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ロバスト Nash 均衡問題の半正定値相補性問題への変換

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Abstract. 本稿では，プレイヤーの戦略と相手の戦略に不確実性が含まれるようなゲームに対して，ロバスト Nash 均衡という概念を導入する．ロバスト Nash 均衡とは，各プレイヤーが，起こりうる最悪の結果を想定して，すなわちロバスト最適化の考え方でもって戦略を決定した際に起こりうる均衡状態のことである．さらに，不確実性集合が球形で表される場合に，各プレイヤーの解くべき最適化問題が半正定値計画問題 (Semi-Definite Programming problem: SDP) として再定式化でき，その結果，ロバスト Nash 均衡問題そのものが半正定値相補性問題 (Semi-Definite Complementarity Problem: SDCP) として帰着できることを示す．

1 Introduction

Robust Nash equilibrium, which attracts much attention recently, is a new concept of equilibrium for games with uncertain data. Hayashi, Yamashita and Fukushima [5], and Aghassi and Bertsimas [1][*1] have proposed the model in which each player makes a decision according to the idea of robust optimization. Aghassi et al. [1] considered the robust Nash equilibrium for N-person games in which each player solves a linear programming (LP) problem. Moreover, they proposed a method for solving the robust Nash equilibrium problem with convex polyhedral uncertainty sets. Hayashi et al. [5] defined the concept of robust Nash equilibria for bimatrix games. Under the assumption that uncertainty sets are expressed by means of the Euclidean or the Frobenius norm, they showed that each player's problem reduces to an SOCP and the robust Nash equilibrium problem can be reformulated as a second-order cone complementarity problem (SOCCP) [3, 4]. In addition, Hayashi et al. [5] studied robust Nash equilibrium problems in which the uncertainty is contained in both opponents' strategies and each player's cost parameters, whereas Aghassi et al. [1] studied only the latter case. More recently, Nishimura, Hayashi and Fukushima [6] extended the

[*1] In [1] a robust Nash equilibrium is called a robust-optimization equilibrium.

definition of robust Nash equilibria in [1] and [5] to the N-person non-cooperative games with nonlinear cost functions. In particular, they showed existence of robust Nash equilibria under the milder assumptions and gave some sufficient conditions for uniqueness of the robust Nash equilibrium. In addition, they reformulated certain classes of robust Nash equilibrium problems to SOCCPs. However, Hayashi et al. [5] and Nishimura et al. [6] have only dealt with the case where the uncertainty is contained in either opponents' strategies or each player's cost parameters, in reformulating the robust Nash equilibrium problem as an SOCCP.

In this paper, we first focus on a special class of linear programs (LPs) with uncertain data. To such a problem, we reformulate its robust counterpart as an SDP. Especially, when the uncertainty sets are spherical, we show that those two problems are equivalent. We then show that the robust Nash equilibrium problem in which uncertainty is contained in both opponents' strategies and each player's cost parameters can be reduced to a semidefinite complementarity problem (SDCP) [2, 8].

Throughout the paper, we use the following notations. For a set X, $\mathcal{P}(X)$ denotes the set consisting of all subsets of X. $\mathbb{R}^n_+$ denotes the nonnegative orthant in $\mathbb{R}^n$, that is, $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n \mid x_i \geq 0\ (i = 1, \ldots, n)\}$. $\mathcal{S}^n$ denotes the set of $n \times n$ real symmetric matrices. $\mathcal{S}^n_+$ denotes the cone of positive semidefinite matrices in $\mathcal{S}^n$. For a vector $x \in \mathbb{R}^n$, $\|x\|$ denotes the Euclidean norm defined by $\|x\| := \sqrt{x^\top x}$. For a matrix $M = (M_{ij}) \in \mathbb{R}^{m \times n}$, $\|M\|_F$ is the Frobenius norm defined by $\|M\|_F := (\sum_{i=1}^m \sum_{j=1}^n (M_{ij})^2)^{1/2}$, $\|M\|_2$ is the ℓ_2-norm defined by $\|M\|_2 := \max_{x \neq 0} \|Mx\|/\|x\|$, and $\ker M$ denotes the kernel of matrix M, i.e., $\ker M := \{x \in \mathbb{R}^n \mid Mx = 0\}$. $B(x, r)$ denotes the closed sphere with center x and radius r, i.e., $B(x, r) := \{y \in \mathbb{R}^n \mid \|y - x\| \leq r\}$. For a problem (P), val(P) denotes the optimal value.

