}$ are given, respectively, by

$$(A_\tau)_{ij} := \frac{1}{1-e^{-(\bar{a}_i+a_j)\tau}}, \quad (B_\tau)_{ij} := \frac{e^{(\bar{a}_i+a_j)\tau}}{1-e^{-(\bar{a}_i+a_j)\tau}}. \tag{4.26}$$

Theorem 4.1 says that the left-hand sides of (4.24) and (4.25) in Proposition 4.2 coincide. On the other hand, we can show the following theorem stating that the right-hand sides of (4.24) and (4.25) also coincide.

Theorem 4.2 $e^{\alpha\tau} = \lambda_{\max}^{1/2}(A_\tau^{-1}B_\tau)$.

Proof of Theorem 4.2: In Sung and Hara [73], Proposition 4.2 is obtained by using the following inequalities[2], and by showing that the right-hand side of (4.28) is equal to $e^{\alpha\tau}$ and the right-hand side of (4.30) is equal to $\lambda_{\max}^{1/2}(A_\tau^{-1}B_\tau)$.

$$\inf_{C_d \in \mathcal{C}} \|S_d\|_\infty = \inf_{q \in \mathbf{R}_-} \|dy - d^i d^o \lambda \widehat{n}^i n^o q\|_\infty = \inf_{q' \in \widehat{n}^i n^o d^o \mathbf{R}_-} \|dy - \lambda d^i q'\|_\infty \tag{4.27}$$

$$\geq \inf_{q' \in \mathbf{R}_-} \|dy - \lambda d^i q'\|_\infty = \inf_{q' \in \mathbf{R}_-} \|d^o y - \lambda q'\|_\infty, \tag{4.28}$$

$$\inf_{C_d \in \mathcal{C}} \|T_d\|_\infty = \inf_{q \in \mathbf{R}_-} \|nx + \lambda \widehat{n}^i n^o d^i d^o q\|_\infty = \inf_{q' \in \widehat{n}^i n^o d^o \mathbf{R}_-} \|nx + \lambda d^i q'\|_\infty \tag{4.29}$$

$$\geq \inf_{q' \in \mathbf{R}_-} \|nx + \lambda d^i q'\|_\infty = \inf_{q' \in \mathbf{R}_-} \|\widehat{n}^i n^o x + d^i q'\|_\infty. \tag{4.30}$$

On the other hand, we prove Theorem 4.2 by showing that the *left*-hand side of (4.28) and that of (4.30) coincide. In the following, we only show the outline of the proof because the proof is similar to the proof of Theorem 4.1.

The left-hand side of (4.28) coincides with the right-hand side of (4.18) if we replace n^i by λ. In a similar manner, the left-hand side of (4.30) coincides with the right-hand side of (4.19) by replacing n^i by λ. Therefore, we can show that $\inf_{q' \in \mathbf{R}_-} \|dy - \lambda d^i q'\|_\infty = \inf_{q' \in \mathbf{R}_-} \|nx + \lambda d^i q'\|_\infty$, if we rewrite the arguments starting from (4.20) of the proof of Theorem 4.1 by replacing n^i by λ and setting $l = 1$, $\beta_1 = 0$. This completes the proof.

Remark 4.3 If $P_d(1/\lambda)$ does not have unstable zeros except for the origin, the set $\widehat{n}^i n^o d^o \mathbf{R}_-$ coincides with the set $\mathbf{R}_-$. In this case, from the above inequalities, we can show that the inequalities of (4.24) and (4.25) turn out to be equalities, that is, the right-hand side of (4.24) and that of (4.25) become the best achievable performance of the naively discretized system Σ_d in the SR problem and that in the CSR problem, respectively.

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[2]Although the derivation of these inequalities is found in the earlier work [73], we briefly summarize the derivation for the self-containedness of this thesis. The inequalities in (4.28) and (4.30) are due to the fact that $\widehat{n}^i n^o d^o \mathbf{R}_- \subset \mathbf{R}_-$. The equalities in (4.28) and (4.30) are due to the facts that d^i and λ are both inner.

4.3 Consideration of SR and CSR problems for Sampled-Data Systems Using Doubly Sensitivity-Preserving Discretized Systems

In this section, we turn back to the SR/CSR problems of the sampled-data system Σ and establish that these problems can be related to some sensitivity reduction problems of the naively discretized system Σ_d with some frequency-dependent weights. For our consideration of the SR and CSR problems for the sampled-data system Σ, we use a discrete-time feedback system whose internal stability is equivalent to that of the original sampled-data system Σ and whose frequency response of the sensitivity (respectively, complementary sensitivity) coincides with that of Σ at each angular frequency. We call this discrete-time system the doubly sensitivity-preserving (DSP) discretized system $\widehat{\Sigma}$. The precise definition of the DSP discretized system will be given later.

Using the DSP discretized system $\widehat{\Sigma}$, we establish a link between the CSR problem of the sampled-data system Σ and that of the naively discretized system Σ_d. This is accomplished by introducing a key notion called the aliasing factor, and its role in the study of the best achievable performance of the sampled-data system Σ for the CSR problem is also discussed here. Then, we study the SR problem of the sampled-data system Σ and establish a similar link to a sort of *mixed* sensitivity reduction problem of the naively discretized system Σ_d.

4.3.1 Derivation of Doubly Sensitivity-Preserving Discretized System $\widehat{\Sigma}$

It is generally understood that the naively discretized plant P_d introduced in the preceding section is useless for studying the SR/CSR problems of the original sampled-data system Σ. For studying such problems directly, we introduce in this section a more elaborate and useful discretized plant that we call the doubly sensitivity-preserving (DSP) discretized plant, denoted by $\widehat{P}$. We also introduce what we call the DSP discretized hold, denoted by $\widehat{H}$, and consider as in Figure 4.3 the closed-loop system consisting of $\widehat{P}$, $\widehat{H}$ and the same discrete-time controller C_d as in the original sampled-data system Σ; we call the latter the DSP discretized system and denote it by $\widehat{\Sigma}$ (the rationale for the term DSP will become clear later). In this section, we establish that $\widehat{\Sigma}$ is stable if and only if the original sampled-data system Σ is stable and that the frequency response gain of the sensitivity (respectively, complementary sensitivity) of $\widehat{\Sigma}$ coincides with that of Σ at each angular frequency. These properties imply that the SR/CSR problems of the sampled-data system Σ
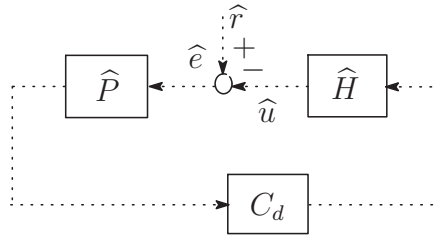
Figure 4.3: Doubly sensitivity-preserving (DSP) discretized system $\widehat{\Sigma}$.

can be reduced equivalently to the corresponding *discrete-time* problems for the DSP discretized system $\widehat{\Sigma}$. The DSP discretized plant $\widehat{P}$ is thus very important in its own right. However, what makes the discussions in this chapter much more significant is that we can in fact reveal a relationship between these equivalent SR and CSR problems in terms of $\widehat{\Sigma}$ and a sort of sensitivity reduction problem of the naively discretized system Σ_d. What is surprising in such a relationship is that the latter system has been obtained by completely ignoring the intersample behavior of Σ, and thus at a glance it is irrelevant to the SR and CSR problems of the *sampled-data system Σ* (or $\widehat{\Sigma}$). The discussions on such a relationship between Σ (or $\widehat{\Sigma}$) and Σ_d as well as its implications form the core of this chapter; however, they will be deferred to Secs. 4.3.2, 4.3.3, and 4.4, and to prepare some fundamental results, we confine our attention in this section to the derivation of the DSP discretized plant $\widehat{P}$ and hold $\widehat{H}$.

In the following, we suppose that the state-space realization of the continuous-time plant P is given by

$$P(s) = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right], \tag{4.31}$$

and the hold function $h(t)$ of the hold circuit $\mathcal{H}$ is given by $h(t) = C_H e^{A_H t} B_H$, where we assume that (A_H, B_H) is controllable and (C_H, A_H) is observable. Now, we begin with a key lemma.

Lemma 4.1 Let $\widehat{B}$ and $\widehat{C}_H$ be matrices satisfying

$$\begin{bmatrix} \widehat{B} \\ \widehat{C}_H^T \end{bmatrix} \begin{bmatrix} \widehat{B}^T & \widehat{C}_H \end{bmatrix} = \int_0^\tau \Gamma(t)\Gamma^T(t)dt, \tag{4.32}$$

$$\Gamma(t) = \begin{bmatrix} e^{A(\tau-t)} & 0 \\ 0 & e^{A_H^T t} \end{bmatrix} \begin{bmatrix} B \\ C_H^T \end{bmatrix}.$$

Then, for

$$\widehat{P}(z) = \left[\begin{array}{c|c} \widehat{A} & \widehat{B} \\ \hline C & 0 \end{array}\right], \quad \widehat{H}(z) = \widehat{C}_H B_H \tag{4.33}$$

with $\widehat{A} = e^{A\tau}$, the following relation holds:

$$\frac{1}{\tau}\begin{bmatrix} \underline{P}(j\varphi) \\ \underline{H}^*(j\varphi) \end{bmatrix}\begin{bmatrix} \underline{P}^*(j\varphi) & \underline{H}(j\varphi) \end{bmatrix} = \begin{bmatrix} \widehat{P}(e^{j\varphi\tau}) \\ \widehat{H}^*(e^{j\varphi\tau}) \end{bmatrix}\begin{bmatrix} \widehat{P}^*(e^{j\varphi\tau}) & \widehat{H}(e^{j\varphi\tau}) \end{bmatrix} \quad \left(\forall \varphi \in \mathcal{I}_0 = \left(-\frac{\pi}{\tau}, \frac{\pi}{\tau}\right]\right).$$

$$(4.34)$$

Proof of Lemma 4.1: The proof is essentially based on the well-known fact that the right-hand side of (4.7), known as the impulse modulation of $P(s)H(s)$, is equivalent to the usual discretization of $P(s)H(s)$ given by (4.8) evaluated on the unit circle. That is, we first note that the left-hand side of (4.34) can be regarded as a sort of impulse modulation since $P^*(j\varphi_m) = P^\sim(j\varphi_m)$ and $H^*(j\varphi_m) = H^\sim(j\varphi_m)$. Then, we see that it can be represented as the left-hand side of the following equation evaluated at $z = e^{j\varphi\tau}$.

$$\mathcal{Z}\left[\begin{bmatrix} P(s) \\ H^\sim(s) \end{bmatrix}\begin{bmatrix} P^\sim(s) & H(s) \end{bmatrix}\right] = \begin{bmatrix} \widehat{P}(z) \\ \widehat{H}^\sim(z) \end{bmatrix}\begin{bmatrix} \widehat{P}^\sim(z) & \widehat{H}(z) \end{bmatrix}. \quad (4.35)$$

Therefore, it suffices to prove the above relation, and it follows readily by applying essentially the same arguments as in the proofs of $\mathcal{Z}\left[P_{21}(s)P_{21}^\sim(s)\right] = \Pi_{21}(z)\Pi_{21}^\sim(z)$ and $\mathcal{Z}\left[H^\sim(s)P_{12}^\sim(s)P_{12}(s)H(s)\right] = \Pi_{12}^\sim(z)\Pi_{12}(z)$ in Sec. 2.4. This completes the proof.

In the following, $\widehat{P}(z)$ and $\widehat{H}(z)$ given by (4.33) are called the doubly sensitivity-preserving (DSP) discretized plant and the DSP discretized hold, respectively, for reasons to become clear shortly. Regarding $\widehat{P}(z)$ and $\widehat{H}(z)$, note that $\widehat{H}(z)$ is in fact a constant matrix independent of z (see (4.33)) and that $\widehat{P}\widehat{H} = P_d$ by (4.7) and the $(1,2)$-component of (4.35). From this observation, we first introduce the alternative representation of the naively discretized system Σ_d shown in Figure 4.4, which is equivalent to that in Figure 4.2. We then see that the system $\widehat{\Sigma}$ shown in Figure 4.3, which we mentioned before, is only slightly modified from the naively discretized system in Figure 4.4 and looks more similar to that in Figure 4.1 with respect to the locations of the external inputs. This rearrangement of the naively
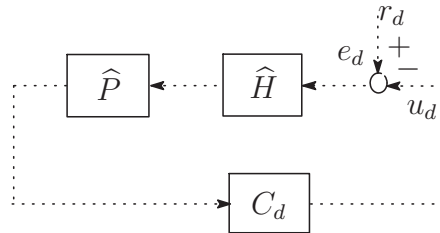
Figure 4.4: Another representation of the naively discretized system Σ_d.

discretized system Σ_d in terms of the DSP discretized plant $\widehat{P}$ and hold $\widehat{H}$, including the modified treatment of the external inputs, is a very important idea that plays the key role in the overall discussion of this chapter. In the following arguments, the discrete-time system $\widehat{\Sigma}$ shown in Figure 4.3 is referred to as the DSP discretized system.

In this section, we begin by establishing that the norm of the sensitivity (respectively, complementary sensitivity) of the sampled-data system Σ at each angular frequency can be computed as that of the DSP discretized system $\widehat{\Sigma}$; this property clearly validates the term DSP. Here, the sensitivity $\widehat{S}$ (respectively, the complementary sensitivity $\widehat{T}$) of $\widehat{\Sigma}$ is defined precisely as the transfer matrix from $\widehat{r}$ to $\widehat{e}$ (respectively, $\widehat{u}$). They are given respectively by

$$\widehat{S} = I - \widehat{H}C_d \left(I + \widehat{P}\widehat{H}C_d \right)^{-1} \widehat{P} = I - \widehat{H}C_d \left(I + P_d C_d \right)^{-1} \widehat{P}$$

$$= I - \widehat{H}K_d\widehat{P}, \tag{4.36}$$

$$\widehat{T} = \widehat{H}C_d \left(I + \widehat{P}\widehat{H}C_d \right)^{-1} \widehat{P} = \widehat{H}K_d\widehat{P} \tag{4.37}$$

where K_d is given by (4.6).

We can now state an important theorem giving the relationship between the (complementary) sensitivity of the sampled-data system Σ and that of the DSP discretized system $\widehat{\Sigma}$.

Theorem 4.3 The DSP discretized system $\widehat{\Sigma}$ is stable if and only if the original sampled-data system Σ is stable. Furthermore, if $\widehat{P}(z)$ and $\widehat{H}(z)$ given by (4.33) satisfy the condition

$$\nexists\varphi \in \mathcal{I}_0 = (-\pi/\tau, \pi/\tau] \text{ such that } \begin{bmatrix} \widehat{P}^* & \widehat{H} \end{bmatrix} \text{ is of full-row rank}, \tag{4.38}$$

then

$$\|\widehat{S}(e^{j\varphi\tau})\| = \|\underline{S}(j\varphi)\|_{l^2/l^2} \quad (\forall\varphi \in \mathcal{I}_0), \tag{4.39}$$

$$\|\widehat{T}(e^{j\varphi\tau})\| = \|\underline{T}(j\varphi)\|_{l^2/l^2} \quad (\forall\varphi \in \mathcal{I}_0). \tag{4.40}$$

Remark 4.4 In Braslavsky et al. [12], a finite-dimensional computation method for $\|\underline{T}(j\varphi)\|_{l^2/l^2}$ and $\|\underline{S}(j\varphi)\|_{l^2/l^2}$ was established, but it involves no viewpoint of "an equivalent discrete-time system" as in the DSP discretized system $\widehat{\Sigma}$. Although such a viewpoint was provided in another method by Hagiwara et al. [29], as far as the computation of $\|\underline{T}(j\varphi)\|_{l^2/l^2}$ is concerned, no such viewpoint that can be

applied to the computation of $\|\underline{S}(j\varphi)\|_{l^2/l^2}$ has been provided in existing methods, e.g., Braslavsky [12], Ito et al. [48], and Yamamoto and Khargonekar [81]. In contrast, Theorem 4.3 clearly establishes that the (complementary) sensitivity of the sampled-data system Σ can be studied through an "equivalent system" (i.e., the DSP discretized system $\widehat{\Sigma}$), which is common for the treatment of the sensitivity and that for the complementary sensitivity.

Proof of Theorem 4.3: It is well known that Σ is stable if and only if Σ_d is stable. Hence, the first assertion is obvious since $P_d = \widehat{P}\widehat{H}$. Also, the proof of (4.40) is essentially the same as the arguments in Hagiwara et al. [29]; this follows readily by applying the equalities about the $(1,1)$ and $(2,2)$-components of (4.34) to (4.2). Accordingly, the details are omitted. Therefore, it suffices to show that (4.39) holds under the assumption (4.38).

From (4.1), we obtain $\underline{S}^*\underline{S} = \underline{I} + \underline{X}$ for each $\varphi \in \mathcal{I}_0$, where

$$\underline{X} = \frac{1}{\tau}\begin{bmatrix} \underline{P}^* & \underline{H} \end{bmatrix} \begin{bmatrix} K_d^* \cdot \frac{1}{\tau}\underline{H}^*\underline{H} \cdot K_d & -K_d^* \\ -K_d & 0 \end{bmatrix} \begin{bmatrix} \underline{P} \\ \underline{H}^* \end{bmatrix}. \tag{4.41}$$

Since $\underline{X}$ is compact (in particular, finite-rank), we have $\|\underline{S}\|_{l^2/l^2}^2 = \max(\lambda_{\max}(\underline{S}^*\underline{S}), 1) = 1 + \max(\lambda_{\max}(\underline{X}), 0)$. On the other hand, from (4.36), we obtain $\widehat{S}^*\widehat{S} = I + \widehat{X}$ for each $\varphi \in \mathcal{I}_0$, where

$$\widehat{X} = \begin{bmatrix} \widehat{P}^* & \widehat{H} \end{bmatrix} \begin{bmatrix} K_d^*\widehat{H}^*\widehat{H}K_d & -K_d^* \\ -K_d & 0 \end{bmatrix} \begin{bmatrix} \widehat{P} \\ \widehat{H}^* \end{bmatrix}. \tag{4.42}$$

Thus, we obtain, $\|\widehat{S}\|^2 = \lambda_{\max}(\widehat{S}^*\widehat{S}) = 1 + \lambda_{\max}(\widehat{X})$. Consequently, to establish (4.39), it is enough to show that $\max(\lambda_{\max}(\underline{X}), 0) = \lambda_{\max}(\widehat{X})$ under the assumption (4.38). To show this, we prove the following three equalities in order:

$$\max(\lambda_{\max}(\underline{X}), 0)$$

$$= \max\left\{ \lambda_{\max}\left(\frac{1}{\tau}\begin{bmatrix} \underline{P} \\ \underline{H}^* \end{bmatrix}\begin{bmatrix} \underline{P}^* & \underline{H} \end{bmatrix}\begin{bmatrix} K_d^* \cdot \frac{1}{\tau}\underline{H}^*\underline{H} \cdot K_d & -K_d^* \\ -K_d & 0 \end{bmatrix} \right), 0 \right\},$$

$$= \max\left\{ \lambda_{\max}\left(\begin{bmatrix} \widehat{P} \\ \widehat{H}^* \end{bmatrix}\begin{bmatrix} \widehat{P}^* & \widehat{H} \end{bmatrix}\begin{bmatrix} K_d^* \cdot \widehat{H}^*\widehat{H} \cdot K_d & -K_d^* \\ -K_d & 0 \end{bmatrix} \right), 0 \right\},$$

$$= \lambda_{\max}(\widehat{X}). \tag{4.43}$$

The second equality of (4.43) is a direct consequence of (4.34), so it is enough to show the first and third equalities; they both rely on the well-known fact that when XY and YX are both well-defined operators or matrices, the spectrum of XY can possibly be a larger set than that of YX (or vice versa) by including just one

additional point that can exist nowhere else than at the origin. The first equality of (4.43) follows immediately from this fact. Similarly, the third equality also follows from this fact if we note that the last quantity in (4.43) is nonnegative under the assumption (4.38), since the matrix $\widehat{X}$ given by (4.42) has an eigenvalue at the origin. Hence, the proof is completed.

Remark 4.5 If $\begin{bmatrix} \widehat{P}^* & \widehat{H} \end{bmatrix}$ is of full-row rank for some φ, we can modify $\widehat{P}$ and $\widehat{H}$ into $\begin{bmatrix} \widehat{P} & 0 \end{bmatrix}$ and $\begin{bmatrix} \widehat{H} \\ 0 \end{bmatrix}$ within the standing constraint on $\widehat{P}$ and $\widehat{H}$ given by (4.35). Consequently, the condition (4.38) is not restrictive, but it is very important in making the last quantity in (4.43) as it is, rather than making it $\max(\lambda_{\max}(\widehat{X}), 0)$. In other words, without the condition (4.38), we would have to introduce such an "additional max-operation," which would then prevent us from associating the computation of $\|\underline{S}(j\varphi)\|_{l^2/l^2}$ directly with the DSP discretized system $\widehat{\Sigma}$. Regarding such a max-operation, we give some further remarks on the arguments in the earlier work [12]. Although no such max-operation can be found there, this is due to a flaw in the arguments. More precisely, equation (19) of that work [12] can be validated only under the nonsingularity assumption of the matrix M therein, but such an assumption is not actually made nor is it satisfied automatically. Since those early arguments [12] are intrinsically irrelevant to such a rank condition as (4.38) in this thesis, correcting the flaw leads to the same additional max-operation with 0 in their arguments.

Remark 4.6 When (A, B) is controllable, $\widehat{B}$ has full-row rank. Therefore, when C is of full-row rank, it follows that $\widehat{P}(z)$ becomes right-invertible except at a point of infinity, i.e., $\widehat{P}(z)$ has no finite zeros. Similarly, when (C_H, A_H) is observable, $\widehat{C}_H$ has full-column rank, and thus if B_H is of full-column rank, then $\widehat{H}$ is left-invertible; since $\widehat{H}$ is independent of z, $\widehat{H}$ has no zeros. If both P and $\mathcal{H}$ are scalar systems satisfying all of the assumptions mentioned above, $\widehat{P}(z)$ and $\widehat{H}$ become a row vector and a column vector, respectively.

Remark 4.7 The idea of the DSP discretized system arose from a desire to relate the the equation (4.41) relevant to sampled-data systems with the equation (4.42) about discrete-time systems. Such a desire leads in a straightforward fashion to considering the discrete-time systems $\widehat{P}$ and $\widehat{H}$ satisfying (4.34) in Lemma 4.1. In this sense, the introduction of the DSP discretized system could be said to be a natural consequence of the fact that the FR-operator representations of the sensitivity and complementary sensitivity directly reflect the way the components in the sampled-data system Σ are connected with each other. This is one of the advantages of the FR-operator approach.

By Theorem 4.3, the DSP discretized system $\widehat{\Sigma}$ shown in Figure 4.3 is equivalent to the sampled-data system Σ in Figure 4.1 when we consider the SR/CSR problems. In this sense, the DSP discretized system $\widehat{\Sigma}$ is much more important than the naively discretized system Σ_d, which does not take the intersample behavior into account. However, we can easily see that the only difference between $\widehat{\Sigma}$ and Σ_d lies in the evaluation points for sensitivity and complementary sensitivity. This might suggest that even the naively discretized system Σ_d could be related somehow to the SR and CSR problems of the original sampled-data system Σ. This is indeed the case, and the discussions in the following part of this chapter are devoted to clarifying such a relationship.

4.3.2 Relationship Between CSR Problems of DSP Discretized System $\widehat{\Sigma}$ and Naively Discretized System Σ_d

In this subsection, we clarify that there exists a clear relationship between two seemingly irrelevant problems, i.e., the CSR problem of the sampled-data system Σ in Figure 4.1, which completely takes into account the intersample behavior, and that of the naively discretized system Σ_d in Figure 4.2, which is inherently free from the concept of intersample behavior. This is accomplished by noting the equivalence of Σ and the DSP discretized system $\widehat{\Sigma}$ with respect to the CSR problem and by introducing a key notion called the aliasing factor ψ, which describes the frequency-dependent effects of the difference in the locations of the external inputs in $\widehat{\Sigma}$ and Σ_d. The significance of the aliasing factor is also demonstrated through a further study on the relationship between the best achievable performances associated with the CSR problems of Σ and Σ_d.

In the following, we clarify the relationship between the CSR problems of the sampled-data system Σ (or equivalently, the DSP discretized system $\widehat{\Sigma}$) and the naively discretized system Σ_d under the same assumptions in Sec. 4.2, i.e., Assumption 4.1. From Assumption 4.1, together with the observability of (C_H, A_H), it follows that $\widehat{P}(z)$ and $\widehat{H}$ given by (4.33) become a row vector with no finite zeros and a constant column vector with no zeros, respectively (see Remark 4.6). In the rest of this section, we use $\lambda := 1/z$ instead of z to describe transfer functions of discrete-time systems.

Let us consider a coprime factorization of $\widehat{P}$ on $\mathbf{R}_-$. Since all poles of P_d and those of $\widehat{P}$ coincide[3], d, the denominator of P_d in (4.14), can also be chosen as a

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[3]Both of them are eigenvalues of $\widehat{A}^{-1}$, where $\widehat{A} = e^{A\tau}$, which appears in (4.33).

denominator of $\widehat{P}$. Let N_P be given by $N_P = \widehat{P}d$. Then, $N_P \in \mathbf{R}_-$ and

$$\widehat{P} = N_P/d, \tag{4.44}$$

is a coprime factorization of $\widehat{P}$ on $\mathbf{R}_-$ because $\widehat{P}$, or equivalently, N_P has no unstable zeros except for the origin, and thus N_P and d do not share the same unstable zeros. It also follows that

$$n = N_P \widehat{H} \tag{4.45}$$

since $P_d = n/d$ and $P_d = \widehat{P}\widehat{H}$. From (4.6), (4.13), (4.36) and (4.37), we obtain the following equations for the DSP discretized system $\widehat{\Sigma}$.

$$\widehat{S} = I - \widehat{H}(x + dq)N_P, \tag{4.46}$$
$$\widehat{T} = \widehat{H}(x + dq)N_P. \tag{4.47}$$

The aliasing factor ψ

We first introduce the inner-outer factorization of $\widehat{H}$ and the co-inner-outer factorization of N_P given respectively by

$$\widehat{H} = H^i \cdot h^o, \ N_P = n_P^{co} \cdot N_P^{ci}, \tag{4.48}$$

where H^i and N_P^{ci} are a column vector and a row vector, respectively. For more on the reason that h^o and n_P^{co} can be taken as scalars, see Fact 17 on p. 165 of Vidyasagar [75] for details. Note that h^o and n_P^{co} are units on $\mathbf{R}_-$ (see Remark 4.6). From (4.47) and (4.48), the H^∞ norm of the complementary sensitivity $\widehat{T}$ is given as follows:

$$\|\widehat{T}\|_\infty = \|H^i h^o(x + dq)n_P^{co}N_P^{ci}\|_\infty = \|h^o(x + dq)n_P^{co}\|_\infty = \|n_P^{co}h^o(x + dq)\|_\infty. \tag{4.49}$$

From (4.49), together with (4.17), we conclude that the H^∞ norm of the complementary sensitivity of the sampled-data system Σ and that of the naively discretized system Σ_d are related by

$$\|\underline{T}\|_\infty = \|\widehat{T}\|_\infty = \left\|\frac{n_P^{co}h^o}{n}T_d\right\|_\infty = \left\|\frac{n_P^{co}h^o}{n^i n^o}T_d\right\|_\infty = \left\|\frac{n_P^{co}h^o}{n^o}T_d\right\|_\infty, \tag{4.50}$$

where n^i and n^o correspond to the inner-outer factorization $n = n^i n^o$. In (4.50), $n_P^{co}h^o/n^o$ is an important factor that relates the CSR problem of the sampled-data system Σ and that of the naively discretized system Σ_d. In other words, this function can be regarded as the frequency-dependent factor with which the influence of aliasing, ignored under the treatment of the sampled-data system Σ as the naively

discretized system Σ_d, can be recovered in the treatment through Σ_d. This observation provides us with a very important viewpoint in the following study of the CSR (or SR) problem of the sampled-data system Σ, as well as its best achievable performances. We define the inverse of this function as the *aliasing factor* [4], which is denoted by ψ:

$$\psi := \frac{n^o}{n_P^{co} h^o}. \tag{4.51}$$

Summarizing the above, we obtain the following theorem on the relationship between the CSR problem of the sampled-data system Σ and that of the naively discretized system Σ_d.

Theorem 4.4 For the CSR problem, the following two conditions are equivalent:

(i) C_d is a stabilizing controller for Σ s.t. $\|\underline{T}\|_\infty < \gamma$.

(ii) C_d is a stabilizing controller for Σ_d s.t. $\left\|\dfrac{1}{\psi} T_d\right\|_\infty < \gamma$.

To clarify further relationships between the CSR problems of Σ and Σ_d, it is important to clarify the properties of the aliasing factor ψ. The following theorem gives a key answer to this issue.

Theorem 4.5 The aliasing factor ψ is a unit such that $|\psi(\lambda)| \le 1$ ($|\lambda| \le 1$), i.e., $\|\psi\|_\infty \le 1$.

Proof of Theorem 4.5: ψ is outer because n_P^{co} and h^o are units (recall Remark 4.6). Since the outer function n^o is also a unit by condition (vi) in Assumption 4.1, it follows readily that ψ is a unit. From (4.14), (4.44), (4.48), and the fact that $\widehat{P}\widehat{H} = P_d$, we obtain

$$\begin{bmatrix} \widehat{P} \\ (\widehat{H})^\sim \end{bmatrix} \begin{bmatrix} (\widehat{P})^\sim & \widehat{H} \end{bmatrix} = \begin{bmatrix} \frac{n_P^{co}}{d} \cdot \frac{n_P^{co\sim}}{d^\sim} & \frac{n}{d} \\ \frac{n^\sim}{d^\sim} & (h^o)^\sim h^o \end{bmatrix}. \tag{4.52}$$

From this together with the definition of the aliasing factor ψ, we obtain

$$\psi \cdot \psi^\sim = \frac{\left(\widehat{P} \cdot \widehat{H}\right) \cdot \left(\widehat{P} \cdot \widehat{H}\right)^\sim}{\left(\widehat{P} \cdot (\widehat{P})^\sim\right) \cdot \left((\widehat{H})^\sim \cdot \widehat{H}\right)}. \tag{4.53}$$

Since ψ is outer, it is analytic on the closed unit disc. Therefore, from the maximum modulus principle, the maximum value of $|\psi|$ is attained on the unit circle. From

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[4]The reason for defining the aliasing factor as the inverse of the function $n_P^{co} h^o / n^o$ is that it is helpful for deriving the analytic properties of the aliasing factor.

the right-hand side of (4.53) and Schwarz's inequality, we obtain $|\psi| \leq 1$ on the unit circle (since $(\cdot)^{\sim} = (\cdot)^*$ on the unit circle). Therefore, when $|\lambda| \leq 1$,

$$|\psi(\lambda)| \leq \max_{\varphi} |\psi(e^{j\varphi\tau})| \leq 1. \tag{4.54}$$

This completes the proof.

It should be noted from (4.53) together with (4.35) that the aliasing factor ψ is determined by $P(s)$ and $H(s)$ and does not in fact depend on the specific choice of the factors $\widehat{P}$ and $\widehat{H}$ nor on the specific coprime factorizations in (4.14) and (4.48). More precisely, ψ is determined uniquely up to the multiplication by a constant inner function (i.e., a complex number with modulus 1), since ψ is outer; this indeed permits us to call it *the* aliasing factor.

It should be noted that the aliasing factor ψ is closely related to the *fidelity index* Φ_d introduced in Braslavsky et al. [12], defined only on the unit circle by

$$\Phi_d^2(e^{j\varphi\tau}) = \frac{\left(\displaystyle\sum_{m=-\infty}^{\infty} |P(j\varphi_m)|^2 \right) \left(\displaystyle\sum_{m=-\infty}^{\infty} |H(j\varphi_m)|^2 \right)}{\left| \displaystyle\sum_{m=-\infty}^{\infty} P(j\varphi_m)H(j\varphi_m) \right|^2}. \tag{4.55}$$

From (4.35) and (4.53), we see that $\Phi_d^2 = 1/(\psi^*\psi)$ holds on the unit circle. Therefore, the aliasing factor can be regarded as the analytic extension, from the unit circle to the complex plane, of the inverse of the square root of the fidelity index.

Remark 4.8 Actually, the fidelity index in the earlier work [12] was introduced for the feedback system in which the external input is located at the output-side of the plant. On the other hand, as shown in Figure 4.1, the external input of our system is located at the input-side of the plant. Therefore, the definition (4.55) of the fidelity index of our system has been slightly modified from the original one to match the context here.

Even though we considered the H^∞ norm in (4.50) or Theorem 4.4, it is obvious that we can develop parallel arguments on the frequency response gain, by which we have $\|\underline{T}\|_{l^2/l^2} = \|\widehat{T}\| = \|\frac{1}{\psi}T_d\|$. This implies that the result in the earlier work [12] can also be recovered with the use of the aliasing factor.

Relationship between the best achievable performances of Σ and Σ_d

Next, we consider the relationship between the CSR problem of the sampled-data system Σ and that of the naively discretized system Σ_d through the property of the aliasing factor ψ.

In the naively discretized system Σ_d, we have

$$\|T_d\|_\infty = \|n(x + dq)\|_\infty = \|n^i n^o (x + d^i d^o q)\|_\infty$$
$$= \|n^o (x + d^i q')\|_\infty, \tag{4.56}$$

where d^i and d^o correspond to the inner-outer factorization of $d = d^i d^o$. Hence, as is well known (e.g., Vidyasagar [75]), the CSR problem of Σ_d, i.e., the problem of minimizing the H^∞ norm of T_d, can essentially be reduced to the problem of finding an interpolation function that is analytic on the closed unit disc and attains, at the zeros of d^i, the same value as $n^o x$, while possessing as small an H^∞ norm as possible.

On the other hand, it follows from Theorem 4.3 that we have the following relation for the sampled-data system Σ:

$$\|\underline{T}\|_\infty = \|\widehat{T}\|_\infty = \left\|\frac{1}{\psi} T_d\right\|_\infty = \left\|\frac{1}{\psi} n(x + dq)\right\|_\infty = \left\|\frac{1}{\psi} n^o (x + d^i q')\right\|_\infty \tag{4.57}$$

Due to the difference between (4.56) and (4.57) by the factor $1/\psi$, we can see that the interpolation function in the CSR problem of the sampled-data system Σ should satisfy the same analyticity constraint but has to attain, at an interpolation point, a value whose magnitude is larger (or no smaller) than that for Σ_d by the factor $|1/\psi|$ (≥ 1), since $|\psi| \leq 1$ inside the unit circle by Theorem 4.5. Hence, in general, the H^∞ norm of the interpolation function would naturally degrade (i.e., become larger) in the case of Σ when compared with the case of Σ_d. We thus arrive at the following theorem immediately.

Theorem 4.6 $\inf_{C_d \in \mathcal{C}} \|\underline{T}\|_\infty \geq \inf_{C_d \in \mathcal{C}} \|T_d\|_\infty$

This theorem says that analyzing the best achievable performance with respect to the CSR problem of the sampled-data system Σ would lead to a "too optimistic" result if it were carried out in terms of the CSR problem of the naively discretized system Σ_d that completely ignores the intersample behavior. A rigorous treatment should use the DSP discretized system $\widehat{\Sigma}$ instead. This consequence itself is well known (e.g., Braslavsky [12], Hara et al. [38]), but the contribution of this thesis is new in the sense that the result has been derived through an important property of the aliasing factor ψ, and thus the significance of the aliasing factor has been demonstrated. The significance of the aliasing factor will be revealed further in the remaining discussions.

4.3.3 Relationship Between SR Problems of DSP Discretized System $\widehat{\Sigma}$ and Naively Discretized System Σ_d

In the preceding subsection, we established that the seemingly irrelevant CSR problem of the naively discretized system Σ_d has a clear link to that of the sampled-data

system Σ through the frequency-dependent weighting by the aliasing factor ψ. In this subsection, we proceed to a consideration of the SR problem of Σ and establish that it also has a clear link to another sort of sensitivity reduction problem of Σ_d, i.e., a *mixed* sensitivity reduction problem with no weighting on the sensitivity but some frequency-dependent weighting on the complementary sensitivity; the latter frequency-dependent weighting is explicitly determined again through the aliasing factor.

Theorem 4.7 For the SR problem, the following two conditions are equivalent when $\gamma > 1$:

(i) C_d is a stabilizing controller for Σ s.t. $\|\underline{S}\|_\infty < \gamma$.

(ii) C_d is a stabilizing controller for Σ_d s.t.
$$\left\| \begin{bmatrix} S_d & \frac{\gamma}{\sqrt{\gamma^2-1}} \frac{1}{\psi}(1 - \psi\psi^{\tilde{}})^{1/2} T_d \end{bmatrix} \right\|_\infty < \gamma.$$

In the above theorem, we denote by $(1 - \psi\psi^{\tilde{}})^{1/2}$ an outer function ξ such that[5] $\xi\xi^{\tilde{}} = 1 - \psi\psi^{\tilde{}}$. As stated in the proof of Theorem 4.3, we have $\|\underline{S}\|^2_{l^2/l^2} = \max(\lambda_{\max}(\underline{S}^*\underline{S}), 1)$ and thus we always have $\|\underline{S}\|_\infty \geq 1$. Hence, the assumption $\gamma > 1$ in the above theorem leads to no loss of generality when we refer to the condition $\|\underline{S}\|_\infty < \gamma$. The following part of this subsection is devoted to the proof of the above theorem.

Let us observe that $\|\underline{S}\|_\infty = \|\widehat{S}\|_\infty$ by Theorem 4.3, and thus consider the condition

$$\|\widehat{S}\|_\infty = \|I - \widehat{H}(x + dq)N_P\|_\infty < \gamma \tag{4.58}$$

for a given $\gamma > 1$, where we used (4.46) for the representation of $\widehat{S}$. For H^i in (4.48), we introduce a complementary inner matrix X such that $\overline{H}^i = \begin{bmatrix} H^i & X \end{bmatrix}$ is square and inner. It is shown in Vidyasagar [75] that any non-square inner matrix has a complementary inner matrix. $\overline{H}^i$ is called a squared inner matrix for H^i. Similarly, we can take Y such that $\overline{N}_P^{ci} = \begin{bmatrix} N_P^{ci} \\ Y \end{bmatrix}$ is square and co-inner. $\overline{N}_P^{ci}$ is referred to as a squared co-inner matrix for N_P^{ci}.

Using $\overline{H}^i$ and $\overline{N}_P^{ci}$, we factorize $\widehat{H}$ and N_P, respectively, as follows:

$$\widehat{H} = \overline{H}^i \begin{bmatrix} h^o \\ 0 \end{bmatrix}, \quad N_P = \begin{bmatrix} n_P^{co} & 0 \end{bmatrix} \overline{N}_P^{ci}. \tag{4.59}$$

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[5] Such an outer exists since $1 - \psi\psi^{\tilde{}} \geq 0$ on the unit circle. See Theorem 4.5 of this dissertation and Lemma 2 on p. 212 of Vidyasagar [75].

Since $\overline{H}^i$ is a square matrix satisfying $(\overline{H}^i)^\sim \overline{H}^i = I$, the equality $\overline{H}^i(\overline{H}^i)^\sim = I$ also holds. Similarly, we obtain $(\overline{N}_P^{ci})^\sim \overline{N}_P^{ci} = \overline{N}_P^{ci}(\overline{N}_P^{ci})^\sim = I$. These facts will often be used later.

From (4.59), the condition (4.58) can be rewritten as

$$\left\| (\overline{H}^i)^\sim (\overline{N}_P^{ci})^\sim - \begin{bmatrix} h^o \\ 0 \end{bmatrix} (x + dq) \begin{bmatrix} n_P^{co} & 0 \end{bmatrix} \right\|_\infty < \gamma. \tag{4.60}$$

Let V and W be the matrices given by

$$V = \begin{bmatrix} 1 & 0 \end{bmatrix} (\overline{H}^i)^\sim (\overline{N}_P^{ci})^\sim - h^o(x + dq) \begin{bmatrix} n_P^{co} & 0 \end{bmatrix}$$
$$= \left\{ \begin{bmatrix} 1 & 0 \end{bmatrix} - h^o(x + dq) N_P \overline{H}^i \right\} (\overline{H}^i)^\sim (\overline{N}_P^{ci})^\sim, \tag{4.61}$$
$$W = \begin{bmatrix} 0 & I \end{bmatrix} (\overline{H}^i)^\sim (\overline{N}_P^{ci})^\sim, \tag{4.62}$$

respectively, where the second equality in (4.61) is obtained by $\begin{bmatrix} n_P^{co} & 0 \end{bmatrix} = N_P(\overline{N}_P^{ci})^\sim$, which is derived from (4.59). Note that the matrix in the left-hand side of (4.60) is expressed as $\begin{bmatrix} V \\ W \end{bmatrix}$. Here, we rewrite (4.60) by using the following fact shown in Vidyasagar [75]: when $\gamma^2 I - W^\sim W$ is positive definite on the unit circle, $\left\| \begin{bmatrix} V \\ W \end{bmatrix} \right\|_\infty < \gamma$ is equivalent to $\|VZ\|_\infty < 1$, where Z is an outer matrix satisfying $ZZ^\sim = (\gamma^2 I - W^\sim W)^{-1}$. The existence of such Z is assured by the positive definiteness of $\gamma^2 I - W^\sim W$ on the unit circle. Before applying this fact, we introduce the following two lemmas:

Lemma 4.2 For W given by (4.62), $\gamma^2 I - W^\sim W$ is positive definite on the unit circle if and only if $\gamma > 1$.

Lemma 4.3 When $\gamma > 1$, we have

$$(\overline{H}^i)^\sim (\overline{N}_P^{ci})^\sim (\gamma^2 I - W^\sim W)^{-1} \overline{N}_P^{ci} \overline{H}^i = \begin{bmatrix} \frac{1}{\gamma} & 0 \\ 0 & \frac{1}{\sqrt{\gamma^2-1}} I \end{bmatrix}^2. \tag{4.63}$$

Proof of Lemma 4.2: Since $\gamma > 0$, the positive definiteness of $\gamma^2 I - W^\sim W$ on the unit circle is equivalent to that of $\gamma^2 I - WW^\sim$ there. From $(\overline{N}_P^{ci})^\sim \overline{N}_P^{ci} = I$ and $(\overline{H}^i)^\sim \overline{H}^i = I$, it follows from (4.62) that $\gamma^2 I - WW^\sim = (\gamma^2 - 1)I$. Hence, the assertion follows immediately.

Proof of Lemma 4.3: Since $\overline{H}^i$ is a squared inner matrix and $\overline{N}_P^{ci}$ is a squared co-inner matrix, we obtain the following from (4.62):

$$(\overline{H}^i)^\sim(\overline{N}_P^{ci})^\sim(\gamma^2 I - W^\sim W)^{-1}\overline{N}_P^{ci}\overline{H}^i = \left\{(\overline{H}^i)^\sim(\overline{N}_P^{ci})^\sim\left(\gamma^2 I - W^\sim W\right)\overline{N}_P^{ci}\overline{H}^i\right\}^{-1}$$

$$= \left(\gamma^2 I - (\overline{H}^i)^\sim(\overline{N}_P^{ci})^\sim W^\sim W\overline{N}_P^{ci}\overline{H}^i\right)^{-1} = \left(\gamma^2 I - \begin{bmatrix} 0 \\ I \end{bmatrix}\begin{bmatrix} 0 & I \end{bmatrix}\right)^{-1}. \quad (4.64)$$

Hence, the assertion follows immediately.

Combining the above arguments, we can readily see that when $\gamma > 1$, the condition (4.60) is equivalent to the following condition:

$$\left\|\left\{\begin{bmatrix} 1 & 0 \end{bmatrix} - h^o(x + dq)N_P\overline{H}^i\right\}\begin{bmatrix} \frac{1}{\gamma} & 0 \\ 0 & \frac{1}{\sqrt{\gamma^2-1}}I \end{bmatrix}\right\|_\infty < 1. \quad (4.65)$$

Here, since $N_P\widehat{H} = n$ (recall (4.45)), it follows that

$$h^o N_P\overline{H}^i = \begin{bmatrix} h^o N_P H^i & h^o N_P X \end{bmatrix} = \begin{bmatrix} n & h^o N_P X \end{bmatrix}. \quad (4.66)$$

From this, together with the Bezout identity (4.15), the condition (4.65) can be written into

$$\left\|\begin{bmatrix} \frac{1}{\gamma}d(y - nq) & -\frac{1}{\sqrt{\gamma^2-1}}(x + dq)h^o N_P X \end{bmatrix}\right\|_\infty < 1. \quad (4.67)$$

Concerning $h^o N_P X$ in the above, we have the following result:

Lemma 4.4 $h^o N_P X X^\sim(N_P)^\sim(h^o)^\sim = \frac{n}{\psi}(1 - \psi\psi^\sim)\left(\frac{n}{\psi}\right)^\sim.$

Proof of Lemma 4.4: From $\overline{H}^i(\overline{H}^i)^\sim = I$, we have $H^i(H^i)^\sim + X(X)^\sim = I$. Hence, it follows that

$$h^o N_P X X^\sim(N_P)^\sim(h^o)^\sim = h^o N_P(N_P)^\sim(h^o)^\sim - h^o N_P H^i(H^i)^\sim(N_P)^\sim(h^o)^\sim$$

$$= h^o n_P^{co}(n_P^{co})^\sim(h^o)^\sim - (N_p\widehat{H})(N_p\widehat{H})^\sim$$

$$= \frac{n h^o n_P^{co}(n_P^{co})^\sim(h^o)^\sim n^\sim}{n^o(n^o)^\sim} - nn^\sim = \frac{n}{\psi}(1 - \psi\psi^\sim)\left(\frac{n}{\psi}\right)^\sim.$$

Thus, the assertion follows immediately.