2 Preliminary: SDP reformulation technique

In this section, we review the SDP reformulation technique for a class of robust LPs discussed in [7]. Consider the following uncertain LP:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & (\hat{\gamma}^0)^\top (\hat{A}^0 x + \hat{b}^0) \\ \text{subject to} & (\hat{\gamma}^i)^\top (\hat{A}^i x + \hat{b}^i) \leq 0 \quad (i = 1, \ldots, K) \\ & x \in \Omega, \end{array} \tag{2.1}$$

where Ω is a given closed convex set with no uncertainty. Let $\mathcal{U}_i$ and $\mathcal{V}_i$ be the uncertainty sets for $\hat{\gamma}^i \in \mathbb{R}^{m_i}$ and $(\hat{A}^i, \hat{b}^i) \in \mathbb{R}^{m_i \times (n+1)}$ satisfying the following assumption.

Assumption 1. *For $i = 0, 1, \ldots, K$, the uncertainty sets $\mathcal{U}_i$ and $\mathcal{V}_i$ are expressed as*

$$\mathcal{U}_i := \left\{ (\hat{A}^i, \hat{b}^i) \;\middle|\; (\hat{A}^i, \hat{b}^i) = (A^{i0}, b^{i0}) + \sum_{j=1}^{s_i} u_j^i (A^{ij}, b^{ij}),\ (u^i)^\top u^i \le 1 \right\},$$
$$\mathcal{V}_i := \left\{ \hat{\gamma} \;\middle|\; \hat{\gamma} = \gamma^{i0} + \sum_{j=1}^{t_i} v_j^i \gamma^{ij},\ (v^i)^\top v^i \le 1 \right\},$$

respectively, where $A^{ij} \in \mathbb{R}^{m_i \times n}, b^{ij} \in \mathbb{R}^{m_i}$ $(j = 0, 1, \ldots, s_i)$ and $\gamma^{ij} \in \mathbb{R}^{m_i}$ $(j = 1, \ldots, t_i)$ are given matrices and vectors.

Then, the robust counterpart (RC) for (2.1) can be written as

$$\begin{array}{lll} \underset{x}{\text{minimize}} & \displaystyle\sup_{(\hat{A}^0, \hat{b}^0) \in \mathcal{U}_0,\ \hat{\gamma}^0 \in \mathcal{V}_0} (\hat{\gamma}^0)^\top (\hat{A}^0 x + \hat{b}^0) & \\ \text{subject to} & (\hat{\gamma}^i)^\top (\hat{A}^i x + \hat{b}^i) \le 0 \quad \forall (\hat{A}^i, \hat{b}^i) \in \mathcal{U}_i,\ \forall \hat{\gamma}^i \in \mathcal{V}_i \quad (i = 1, \ldots, K) & \\ & x \in \Omega. & \end{array} \tag{2.2}$$

According to the reformulation technique in [7], we introduce the following SDP related to RC (2.2):

$$\begin{array}{ll} \underset{x, \alpha, \beta, \lambda_0}{\text{minimize}} & -\lambda_0 \\ \text{subject to} & \begin{bmatrix} P_0^0(x) & q^0(x) \\ q^0(x)^\top & r^0(x) - \lambda_0 \end{bmatrix} \succeq \alpha_0 \begin{bmatrix} P_1^0 & 0 \\ 0 & 1 \end{bmatrix} + \beta_0 \begin{bmatrix} P_2^0 & 0 \\ 0 & 1 \end{bmatrix}, \\ & \begin{bmatrix} P_0^i(x) & q^i(x) \\ q^i(x)^\top & r^i(x) \end{bmatrix} \succeq \alpha_i \begin{bmatrix} P_1^i & 0 \\ 0 & 1 \end{bmatrix} + \beta_i \begin{bmatrix} P_2^i & 0 \\ 0 & 1 \end{bmatrix} \quad (i = 1, \ldots, K), \\ & \alpha = (\alpha_0, \alpha_1, \ldots, \alpha_K) \in \mathbb{R}_+^{K+1}, \quad \beta = (\beta_0, \beta_1, \ldots, \beta_K) \in \mathbb{R}_+^{K+1}, \\ & \lambda_0 \in \mathbb{R}, \quad x \in \Omega, \end{array} \tag{2.3}$$

where $P_0^i(x), q^i(x)$ and $r^i(x)$ are defined by

$$\begin{array}{l} P_0^i(x) = -\dfrac{1}{2} \begin{bmatrix} 0 & (\Gamma_i^\top \Phi_i(x))^\top \\ \Gamma_i^\top \Phi_i(x) & 0 \end{bmatrix}, \quad q^i(x) = -\dfrac{1}{2} \begin{bmatrix} \Phi_i(x)^\top \gamma^i \\ \Gamma_i^\top (A^{i0} x + b^{i0}) \end{bmatrix}, \\ r^i(x) = -(\gamma^i)^\top (A^{i0} x + b^{i0}), \quad P_1^i = \begin{bmatrix} -I_{s_i} & 0 \\ 0 & 0 \end{bmatrix}, \quad P_2^i = \begin{bmatrix} 0 & 0 \\ 0 & -I_{t_i} \end{bmatrix}, \\ \Gamma_i = \begin{bmatrix} \gamma^{i1} & \cdots & \gamma^{it} \end{bmatrix}, \quad \Phi_i(x) = \begin{bmatrix} A^{i1} x + b^{i1} & \cdots & A^{is_i} x + b^{is_i} \end{bmatrix}. \end{array} \tag{2.4}$$

Then, we can show that RC (2.2) and SDP (2.3) are equivalent under the following assumption:

Assumption 2. *Let* $z^* := (x^*, \alpha^*, \beta^*, \lambda_0^*)$ *be an optimum of SDP* (2.3). *Then, there exists* $\varepsilon > 0$ *such that*

$$\dim(\ker(P_0^i(x) - \alpha_i P_1^i - \beta_i P_2^i)) \neq 1 \ (i = 0, 1, \dots, K)$$

for all $(x, \alpha, \beta, \lambda_0^*) \in B(z^*, \varepsilon)$.

Theorem 2.1. *Suppose that Assumption 1 holds, and* $(x^*, \alpha^*, \beta^*, \lambda_0^*)$ *be the optimum of SDP* (2.3), *then* x^* *is feasible in RC*(2.2) *and* val (2.3) *is an upper bound of* val (2.2). *Moreover,* x^* *solves RC*(2.2) *if Assumption 2 further holds.*

When the uncertainty sets $\mathcal{U}_i$ and $\mathcal{V}_i$ are spherical, Assumption 2 also holds automatically.

Assumption 3. *Suppose that Assumption 1 holds. Moreover, for each* $i = 0, 1, \dots, K$, *matrices* (A^{ij}, b^{ij}) $(j = 1, \dots, m_i(n+1))$ *and vectors* γ^{ij} $(j = 1, \dots, t_i)$ $(t_i \geq 2)$ *satisfy the following.*

- *For* $(k, l) \in \{1, \dots, m_i\} \times \{1, \dots, n+1\}$,

$$(A^{ij}, b^{ij}) = \rho_i e_k^{(m_i)} (e_l^{(n+1)})^\top \quad \text{with } j := m_i l + k,$$

  where ρ_i *is a given nonnegative constant, and* $e_r^{(p)}$ *is a unit vector with 1 at* r*-th element and 0 elsewhere.*
- *For any* $(k, l) \in \{1, \dots, t_i\} \times \{1, \dots, t_i\}$,

$$(\gamma^{ik})^\top \gamma^{il} = \sigma_i^2 \delta_{kl},$$

  where σ_i *is a given nonnegative constant, and* δ_{kl} *denotes Kronecker's delta, i.e.,* $\delta_{kl} = 0$ *for* $k \neq l$ *and* $\delta_{kl} = 1$ *for* $k = l$.