By Lemma 4.4, we can replace $h^o N_P X$ in (4.67) by $\frac{1}{\psi}(1 - \psi\psi^\sim)^{1/2}n$. Thus, we obtain the equivalent condition

$$\left\|\begin{bmatrix} \frac{1}{\gamma}d(y - nq) & \frac{1}{\sqrt{\gamma^2-1}}\frac{1}{\psi}(1 - \psi\psi^\sim)^{1/2}n(x + dq) \end{bmatrix}\right\|_\infty < 1 \quad (4.68)$$

to the condition $\|\underline{S}\|_\infty < \gamma$ provided that $\gamma > 1$. Now, the assertion of Theorem 4.7 follows immediately from (4.16) and (4.17).

From Theorem 4.7, we obtain the following result, a counterpart to Theorem 4.6.

Theorem 4.8 $\inf_{C_d \in \mathcal{C}} \|\underline{S}\|_\infty \geq \inf_{C_d \in \mathcal{C}} \|S_d\|_\infty$

Proof of Theorem 4.8: From Theorem 4.7, we see that for every $C_d \in \mathcal{C}$ and any $\gamma > 1$, $\|\underline{S}\|_\infty < \gamma$ implies $\|S_d\|_\infty < \gamma$ because

$$\|S_d\|_\infty \leq \left\| \left[S_d \quad \frac{\gamma}{\sqrt{\gamma^2-1}} \frac{1}{\psi}(1 - \psi\psi^\sim)^{1/2} T_d \right] \right\|_\infty. \tag{4.69}$$

By this, we have $\|\underline{S}\|_\infty \geq \|S_d\|_\infty$ for every $C_d \in \mathcal{C}$. This yields the assertion of Theorem 4.8 immediately.

Theorem 4.8 says that analyzing the best achievable performance with respect to the SR problem of the sampled-data system Σ would lead to a "too optimistic" result if it were carried out in terms of the SR problem about the naively discretized system Σ_d.

It is obvious that we can develop parallel arguments on the frequency response gain. Indeed, we can show that the frequency response gain of $\underline{S}$ is given by

$$\|\underline{S}\|_{l^2/l^2} = A_+ + A_-, \tag{4.70}$$

where

$$A_+ = \frac{1}{2}\sqrt{\left(\frac{1}{|\psi|^2} - 1\right) \|T_d\|^2 + (\|S_d\| + 1)^2}, \tag{4.71}$$

$$A_- = \frac{1}{2}\sqrt{\left(\frac{1}{|\psi|^2} - 1\right) \|T_d\|^2 + (\|S_d\| - 1)^2}. \tag{4.72}$$

This expression of $\|\underline{S}\|_{l^2/l^2}$ is nothing but the restatement of the one given in Braslavsky et al. [12] with the fidelity index instead of the aliasing factor.

4.4 Relationship Between Best Achievable Performance in SR Problem and That in CSR Problem for Sampled-Data Systems

In Sec. 4.2, we showed that the best achievable performances in the SR and CSR problems of naively discretized systems coincide with each other. In this section, we show that the same result also holds for sampled-data systems. Through the derivation of this result, we further demonstrate the importance of the preceding arguments with the DSP discretized system, the aliasing factor, and the coprime factorization treatment.

Since we have clarified that the H^∞ norm of the sensitivity (respectively, complementary sensitivity) of the sampled-data system Σ can be dealt with through the DSP discretized system $\widehat{\Sigma}$, the best achievable performance in the SR (respectively, CSR) problem of the sampled-data system Σ can be studied through $\widehat{\Sigma}$. By slightly extending the preceding arguments on the coprime factorization for $\widehat{\Sigma}$, we show the following result, which says that, in the sampled-data system Σ, the best achievable performance in the SR problem is equal to that in the CSR problem.

Theorem 4.9 $\inf_{C_d \in \mathcal{C}} \|\underline{S}\|_\infty = \inf_{C_d \in \mathcal{C}} \|\underline{T}\|_\infty$

Proof of Theorem 4.9: Let $\alpha_i := e^{-a_i \tau}$ $(i = 1, \cdots, m)$ and β_j $(i = 1, \cdots, l)$ be unstable poles and zeros of $P_d(1/\lambda)$, respectively. Then, we have $|\alpha_i| < 1$, $d^i(\alpha_1) = \cdots = d^i(\alpha_m) = 0$ and $|\beta_j| < 1$, $n^i(\beta_1) = \cdots = n^i(\beta_l) = 0$. (Note that $P_d(1/\lambda) = n(\lambda)/d(\lambda)$ has no poles or zeros on the unit circle due to conditions (iii) and (vi) in Assumption 4.1.)

From (4.68) and (4.57), together with the inner-outer factorizations $d = d^i d^o$ and $n = n^i n^o$, $\inf_{C_d \in \mathcal{C}} \|\underline{S}\|_\infty$ and $\inf_{C_d \in \mathcal{C}} \|\underline{T}\|_\infty$ can be restated, respectively, as

$$\inf_{q' \in \mathbf{R}_-} \left\{ \gamma > 0 \ \middle| \ \left\| \left[\frac{dy}{\gamma} \quad \frac{1}{\sqrt{\gamma^2-1}} \phi n x \right] - n^i d^i \left[\frac{1}{\gamma} \quad -\frac{\phi}{\sqrt{\gamma^2-1}} \right] q' \right\|_\infty < 1 \right\},$$

$$\inf_{q' \in \mathbf{R}_-} \left\{ \gamma > 0 \ \middle| \ \left\| \frac{nx}{\gamma\psi} + n^i d^i q' \right\|_\infty < 1 \right\},$$

where $q' = n^o d^o q \in \mathbf{R}_-$[6] and ϕ is an outer function satisfying $\phi\tilde{}\phi = \frac{1}{\psi\tilde{}\psi} - 1$; we have also used the fact that ψ is a unit. The existence of ϕ is due to Theorem 4.5. Here, note from (4.15) that $dy = 0$ and $nx = 1$ for λ such that $d(\lambda) = 0$ while $dy = 1$ and $nx = 0$ for λ such that $n(\lambda) = 0$. Hence, it follows that the computation of the best achievable performance $\inf_{C_d \in \mathcal{C}} \|\underline{S}\|_\infty$ reduces to Problem 4.1 with $\Xi_i = \left[0 \quad \frac{\phi(\alpha_i)}{\sqrt{\gamma^2-1}} \right]$ $(i = 1, \ldots, m)$ and $\Upsilon_j = \left[\frac{1}{\gamma} \quad 0 \right]$ $(j = 1, \ldots, l)$. We refer to this problem as Problem-$\underline{S}$. In a similar fashion, the computation of the best achievable performance $\inf_{C_d \in \mathcal{C}} \|\underline{T}\|_\infty$ reduces to Problem 4.1 with $\Xi_i = \frac{1}{\gamma\psi(\alpha_i)}$ $(i = 1, \ldots, m)$ and $\Upsilon_j = 0$ $(j = 1, \ldots, l)$. We refer to this problem as Problem-$\underline{T}$.

As far as the computation of the infimum of γ is concerned, we may replace Ξ_i and Υ_j in Problem-$\underline{S}$ by their transpose (Vidyasagar [75]). Using this fact and applying Proposition 4.1, the computation of the infimum of γ for Problem-$\underline{S}$ reduces to the problem of finding the infimum of $\gamma > 1$ such that the Pick matrix

$$Q_{\underline{S}} = \begin{bmatrix} X & Y \\ Y^* & (1-\gamma^{-2})Z \end{bmatrix} \tag{4.73}$$

---

[6]Note that d^o and n^o are units due to conditions (iii) and (vi) in Assumption 4.1, respectively. (See Remark 4.2.)

is positive definite, where the (i, j)-elements of X, Y, and Z (denoted, respectively, by X_{ij}, Y_{ij}, and Z_{ij}) are given by $X_{ij} = \frac{1}{1 - \overline{\alpha}_i \alpha_j} \left(1 - \frac{\phi^*(\alpha_i)\phi(\alpha_j)}{\gamma^2 - 1} \right)$ $(i, j = 1, \cdots, m)$, $Y_{ij} = \frac{1}{1 - \overline{\alpha}_i \beta_j}$ $(i = 1, \cdots, m; j = 1, \cdots, l)$, $Z_{ij} = \frac{1}{1 - \overline{\beta}_i \beta_j}$ $(i, j = 1, \cdots, l)$, respectively. The positive definiteness of $Q_{\underline{S}}$ is obviously equivalent to the positive definiteness of the following matrix $Q'_{\underline{S}}$:

$$Q'_{\underline{S}} = \begin{bmatrix} (1 - \gamma^{-2})X & Y \\ Y^* & Z \end{bmatrix}. \tag{4.74}$$

Here, note that the (i, j)-element of $(1 - \gamma^{-2})X$ can be represented as

$$(1 - \gamma^{-2})X_{ij} = \frac{1}{1 - \overline{\alpha}_i \alpha_j} \left(1 - \frac{1}{\gamma^2} \begin{bmatrix} 1 \\ \phi(\alpha_i) \end{bmatrix}^* \begin{bmatrix} 1 \\ \phi(\alpha_j) \end{bmatrix} \right). \tag{4.75}$$

Observing that the above matrix has essentially the same form as Q in (4.10), we see from Proposition 4.1 that the problem of finding the infimum of γ such that $Q'_{\underline{S}}$ is positive definite is equivalent to the interpolation problem with $\Xi_i = \frac{1}{\gamma} \begin{bmatrix} 1 \\ \phi(\alpha_i) \end{bmatrix}$ $(i = 1, \ldots, m)$ and $\Upsilon_j = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $(j = 1, \ldots, l)$. Note that each value of Ξ_i or Υ_j to be interpolated in this modified interpolation problem is $\begin{bmatrix} \psi \\ \phi\psi \end{bmatrix}$ times that of the corresponding scalar value in the interpolation problem associated with Problem-$\underline{T}$. However, it follows from the relation $\phi\tilde{\phi} = \frac{1}{\psi\tilde{\psi}} - 1$ that $\begin{bmatrix} \psi \\ \phi\psi \end{bmatrix}$ is inner. Therefore, the modified interpolation problem derived above for the computation of $\inf_{C_d \in \mathcal{C}} \|\underline{S}\|_\infty$ is essentially equivalent to that associated with Problem-$\underline{T}$. Hence Problem-$\underline{S}$ shares the same infimum as Problem-$\underline{T}$. This completes the proof.

Remark 4.9 One might suspect that Theorem 4.9 could be trivial and shown immediately if we used some sort of symmetry between $\widehat{S}$ and $\widehat{T}$, which is evoked by the relation $\widehat{S} + \widehat{T} = I$. More precisely, if the set $\widehat{\mathcal{S}} = \left\{ \widehat{S} = I - \widehat{H}(x + dq)N_P \mid q \in \mathbf{R}_- \right\}$ were to coincide with $\widehat{\mathcal{T}} = \left\{ \widehat{T} = \widehat{H}(x + dq)N_P \mid q \in \mathbf{R}_- \right\}$, then the idea would indeed be correct, but these sets do not in fact coincide for the following reason. Each $\widehat{T} \in \widehat{\mathcal{T}}$ has an eigenvalue at the origin for all angular frequencies, since $\widehat{H}$ is a column vector and $\widehat{P}$ is a row vector. On the other hand, $\widehat{S} \in \widehat{\mathcal{S}}$ has an eigenvalue at the origin if and only if $\widehat{T} = \widehat{H}(x + dq)N_P$ has an eigenvalue at 1, or equivalently, $T_d = N_P\widehat{H}(x + dq) = 1$ by (4.17) and (4.45) (and thus $S_d = 0$). Therefore, $\widehat{S} \in \widehat{\mathcal{S}}$ has an eigenvalue at the origin only at the angular frequencies such that $S_d = 0$. Thus, we have $\widehat{\mathcal{S}} \neq \widehat{\mathcal{T}}$.

4.5   Numerical Example

In this section, we verify the results obtained in the preceding sections through a numerical example. Let us consider the sampled-data system Σ of Figure 4.1, in which the transfer function $P(s)$ of the continuous-time plant P and the hold function $h(t)$ of the hold circuit $\mathcal{H}$ are given, respectively, by

$$P(s) = \frac{s + 0.9}{(s - 0.2)(s - 1.8)(s + 5)}, \tag{4.76}$$

$$h(t) = e^{-t}. \tag{4.77}$$

We compute the best achievable performances in the SR and CSR problems of the sampled-data system Σ and those of the naively discretized system Σ_d, together with the lower bounds of the best achievable performances of the SR and CSR problems of the naively discretized system Σ_d given by Sung and Hara [73]. For the computation of the best achievable performances of the sampled-data system Σ, we employ DSP discretized system $\widehat{\Sigma}$ and use a MATLAB function `dhinflmi`. We also use this function for the computation of the best achievable performances of the naively discretized system Σ_d. The results are shown in Figure 4.5 for the sampling period $\tau \in [0.1, 1]$.

In Figure 4.5, solid lines, dash-dot lines, and dash-dash lines are plots for the sampled-data system Σ, the naively discretized system Σ_d, and lower bounds, respectively. The lines with circle and the lines with cross indicate that they are plots for SR problem and CSR problem, respectively. From Figure 4.5, we can observe that $\inf_{C_d \in \mathcal{C}} \|\underline{S}\|_\infty = \inf_{C_d \in \mathcal{C}} \|\underline{T}\|_\infty$, $\inf_{C_d \in \mathcal{C}} \|S_d\|_\infty = \inf_{C_d \in \mathcal{C}} \|T_d\|_\infty$, and the lower bound of the best achievable performance of the naively discretized system Σ_d in the SR problem and that in the CSR problem coincide. These results correspond to Theorems 4.9, 4.1, and 4.2, respectively. We can also observe from Figure 4.5 that $\inf_{C_d \in \mathcal{C}} \|\underline{T}\|_\infty \geq \inf_{C_d \in \mathcal{C}} \|T_d\|_\infty$ and $\inf_{C_d \in \mathcal{C}} \|\underline{S}\|_\infty \geq \inf_{C_d \in \mathcal{C}} \|S_d\|_\infty$, which correspond to Theorems 4.6 and 4.8, respectively. Thus, we have verified that all of the results for the best achievable performances, as well as the lower bounds, in the SR and CSR problems of the sampled-data systems Σ and the naively discretized system Σ_d hold for this example.

In this example, note that the best achievable performance in the SR problem (or CSR problem) of the naively discretized system Σ_d is equal to its lower bound when the sampling period τ is less than 0.7, whereas it differs from its lower bound when τ exceeds 0.7. This is due to the fact that $P_d(z)$ does not have unstable zeros except for the point at infinity when τ is less than 0.7, while it has an unstable zero when τ exceeds 0.7. See Remark 4.3 and Figure 4.6, in which two finite zeros of $P_d(z)$ are plotted for $\tau \in [0.1, 1]$.
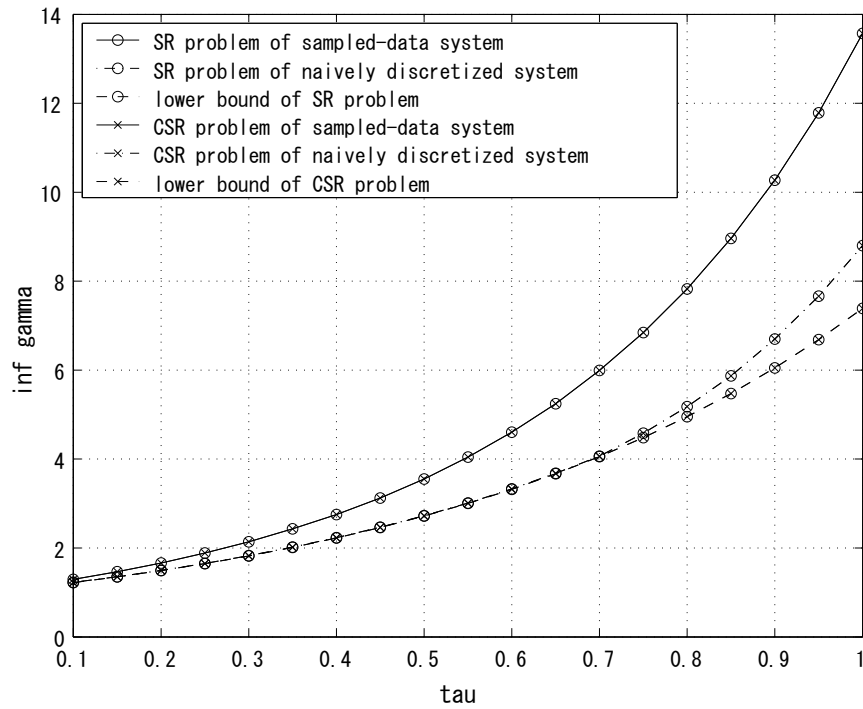
Figure 4.5: Best achievable performances in SR and CSR problems of sample-data system Σ and those of naively discretized system Σ_d, together with their lower bounds.
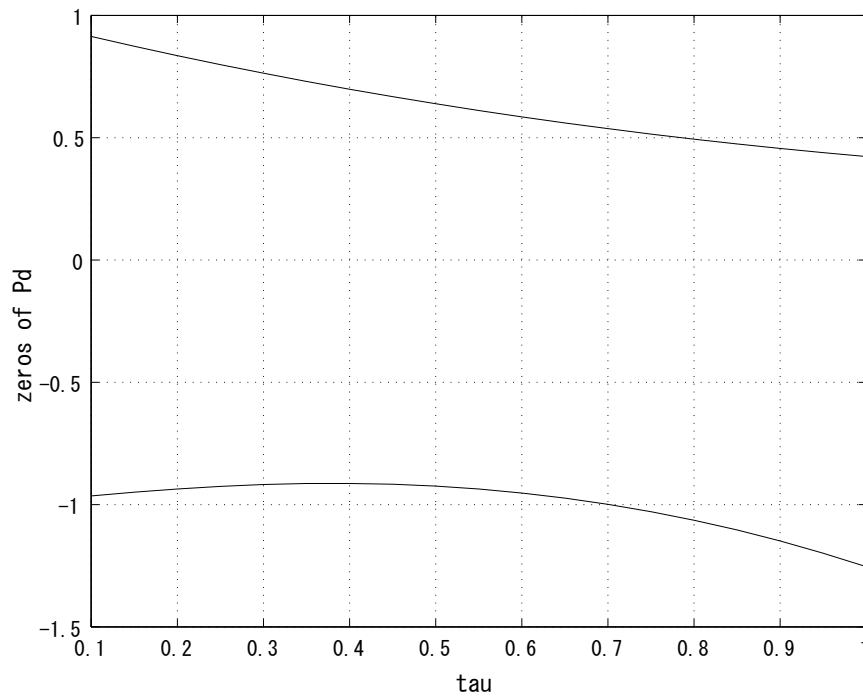
Figure 4.6: Zeros of naively discretized plant $P_d(z)$.

4.6 Summary

In this chapter, we studied some aspects of the sensitivity reduction (SR) and complementary sensitivity reduction (CSR) problems of sampled-data systems. We first considered the SR and CSR problems of the conventional 'hold equivalent' discretized system Σ_d (called the naively discretized system), which are much easier problems than those of the original sampled-data system Σ, and showed that the best achievable performance in the SR problem coincides with that in the CSR problem. We also showed that the lower bound of the best achievable performance obtained by Sung and Hara [73] in the SR problem coincides with that in the CSR problem.

Next, we considered the SR and CSR problems of sampled-data systems and showed that these problems can be reduced to equivalent discrete-time problems by introducing the doubly sensitivity-preserving (DSP) discretized system $\widehat{\Sigma}$. Through the coprime factorization treatment of $\widehat{\Sigma}$, we further introduced an important function called the aliasing factor. We then showed that the naively discretized system Σ_d can also be used for the SR/CSR problems of the sampled-data system Σ, provided that appropriate frequency-dependent weights constructed from the aliasing factor are applied on Σ_d. This should be very interesting because Σ_d has generally been considered useless for the study of the SR/CSR problems of the sampled-data system Σ, since the intersample behavior of Σ is completely ignored in Σ_d. We then showed that a relation between the best achievable performance in the SR (or CSR) problem of $\widehat{\Sigma}$ (and thus Σ) and that of Σ_d can be proved through clarifying an analytic property of the aliasing factor. We also provided an interesting property that the best achievable performance for the SR problem of Σ and that for the CSR problem of Σ coincide with each other.

We finally remark that similar results can also be derived when the (complementary) sensitivity defined on the input side of the plant is dealt with, as opposed to the treatment in this chapter, which is about the (complementary) sensitivity defined on the output side. We hope that the study developed in this chapter could provide a fundamental basis for a further study on the best achievable performances of sampled-data systems, in addition to the existing study on the H^2 tracking performance of sampled-data systems conducted in [15].

Chapter 5

Frequency Response of Sampled-Data Systems Using Time-Sharing Multirate Sample-Hold Controllers and Application to Reliable Stabilization

In the preceding sections, we considered several problems related to the frequency-domain characteristics of sampled-data systems under the assumption that the continuous-time plant and the discrete-time controller are connected with the generalized hold and the ideal sampler. In this chapter, we consider the sampled-data system using a different type of sample-hold scheme called "time-sharing multirate sample-hold scheme." This is a special kind of control scheme that uses a multirate sample-hold scheme, where "multirate" implies that the sampler and the hold operate with different periods, and "time-sharing" implies that the manipulation of the plant input and the detection of the plant output work at separate time intervals. We refer to the controllers using the time-sharing multirate sample-hold scheme as time-sharing multirate sample-hold controllers.

In this chapter, we define the frequency response of the sampled-data system using time-sharing multirate sample-hold controllers by giving the FR-operator representation of the sampled-data system. Then, we derive some methods for computing the frequency response gain of the sampled-data system. We also show that such types of controllers can be applied to solving the reliable stabilization problem,

in which we are to find a set of controllers that stabilize a given plant when they all act together, as well as when any one of them fails.

5.1 Sampled-Data System Using Time-Sharing Multirate Sample-Hold Controllers

In this section, we introduce a new class of control scheme called the time-sharing multirate sample-hold scheme. The feature of this class of controllers is that the multirate hold and the multirate sampler employed satisfy what we call the time-sharing condition. Next, to give a reasonable justification for introducing the time-sharing multirate sample-hold scheme, we consider the open-loop system discretized with a generalized hold and a generalized sampler that does not necessarily satisfy the time-sharing condition. Considering the condition that guarantees a certain desirable property of the discretized system, we are led to a useful notion called the "orthogonality condition." Consequently, time-sharing multirate sample-hold controllers show themselves to be a natural class of controllers satisfying this useful orthogonality condition, which in turn gives a justification for the introduction of the time-sharing multirate sample-hold scheme.