Theorem 2.2. *Suppose Assumption* 3 *holds. Then,* x^* *solves RC*(2.2) *if and only if there exists* $(\alpha^*, \beta^*, \lambda_0^*)$ *such that* $(x^*, \alpha^*, \beta^*, \lambda_0^*)$ *is an optimal solution of SDP*(2.3).

Note that Assumption 3 claims that $\mathcal{U}_i$ is an $m_i(n+1)$-dimensional sphere with radius ρ_i in the $m_i(n+1)$-dimensional space and $\mathcal{V}_i$ is a t_i-dimensional sphere with radius σ_i in the m_i-dimensional space, i.e.,

$$\mathcal{U}_i = \{(\hat{A}^i, \hat{b}^i) \mid (\hat{A}^i, \hat{b}^i) = (A^{i0}, b^{i0}) + (\delta A^i, \delta b^i), \ \|(\delta A^i, \delta b^i)\|_F \le \rho_i\} \subset \mathbb{R}^{m_i(n+1)},$$
$$\mathcal{V}_i = \{\hat{\gamma}^i \mid \hat{\gamma}^i = \gamma^{i0} + \delta\gamma^i, \ \|\delta\gamma^i\| \le \sigma_i, \delta\gamma^i \in \operatorname{span}\{\gamma^{ij}\}_{j=1}^{t_i}\} \subset \mathbb{R}^{m_i}.$$

3 SDCP reformulation of robust Nash equilibrium problems

In this section, we apply the idea in the previous section to the robust Nash equilibrium problem, and show that it can be reduced to a semidefinite complementarity problem (SDCP) under some assumptions.

Consider an N-person non-cooperative game in which each player tries to minimize his own cost. Let $x^i \in \mathbb{R}^{m_i}$, $S_i \subseteq \mathbb{R}^{m_i}$, and $f_i : \mathbb{R}^{m_1} \times \cdots \times \mathbb{R}^{m_N} \to \mathbb{R}$ be player i's strategy, strategy set, and cost function, respectively. Moreover, denote

$$\mathcal{I} := \{1, \dots, N\}, \quad \mathcal{I}_{-i} := \mathcal{I} \setminus \{i\}, \quad m := \sum_{j \in \mathcal{I}} m_j, \quad m_{-i} := \sum_{j \in \mathcal{I}_{-i}} m_j,$$
$$x := (x^j)_{j \in \mathcal{I}} \in \mathbb{R}^m, \quad x^{-i} := (x^j)_{j \in \mathcal{I}_{-i}} \in \mathbb{R}^{m_{-i}},$$
$$S := \prod_{j \in \mathcal{I}} S_j \subseteq \mathbb{R}^m, \quad S_{-i} := \prod_{j \in \mathcal{I}_{-i}} S_j \subseteq \mathbb{R}^{m_{-i}}.$$

When the complete information is assumed, each player i decides his own strategy by solving the following optimization problem with the opponents' strategies x^{-i} fixed:

$$\begin{array}{ll} \underset{x^i}{\text{minimize}} & f_i(x^i, x^{-i}) \\ \text{subject to} & x^i \in S_i. \end{array} \tag{3.1}$$

A tuple $(\overline{x}^1, \overline{x}^2, \dots, \overline{x}^N)$ satisfying $\overline{x}^i \in \operatorname{argmin}_{x^i \in S_i} f_i(x^i, \overline{x}^{-i})$ for each player $i = 1, \dots, N$ is called a Nash equilibrium. In other words, if each player i chooses the strategy $\overline{x}^i$, then no player has an incentive to change his own strategy. The Nash equilibrium is well-defined only when each player can estimate his opponents' strategies and can evaluate his own cost exactly. In the real situation, however, any information may contain uncertainty such as observation errors or estimation errors. Thus, we focus on games with uncertainty.

To deal with such uncertainty, we introduce uncertainty sets U_i and $X_i(x^{-i})$, and assume the following statements for each player $i \in \mathcal{I}$:

(A) Player i's cost function involves a parameter $\hat{u}^i \in \mathbb{R}^{s_i}$, i.e., it can be expressed as $f_i^{\hat{u}^i} : \mathbb{R}^{m_i} \times \mathbb{R}^{m_{-i}} \to \mathbb{R}$. Although player i does not know the exact value of $\hat{u}^i$ itself, he can estimate that it belongs to a given nonempty set $U_i \subseteq \mathbb{R}^{s_i}$.
(B) Although player i knows his opponents' strategies x^{-i}, his actual cost is evaluated with x^{-i} replaced by $\hat{x}^{-i} = x^{-i} + \delta x^{-i}$, where δx^{-i} is a certain error or noise. Player i cannot know the exact value of $\hat{x}^{-i}$. However, he can estimate that $\hat{x}^{-i}$ belongs to a certain nonempty set $X_i(x^{-i})$.

Under these assumptions, each player encounters the difficulty of addressing the following family of problems involving uncertain parameters $\hat{u}^i$ and $\hat{x}^{-i}$:

$$\begin{array}{ll} \underset{x^i}{\text{minimize}} & f_i^{\hat{u}^i}(x^i, \hat{x}^{-i}) \\ \text{subject to} & x^i \in S_i, \end{array} \tag{3.2}$$

where $\hat{u}^i \in U_i$ and $\hat{x}^{-i} \in X_i(x^{-i})$. To overcome such a difficulty, we further assume that each player chooses his strategy according to the following criterion of rationality:

(C) Player i tries to minimize his worst cost under assumptions (A) and (B).

From assumption (C), each player considers the worst cost function $\tilde{f}_i : \mathbb{R}^{m_i} \times \mathbb{R}^{m_{-i}} \to (-\infty, +\infty]$ defined by

$$\tilde{f}_i(x^i, x^{-i}) := \sup\{f_i^{\hat{u}^i}(x^i, \hat{x}^{-i}) \mid \hat{u}_i \in U_i, \hat{x}^{-i} \in X_i(x^{-i})\}, \tag{3.3}$$

and then solves the following worst cost minimization problem:

$$\begin{array}{ll} \underset{x^i}{\text{minimize}} & \tilde{f}_i(x^i, x^{-i}) \\ \text{subject to} & x^i \in S_i. \end{array} \tag{3.4}$$

Note that, for fixed x^{-i}, (3.4) is nothing other than the robust counterpart of the uncertain cost minimization problem (3.2). Also, (3.4) can be regarded as a complete information game with cost functions $\tilde{f}_i$. Based on the above discussions, we define the robust Nash equilibrium.

Definition 3.1. Let $\tilde{f}_i$ be defined by (3.3) for $i = 1, \dots, N$. A tuple $(\overline{x}^i)_{i \in \mathcal{I}}$ is called a robust Nash equilibrium of game (3.2), if $\bar{x}^i \in \operatorname{argmin}_{x^i \in S_i} \tilde{f}_i(x^i, \bar{x}^{-i})$ for all i, i.e., a Nash equilibrium of game (3.4). The problem of finding a robust Nash equilibrium is called a robust Nash equilibrium problem.

Now, we focus on the games in which each player takes mixed strategy and minimizes a convex quadratic cost function with respect to his own strategy. For such games, we will show that each player's optimization problem can be reformulated as an SDP, and the robust Nash equilibrium problem reduces to an SDCP.

Originally, SDCP [2, 8] is a problem of finding, for a given mapping $F : \mathcal{S}^n \times \mathcal{S}^n \times \mathbb{R}^m \to \mathcal{S}^n \times \mathbb{R}^m$, a triple $(X, Y, z) \in \mathcal{S}^n \times \mathcal{S}^n \times \mathbb{R}^m$ such that

$$\mathcal{S}^n_+ \ni X \perp Y \in \mathcal{S}^n_+, \quad F(X, Y, z) = 0,$$

where $X \perp Y$ means $\operatorname{tr}(XY) = 0$. SDCP can be solved by some modern algorithms such as a non-interior continuation method [2].

In the remainder of this section, the cost functions and the strategy sets satisfy the followings.