5.1.1 Time-sharing multirate sample-hold scheme

Let us consider the system depicted in Figure 5.1, where solid lines represent continuous-time signals and dashed lines represent discrete-time signals. P is a controllable, observable, linear time-invariant plant described by

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{5.1}$$
$$y(t) = Cx(t), \tag{5.2}$$

where $x(t) \in \mathbf{R}^n$ is the state, $u(t) \in \mathbf{R}^m$ is the input, $y(t) \in \mathbf{R}^l$ is the output, and A, B, C are real matrices of appropriate dimensions. $\mathcal{GH}$ and $\mathcal{GS}$ are generalized holds and generalized samplers, respectively, and they act according to the following equations:

$$u(t) = F_1(t - k\tau)\rho[k] + F_2(t - k\tau)u_d[k] \qquad k\tau \leq t < (k+1)\tau \tag{5.3}$$

$$\rho[k] = \int_{(k-1)\tau}^{k\tau} G_1(t - (k-1)\tau)y(t)dt, \ y_d[k] = \int_{(k-1)\tau}^{k\tau} G_2(t - (k-1)\tau)y(t)dt. \tag{5.4}$$
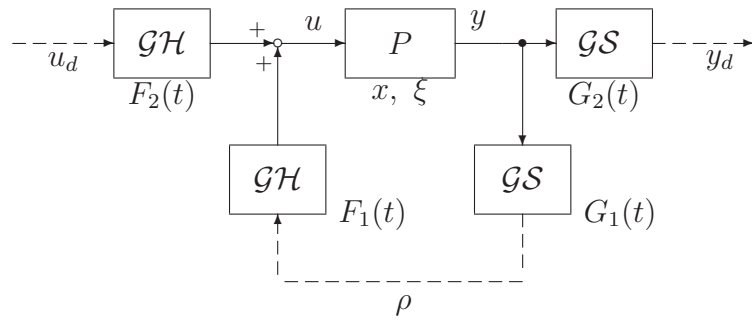
Figure 5.1: Sampled-data system using time-sharing multirate sample-hold controllers.
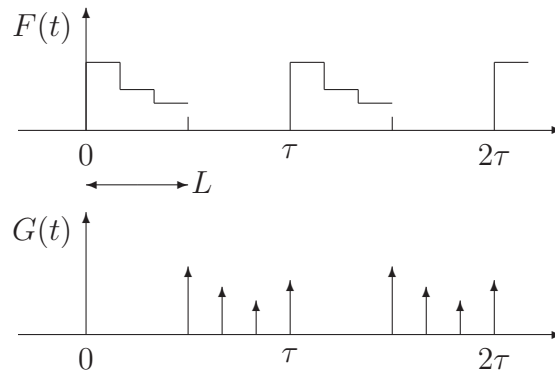
Figure 5.2: Hold function $F(t)$ and sampling function $G(t)$ ($N_i = 3$, $N_o = 4$).

Here, τ is a basic period for the control, which we call the frame period. In particular, in the following, we consider the hold functions $F_i(t)$ and the sampling functions $G_i(t)$ ($i = 1, 2$) of the form

$$F_i(t) = \begin{cases} F_{ij} & j\tau_i \leq t < (j+1)\tau_i \quad (j = 0, \cdots, N_i - 1) \\ 0 & L \leq t < \tau \end{cases} \tag{5.5}$$

$$G_i(t) = \begin{cases} 0 & 0 < t < L \\ \sum_{j=0}^{N_o - 1} G_{ij} \delta\left(t - (L + j\tau_o)\right) & L \leq t \leq \tau \end{cases} \tag{5.6}$$

where $\tau_i = L/N_i$ and $\tau_o = (\tau - L)/(N_o - 1)$ (see Figure 5.2). As seen from (5.5) and (5.6), at each interval of their operations, the hold functions change their outputs N_i times with period τ_i and the generalized samplers detect their inputs N_o times with period τ_o. Such a sample-hold scheme that uses the different periods τ_i and τ_o for manipulating inputs and detecting outputs, respectively, is called a multirate control scheme. The integers N_i and N_o are called, respectively, the input multiplicity and the output multiplicity. Furthermore, (5.5) and (5.6) imply that the manipulation of the plant input and the detection of the plant output work at separate time intervals,

i.e., $[k\tau, k\tau + L)$ and $[k\tau + L, (k+1)\tau]$, respectively. Therefore, we call the above control scheme the "time-sharing multirate sample-hold scheme," and the above conditions (5.5) and (5.6) the "time-sharing condition" between the multirate holds and the multirate samplers. The above scheme and condition were first introduced in Ito et al. [43].

Now, we derive the discrete-time state equation and the output equation of the system in Figure 5.1. From the time-sharing condition, the plant input is given by

$$u(k\tau + t) = \begin{cases} F_1(t)\rho[k] + F_2(t)u_d[k] & 0 \le t < L \\ 0 & L \le t < \tau \end{cases} \tag{5.7}$$

Hence, defining $\xi[k] := x(k\tau)$, we have

$$\xi^L[k] := x(k\tau + L)$$
$$= e^{AL}\xi[k] + \int_0^L e^{A(L-t)}BF_1(t)dt\rho[k] + \int_0^L e^{A(L-t)}BF_2(t)dtu_d[k] \tag{5.8}$$
$$\xi[k+1] = e^{A(\tau-L)}\xi^L[k] \tag{5.9}$$
$$y(t) = Ce^{A(t-(k\tau+L))}\xi^L[k], \quad k\tau + L \le t \le (k+1)\tau. \tag{5.10}$$

Since the hold functions are given by (5.5), it follows from (5.8) and (5.9) that

$$\xi[k+1] = \overline{A}\xi[k] + \overline{F}_1\rho[k] + \overline{F}_2 u_d[k], \tag{5.11}$$

where

$$\overline{A} = e^{A\tau}, \ \overline{F}_i = e^{A(\tau-L)}\left[A_i^{N_i-1}B_i, \cdots, B_i\right]\begin{bmatrix} F_{i,0} \\ \vdots \\ F_{i,N_i-1} \end{bmatrix}, \ A_i = e^{A\tau_i}, \ B_i = \int_0^{\tau_i} e^{At}Bdt. \tag{5.12}$$

On the other hand, since the sampler functions are given by (5.6), it follows from (5.9) and (5.10) that

$$\rho[k] = \overline{G}_1\xi[k], \quad y_d[k] = \overline{G}_2\xi[k], \tag{5.13}$$

where

$$\overline{G}_i = \left[G_{i,0}, \cdots, G_{i,N_o-1}\right]\begin{bmatrix} C \\ \vdots \\ CA_o^{N_o-1} \end{bmatrix}e^{-A(\tau-L)}, \quad A_o = e^{A\tau_o}. \tag{5.14}$$

Combining the above equations leads to the following fundamental theorem.

Theorem 5.1 The state equation for the closed-loop system with the multirate holds (5.5) and the multirate samplers (5.6) satisfying the time-sharing condition is given by

$$\xi(k+1) = (\overline{A} + \overline{F}_1\overline{G}_1)\xi[k] + \overline{F}_2 u_d[k] \qquad (5.15)$$

$$y_d[k] = \overline{G}_2\xi[k]. \qquad (5.16)$$

Here, noting that (A, B, C) is controllable and observable, it follows from well-known results (e.g., Hagiwara and Araki [25]) that the matrices $\overline{F}_1$, $\overline{G}_1$, $\overline{F}_2$ and $\overline{G}_2$ can be made to coincide with any prescribed matrices by suitably choosing the multirate holds and multirate samplers, provided that the input multiplicity N_i and the output multiplicity N_o are large enough. Note in particular that the numbers of columns of $\overline{F}_1$ and $\overline{F}_2$, as well as the numbers of rows of $\overline{G}_1$ and $\overline{G}_2$, can be changed arbitrarily. Hence, the above theorem is a very powerful result, since the state transition matrix as well as the input and output matrices can be adjusted arbitrarily. Thus, for example, arbitrary pole/zero assignment can be achieved very easily. This result will be especially useful when $y_d[k]$ is used as an artificial output for the upper-level feedback (e.g., simultaneous pole assignment of Araki et al. [5], Araki et al. [6], Hagiwara et al. [32]) to be applied on the top of the system given by (5.15) and (5.16).

5.1.2   Control systems with a generalized hold and a generalized sampler

This subsection aims to give a reasonable justification for introducing the time-sharing multirate sample-hold scheme described in the preceding subsection. In fact, the time-sharing multirate sample-hold scheme was inspired by the following observations.

It is known that when we consider the pole assignment problem using generalized samplers (or multirate output feedback) (e.g., Hagiwara and Araki [26]), some additional rank condition is required on the plant, although such a condition is not required when we use generalized holds (or multirate input feedback) (e.g., Kabamba [55], Araki and Hagiwara [3]). Therefore, parallel results cannot be obtained for manipulation and detection. Thus, it would be interesting to investigate why such additional rank conditions arise in the studies with generalized samplers. To this question, the notion of the "orthogonality condition" between generalized holds and generalized samplers turns out to play an important role, as we discuss in the following. Actually, this orthogonality condition is always satisfied by controllers
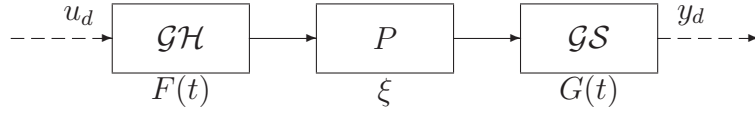
Figure 5.3: Sampled-data control system using a generalized sampler and a generalized hold (open-loop).

with generalized holds, but not by controllers with generalized samplers. Hence, it can be regarded that the orthogonality condition has a close connection with the ability to attain difficult control purposes without strong assumptions on the plant. To meet that condition without introducing any coupled constraints between the multirate holds and multirate samplers to be employed, it will be clearly shown that the time-sharing multirate sample-hold scheme is very natural.

In the remaining part of this subsection, we discuss the orthogonality condition mentioned above. To this end, let us consider an open-loop system with a generalized hold (5.3) and a generalized sampler (5.4) as shown in Figure 5.3, without imposing the time-sharing condition given by (5.5) and (5.6), and derive the equation of the discretized system.

In this case, the inter-sample behavior of the state vector $x(t)$ is given by

$$x(k\tau + \theta) = e^{A\theta}\xi[k] + \int_0^\theta e^{A(\theta-\sigma)}BF(\sigma)d\sigma u_d[k] \quad (0 \leq \theta \leq \tau) \tag{5.17}$$

so that the transition of the state between the sampling instants is given by

$$\xi[k+1] = e^{A\tau}\xi[k] + \int_0^\tau e^{A(\tau-\sigma)}BF(\sigma)d\sigma u_d[k]. \tag{5.18}$$

It follows from (5.17) and (5.18) that

$$x(k\tau + \theta) = e^{-A(\tau-\theta)}\xi[k+1] - \int_\theta^\tau e^{A(\theta-\sigma)}BF(\sigma)d\sigma u_d[k] \quad (0 \leq \theta \leq \tau). \tag{5.19}$$

By (5.19), the inter-sample output of the plant in the interval $[(k-1)\tau, k\tau]$ is described by

$$y((k-1)\tau + \theta) = Ce^{-A(\tau-\theta)}\xi[k] - \int_\theta^\tau Ce^{A(\theta-\sigma)}BF(\sigma)d\sigma u_d[k-1] \quad (0 \leq \theta \leq \tau). \tag{5.20}$$

Substituting (5.20) into (5.4) leads to the discrete-time output equation given by

$$y_d[k] = \int_0^\tau G(\theta)Ce^{-A(\tau-\theta)}d\theta \xi[k] - \int_0^\tau \int_\theta^\tau G(\theta)Ce^{A(\theta-\sigma)}BF(\sigma)d\sigma d\theta u_d(k-1). \tag{5.21}$$

Summarizing the above equations, the discrete-time open-loop equation becomes

$$\xi(k+1) = \overline{A}\xi[k] + \overline{B}u_d[k] \tag{5.22}$$

$$y_d[k] = \overline{C}\xi[k] + \overline{D}u_d(k-1), \tag{5.23}$$

where

$$\overline{A} = e^{A\tau} \tag{5.24}$$

$$\overline{B} = \int_0^\tau e^{A(\tau-\sigma)}BF(\sigma)d\sigma \tag{5.25}$$

$$\overline{C} = \int_0^\tau G(\theta)Ce^{-A(\tau-\theta)}d\theta \tag{5.26}$$

$$\overline{D} = -\int_0^\tau\int_\theta^\tau G(\theta)Ce^{A(\theta-\sigma)}BF(\sigma)d\sigma d\theta. \tag{5.27}$$

Note that the discrete-time output equation turns out to include the term $\overline{D}u_d(k-1)$, although the original plant does not include such a kind of terms. What is more, it presents some delayed effect on the output from the input. Therefore, when we connect a feedback controller between $\rho(\cdot)$ and $v(\cdot)$, we must view the above open-loop discretized plant as having the augmented state vector $[\xi[k]^T \ u_d(k-1)^T]^T$, unless $\overline{D} = 0$.

In the case of the standard discretized plant with a generalized hold and the impulsive detector (i.e., the ideal sampler), the condition $\overline{D} = 0$ is satisfied automatically, as is well known. On the other hand, in the case of the system with the generalized sampler and the zero-order hold, we have $\overline{D} \neq 0$ in general. We can interpret that this difference gives rise to the difference in the order of the discretized plant, which in turn results in different conditions required in achieving similar control purposes with generalized holds on one hand and generalized samplers the other.

From the above consideration, it can be expected that the condition

$$\overline{D} = 0 \tag{5.28}$$

plays an important role in attaining some difficult control purposes under mild conditions when we use generalized holds and generalized samplers at the same time. We refer to the above condition as the "orthogonality condition between a generalized hold and a generalized sampler," or simply the "orthogonality condition."

There are many ways to choose $F(t)$ and $G(t)$ that satisfy the orthogonality condition. One of the easiest ways to satisfy it is to have the manipulation and detection work at separate intervals. This leads to the idea of the time-sharing multirate sample-hold controllers proposed in Sec. 5.1.1.

5.2 Frequency Response of Sampled-data Systems Using Time-Sharing Multirate Sample-Hold Controllers

In this section, we give the FR-operator representation for the sampled-data system using time-sharing multirate sample-hold controllers and define the frequency response of the sampled-data system. We also provide some methods for computing its frequency response gain.

Let us consider the sampled-data system shown in Figure 5.4, which is a slightly modified version of the sampled-data system shown in Figure 2.3. Here, P and $\overline{C}_d$ are a finite-dimensional linear time-invariant (FDLTI) continuous-time generalized plant and an FDLTI discrete-time controller, respectively. $\overline{\mathcal{H}}$ and $\overline{\mathcal{S}}$ are the multirate hold and the multirate sampler, respectively, with the frame period τ, which satisfy the time-sharing condition. To be more specific, the multirate hold $\overline{\mathcal{H}}$ maps the discrete-time signal $\overline{u}_d$ given by

$$\overline{u}_d[k] = \begin{bmatrix} \overline{u}_{d,0}^T[k] & \overline{u}_{d,1}^T[k] & \cdots, \overline{u}_{d,N_i-1}^T[k] \end{bmatrix}^T \tag{5.29}$$

to the continuous-time signal

$$u(t) = \begin{cases} \overline{u}_{d,j}[k] & (k\tau + j\tau_i \le t < k\tau + (j+1)\tau_i; \ j = 0, 1, \cdots, N_i - 1), \\ 0 & (L \le t \le \tau), \end{cases} \tag{5.30}$$

where N_i is the input multiplicity and $\tau_i = L/N_i$. The multirate sampler $\overline{\mathcal{S}}$ maps the continuous-time signal y to discrete-time signal $\overline{y}_d$ by

$$\overline{y}_d[k] = \begin{bmatrix} \overline{y}_{d,0}^T[k] & \overline{y}_{d,1}^T[k] & \cdots & \overline{y}_{d,N_o-1}^T[k] \end{bmatrix}^T, \tag{5.31}$$

$$\overline{y}_{d,j}[k] = y\big((k-1)\tau + L + j\tau_o\big) \ (j = 0, 1, \cdots, N_o - 1), \tag{5.32}$$

where N_o is the output multiplicity and $\tau_o = (\tau - L)/(N_o - 1)$ (see Figure 5.5).

Remark 5.1 The definitions of the multirate hold and the multirate sampler in this section are slightly different from those in Sec. 5.1. To see this, consider the sampled-data system shown in Figure 5.1 without feedback loop (or equivalently, $F_1(t) = 0$ and $G_1(t) = 0$). The input to the multirate hold and the output from the multirate sampler in this section are $\overline{u}_d$ and $\overline{y}_d$, respectively, which consist of inputs and outputs at multiple instants, while in Sec. 5.1, the corresponding signals are u_d and y_d, which consist of the input and output at a single instant.
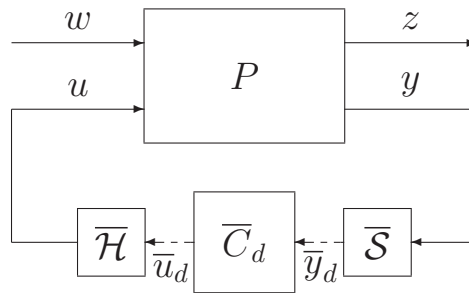
Figure 5.4: Sampled-data system using time-sharing multirate sample-hold controllers.
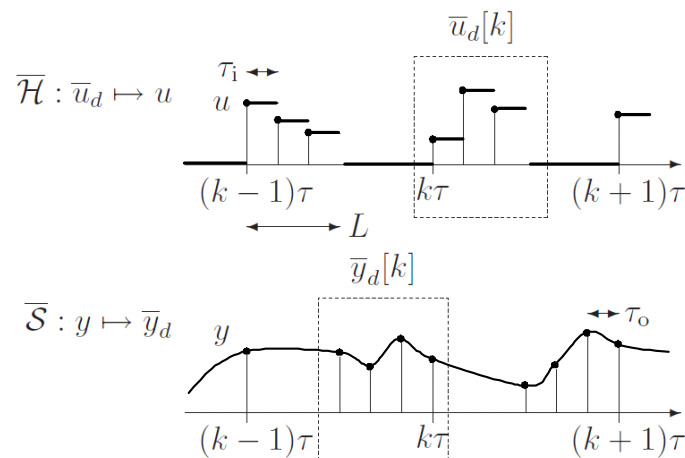
Figure 5.5: Time-chart of time-sharing multirate sample-hold scheme.

5.2.1 Representation via FR-operator

In this section, we give the FR-operator representation for the sampled-data system using time-sharing multirate sample-hold controllers shown in Figure 5.4 and define the frequency response for the sampled-data system. Here, as in the case of Sec. 2.2, we suppose that the transfer matrix of the generalized plant P is given by

$$P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & D_{11} & D_{12} \\ C_2 & 0 & 0 \end{array} \right], \tag{5.33}$$

where $P_{ij}(s)$'s are component blocks of $P(s)$ divided according to the sizes of the signals z, y, w and u, and that the transfer matrix of the discrete-time controller $\overline{C}_d$ is given by $\overline{C}_d(z)$. Before deriving the FR-operator representation of the sampled-data system shown in Figure 5.4, we first consider the transfer characteristics of the

multirate hold $\overline{\mathcal{H}}$ and the multirate sampler $\overline{\mathcal{S}}$.

Now, we begin by considering the multirate hold $\overline{\mathcal{H}}$. To obtain the transfer characteristics of $\overline{\mathcal{H}}$, let us introduce the discrete-time signal $\overline{u}_d[k]$ such that $\overline{u}_d[0] = \begin{bmatrix} \overline{u}_{d,0}^T & \cdots & \overline{u}_{d,N_i-1}^T \end{bmatrix}^T$ and $\overline{u}_d[k] = 0$ $(k \neq 0)$. In this case, according to (5.30), the output u of the multirate hold becomes

$$u(t) = \begin{cases} \overline{u}_{d,j} & (j\tau_i \leq t < (j+1)\tau_i; \; j = 0, 1, \cdots, N_i - 1), \\ 0 & (\text{otherwise}). \end{cases} \tag{5.34}$$

From (5.34), the Laplace transform of $u(t)$ is given by

$$\int_0^\infty u(t)e^{-st}dt = \int_0^{N_i\tau} u(t)e^{-st}dt = \sum_{j=0}^{N_i-1} \int_{j\tau_i}^{(j+1)\tau_i} \overline{u}_{d,j}e^{-st}dt \tag{5.35}$$

$$= \sum_{j=0}^{N_i-1} \int_0^{\tau_i} e^{-s(t+j\tau_i)}dt\overline{u}_{d,j} = \sum_{j=0}^{N_i-1} \frac{1-e^{-\tau_i s}}{s}e^{-j\tau_i s}\overline{u}_{d,j} \tag{5.36}$$

$$= \begin{bmatrix} I & \cdots & e^{-(N_i-1)\tau_i s}I \end{bmatrix} H_{\tau_i}(s) \begin{bmatrix} \overline{u}_{d,0}^T & \cdots & \overline{u}_{d,N_i-1}^T \end{bmatrix}^T \tag{5.37}$$

$$= \overline{H}(s)\overline{u}_d[0], \tag{5.38}$$

where

$$\overline{H}(s) = \overline{D}_i(s)H_{\tau_i}(s), \; \overline{D}_i(s) = \begin{bmatrix} I & \cdots & e^{-(N_i-1)\tau_i s}I \end{bmatrix}, \; H_{\tau_i}(s) = \frac{1-e^{-\tau_i s}}{s}I. \tag{5.39}$$

Thus, we obtain the transfer matrix of the multirate hold $\overline{\mathcal{H}}$ as $\overline{H}(s)$ given in (5.39).

Next, we consider the transfer characteristic of the multirate sampler $\overline{\mathcal{S}}$. From (5.31) and (5.32), together with the relation $\tau_o = (\tau - L)/(N_o - 1)$, we have

$$\overline{y}_d[k] = \begin{bmatrix} y((k-1)\tau + L) \\ \vdots \\ y((k-1)\tau + L + (N_o-1)\tau_o) \end{bmatrix} = \begin{bmatrix} y(k\tau - (N_o-1)\tau_o)) \\ \vdots \\ y(k\tau) \end{bmatrix}. \tag{5.40}$$

This implies that $\overline{y}_d[k]$ is obtained by sampling the continuous-time signal $\overline{y}(t)$ given by

$$\overline{y}(t) = \begin{bmatrix} y^T(t - (N_o-1)\tau_o)) & \cdots & y^T(t) \end{bmatrix}^T \tag{5.41}$$

with the sampling period τ. From the above observation, the multirate sampler is regarded as the time-delay components $\overline{D}_o$ with the transfer matrix $\overline{D}_o(s) = \begin{bmatrix} e^{-(N_o-1)\tau_o s}I & \cdots & I \end{bmatrix}^T$ followed by the ideal sampler with the sampling period τ.

From the above observations, we can obtain the sampled-data system shown in Figure 5.6 as an equivalent expression of the sampled-data system shown in

Figure 5.4, where $\mathcal{H}_{\tau_i}$ is the zero-order hold with the period τ_i and $\mathcal{S}$ is the ideal sampler with the sampling period τ. Thus, it readily follows that the FR-operator representation of the sampled-data system of Figure 5.4 is obtained by replacing the transfer matrices $P_{ij}(s)$ appearing in the FR-matrix given by (2.3)-(2.8) with the corresponding components $\overline{P}_{ij}(s)$ of the generalized plant $\overline{P}(s)$ given by

$$\overline{P}(s) = \begin{bmatrix} \overline{P}_{11}(s) & \overline{P}_{12}(s) \\ \overline{P}_{21}(s) & \overline{P}_{22}(s) \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & \overline{D}_o(s) \end{bmatrix} \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & \overline{D}_i(s) \end{bmatrix}, \quad (5.42)$$

and by replacing $H(s)$ in (2.5) and (2.8) with $H_{\tau_i}(s)$. Consequently, we obtain the FR-matrix $\underline{Q}(j\varphi)$ for the sampled-data system of Figure 5.4 as

$$\underline{Q}(j\varphi) = \overline{\underline{P}}_{11}(j\varphi) + \frac{1}{\tau}\overline{\underline{P}_{12}H_{\tau_i}}(j\varphi)\overline{\Lambda}(e^{j\varphi\tau})\overline{\underline{P}}_{21}(j\varphi), \quad (5.43)$$

where

$$\overline{\underline{P}}_{11}(j\varphi) = \underline{P}_{11}(j\varphi) = \text{block diag}[\cdots, P_{11}(j\varphi_{-1}), P_{11}(j\varphi_0), P_{11}(j\varphi_1), \cdots], \quad (5.44)$$

$$\overline{\underline{P}_{12}H_{\tau_i}}(j\varphi) = \begin{bmatrix} \vdots \\ \overline{P}_{12}(j\varphi_{-1})H_{\tau_i}(j\varphi_{-1}) \\ \overline{P}_{12}(j\varphi_0)H_{\tau_i}(j\varphi_0) \\ \overline{P}_{12}(j\varphi_1)H_{\tau_i}(j\varphi_1) \\ \vdots \end{bmatrix}, \quad (5.45)$$

$$\overline{\underline{P}}_{21}(j\varphi) = \begin{bmatrix} \cdots & \overline{P}_{21}(j\varphi_{-1}) & \overline{P}_{21}(j\varphi_0) & \overline{P}_{21}(j\varphi_1) & \cdots \end{bmatrix}, \quad (5.46)$$

$$\overline{\Lambda}(e^{j\varphi\tau}) = \overline{C}_d(e^{j\varphi\tau})\bigl(I - \overline{\Pi}_{22}(e^{j\varphi\tau})\overline{C}_d(e^{j\varphi\tau})\bigr)^{-1}, \quad (5.47)$$
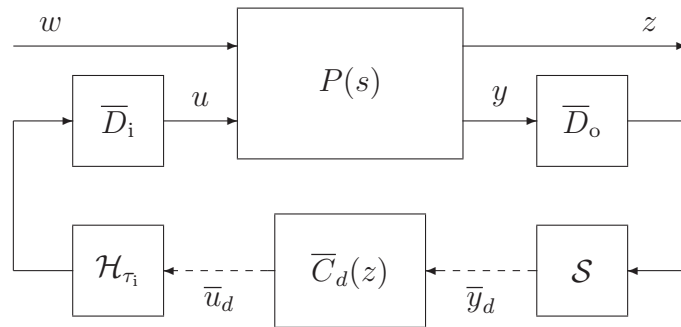
Figure 5.6: Sampled-data system equivalent to Figure 5.4.

$$\overline{\Pi}_{22}(e^{j\varphi\tau}) = \frac{1}{\tau} \sum_{m=-\infty}^{\infty} \overline{P}_{22}(j\varphi_m)H_{\tau_{\mathrm{i}}}(j\varphi_m) = \mathcal{Z}\left[\overline{P}_{22}(s)H_{\tau_{\mathrm{i}}}(s)\right]_{z=e^{j\varphi\tau}}, \tag{5.48}$$

$$= \mathcal{Z}\left[\overline{D}_{\mathrm{o}}(s)P_{22}(s)\overline{D}_{\mathrm{i}}(s)H_{\tau_{\mathrm{i}}}(s)\right]_{z=e^{j\varphi\tau}}, \tag{5.49}$$

$$\varphi_m = \varphi + m\omega_s \ (m = 0, \pm 1, \cdots), \quad \omega_s = \frac{2\pi}{\tau}. \tag{5.50}$$

Using the FR-matrix $\underline{Q}(j\varphi)$, we define the frequency response gain of the sampled-data system of Figure 5.4 at angular frequency φ as $\|\underline{Q}(j\varphi)\|_{l^2/l^2}$. The H^∞ norm of the sampled-data system is defined by $\|\underline{Q}(j\varphi)\|_\infty := \max_\varphi \|\underline{Q}(j\varphi)\|_{l^2/l^2}$.

5.2.2 Computation method for frequency response gain

In this section we provide two methods for computing the frequency response gain of sampled-data systems using the time-sharing multirate sample-hold controllers. The first method is a general method based on a bisection algorithm. The second method is based on the norm equivalent discrete-time system, which can be applied under the condition $P_{11}(s) = 0$. We begin with the bisection method.

In Sec. 3.2, we gave a bisection method for computing the frequency response gain of sampled-data systems by using the infinite-dimensional congruent transformation. A parallel method can be obtained by using the same approach. Here, we assume that the generalized plant given by (5.33) satisfies $D_{11} = 0$, as in the case of Sec. 3.2. A key result for the frequency response gain computation is as follows, which corresponds to Theorem 3.1 in Sec. 3.2:

Theorem 5.2 Suppose that γ is not a singular value of $\overline{P}_{11}(j\varphi)$. Then, the following conditions are equivalent:

(i) $\|\underline{Q}(j\varphi)\|_{l^2/l^2} < \gamma,$ \hfill (5.51)

(ii) $\mathrm{N}\left(\overline{F}(\varphi, \gamma)\right) = (0, \bar{l} - \nu),$ \hfill (5.52)

where $\bar{l}$ and ν are the number of outputs of $\overline{\Pi}_{22}(s)$ and the number of singular values of $\overline{P}_{11}(j\varphi)$ larger than γ, respectively, and $\overline{F}(\varphi, \gamma)$ is a finite-dimensional matrix given by

$$\overline{F}(\varphi, \gamma) := \begin{bmatrix} 0 & I - \overline{\Pi}_{22}(e^{j\varphi\tau})\overline{C}_d(e^{j\varphi\tau}) \\ I - \overline{C}_d^*(e^{j\varphi\tau})\overline{\Pi}_{22}^*(e^{j\varphi\tau}) & 0 \end{bmatrix}$$
$$-\frac{1}{\tau}\begin{bmatrix} \underline{P}_{21}(j\varphi) & 0 \\ 0 & \overline{C}_d^*(e^{j\varphi\tau})\underline{P}_{12}H_{\tau_{\mathrm{i}}}^*(j\varphi) \end{bmatrix}\begin{bmatrix} \gamma I & -\underline{P}_{11}^*(j\varphi) \\ -\underline{P}_{11}(j\varphi) & \gamma I \end{bmatrix}^{-1}$$
$$\times\begin{bmatrix} \underline{P}_{21}^*(j\varphi) & 0 \\ 0 & \underline{P}_{12}H_{\tau_{\mathrm{i}}}(j\varphi)\overline{C}_d(e^{j\varphi\tau}) \end{bmatrix}. \tag{5.53}$$

By Theorem 5.2, whether $\|\underline{Q}(j\varphi)\|_{l^2/l^2} < \gamma$ holds can be determined by counting the number of singular values of $\overline{P}_{11}(j\varphi)$ (or equivalently, $\underline{P}_{11}(j\varphi)$) larger than γ, the number of negative eigenvalues of the finite-dimensional matrix $\overline{F}(\varphi, \gamma)$, and the multiplicity of the zero eigenvalue of $\overline{F}(\varphi, \gamma)$. A method for computing the first number has already been provided in Sec. 3.2.2. For the second and third numbers, we provide a method for computing $\overline{F}(\varphi, \gamma)$ in the following.

Observe that (5.53) can be rewritten as

$$\overline{F}(\varphi, \gamma) = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} - \begin{bmatrix} I & 0 \\ 0 & \overline{C}_d^*(e^{j\varphi\tau}) \end{bmatrix} \overline{\Phi}_\gamma(e^{j\varphi\tau}) \begin{bmatrix} I & 0 \\ 0 & \overline{C}_d(e^{j\varphi\tau}) \end{bmatrix}, \tag{5.54}$$

where

$$\overline{\Phi}_\gamma(e^{j\varphi\tau}) := \begin{bmatrix} 0 & \overline{\Pi}_{22}(e^{j\varphi\tau}) \\ \overline{\Pi}_{22}^*(e^{j\varphi\tau}) & 0 \end{bmatrix} + \frac{1}{\tau} \begin{bmatrix} \overline{P}_{21}(j\varphi) & 0 \\ 0 & \overline{P}_{12}H_{\tau_i}^*(j\varphi) \end{bmatrix}$$

$$\times \begin{bmatrix} \gamma I & -\overline{P}_{11}^*(j\varphi) \\ -\overline{P}_{11}(j\varphi) & \gamma I \end{bmatrix}^{-1} \begin{bmatrix} \overline{P}_{21}^*(j\varphi) & 0 \\ 0 & \underline{P}_{12}H_{\tau_i}(j\varphi) \end{bmatrix}. \tag{5.55}$$

For the computation of $\overline{\Phi}_\gamma(e^{j\varphi\tau})$, we have the following theorem, which corresponds to Theorem 3.3 in Sec. 3.2:

Theorem 5.3 $\overline{\Phi}_\gamma(e^{j\varphi\tau})$ is the frequency pulse-transfer function of the discrete-time system given by

$$\overline{\Phi}_\gamma(z) = \begin{bmatrix} \overline{\Phi}_{\gamma 11}(z) & \overline{\Phi}_{\gamma 12}(z) \\ \overline{\Phi}_{\gamma 21}(z) & \overline{\Phi}_{\gamma 22}(z) \end{bmatrix} = \left[\begin{array}{c|cc} \overline{A} & \overline{B}_1 & \overline{B}_2 \\ \hline \overline{C}_1 & \overline{D}_{11} & 0 \\ \overline{C}_2 & \overline{D}_{12} & \overline{D}_{22} \end{array} \right], \tag{5.56}$$

where

$$\overline{A} = e^{E\tau}, \quad \overline{B}_1 = E_i^{N_i} \begin{bmatrix} E_o^{N_o-1}F_1 & \cdots & E_oF_1 & F_1 \end{bmatrix}, \tag{5.57}$$

$$\overline{B}_2 = \begin{bmatrix} E_i^{N_i-1}F_{2i} & \cdots & E_iF_{2i} & F_{2i} \end{bmatrix}, \tag{5.58}$$

$$\overline{C}_1 = \begin{bmatrix} G_1 \\ G_1E_o \\ \vdots \\ G_1E_o^{N_o-1} \end{bmatrix}, \quad \overline{C}_2 = \begin{bmatrix} G_{2i} \\ G_{2i}E_i \\ \vdots \\ G_{2i}E_i^{N_i-1} \end{bmatrix} E_o^{N_o-1}, \tag{5.59}$$

$$E_o = e^{E\tau_o}, \quad \begin{bmatrix} E_i & F_{2i} & 0 \\ 0 & I & 0 \\ G_{2i} & D_i & I \end{bmatrix} = \exp\left\{ \begin{bmatrix} E & F_2 & 0 \\ 0 & 0 & 0 \\ G_2 & D & 0 \end{bmatrix} \tau_i \right\}, \tag{5.60}$$

$$E = \begin{bmatrix} -A^T & -\frac{1}{\gamma}C_1^TC_1 \\ \frac{1}{\gamma}B_1B_1^T & A \end{bmatrix}, \quad F_1 = \begin{bmatrix} -C_2^T \\ 0 \end{bmatrix}, \quad F_2 = \begin{bmatrix} -\frac{1}{\gamma}C_1^TD_{12} \\ B_2 \end{bmatrix}, \tag{5.61}$$

$$G_1 = \begin{bmatrix} 0 & C_2 \end{bmatrix}, \quad G_2 = \begin{bmatrix} B_2^T & \frac{1}{\gamma}D_{12}^TC_1 \end{bmatrix}, \quad D = \frac{1}{\gamma}D_{12}^TD_{12}, \tag{5.62}$$

and the (m,n)-block components of $\overline{D}_{11}$, $\overline{D}_{21}$, and $\overline{D}_{22}$ (denoted by $\overline{D}_{11mn}$, $\overline{D}_{12mn}$, and $\overline{D}_{22mn}$) are given by

$$\overline{D}_{11mn} = \begin{cases} 0 & (m \le n) \\ G_1 E_{\text{o}}^{m-n} F_1 & (m > n) \end{cases} \quad (m = 0, \cdots, N_{\text{o}} - 1; n = 0, \cdots, N_{\text{o}} - 1), \quad (5.63)$$

$$\overline{D}_{21mn} = G_{2\text{i}} E_{\text{i}}^{m} E_{\text{o}}^{N_{\text{o}}-1-n} F_1 \quad (m = 0, \cdots, N_{\text{i}} - 1; n = 0, \cdots, N_{\text{o}} - 1), \quad (5.64)$$

$$\overline{D}_{22mn} = \begin{cases} 0 & (m < n) \\ D_{\text{i}} & (m = n) \\ G_{2\text{i}} E_{\text{i}}^{m-n-1} F_{2\text{i}} & (m > n) \end{cases}$$

$$(m = 0, \cdots, N_{\text{i}} - 1; n = 0, \cdots, N_{\text{i}} - 1). \quad (5.65)$$

By Theorem 5.3, we can compute the finite-dimensional matrix $\overline{F}(\varphi, \gamma)$, and thus by Theorem 5.2, we can compute the frequency response gain of sampled-data systems using time-sharing multirate sample-hold scheme. To prove Theorem 5.3, we need the following lemma:

Lemma 5.1 Let $-\tau \le a \le \tau$. Then,

$$\mathcal{Z}\left[C(sI - A)^{-1} B e^{as} \right] = \begin{cases} C e^{A\tau_1} (zI - \widehat{A})^{-1} e^{A\tau_2} B & (-\tau \le a < 0) \\ C e^{A\tau_1} (zI - \widehat{A})^{-1} e^{A\tau_2} B + C e^{Aa} B & (0 \le a \le \tau) \end{cases} \quad (5.66)$$

where $\widehat{A} = e^{A\tau}$ and τ_i $(i = 1, 2)$ are any real numbers satisfying $\tau_1 + \tau_2 = \tau + a$.

Proof of Lemma 5.1: Note that the inverse z-transform of $C(sI - A)^{-1} B e^{as}$ is $C e^{A(t+a)} B \mathbf{u}(t + a)$, where $\mathbf{u}(t)$ is the unit step function. From this, we have

$$\mathcal{Z}\left[C(sI - A)^{-1} B e^{as} \right] = \begin{cases} \sum_{k=1}^{\infty} C e^{A(k\tau+a)} B z^{-k} & (-\tau \le a < 0) \\ \sum_{k=0}^{\infty} C e^{A(k\tau+a)} B z^{-k} & (0 \le a \le \tau) \end{cases}. \quad (5.67)$$

This yields (5.66).

Proof of Theorem 5.3: From the impulse modulation formula, the right-hand side of (5.55) can be rewritten as

$$\begin{aligned} \overline{\Phi}_\gamma(e^{j\varphi\tau}) &= \frac{1}{\tau} \sum_{m=-\infty}^{\infty} \begin{bmatrix} \overline{D}_{\text{o}}(j\varphi_m) & 0 \\ 0 & H_{\tau_{\text{i}}}^{*}(j\varphi_m)\overline{D}_{\text{i}}^{*}(j\varphi_m) \end{bmatrix} T(j\varphi_m) \begin{bmatrix} \overline{D}_{\text{o}}^{*}(j\varphi_m) & 0 \\ 0 & \overline{D}_{\text{i}}(j\varphi_m)H_{\tau_{\text{i}}}(j\varphi_m) \end{bmatrix} \\ &= \mathcal{Z}\left[\begin{bmatrix} \overline{D}_{\text{o}}(s) & 0 \\ 0 & H_{\tau_{\text{i}}}\tilde{}(s)\overline{D}_{\text{i}}\tilde{}(s) \end{bmatrix} T(s) \begin{bmatrix} \overline{D}_{\text{o}}\tilde{}(s) & 0 \\ 0 & \overline{D}_{\text{i}}(s)H_{\tau_{\text{i}}}(s) \end{bmatrix} \right]_{z=e^{j\varphi\tau}}, \quad (5.68) \end{aligned}$$

where

$$T(s) = \begin{bmatrix} 0 & P_{22}(s) \\ P_{22}\tilde{}(s) & 0 \end{bmatrix} + \begin{bmatrix} P_{21}(s) & 0 \\ 0 & P_{12}\tilde{}(s) \end{bmatrix} \begin{bmatrix} \gamma I & -P_{11}\tilde{}(s) \\ -P_{11}(s) & \gamma I \end{bmatrix}^{-1} \begin{bmatrix} P_{21}\tilde{}(s) & 0 \\ 0 & P_{12}(s) \end{bmatrix}. \tag{5.69}$$

The state-space representation of $T(s)$ was derived in Sec. 3.2.3. It was given by

$$T(s) = \left[\begin{array}{c|cc} E & F_1 & F_2 \\ \hline G_1 & 0 & 0 \\ G_2 & 0 & D \end{array} \right], \tag{5.70}$$

where E, F_1, F_2, G_1, G_2, and D are given by (5.61) and (5.62). Here, note that

$$\begin{bmatrix} \overline{D}_o\tilde{}(s) & 0 \\ 0 & \overline{D}_i(s) H_{\tau_i}(s) \end{bmatrix} = \left[\begin{array}{c|cc} 0 & 0 & I \\ 0 & I & 0 \\ I & 0 & 0 \end{array} \right] \begin{bmatrix} \overline{D}_o\tilde{}(s) & 0 \\ 0 & (1 - e^{-\tau_i s})\overline{D}_i(s) \end{bmatrix}, \tag{5.71}$$

and

$$\begin{bmatrix} \overline{D}_o(s) & 0 \\ 0 & H_{\tau_i}\tilde{}(s)\overline{D}_i\tilde{}(s) \end{bmatrix} = \begin{bmatrix} \overline{D}_o(s) & 0 \\ 0 & (1 - e^{\tau_i s})\overline{D}_i\tilde{}(s) \end{bmatrix} \left[\begin{array}{c|cc} 0 & 0 & I \\ 0 & I & 0 \\ -I & 0 & 0 \end{array} \right]. \tag{5.72}$$

Therefore, from (5.68), (5.70), (5.71), and (5.72), together with Lemma 2.1, we obtain

$$\overline{\Phi}_\gamma(z) = \mathcal{Z}\left[\begin{bmatrix} \overline{D}_o(s) & 0 \\ 0 & H_{\tau_i}\tilde{}(s)\overline{D}_i\tilde{}(s) \end{bmatrix} T(s) \begin{bmatrix} \overline{D}_o\tilde{}(s) & 0 \\ 0 & \overline{D}_i(s)H_{\tau_i}(s) \end{bmatrix} \right]$$

$$= \mathcal{Z}\left[\begin{bmatrix} \overline{D}_o(s) & 0 \\ 0 & (1 - e^{\tau_i s})\overline{D}_i\tilde{}(s) \end{bmatrix} \left[\begin{array}{c|cc} \check{A} & \check{B}_1 & \check{B}_2 \\ \hline \check{C}_1 & 0 & 0 \\ \check{C}_2 & 0 & 0 \end{array} \right] \begin{bmatrix} \overline{D}_o\tilde{}(s) & 0 \\ 0 & (1 - e^{-\tau_i s})\overline{D}_i(s) \end{bmatrix} \right], \tag{5.73}$$

where

$$\check{A} = \begin{bmatrix} E & F_2 & 0 \\ 0 & 0 & 0 \\ G_2 & D & 0 \end{bmatrix}, \quad \check{B}_1 = \begin{bmatrix} F_1 \\ 0 \\ 0 \end{bmatrix}, \quad \check{B}_2 = \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix}, \quad \check{C}_1 = \begin{bmatrix} G_1 & 0 & 0 \end{bmatrix}, \quad \check{C}_2 = \begin{bmatrix} 0 & 0 & -I \end{bmatrix}.$$

By applying Lemma 5.1 to (5.73), we obtain a state-space representation of $\overline{\Phi}_\gamma(z)$ as

$$\overline{\Phi}_\gamma(z) = \begin{bmatrix} \overline{\Phi}_{\gamma 11}(z) & \overline{\Phi}_{\gamma 12}(z) \\ \overline{\Phi}_{\gamma 21}(z) & \overline{\Phi}_{\gamma 22}(z) \end{bmatrix} = \left[\begin{array}{c|cc} \check{A} & \check{B}_1 & \check{B}_2 \\ \hline \check{C}_1 & \check{D}_{11} & 0 \\ \check{C}_2 & \check{D}_{12} & \check{D}_{22} \end{array} \right], \tag{5.74}$$

where

$$\check{A} = e^{\check{A}\tau}, \quad \check{B}_1 = e^{\check{A}L}\left[\check{A}_{\mathrm{o}}^{N_{\mathrm{o}}-1}\check{B}_1 \quad \cdots \quad \check{A}_{\mathrm{o}}\check{B}_1 \quad \check{B}_1\right], \quad \check{A}_{\mathrm{o}} = e^{\check{A}\tau_{\mathrm{o}}}, \tag{5.75}$$

$$\check{B}_2 = (I - \check{A}_{\mathrm{i}}^{-1})\check{A}_{\mathrm{i}}\left[\check{A}_{\mathrm{i}}^{N_{\mathrm{i}}-1}\check{B}_2 \quad \cdots \quad \check{A}_{\mathrm{i}}\check{B}_2 \quad \check{B}_2\right], \quad \check{A}_{\mathrm{i}} = e^{\check{A}\tau_{\mathrm{i}}}, \tag{5.76}$$

$$\check{C}_1 = \begin{bmatrix} \check{C}_1 \\ \check{C}_1\check{A}_{\mathrm{o}} \\ \vdots \\ \check{C}_1\check{A}_{\mathrm{o}}^{N_{\mathrm{o}}-1} \end{bmatrix}, \quad \check{C}_2 = \begin{bmatrix} \check{C}_2 \\ \check{C}_2\check{A}_{\mathrm{i}} \\ \vdots \\ \check{C}_2\check{A}_{\mathrm{i}}^{N_{\mathrm{i}}-1} \end{bmatrix}(I - \check{A}_{\mathrm{i}})e^{\check{A}(\tau-L)}, \tag{5.77}$$

and the (m,n)-block components of $\overline{D}_{11}$, $\overline{D}_{21}$, and $\overline{D}_{22}$ (denoted by $\overline{D}_{11mn}$, $\overline{D}_{12mn}$, and $\overline{D}_{22mn}$) are given by

$$\check{D}_{11mn} = \begin{cases} 0 & (m \le n) \\ \check{C}_1\check{A}_{\mathrm{o}}^{m-n}\check{B}_1 & (m > n) \end{cases} \quad (m = 0, \cdots, N_{\mathrm{o}}-1; n = 0, \cdots, N_{\mathrm{o}}-1), \tag{5.78}$$

$$\check{D}_{21mn} = \check{C}_2\check{A}_{\mathrm{i}}^{m}(I - \check{A}_{\mathrm{i}})\check{A}_{\mathrm{o}}^{(N_{\mathrm{o}}-1-n)}\check{B}_1 \quad (m = 0, \cdots, N_{\mathrm{i}}-1; n = 0, \cdots, N_{\mathrm{o}}-1), \tag{5.79}$$

$$\check{D}_{22mn} = \begin{cases} 0 & (m < n) \\ -\check{C}_2\check{A}_{\mathrm{i}}\check{B}_2 & (m = n) \\ \check{C}_2(2I - \check{A}_{\mathrm{i}})\check{A}_{\mathrm{i}}\check{B}_2 & (m = n+1) \\ \check{C}_2(I - \check{A}_{\mathrm{i}})\check{A}_{\mathrm{i}}^{m-n}(I - \check{A}_{\mathrm{i}}^{-1})\check{B}_2 & (m > n+1) \end{cases}$$

$$(m = 0, \cdots, N_{\mathrm{i}}-1; n = 0, \cdots, N_{\mathrm{i}}-1). \tag{5.80}$$

For the above derivation, we used $\check{C}_1\check{B}_1 = 0$, $\check{C}_1\check{B}_2 = 0$, and $\check{C}_2\check{B}_2 = 0$.