(i) Player i's cost function $f_i^{\hat{u}^i}$ is defined by*2

$$f_i(x^i, x^{-i}) = \frac{1}{2}(x^i)^\top \hat{A}_{ii} x^i + \sum_{j \in \mathcal{I}_{-i}} (x^i)^\top \hat{A}_{ij} x^j, \tag{3.5}$$

where $\hat{A}_{ij} \in \mathbb{R}^{m_i \times m_j}$ $(j \in \mathcal{I}_{-i})$ are given constants involving uncertainties.

(ii) Player i takes mixed strategy, i.e.,

$$S_i = \{x^i \in \mathbb{R}^{m_i} \mid x^i \geq 0,\ \mathbf{1}_{m_i}^\top x^i = 1\} \tag{3.6}$$

where $\mathbf{1}_{m_i}$ denotes $(1, 1, \ldots, 1)^\top \in \mathbb{R}^{m_i}$.

(iii) $m_i \geq 3$ for all $i \in \mathcal{I}$.

We call $\hat{A}_{ij}$ a cost matrix. Note that these constants correspond to the cost function parameter $\hat{u}^i$, i.e.,

$$\hat{u}^i = \operatorname{vec}\begin{bmatrix}\hat{A}_{i1} & \ldots, \hat{A}_{iN}\end{bmatrix} \in \mathbb{R}^{m_i m}$$

where vec denotes the vectorization operator that creates an nm-dimensional vector

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*2 Although we can consider the additional term $c^\top x$, for simplicity, we omit the term.

$[(p_1^c)^\top \cdots (p_m^c)^\top]^\top$ from a matrix $P \in \Re^{n\times m}$ with column vectors $p_1^c, \ldots, p_m^c \in \mathbb{R}^n$.

For the robust Nash equilibrium problem with the above cost functions and strategy sets, Hayashi et al. [5] and Nishimura et al. [6] showed that it can be reformulated as an SOCCP. Since the SOCCP can be solved by some existing algorithms, we can calculate the robust Nash equilibria efficiently. However, they have only dealt with the case where the uncertainty is contained in either opponents' strategies or each player's cost matrices and vectors.

In this subsection, we consider the case where each player cannot exactly estimate both the cost matrices and the opponents' strategies. For such a case, we first show the existence of a robust Nash equilibrium, and then, prove that the robust Nash equilibrium problem can be reformulated as an SDCP. To this end, we make the following assumption.

Assumption 4. *For each $i \in \mathcal{I}$, the uncertainty sets $X_i(\cdot)$ and U_i are given as follows.*

(a) $X_i(x^{-i}) = \prod_{j\in\mathcal{I}_{-i}} X_{ij}(x^j)$, *where* $X_{ij}(x^j) = \{x^j + \delta x^{ij} \mid \|\delta x^{ij}\| \le \sigma_{ij},\ \mathbf{1}_{m_j}^\top \delta x^j = 0\ (j \in \mathcal{I}_{-i})\}$ *for some nonnegative scalar* σ_{ij}.

(b) $U_i = \prod_{j\in\mathcal{I}_{-i}} D_{ij}$, *where* $D_{ij} := \{A_{ij} + \delta A_{ij} \in \mathbb{R}^{m_i\times m_j} \mid \|\delta A_{ij}\|_F \le \rho_{ij}\}$ *for some nonnegative scalar* ρ_{ij}. *Moreover,* $A_{ii} + \rho_{ii} I$ *is symmetric and positive semidefinite.*

Assumption 4 claims that $X_{ij}(x^j)$ is the closed sphere with center x^j and radius σ_{ij} in the subspace $\{x \in \mathbb{R}^{m_j} \mid \mathbf{1}_{m_j}^\top x = 0\}$, and D_{ij} is also the closed sphere with center A_{ij} and radius ρ_{ij}. Note that Assumption 4 is milder than the assumptions made by Hayashi et al. [5] and Nishimura et al. [6]. Indeed, Assumption 4 with either $\rho_{ij} = 0$ or $\sigma_{ij} = 0$ for all $(i, j) \in \mathcal{I}\times\mathcal{I}$ corresponds to their assumptions.