In the following, we only show that $\overline{\Phi}_{\gamma22}(z)$ given by (5.74)-(5.80) coincides with $\overline{\Phi}_{\gamma22}(z)$ given by (5.56)-(5.65). For $\overline{\Phi}_{\gamma11}(z)$, $\overline{\Phi}_{\gamma12}(z)$, and $\overline{\Phi}_{\gamma21}(z)$, their derivations are carried out in a similar manner, and so they are omitted. From (5.75), (5.76), (5.77), and (5.80), the (m,n)-block component of $\overline{\Phi}_{\gamma22}(z)$, denoted by $\overline{\Phi}_{\gamma22mn}(z)$, is given by

$$\overline{\Phi}_{\gamma22mn}(z) = \left[\begin{array}{c|c} e^{\check{A}\tau} & \check{A}_{\mathrm{i}}^{N_{\mathrm{i}}-1-n}(\check{A}_{\mathrm{i}} - I)\check{B}_2 \\ \hline \check{C}_2(I - \check{A}_{\mathrm{i}})\check{A}_{\mathrm{i}}^{m}e^{\check{A}(\tau-L)} & \check{D}_{22mn} \end{array}\right], \tag{5.81}$$

where we used the relation $(\check{A}_{\mathrm{i}} - I)\check{A}_{\mathrm{i}}^{k} = \check{A}_{\mathrm{i}}^{k}(\check{A}_{\mathrm{i}} - I)$. Now, let us introduce the following matrix exponentiation:

$$\begin{bmatrix} E(t) & F_2(t) & 0 \\ 0 & I & 0 \\ G_2(t) & D(t) & I \end{bmatrix} = e^{\check{A}t} = \exp\left\{\begin{bmatrix} E & F_2 & 0 \\ 0 & 0 & 0 \\ G_2 & D & 0 \end{bmatrix}t\right\}. \tag{5.82}$$

From (5.60), (5.81), (5.82), and Lemma 2.2, we obtain

$$
\overline{\Phi}_{\gamma 22mn}(z) =
\left[
\begin{array}{ccc|c}
E(\tau) & * & 0 & E\left((N_{\mathrm{i}}-1-n)\tau_{\mathrm{i}}\right)F_2(\tau_{\mathrm{i}}) \\
0 & * & 0 & 0 \\
* & * & * & * \\
\hline
G_2(\tau_{\mathrm{i}})E(m\tau_{\mathrm{i}})E(\tau-L) & * & 0 & \check{D}_{22mn}
\end{array}
\right]
\tag{5.83}
$$

$$
=
\left[
\begin{array}{c|c}
e^{E\tau} & E_{\mathrm{i}}^{N_{\mathrm{i}}-1-n}F_{2\mathrm{i}} \\
\hline
G_{2\mathrm{i}}E_{\mathrm{i}}^{m}E_{\mathrm{o}}^{N_{\mathrm{o}}-1} & \check{D}_{22mn}
\end{array}
\right].
\tag{5.84}
$$

As for $\check{D}_{22mn}$, we can obtain

$$
-\check{C}_2\check{A}_{\mathrm{i}}\check{B}_2 = D(\tau_{\mathrm{i}}) = D_{\mathrm{i}}, \ \check{C}_2(2I-\check{A}_{\mathrm{i}})\check{A}_{\mathrm{i}}\check{B}_2 = G_2(\tau_{\mathrm{i}})F_2(\tau_{\mathrm{i}}) = G_{2\mathrm{i}}F_{2\mathrm{i}},
\tag{5.85}
$$

$$
\check{C}_2(I-\check{A}_{\mathrm{i}})\check{A}_{\mathrm{i}}^{m-n}(I-\check{A}_{\mathrm{i}}^{-1})\check{B}_2 = G_2(\tau_{\mathrm{i}})E(\tau_{\mathrm{i}})^{m-n-1}F_2(\tau_{\mathrm{i}}) = G_{2\mathrm{i}}E_{\mathrm{i}}^{m-n-1}F_{2\mathrm{i}}.
\tag{5.86}
$$

The equations (5.82)-(5.86) imply that $\overline{\Phi}_{\gamma 22}(z)$ given by (5.74)-(5.80) coincides with $\overline{\Phi}_{\gamma 22}(z)$ given by (5.56)-(5.65).

We can also derive a norm equivalent discrete-time system when the generalized plant shown in Figure 5.4 satisfies $P_{11}(s) = 0$.

Theorem 5.4 Consider the sampled-data system shown in Figure 5.4. Suppose that $P_{11}(s) = 0$. Then, the frequency response gain $\|Q(j\varphi)\|_{l^2/l^2}$ coincides with $\bar{\sigma}(\widehat{Q}(e^{j\varphi\tau}))$ for every φ, where $\widehat{Q}(z) = \mathcal{F}_l(\overline{\Pi}(z), \overline{C}_d(z))$ and $\overline{\Pi}(z)$ is a generalized plant given by

$$
\overline{\Pi}(z) =
\left[
\begin{array}{c|cc}
\widehat{A} & \overline{B}_{1d} & \overline{B}_{2d} \\
\hline
\overline{C}_{1d} & 0 & \overline{D}_{12d} \\
\overline{C}_{2d} & \overline{D}_{21d} & 0
\end{array}
\right],
\tag{5.87}
$$

where

$$
\widehat{A} = e^{A\tau}, \ \overline{B}_{2d} = \left[A_{\mathrm{i}}^{N_{\mathrm{i}}-1}B_{\mathrm{i}} \ \cdots \ A_{\mathrm{i}}B_{\mathrm{i}} \ B_{\mathrm{i}}\right], \ A_{\mathrm{i}} = e^{A\tau_{\mathrm{i}}}, \ B_{\mathrm{i}} = \int_0^{\tau_{\mathrm{i}}} e^{At}B_2 dt,
\tag{5.88}
$$

$$
\overline{C}_{2d} = \left[C_2^T \ (C_2 A_{\mathrm{o}})^T \ \cdots (C_2 A_{\mathrm{o}}^{N_{\mathrm{o}}-1})^T\right]^T, \ A_{\mathrm{o}} = e^{A\tau_{\mathrm{o}}},
\tag{5.89}
$$

and $\overline{B}_{1d}$, $\overline{C}_{1d}$, $\overline{D}_{12d}$, and $\overline{D}_{21d}$ are the matrices satisfying

$$
\left[\begin{array}{c}\overline{B}_{1d} \\ \overline{D}_{21d}\end{array}\right]
\left[\overline{B}_{1d}^T \ \overline{D}_{21d}^T\right] =
\left[\begin{array}{cc}\overline{W}_{11} & \overline{W}_{12} \\ \overline{W}_{12}^T & \overline{W}_{22}\end{array}\right], \
\left[\begin{array}{c}\overline{C}_{1d}^T \\ \overline{D}_{12d}^T\end{array}\right]
\left[\overline{C}_{1d} \ \overline{D}_{12d}\right] =
\left[\begin{array}{cc}\overline{V}_{11} & \overline{V}_{12} \\ \overline{V}_{12}^T & \overline{V}_{22}\end{array}\right].
\tag{5.90}
$$

In (5.90), $\overline{W}_{11}$ and $\overline{W}_{12}$ are given by

$$
\overline{W}_{11} = W(\tau), \ W(t) = \int_0^t e^{A\theta}B_1 B_1^T e^{A^T\theta}d\theta,
\tag{5.91}
$$

$$
\overline{W}_{12} = \left[0 \ e^{A(\tau-\tau_{\mathrm{o}})}W(\tau_{\mathrm{o}})C_2^T \ \cdots \ e^{A(\tau-(N_{\mathrm{o}}-1)\tau_{\mathrm{o}})}W((N_{\mathrm{o}}-1)\tau_{\mathrm{o}})C_2^T\right],
\tag{5.92}
$$

and the (m, n)-block component of $\overline{W}_{22}$ (denoted by $\overline{W}_{22mn}$) is given by

$$
\overline{W}_{22mn} = \begin{cases} C_2 W(m\tau_\mathrm{o}) A_\mathrm{o}^{(n-m)T} C_2^T & (m \leq n) \\ C_2 A_\mathrm{o}^{(m-n)} W(n\tau_\mathrm{o}) C_2^T & (m > n) \end{cases}
$$

$$
(m = 0, \cdots, N_\mathrm{o} - 1; n = 0, \cdots, N_\mathrm{o} - 1). \qquad (5.93)
$$

The matrices $\overline{V}_{11}$ and $\overline{V}_{12}$ are given by

$$
\overline{V}_{11} = V(\tau), \ V(t) = \int_0^t e^{A^T \theta} C_1^T C_1 e^{A\theta} d\theta, \qquad (5.94)
$$

$$
\overline{V}_{12} = e^{A(\tau - L)} \left[V_{12\mathrm{i}} + A_\mathrm{i} V((N_\mathrm{i} - 1)\tau_\mathrm{i}) B_\mathrm{i} \quad \cdots \quad A_\mathrm{i}^{N_\mathrm{i}-1} (V_{12\mathrm{i}} + A_\mathrm{i} V(0) B_\mathrm{i}) \right], \qquad (5.95)
$$

and the (m, n)-block component of $\overline{V}_{22}$ (denoted by $\overline{V}_{22mn}$) is given by

$$
\overline{V}_{22mn} = \begin{cases} B_\mathrm{i}^T A_\mathrm{i}^{(n-m-1)T} V_{12\mathrm{i}} + B_\mathrm{i}^T A_\mathrm{i}^{(n-m)T} V((N_\mathrm{i} - 1 - n)\tau_\mathrm{i}) B_\mathrm{i} & (m < n) \\ V_{22\mathrm{i}} + B_\mathrm{i}^T V((N_\mathrm{i} - 1 - m)\tau_\mathrm{i}) B_\mathrm{i} & (m = n) \\ V_{12\mathrm{i}}^T A_\mathrm{i}^{(m-n-1)} B_\mathrm{i} + B_\mathrm{i}^T V((N_\mathrm{i} - 1 - m)\tau_\mathrm{i}) A_\mathrm{i}^{(m-n)} B_\mathrm{i} & (m > n) \end{cases}
$$

$$
(m = 0, \cdots, N_\mathrm{i} - 1; n = 0, \cdots, N_\mathrm{i} - 1), \qquad (5.96)
$$

where $V_{11\mathrm{i}}$, $V_{12\mathrm{i}}$, and $V_{22\mathrm{i}}$ are given by

$$
\begin{bmatrix} V_{11\mathrm{i}} & V_{12\mathrm{i}} \\ V_{12\mathrm{i}}^T & V_{22\mathrm{i}} \end{bmatrix} = \int_0^{\tau_\mathrm{i}} \exp\left(\begin{bmatrix} A^T & 0 \\ B_2^T & 0 \end{bmatrix} t \right) \begin{bmatrix} C_1^T \\ D_{12}^T \end{bmatrix} \begin{bmatrix} C_1 & D_{12} \end{bmatrix} \exp\left(\begin{bmatrix} A & B_2 \\ 0 & 0 \end{bmatrix} t \right) dt. \quad (5.97)
$$

Theorem 5.4 is shown by applying techniques similar to those used in Sec. 2.4 and the proof of Theorem 5.3, and so the proof is omitted.

Here, note that the generalized plant $\overline{\overline{\Pi}}(z)$ given by (5.87) is not only used for computing the frequency response gain of the sampled-data system shown in Figure 5.4 but also for solving H^∞ problems of the sampled-data system. This is because the stability of the original sampled-data system and that of the norm equivalent discrete-time system composed of $\overline{\overline{\Pi}}(z)$ and $\overline{C}_d(z)$ are also equivalent, which is shown in a similar manner to the standard case that uses the zero-order hold and the ideal sampler.

5.3　Reliable Stabilization Using Time-Sharing Multirate Sample-Hold Scheme

As an application of the time-sharing multirate sample-hold controller, we consider the reliable stabilization problem. A general formulation of the reliable stabilization
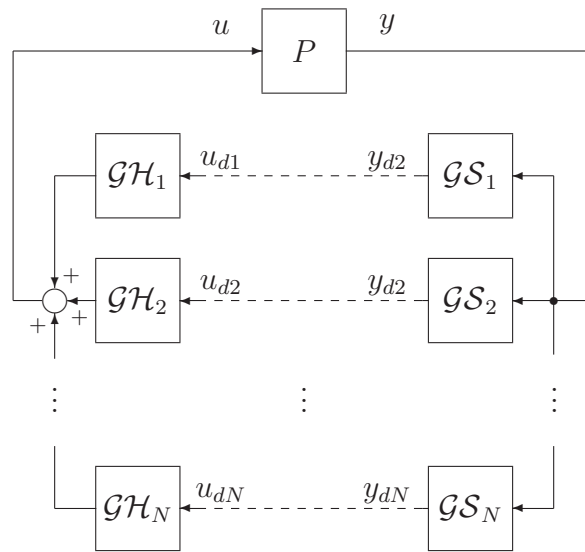
Figure 5.7: Reliable stabilization using time-sharing multirate sample-hold controllers

problem is as follows: for a given plant P and an integer N, find a set of N controllers $C_1, \cdots, C_N$ that stabilize P when they all act together, as well as when one of them fails. In the following, we assume that each C_i consists of a multirate hold $\mathcal{GH}_i$ and a multirate sampler $\mathcal{GS}_i$, with the frame period τ, that are connected directly (Figure 5.7). We further assume that $\mathcal{GH}_i$ and $\mathcal{GS}_i$ satisfy the time-sharing condition for each i (with L replaced by L_i, while τ does not depend on i). For simplicity, however, we assume in the following that τ and L_i $(i = 1, \cdots, N)$ are given, unless otherwise stated. Their values are in fact not crucial in the arguments to follow in the sense that no conditions are imposed on them, and thus we can readily extend the results to the case where τ and L_i $(i = 1, \cdots, N)$ can also be chosen as design parameters.

Remark 5.2 In this section, we only deal with a stabilization problem, so we do not consider those holds and samplers corresponding to the multirate hold with the hold function $F_2(t)$ and the multirate sampler with the sampling function $G_2(t)$ in Figure 5.1. Therefore, in the following, when we refer to multirate holds and samplers, they correspond to those in Figure 5.1 with the hold function $F_1(t)$ and the sampling function $G_1(t)$. In view of this, the subscript i $(i = 1, \cdots, N)$ in this section is used to indicate the index of the controllers and their parameters such as their hold and sampling functions, which obviates the use of double subscripts such as $F_{1i}(t)$ and $G_{1i}(t)$.

5.3.1 Reliable stabilization against 1-of-N controller failure

Now, we state the reliable stabilization problem studied in this section.

Problem 5.1 Given an integer $N(\geq 2)$, find a set of N time-sharing multirate sample-hold controllers that stabilize a given unstable plant P when they all act together, as well as when any one of them fails.

We refer to Problem 5.1 as "reliable stabilization against 1-of-N controller failure."

Extending (5.15), it is easy to see that the closed-loop state equation, when all the controllers are working normally, is given by

$$\xi[k+1] = (\overline{A} + \overline{F}_1\overline{G}_1 + \cdots + \overline{F}_N\overline{G}_N)\xi[k]. \tag{5.98}$$

Here, we assume that the transfer characteristic of a controller becomes 0 when it goes into malfunction. Therefore, if the ith controller fails, then the term $\overline{F}_i\overline{G}_i$ vanishes from the above equation.

Now, let us assume that P is given by (5.1) and (5.2), and is controllable and observable. Then $\overline{F}_i$ and $\overline{G}_i$ $(i = 1, \cdots, N)$ can be made arbitrary matrices by a suitable choice of $F_i(t)$ and $G_i(t)$ $(i = 1, \cdots, N)$ (including the number of the column of $\overline{F}_i$ and that of the row of $\overline{G}_i$). Therefore, $Q_i := \overline{F}_i\overline{G}_i$ can be made arbitrary matrices. From this, Problem 5.1 can be restated as follows:

Problem 5.2 Given $\overline{A} = e^{A\tau} \in \mathbf{R}^{n \times n}$ and an integer $N(\geq 2)$, find Q_i $(i = 1, \cdots, N)$ that make the following $N + 1$ matrices all stable.

$$S := \overline{A} + Q_1 + \cdots + Q_N, \tag{5.99}$$
$$S_i := S - Q_i \ (i = 1, \cdots, N). \tag{5.100}$$

Once Q_i $(i = 1, \cdots, N)$ are obtained such that the matrices given by (5.99) and (5.100) are all stable, we can find $\overline{F}_i$ and $\overline{G}_i$ such that $Q_i = \overline{F}_i\overline{G}_i$. We can then find $F_i(t)$ and $G_i(t)$ satisfying (5.5), (5.6), (5.12) and (5.14) for each $i = 1, \cdots, N$, provided that we take sufficiently large input and output multiplicities $N_{\mathrm{i}i}$ and $N_{\mathrm{o}i}$, and this leads to one solution to Problem 5.1. Even though there are infinitely many degrees of freedom in the design of $F_i(t)$ and $G_i(t)$ under the specified Q_i for each $i = 1, \cdots, N$, we do not pursue how to exploit that freedom in this thesis. This is because our purpose in this section is to demonstrate the high ability of the time-sharing multirate sample-hold controllers to handle the reliable stabilization problem, and thus it suffices to discuss the existence of the above matrices Q_i $(i = 1, \cdots, N)$.

To solve Problem 5.2, the following lemma obtained by Ito et al. [46] is very useful. The proof will be given in Sec. 5.5.

Lemma 5.2 Suppose $X \in \mathbf{R}^{n \times n}$ and $N \geq 2$. Then, X can be decomposed into the sum of N (discrete-time) stability matrices if and only if $|\text{trace } (X)| < Nn$.

By applying Lemma 5.2, we can obtain the solvability condition of Problem 5.2.

Lemma 5.3 There exist Q_i $(i = 1, \cdots, N)$ satisfying the condition of Problem 5.2 if and only if $|\text{trace } (\overline{A})| < (2N - 1)n$.

Proof of Lemma 5.3: First, we prove the necessity part. Suppose that S and S_i given by (5.99) and (5.100) are stable. Then, $|\text{trace } (S)| < n$ and $|\text{trace } (S_i)| < n$ $(i = 1, \cdots, N)$. Also by (5.99) and (5.100), we obtain $S_1 + \cdots + S_N = \overline{A} + (N-1)S$. Therefore,

$$
\begin{aligned}
|\text{trace } (\overline{A})| &= |\text{trace } (S_1 + \cdots + S_N - (N - 1)S)| \\
&\leq |\text{trace } (S_1)| + \cdots + |\text{trace } (S_N)| + (N - 1)|\text{trace } (S)| \\
&< (2N - 1)n.
\end{aligned}
$$

Next, we prove the sufficiency part. Suppose that $|\text{trace } (\overline{A})| < (2N - 1)n$. By Lemma 5.2, $\overline{A}$ can be decomposed into the sum of $2N - 1$ stability matrices. In particular, in view of the proof of the sufficiency part of Lemma 5.2 given in Sec. 5.5, the decomposition

$$\overline{A} = X_1 + \cdots + X_N + (N - 1)X_0$$

is possible, where X_i $(i = 0, \cdots, N)$ are all stability matrices. Now, let $Q_i = -(X_0 + X_i)$ $(i = 1, \cdots, N)$. Then, from (5.99) and (5.100), we obtain

$$
\begin{aligned}
S &= \overline{A} + Q_1 + \cdots + Q_N \\
&= \{X_1 + \cdots + X_N + (N - 1)X_0\} - \sum_{i=1}^{N}(X_0 + X_i) = -X_0, \quad (5.101)
\end{aligned}
$$

$$S_i = S - Q_i = -X_0 + (X_0 + X_i) = X_i \ (i = 1, \cdots, N). \quad (5.102)$$

This means that S_i $(i = 1, \cdots, N)$ and S are all stability matrices. This completes the proof.

By Lemma 5.3, we can readily obtain the solvability condition of Problem 5.1.

Theorem 5.5 Problem 5.1 is solvable if and only if

$$|\text{trace } (e^{A\tau})| < (2N - 1)n. \quad (5.103)$$

It should be noted that for any A and $N \geq 2$, it is always possible to satisfy the condition of Theorem 5.5 by a suitable choice of τ, since $|\text{trace } (e^{A\tau})|$ tends to n as τ goes to zero. Therefore, if (A, B, C) is controllable and observable, reliable stabilization is always possible by an appropriate choice of the frame period τ when the time-sharing multirate sample-hold scheme is employed. Also note that even if the frame period τ is fixed, reliable stabilization is possible by increasing the number of controllers. This is in sharp contrast to the case of LTI controllers, in which case the solvability condition has been obtained only under the two-controller configuration, and strong stabilizability (e.g., Vidyasagar [75]) of the plant is required in that case (e.g., Minto and Ravi [62]).

5.3.2 Reliable stabilization by decomposing a given controller

In this subsection, we consider a similar problem for reliable stabilization. Namely, suppose that we are given a stabilizing controller with the time-sharing multirate sample-hold scheme, and we study whether it is possible to decompose it into the sum of controllers with the time-sharing multirate sample-hold scheme. Recalling the arguments to reduce Problem 5.1 to Problem 5.2, we can easily see that such a problem can be formulated as follows.

Problem 5.3 Suppose that a time-sharing multirate sample-hold controller stabilizing P is given, and let $Q := \overline{F}\,\overline{G}$ where $\overline{F}$ and $\overline{G}$ correspond to $\overline{F}_1$ and $\overline{G}_1$ of (5.12) and (5.14), respectively (thus, $\overline{A} + \overline{F}\,\overline{G} = e^{A\tau} + Q =: S$ is a stability matrix). Given also an integer N (≥ 2), find Q_i ($i = 1, \cdots, N$) such that the following conditions are satisfied.

(i) $Q = Q_1 + \cdots + Q_N$.
(ii) N matrices in (5.100) are all stable.

Needless to say, condition (i) in the above problem corresponds to the decomposition constraint, while condition (ii) corresponds to the reliable stabilization. Note that the matrix corresponding to (5.99) is stable from the assumption of Problem 5.3. The solvability condition of this problem is given by the following theorem.

Theorem 5.6 Problem 5.3 is solvable if and only if

$$|\text{trace } (e^{A\tau} + \frac{N-1}{N}Q)| < n \qquad (5.104)$$

Proof of Theorem 5.6: First, we show the necessity of the condition (5.104). Let Q_i $(i = 1, \cdots, N)$ satisfy the conditions (i) and (ii) of Problem 5.3, and let S_i $(i = 1, \cdots, N)$ be the stability matrices given by (5.100). Then, we can readily show that

$$S_1 + \cdots + S_N = N\overline{A} + (N-1)Q \tag{5.105}$$

This implies that $N\overline{A} + (N-1)Q$ can be decomposed into the sum of N stability matrices. Hence, by Lemma 5.2, the condition (5.104) follows.

Next, we show the sufficiency of the condition (5.104). If the condition (5.104) holds, again from Lemma 5.2, $N\overline{A} + (N-1)Q$ $(= NS - Q)$ can be decomposed into the sum of N stability matrices $X_1, \cdots, X_N$. Let

$$Q_i := S - X_i \quad (i = 1, \cdots, N) \tag{5.106}$$

Then,

$$Q_1 + \cdots + Q_N = NS - (X_1 + \cdots + X_N) = NS - (NS - Q) = Q,$$
$$S_i = S - Q_i = X_i.$$

These equations, together with the stability of X_i, imply that Q_i $(i = 1, \cdots, N)$ given by (5.106) satisfy the conditions (i) and (ii) of Problem 5.3. This completes the proof.

Summarizing the results in this section, we can see that time-sharing multi-rate sample-hold controllers make it rather easy to solve the reliable stabilization problem. It is important to note the following key fact to this strong result: once Q_i $(i = 1, \cdots, N)$ are obtained, there exist a multirate hold and multirate a sampler that attain Q_i whatever Q_i may be, provided that the time-sharing condition is satisfied.

5.4   Numerical Example

In this section, we give some numerical examples concerning the robust stabilization, which is an application of Theorem 5.4 in Sec. 5.2, and the reliable stabilization using time-sharing multirate sample-hold scheme.

5.4.1   Robust stabilization

First, we briefly review the robust stabilization problem and related issues. For a given nominal plant $G_0(s)$, a stable rational transfer function $\delta(s)$, and a nonnegative

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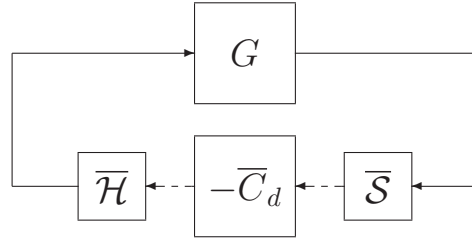
Figure 5.8: Feedback system Σ_{TSMSH}.
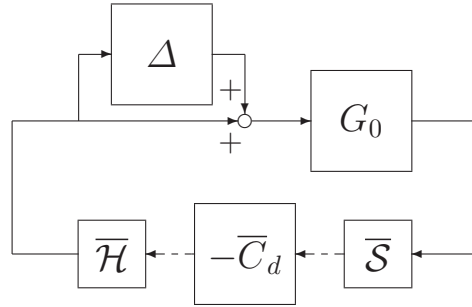
Figure 5.9: Robust stabilization using time-sharing multirate sample-hold controllers.

number k, we introduce a set of continuous-time plants given by

$$\mathcal{G}(G_0, \delta, k) = \big\{ G(s) = G_0(s)(I + \Delta(s)) \mid \Delta(s) \text{ is a stable rational transfer matrix}$$
$$\text{satisfying } \|\Delta(j\varphi)\| \leq k|\delta(j\varphi)| \ (\forall \varphi) \big\}. \quad (5.107)$$

Now, let us consider the feedback system shown in Figure 5.8, where G is a continuous-time plant, $\overline{C}_d$ is a discrete-time controller, and $\overline{H}$ and $\overline{S}$ are the multirate hold and the multirate sampler, respectively, which satisfy the time-sharing condition. The feedback system shown in Figure 5.8 is denoted by Σ_{TSMSH} (TSMSH stands for time-sharing multirate sample-hold). The feedback system Σ_{TSMSH} is said to be *robustly stable with respect to the plant set* $\mathcal{G}(G_0, \delta, k)$ if Σ_{TSMSH} is stable for all $G \in \mathcal{G}(G_0, \delta, k)$. The problem of finding a controller $\overline{C}_d$ that stabilizes all plants $G \in \mathcal{G}(G_0, \delta, k)$ is referred to as the *robust stabilization problem*. The set $\mathcal{G}(G_0, \delta, k)$ is said to be *robustly stabilizable* if there exists a controller $\overline{C}_d$ that stabilizes all plants $G \in \mathcal{G}(G_0, \delta, k)$.

In considering the robust stability of the sampled-data system Σ_{TSMSH} with respect to the plant set $\mathcal{G}(G_0, \delta, k)$, the sampled-data system shown in Figure 5.9 is often introduced. By applying the small gain theorem (e.g., [86]) to the feedback system of Figure 5.9, we can obtain the following result regarding the robust stability of Σ_{TSMSH} with respect to $\mathcal{G}(G_0, \delta, k)$.

Proposition 5.1 Suppose that the feedback system Σ_{TSMSH} is stable when $G = G_0$. Then, Σ_{TSMSH} is robustly stable with respect to $\mathcal{G}(G_0, \delta, k)$ if

$$\|\underline{Q}(j\varphi)\|_\infty < \gamma := 1/k, \tag{5.108}$$

where $\underline{Q}(j\varphi)$ is the FR-matrix associated with the sampled-data system of Figure 5.4 whose generalized plant is given by

$$P(s) = \begin{bmatrix} P_{11}(s) & P_{12}(s) \\ P_{21}(s) & P_{22}(s) \end{bmatrix} = \begin{bmatrix} 0 & I \\ G_0(s)\delta(s) & -G_0(s) \end{bmatrix}. \tag{5.109}$$

Thus, we can reduce the robust stabilization problem of the sampled-data system using a time-sharing sample-hold scheme to the H^∞ problem of the sampled-data system shown in Figure 5.4. Here, note that $P_{11}(s) = 0$. Therefore, we can apply Theorem 5.4 to the robust stabilization problem. It is derived from Proposition 5.1 that, for given G_0 and δ, $\mathcal{G}(G_0, \delta, k)$ is robustly stabilizable by the time-sharing sample-hold scheme if $0 \le k < k^*$ where k^* is given by

$$k^* = \frac{1}{\inf_{\overline{C}_d \in \mathcal{S}_0} \|\underline{Q}(j\varphi)\|_\infty}, \tag{5.110}$$

and $\mathcal{S}_0$ is the set of discrete-time controllers that stabilize G_0. Therefore, k^* can be regarded as a robustly-stabilizing ability of the time-sharing multirate sample-hold controller. Since k^* is the upper bound of k that is derived by the small gain theorem, we refer to k^* as *the guaranteed upper bound of k by the small gain theorem*, or shortly, *the guaranteed upper bound*. Here, it should be noted that k^* is only an upper bound of k and is not the supremum of k for which $\mathcal{G}(G_0, \delta, k)$ is robustly stabilizable because k^* is derived from the sufficient condition in Proposition 5.1.

In the following, we examine the robustly-stabilizing ability of time-sharing multirate sample-hold scheme by using k^* through numerical examples. Let $G_0(s)$ and $\delta(s)$ be given, respectively, by

$$G_0(s) = \frac{s-2}{(s-1)(s+1)}, \quad \delta(s) = \frac{3s+1}{s+1}. \tag{5.111}$$

The guaranteed upper bounds under various situations are shown in Tables 5.1 and 5.2. The guaranteed upper bounds under the continuous-time setting and the standard sampled-data setting (i.e., using the zero-order hold and the ideal sampler) are shown in Table 5.3, where the upper bounds are obtained similarly by using the small gain theorem.

From Tables 5.1 and 5.2, we see that the robustly-stabilizing ability of the time-sharing multirate sample-hold controller becomes higher as the input multiplicity N_{i} and the output multiplicity N_{o} become larger, and that such a tendency becomes

Table 5.1: Guaranteed upper bounds for $\tau = 0.005$ and $\tau = 0.05$.

$\tau = 0.005$				$\tau = 0.05$			
$L = 0.1\tau$	$N_{\mathrm{i}} = 1$	$N_{\mathrm{i}} = 2$	$N_{\mathrm{i}} = 3$	$L = 0.1\tau$	$N_{\mathrm{i}} = 1$	$N_{\mathrm{i}} = 2$	$N_{\mathrm{i}} = 3$
$N_{\mathrm{o}} = 2$	0.05256	0.05256	0.05256	$N_{\mathrm{o}} = 2$	0.05126	0.05126	0.05126
$N_{\mathrm{o}} = 3$	0.05256	0.05256	0.05256	$N_{\mathrm{o}} = 3$	0.05126	0.05126	0.05126
$N_{\mathrm{o}} = 4$	0.05256	0.05256	0.05256	$N_{\mathrm{o}} = 4$	0.05126	0.05126	0.05126
$L = 0.5\tau$	$N_{\mathrm{i}} = 1$	$N_{\mathrm{i}} = 2$	$N_{\mathrm{i}} = 3$	$L = 0.5\tau$	$N_{\mathrm{i}} = 1$	$N_{\mathrm{i}} = 2$	$N_{\mathrm{i}} = 3$
$N_{\mathrm{o}} = 2$	0.1174	0.1174	0.1174	$N_{\mathrm{o}} = 2$	0.1135	0.1135	0.1135
$N_{\mathrm{o}} = 3$	0.1174	0.1174	0.1174	$N_{\mathrm{o}} = 3$	0.1135	0.1135	0.1135
$N_{\mathrm{o}} = 4$	0.1174	0.1174	0.1174	$N_{\mathrm{o}} = 4$	0.1135	0.1135	0.1135
$L = 0.9\tau$	$N_{\mathrm{i}} = 1$	$N_{\mathrm{i}} = 2$	$N_{\mathrm{i}} = 3$	$L = 0.9\tau$	$N_{\mathrm{i}} = 1$	$N_{\mathrm{i}} = 2$	$N_{\mathrm{i}} = 3$
$N_{\mathrm{o}} = 2$	0.1574	0.1574	0.1574	$N_{\mathrm{o}} = 2$	0.1507	0.1508	0.1508
$N_{\mathrm{o}} = 3$	0.1574	0.1574	0.1574	$N_{\mathrm{o}} = 3$	0.1507	0.1508	0.1508
$N_{\mathrm{o}} = 4$	0.1574	0.1574	0.1574	$N_{\mathrm{o}} = 4$	0.1507	0.1508	0.1508
$L = 0.99\tau$	$N_{\mathrm{i}} = 1$	$N_{\mathrm{i}} = 2$	$N_{\mathrm{i}} = 3$	$L = 0.99\tau$	$N_{\mathrm{i}} = 1$	$N_{\mathrm{i}} = 2$	$N_{\mathrm{i}} = 3$
$N_{\mathrm{o}} = 2$	0.1650	0.1650	0.1650	$N_{\mathrm{o}} = 2$	0.1578	0.1578	0.1578
$N_{\mathrm{o}} = 3$	0.1650	0.1650	0.1650	$N_{\mathrm{o}} = 3$	0.1578	0.1578	0.1578
$N_{\mathrm{o}} = 4$	0.1650	0.1650	0.1650	$N_{\mathrm{o}} = 4$	0.1578	0.1578	0.1578

stronger as the frame period τ becomes larger. We can also observe that the effect of the increase of N_{i} becomes higher when L, the interval of the manipulation, becomes larger, whereas the effect of the increase of N_{o} becomes higher when L becomes small. By comparing Tables 5.1, 5.2, and 5.3, we see that the guaranteed upper bound for the time-sharing multirate sample-hold controller approaches that for the standard sampled-data controller as L tends to τ when $N_{\mathrm{i}} = 1$. We can also observe that the robustly stabilizing ability of time-sharing sample-hold controller becomes superior to the standard sampled-data controller as frame period becomes larger.

Figure 5.10 shows the frequency response gain of the FR-matrix $\underline{Q}(j\varphi)$ (i.e., $\|\underline{Q}(j\varphi)\|_{l^2/l^2}$) and $\frac{1}{k^*}$ for the case of $\tau = 0.05$, $L = 0.5\tau$, $N_{\mathrm{i}} = 1$, $N_{\mathrm{o}} = 2$, where the solid line and the dashed line indicate $\|\underline{Q}(j\varphi)\|_{l^2/l^2}$ and $\frac{1}{k^*}$, respectively. From Figure 5.10, we see that $\|\underline{Q}(j\varphi)\|_{l^2/l^2} < \frac{1}{k^*}$ holds for all φ. We can observe such a relation for every example listed in Tables 5.1 and 5.2.

Table 5.2: Guaranteed upper bounds for $\tau = 0.5$ and $\tau = 5$.

	$\tau = 0.5$				$\tau = 5$		
$L = 0.1\tau$	$N_i = 1$	$N_i = 2$	$N_i = 3$	$L = 0.1\tau$	$N_i = 1$	$N_i = 2$	$N_i = 3$
$N_o = 2$	$3.918{\times}10^{-2}$	$3.919{\times}10^{-2}$	$3.919{\times}10^{-2}$	$N_o = 2$	$8.623{\times}10^{-4}$	$8.689{\times}10^{-4}$	$8.702{\times}10^{-4}$
$N_o = 3$	$3.921{\times}10^{-2}$	$3.921{\times}10^{-2}$	$3.921{\times}10^{-2}$	$N_o = 3$	$8.737{\times}10^{-4}$	$8.804{\times}10^{-4}$	$8.817{\times}10^{-4}$
$N_o = 4$	$3.921{\times}10^{-2}$	$3.922{\times}10^{-2}$	$3.922{\times}10^{-2}$	$N_o = 4$	$8.782{\times}10^{-4}$	$8.849{\times}10^{-4}$	$8.862{\times}10^{-4}$
$L = 0.5\tau$	$N_i = 1$	$N_i = 2$	$N_i = 3$	$L = 0.5\tau$	$N_i = 1$	$N_i = 2$	$N_i = 3$
$N_o = 2$	$7.952{\times}10^{-2}$	$7.968{\times}10^{-2}$	$7.971{\times}10^{-2}$	$N_o = 2$	$9.096{\times}10^{-4}$	$10.04{\times}10^{-4}$	$10.74{\times}10^{-4}$
$N_o = 3$	$7.953{\times}10^{-2}$	$7.969{\times}10^{-2}$	$7.972{\times}10^{-2}$	$N_o = 3$	$9.146{\times}10^{-4}$	$10.46{\times}10^{-4}$	$10.80{\times}10^{-4}$
$N_o = 4$	$7.953{\times}10^{-2}$	$7.969{\times}10^{-2}$	$7.972{\times}10^{-2}$	$N_o = 4$	$9.155{\times}10^{-4}$	$10.47{\times}10^{-4}$	$10.81{\times}10^{-4}$
$L = 0.9\tau$	$N_i = 1$	$N_i = 2$	$N_i = 3$	$L = 0.9\tau$	$N_i = 1$	$N_i = 2$	$N_i = 3$
$N_o = 2$	$9.705{\times}10^{-2}$	$9.766{\times}10^{-2}$	$9.777{\times}10^{-2}$	$N_o = 2$	$7.222{\times}10^{-4}$	$9.291{\times}10^{-4}$	$10.08{\times}10^{-4}$
$N_o = 3$	$9.705{\times}10^{-2}$	$9.766{\times}10^{-2}$	$9.777{\times}10^{-2}$	$N_o = 3$	$7.224{\times}10^{-4}$	$9.293{\times}10^{-4}$	$10.08{\times}10^{-4}$
$N_o = 4$	$9.705{\times}10^{-2}$	$9.766{\times}10^{-2}$	$9.777{\times}10^{-2}$	$N_o = 4$	$7.224{\times}10^{-4}$	$9.294{\times}10^{-4}$	$10.08{\times}10^{-4}$
$L = 0.99\tau$	$N_i = 1$	$N_i = 2$	$N_i = 3$	$L = 0.99\tau$	$N_i = 1$	$N_i = 2$	$N_i = 3$
$N_o = 2$	$9.967{\times}10^{-2}$	$10.04{\times}10^{-2}$	$10.06{\times}10^{-2}$	$N_o = 2$	$6.908{\times}10^{-4}$	$9.043{\times}10^{-4}$	$9.921{\times}10^{-4}$
$N_o = 3$	$9.967{\times}10^{-2}$	$10.04{\times}10^{-2}$	$10.06{\times}10^{-2}$	$N_o = 3$	$6.908{\times}10^{-4}$	$9.043{\times}10^{-4}$	$9.921{\times}10^{-4}$
$N_o = 4$	$9.967{\times}10^{-2}$	$10.04{\times}10^{-2}$	$10.06{\times}10^{-2}$	$N_o = 4$	$6.908{\times}10^{-4}$	$9.043{\times}10^{-4}$	$9.921{\times}10^{-4}$

Table 5.3: Guaranteed upper bounds of k for the continuous-time system and the standard sampled-data system.

continuous-time control	standard sampled-data control			
	$\tau = 0.005$	$\tau = 0.05$	$\tau = 0.5$	$\tau = 5$
0.1667	0.1658	0.1585	$9.994{\times}10^{-2}$	$6.876{\times}10^{-4}$
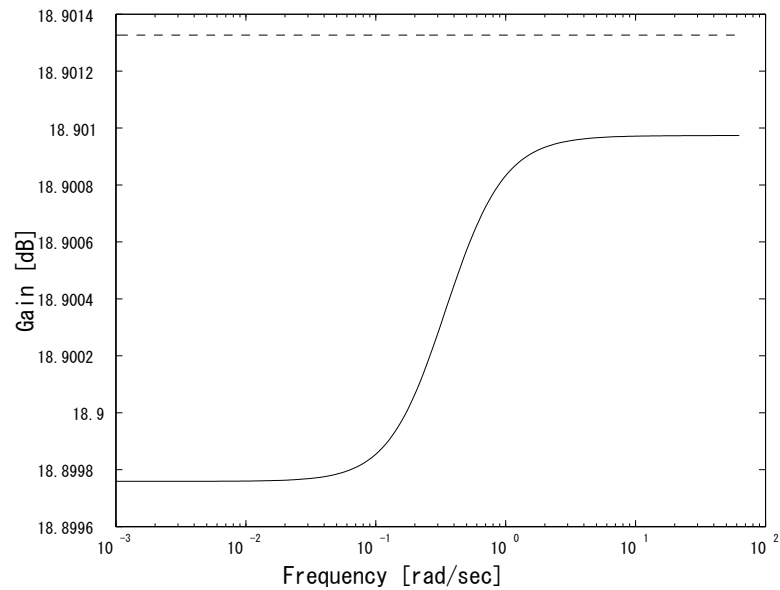
Figure 5.10: Frequency response gain of $\underline{Q}(j\varphi)$ (solid line) and $\frac{1}{k^*}$ (dashed line).

5.4.2 Reliable stabilization

In this subsection, we give a numerical example of the reliable stabilization for the following linear time-invariant plant:

$$P(s) = \frac{8(s-2)(s-4)}{(s-1)(s-3)(s-5)}.$$

It was shown in Minto and Ravi [62] that reliable stabilization under the two-controller configuration using LTI controllers requires strong stabilizability of the plant, that is, the plant must be stabilizable by a stable controller. Youla et al. [84] showed that the strong stabilizability of the plant is equivalent to the condition that the number of poles of the plant between every pair of real unstable zeros of the plant is even, which is referred to as the "parity interlacing property." The above plant does not satisfy this property, and thus the plant is not reliably stabilizable under the two-controller configuration by any LTI controllers.

We apply the results in the preceding section to obtain a set of reliably stabilizing controllers using the time-sharing multirate sample-hold scheme. Clearly, it is possible to study under the general N-controller configuration for any given integer $N \geq 2$, but here we consider the three-controller configuration ($N = 3$), since an example of reliably stabilizing controllers for such a case has not been shown anywhere in whatever control scheme.

Now, let us take the following realization of the plant:

$$\dot{x}(t) = Ax(t) + Bu(t), \; y(t) = Cx(t), \; A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \; B = \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix}, \; C = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}.$$

The frame period is selected to be $\tau = 0.1$ so as to satisfy the condition of Theorem 5.5, i.e., to guarantee the existence of a set of $N(=3)$ controllers to achieve reliable stabilization. Indeed, $|\text{trace }(e^{A\tau})| \approx 4.10 < 15 = (2N-1)n$. However, we proceed in our design according to the idea in Problem 5.3. Namely, first, we find the optimal stabilizing controller Q (i.e., $e^{A\tau} + Q$ is a stability matrix) such that $J = \int_0^\infty \left(x^T(t)x(t) + 0.1u^T(t)u(t) \right) dt$ is minimized, and then we consider to decompose it into three reliably stabilizing controllers.

The problem of finding the optimal controller Q that minimizes J under the time-sharing multirate sample-hold setting is equivalently reduced to a certain discrete-time problem as in the case of using the zero-order hold and the ideal sampler (e.g., Hagiwara and Araki [28] and Furuta [23] on using the zero-order hold and the ideal sampler). By solving the discrete-time problem, we can obtain the optimal Q as

follows:

$$Q = \begin{bmatrix} -0.882 & 7.25 & -5.23 \\ -0.724 & 5.90 & -4.21 \\ -1.34 & 10.8 & -7.63 \end{bmatrix}.$$

In this case, $|\text{trace } (\overline{A} + \frac{N-1}{N}Q)| \approx 2.36 < 3 = n$, so the condition of Theorem 5.6 is satisfied. Therefore, Q can be decomposed into the sum of three matrices Q_i ($i = 1, 2, 3$) satisfying the condition of Problem 5.3. Such Q_i are obtained in accordance with the constructive proof of Theorem 5.6. However, we simply set Q_i as $Q_i = \frac{1}{3}Q$ ($i = 1, 2, 3$) because these Q_i satisfy the condition. This situation corresponds to the case of three identical controllers working in cooperation. Here, we determine $F_i(t)$ and $G_i(t)$ so that $\overline{F}_i = \frac{1}{3}Q_i = \frac{1}{3}Q$ and $\overline{G}_i = I$ ($i = 1, 2, 3$). These matrices are attained by choosing the input multiplicities and the output multiplicities as $N_i = N_o = 3$, and under the condition $L = \tau/2 = 0.05$, the resulting hold function $F_i(t)$ and sampling function $G_i(t)$ are obtained, respectively, as follows:

$$F_i(t) = \begin{cases} \begin{bmatrix} -7.17 & 50.6 & -31.5 \end{bmatrix} & 0 \leq t < 0.05/3 \\ \begin{bmatrix} -0.758 & 7.83 & -6.53 \end{bmatrix} & 0.05/3 \leq t < 0.1/3 \\ \begin{bmatrix} 2.64 & -14.7 & 6.32 \end{bmatrix} & 0.1/3 \leq t < 0.05 \\ \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} & 0.05 \leq t < 0.1 \end{cases} \quad (i = 1, 2, 3),$$

$$G_i(t) = \begin{bmatrix} 227 \\ -465 \\ 238 \end{bmatrix} \delta(t - 0.05) + \begin{bmatrix} -410 \\ 863 \\ -453 \end{bmatrix} \delta(t - 0.075) + \begin{bmatrix} 185 \\ -400 \\ 215 \end{bmatrix} \delta(t - 0.1). \quad (i = 1, 2, 3).$$

The state responses for the initial value $x(0) = [1\ 1\ 1]^T$ are shown in Figure 5.11 and Figure 5.12, where the former corresponds to the case when all controllers work together while the latter the case when one of the three identical controllers fails. The time step size for these simulations is $0.05/30$. These figures show that the state converges to zero quickly when all controllers work together, and the stability of the closed-loop system is guaranteed even when one of the controllers fails.
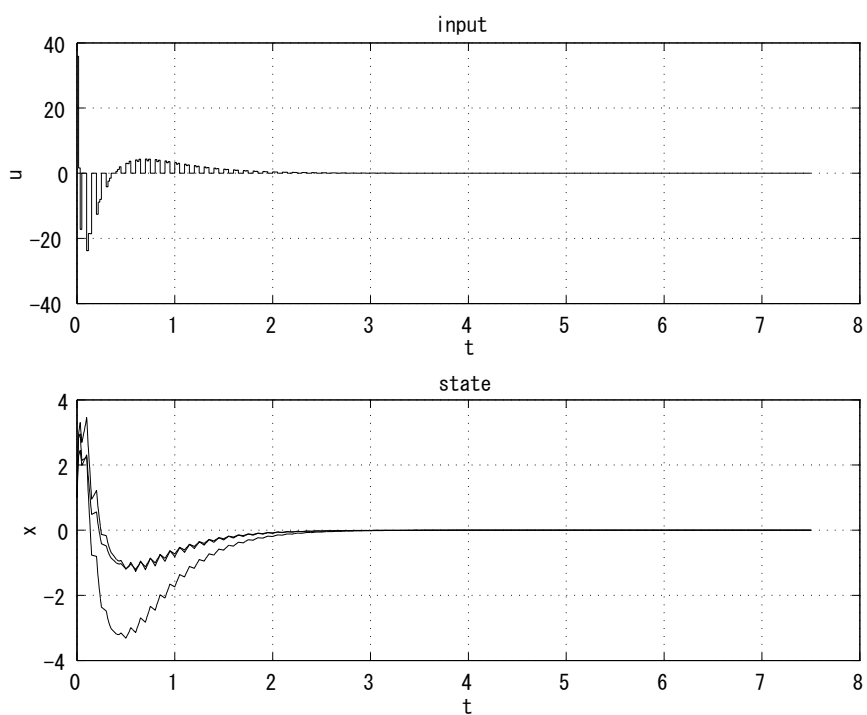
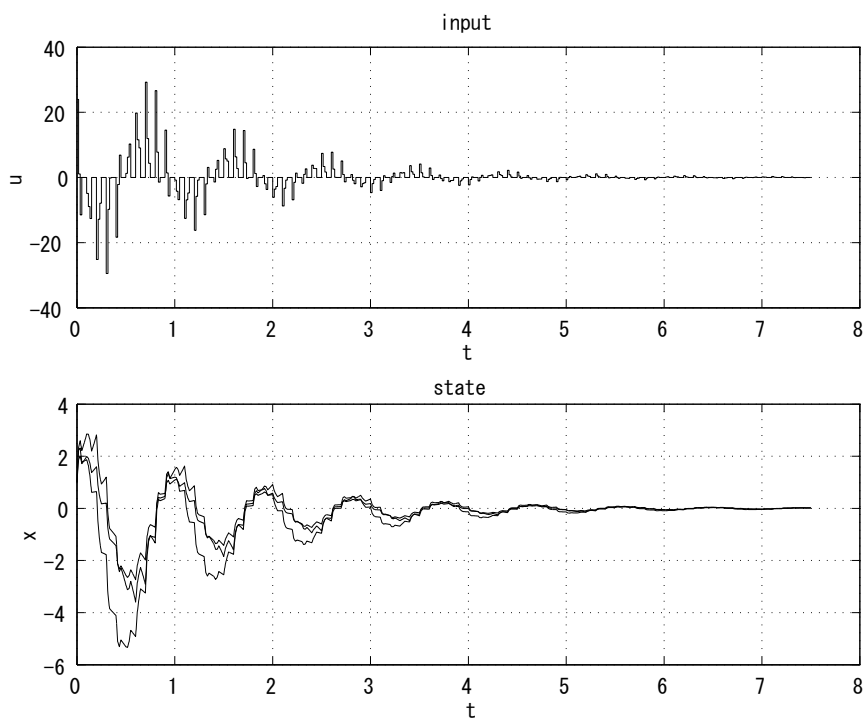Figure 5.11: Response of the input and the state (with all controllers working)

Figure 5.12: Response of the input and the state (with one controller failing)

5.5 Proof of Lemma 5.2

In this subsection, we give the proof of Lemma 5.2. For the proof, we exploit the following properties:

(i) $x \in \mathbf{R}$ can be expressed as the sum of N numbers whose absolute values are less than p, if and only if $|x| < Np$.

(ii) If $X \in \mathbf{R}^{n \times n}$ is a stability matrix, then $|\text{trace}\,(X)| < n$.

In addition to these properties, we need the following two lemmas to prove Lemma 5.2.

Lemma 5.4 Suppose $X \in \mathbf{R}^{2 \times 2}$ and $X - xI_2 \neq 0 \ (\forall x)$. Then, there exists an invertible matrix T that assigns the diagonal elements of the matrix $T^{-1}XT$ arbitrarily within the constraint that their sum equals trace (X).

Lemma 5.5 Suppose $X \in \mathbf{R}^{n \times n}$ and $X - xI_n \neq 0 \ (\forall x)$. Suppose also that p is an arbitrary positive real number. If $|\text{trace}\,(X)| < p$, then there exists an invertible matrix T that makes the absolute value of each diagonal element of $T^{-1}XT$ less than p/n.

In the following, we give the proofs of Lemmas 5.4 and 5.5.

Proof of Lemma 5.4: We first show that the assertion of Lemma 5.4 is true when $n = 2$. Let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. In this case, it is sufficient to show the assertion for the following three cases: (i) $b \neq 0$, (ii) $c \neq 0$, (iii) $b = c = 0$ (Here, $a \neq d$ by assumption). For (i), $T = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$ yields $T^{-1}XT = \begin{bmatrix} a + bt & b \\ * & d - bt \end{bmatrix}$. This means that one of the diagonal elements can be assigned arbitrarily by the appropriate choice of t. (ii) can be reduced to (i) by transposing X. For (iii), $T = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$ yields $T^{-1}XT = \begin{bmatrix} a & 0 \\ (a - d)t & d \end{bmatrix}$. Therefore, (iii) can also be reduced to (ii). This completes the proof.

Proof of Lemma 5.5: For $n = 1$, the assertion is trivial. For $n = 2$, the assertion readily follows from Lemma 5.4 and property (i). In the following, we assume that the assertion of Lemma 5.5 is true for $n = k$ $(k \geq 2)$ and show that this is also true for $n = k+1$. Since $X \neq xI_{k+1}$, there exist i and j such that the matrix $\begin{bmatrix} x_{ii} & x_{ij} \\ x_{ji} & x_{jj} \end{bmatrix}$, which we denote by $\widetilde{X}$, is not a scalar multiple of the identity I_2. By an appropriate

permutation T_0, we obtain

$$X_0 = T_0^{-1} X T_0 = \begin{bmatrix} x_1 & & & & \ast \\ & \ddots & & & \\ & & x_{k-1} & & \\ \ast & & & \widetilde{X} \end{bmatrix} = \begin{bmatrix} x_1 & & & & & \ast \\ & \ddots & & & & \\ & & x_{k-1} & & & \\ & & & x_k & & \\ \ast & & & & x_{k+1} \end{bmatrix}$$

$$(5.112)$$

where $x_k = x_{ii}$, $x_{k+1} = x_{jj}$. Now, let x_k' and x_{k+1}' be the numbers satisfying the following conditions:

$$x_k' + x_{k+1}' = x_k + x_{k+1}, \ |x_{k+1}'| < \frac{p}{k+1}, \ |x_1 + \cdots + x_{k-1} + x_k'| < \frac{p}{k+1}k, \ x_k' \neq x_{k-1}.$$

$$(5.113)$$

Such x_k' and x_{k+1}' always exist because $|\text{trace}\,(X)| < p$. Thus, from Lemma 5.4, there exists a matrix S_1 such that

$$S_1^{-1} \widetilde{X} S_1 = \begin{bmatrix} x_k' & \ast \\ \ast & x_{k+1}' \end{bmatrix}. \tag{5.114}$$

Now, let the transformation T_1 be given by $T_1 = \text{block diag}[I_{k-1}, S_1]$. By this transformation, we obtain

$$T_1^{-1} X_0 T_1 = \begin{bmatrix} x_1 & & & & \\ & \ddots & & & \ast \\ & & x_{k-1} & & \\ & & & x_k' & \\ & \ast & & & x_{k+1}' \end{bmatrix} =: \begin{bmatrix} \widehat{X} & \ast \\ \ast & x_{k+1}' \end{bmatrix}. \tag{5.115}$$

By the second constraint in (5.113), the matrix $\widehat{X} \in \mathbf{R}^{k \times k}$ satisfies $|\text{trace}\,(\widehat{X})| < \frac{p}{k+1}k$. Also, by the third constraint in (5.113), $\widehat{X}$ is not a scalar multiple of the identity. From the above, together with the assumption that the assertion of Lemma 5.5 is true for $n = k$, there exists a transformation T_k that makes the absolute value of each diagonal element of $T_k^{-1} \widehat{X} T_k$ less than $\frac{p}{k+1}$. Therefore, the transformation $T := T_0 T_1 T_k$ makes the absolute value of each diagonal element of $T^{-1} X T$ less than $\frac{p}{k+1}$. This means the assertion of Lemma 5.5 is true also for $n = k+1$. By induction, we can conclude that Lemma 5.5 is true for any integer n. This completes the proof.

Remark 5.3 We can extend Lemma 5.4 to a stronger result. Namely, the following proposition holds.

Proposition 5.2 Suppose $X \in \mathbf{R}^{n \times n}$ and $X - xI_n \neq 0$ ($\forall x$). Then, there exists an invertible matrix T that assigns the diagonal elements of the matrix $T^{-1}XT$ arbitrarily within the constraint that their sum equals trace (X).

The proof of Proposition 5.2 is slightly complicated, but it is proved in a similar way to Lemma 5.5. Therefore, we omit the proof.

Now, we are in a position to prove Lemma 5.2.

Proof of Lemma 5.2: When X is equal to xI_n for some $x \in \mathbf{R}$, Lemma 5.2 follows immediately from property (i). Hence, we assume that X does not satisfy that condition. Since the necessity part of Lemma 5.2 is straightforward, we only prove the sufficiency part.

Now, suppose that X satisfies the condition $|\text{trace } (X)| < Nn$. By this, together with Lemma 5.5, $T^{-1}XT$ can be set as follows by an appropriate choice of the transformation T.

$$T^{-1}XT = \begin{bmatrix} x_1 & & \text{\Large *} \\ & \ddots & \\ \text{\Large *} & & x_n \end{bmatrix}, \quad |x_i| < N \ \ (i = 1, \cdots, n). \tag{5.116}$$

By property (i), each x_i $(i = 1, \cdots, n)$ can be decomposed as[1]

$$x_i = x_i^{(1)} + x_i^{(2)} + \cdots + x_i^{(N)} \quad (i = 1, \cdots, n)$$

where

$$|x_i^{(j)}| < 1 \quad (i = 1, \cdots, n; j = 1, \cdots, N). \tag{5.117}$$

By using the above $x_i^{(j)}$, we decompose the right-hand side of (5.116) as

$$\begin{bmatrix} x_1 & & \text{\Large *} \\ & \ddots & \\ \text{\Large *} & & x_n \end{bmatrix} = \begin{bmatrix} x_1^{(1)} & & \text{\Large *} \\ & \ddots & \\ \text{\Large 0} & & x_n^{(1)} \end{bmatrix} + \cdots + \begin{bmatrix} x_1^{(m)} & & \text{\Large 0} \\ & \ddots & \\ \text{\Large *} & & x_n^{(m)} \end{bmatrix} \tag{5.118}$$

where each matrix in the right-hand side is an upper or lower triangular matrix. By this, together with (5.117), each matrix in the right-hand side of (5.118) is a stability matrix. This means that X can be expressed as the sum of stability matrices.

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[1] One of the simplest ways to carry out such decomposition is to set $x_i^{(j)} = x_i/m$.

5.6   Summary

In this chapter, we gave the FR-operator-based representation for the frequency-domain characteristics of sampled-data systems using the time-sharing multirate sample-hold scheme. First, we introduced the time-sharing multirate sample-hold scheme. This is a special kind of control scheme that uses a multirate sample-hold scheme, where "multirate" implies that the sampler and the hold operate with different periods, and "time-sharing" implies that the manipulation of the plant input and the detection of the plant output work on separate time intervals. We showed that when we use generalized holds together with generalized samplers, the state-space representation of the discretized plant turns out to have a term consisting of a one-step delay in general, which is caused by a certain interference of the hold and sampling actions. We referred to the condition that such a term is equal to zero as the orthogonality condition. In view of this, the time-sharing multirate sample-hold scheme can be regarded as a special class that satisfies the orthogonality condition.

Next, using the notion of FR-operators, we provided a frequency domain representation of the sampled-data system using the time-sharing multirate sample-hold scheme, and then we derived a bisection method for computing the frequency response gain of such sampled-data systems. The derivation was based on the infinite-dimensional congruent transformation approach used in Chapter 3. We also provided a computation method based on the norm equivalent discrete-time system, which can be applied under the condition $P_{11}(s) = 0$. As an application of the time-sharing multirate sample-hold scheme, we addressed the reliable stabilization problem. We showed that the reliable stabilization problem under the N-controller configuration with $N \geq 2$, which is known to be a hard problem for LTI controllers, is solvable by using the time-sharing multirate sample-hold scheme.

Chapter 6

Conclusion

In this thesis, we dealt with three topics involved in understanding the frequency domain characteristics of sampled-data systems by using FR-operators.

  (i) Computation of the frequency response gain of sampled-data systems.

 (ii) Sensitivity and complementary sensitivity reduction problems of sampled-data systems.

(iii) Frequency response of sampled-data systems using a time-sharing multirate sample-hold scheme.

As a concluding chapter, we now summarize the contributions of this thesis.

In Chapter 2, as the fundamental background of this work, we introduced the notion of an FR-operator, which enables us to consider the frequency domain characteristics of sampled-data systems while taking into account the intersample behavior and the influence of aliasing. We then summarized some useful properties of FR-operators, which involve the l^2-boundedness of the FR-operator, an approximative method for frequency response gain computation through finite-dimensional truncation of the FR-matrix, and the relationship between the L^2-induced norm and the H^∞ norm of sampled-data systems.

Next, we provided an exact and direct method for computing the frequency response gain of a class of sampled-data systems satisfying the condition $P_{11}(s) = 0$ under the generalized plant setting. The key result we obtained is that, under the condition $P_{11}(s) = 0$, there exists an 'equivalent discrete-time system' whose frequency response gain coincides with that of the given sampled-data system at every frequency. The technique used for the derivation of the method is fundamental for deriving other related results. These results include a bisection algorithm for computing the frequency response gain of general sampled-data systems (without

the condition $P_{11}(s) = 0$), which is presented in Chapter 3, and an analysis of the sensitivity and complementary sensitivity reduction problems of sampled-data systems, which is presented in Chapter 4.

In Chapter 3, we gave a complete bisection algorithm for computing the frequency response gain of sampled-data systems that do not necessarily satisfy the condition $P_{11}(s) = 0$, which corresponds to topic (i). We showed that the problem of deciding whether the frequency response gain at a given angular frequency is smaller than a given positive number γ is reduced to the problem of counting up the number of negative eigenvalues of a certain block-diagonal, self-adjoint infinite-dimensional matrix. The algorithm can be carried out easily with the state-space matrices of the sampled-data system. We also showed that a similar bisection method can be obtained for the lifting-based representation of sampled-data systems.

The derivation of the algorithm is based on the properties of the infinite-dimensional congruent transformation (i.e., the Schur complement arguments and Sylvester's law of inertia). This contrasts with the well-known arguments on the related issue of the sampled-data H^∞ problem (e.g., Bamieh and Pearson [10], Hayakawa et al. [40]), where the positivity of the operator $\gamma^2 I - \mathcal{D}^*\mathcal{D}$ (or equivalently, $\gamma > \|\mathcal{D}\|$) and the loop-shifting technique play key roles, where $\mathcal{D}$ is the direct feedthrough term in the lifted representation of the sampled-data system. Because of this difference in the underlying tools, we do not need to set the above assumption on γ that was set in the previous attempt towards a bisection algorithm (e.g., Hara et al. [39]); since the frequency response gain can actually be smaller than $\|\mathcal{D}\|$, such an assumption is not appropriate and thus the previous attempt was only partially successful.

In Chapter 4, we clarified several relationships between the SR/CSR problems of a given sampled-data system and those of its 'hold equivalent' discretized system (called the naively discretized system), which correspond to the issues of topic (ii), where SR and CSR stand for sensitivity reduction and complementary sensitivity reduction. We first considered the SR and CSR problems of the naively discretized system Σ_d before considering these problems of the original sampled-data system Σ, and showed that the best achievable performance in the SR problem coincides with that in the CSR problem. We also showed that the lower bound of the best achievable performance obtained by Sung and Hara [73] in the SR problem coincides with that in the CSR problem.

Next, we showed that the SR/CSR problems of sampled-data systems can be reduced to equivalent discrete-time problems by introducing the doubly sensitivity-preserving (DSP) discretized system $\widehat{\Sigma}$. Through the coprime factorization treatment of $\widehat{\Sigma}$, we further introduced an important function called the aliasing factor. We then showed that the naively discretized system Σ_d can also be used for the SR/

CSR problems of the sampled-data system Σ, provided that appropriate frequency-dependent weights constructed from the aliasing factor are applied to Σ_d. We then showed that the relation between the best achievable performance in the SR (or CSR) problem of $\widehat{\Sigma}$ (and thus Σ) and that of Σ_d can be proved through clarifying an analytic property of the aliasing factor. We also derived an interesting property that the best achievable performance for the SR problem of Σ and that for the CSR problem of Σ coincide with each other.

The arguments in this thesis enable us to study the SR/CSR problems of sampled-data systems through equivalent discrete-time problems with a fixed γ-*independent* discretized plant. This contrasts with well-known solution methods for the sampled-data H^∞ problem, in which a γ-*dependent* discretized plant is used. Hence, the arguments are not only important in their own right but might hopefully provide a fundamental basis for further study of the performance limitations regarding the SR/CSR problems of sampled-data systems.

In Chapter 5, we presented the FR-operator-based representation for the frequency-domain characteristics of sampled-data systems using the time-sharing multirate sample-hold scheme, which corresponds to the issues of topic (iii). First, we introduced the time-sharing multirate sample-hold scheme. This is a special kind of control scheme, where "multirate" implies that the sampler and the hold operate with different periods and "time-sharing" implies that the manipulation of the plant input and the detection of the plant output work at separate time intervals. We showed that when we use generalized holds together with generalized samplers, the state-space representation of the discretized plant turns out to have a term consisting of a one-step delay in general, which is caused by a certain interference of the hold and sampling actions. We referred to the condition that such a term is equal to zero as the orthogonality condition. In view of this, the time-sharing multirate sample-hold scheme can be regarded as a special class that satisfies the orthogonality condition.

Next, using the notion of FR-operators, we provided a frequency domain representation of the sampled-data system using the time-sharing multirate sample-hold scheme and derived a bisection method for computing the frequency response gain of such sampled-data systems. The derivation was based on the infinite-dimensional congruent transformation approach used in the study of topic (i). We also provided a computation method based on the norm equivalent discrete-time system, which can be applied under the condition $P_{11}(s) = 0$. As an application of the time-sharing multirate sample-hold scheme, we considered the reliable stabilization problem, in which we are to find a set of controllers that stabilize a given plant when they all act together, as well as when any one of them fails. We showed that the reliable

stabilization problem under the N-controller configuration with $N \geq 2$, which is known to be a hard problem under the use of LTI controllers, is solvable by using the time-sharing multirate sample-hold scheme. In deriving the result, a lemma on the decomposition of a matrix into the sum of stability matrices played a key role.

Through our studies on the above three topics, we have established several fundamental tools and methods for dealing with the frequency domain characteristics of sampled-data systems. The author believes that these tools and methods will be useful for further studies on the frequency domain nature of sampled-data systems and contribute to the development of the frequency domain theory of sampled-data systems. In closing, we describe some future topics and possible extensions of the results obtained in this thesis.

After the bisection method of Ito et al. [48] was presented, new bisection methods were presented by Fujioka [21], Fujioka and Mirkin [22], and Mirkin [63]. These methods are based on a certain kind of projection from infinite-dimensional space to finite-dimensional space, and it was shown that their methods do not need the norm of the singular values of the direct-feedthrough term of the lifted transfer operator, which contrasts with our bisection method presented in Sec. 3.3.1. Clarifying the relationship between their methods and our method is an interesting future topic.

One of the useful tools exploited in this thesis is the infinite-dimensional congruent transformation, which was used for deriving the bisection algorithm for computing the frequency response gain of sampled-data systems. Since the congruent transformation approach yields a different perspective to the problems that are related to some positivity conditions, it can be applied to a wide class of problems on systems and control. Finding useful applications of the congruent transformation approach is also an interesting future topic.

We expect that the best achievable performances in the SR/CSR problems of sampled-data systems can be evaluated more precisely by clarifying additional properties of aliasing factors. For example, by using limiting properties of aliasing factors (i.e., the properties of aliasing factors when the sampling period τ tends to zero), along with the properties of zeros of naively discretized systems, it would be possible for us to clarify the limiting properties of the best achievable performances of sampled-data systems.

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   (This paper contains incorrect equations due to printer's errors after the proof-reading process. The following paper corrects these errors:
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