Under Assumption 4, we rewrite each player i's optimization problem (3.4). Note that the worst cost function $\tilde{f}_i$ can be written as

$$
\begin{aligned}
&\tilde{f}_i(x^i, x^{-i}) \\
&= \max\left\{ \frac{1}{2}(x^i)^\top \hat{A}_{ii} x^i + \sum_{j\in\mathcal{I}_{-i}} (x^i)^\top \hat{A}_{ij}\hat{x}^j \;\middle|\; \begin{array}{l} \hat{A}_{ii} \in D_{ii}, \\ \hat{A}_{ij} \in D_{ij},\ \hat{x}^j \in X_{ij}(x^j)\ (j \in \mathcal{I}_{-i}) \end{array} \right\} \\
&= \max\left\{ \frac{1}{2}(x^i)^\top \hat{A}_{ii} x^i \;\middle|\; \hat{A}_{ii} \in D_{ii} \right\} + \sum_{j\in\mathcal{I}_{-i}} \max\left\{ (x^i)^\top \hat{A}_{ij}\hat{x}^j \;\middle|\; \hat{A}_{ij} \in D_{ij}, \hat{x}^j \in X_{ij}(x^j) \right\} \\
&= \frac{1}{2}(x^i)^\top (A_{ii} + \rho_{ii} I) x^i + \sum_{j\in\mathcal{I}_{-i}} \max\left\{ (\hat{x}^j)^\top \hat{A}_{ij}^\top x^i \;\middle|\; \hat{A}_{ij} \in D_{ij}, \hat{x}^j \in X_{ij}(x^j) \right\},
\end{aligned} \tag{3.7}
$$

where the last equality holds since

$$
\begin{aligned}
\max\left\{ \frac{1}{2}(x^i)^\top \hat{A}_{ii} x^i \;\middle|\; \hat{A}_{ii} \in D_{ii} \right\} &= \frac{1}{2}(x^i)^\top A_{ii} x^i + \max\left\{ \frac{1}{2}(x^i)^\top \delta A_{ii} x^i \;\middle|\; \|\delta A_{ii}\| \le \rho_{ii} \right\} \\
&= \frac{1}{2}(x^i)^\top A_{ii} x^i + \max\left\{ \frac{1}{2}(x^i \otimes x^i)\,\mathrm{vec}(\delta A_{ii}) \;\middle|\; \|\delta A_{ii}\| \le \rho_{ii} \right\} \\
&= \frac{1}{2}(x^i)^\top A_{ii} x^i + \frac{1}{2}\rho_{ii}\|x^i\|^2 \\
&= \frac{1}{2}(x^i)^\top (A_{ii} + \rho_{ii} I) x^i.
\end{aligned}
$$

Hence, each player i's optimization problem (3.4) can be rewritten as follows:

$$
\begin{array}{ll}
\underset{x^i}{\text{minimize}} & \frac{1}{2}(x^i)^\top (A_{ii} + \rho_{ii} I) x^i + \sum_{j\in\mathcal{I}_{-i}} \max\left\{ (\hat{x}^j)^\top \hat{A}_{ij}^\top x^i \;\middle|\; \hat{A}_{ij} \in D_{ij}, \hat{x}^j \in X_{ij}(x^j) \right\} \\
\text{subject to} & \mathbf{1}_{m_i}^\top x^i = 1, \quad x^i \ge 0.
\end{array} \tag{3.8}
$$

Now we show the existence of a robust Nash equilibrium under Assumption 4.

Theorem 3.2. *Suppose that the cost functions and the strategy sets are given by* (3.5) *and* (3.6)*, respectively. Suppose further that Assumption* 4 *holds. Then, there exists at least one robust Nash equilibrium.*

Next we show that problem (3.8) can be rewritten as an SDP. We note that problem (3.8) has a structure analogous to problem (2.2), and $X_{ij}(x^j)$ and D_{ij} satisfy Assumption 3. Indeed, $X_{ij}(x^j)$ can be constructed by the vectors γ^{ijk} ($k = 1, \dots, m_j - 1$) which from orthogonal bases of the subspace $\{x \mid \mathbf{1}_{m_j}^\top x = 0\}$ with $\|\gamma^{ijk}\| = \sigma_{ij}$ for all k. Thus, by Theorem 2.2, problem (3.8) can be rewritten as the following SDP:

$$
\begin{aligned}
\underset{x^i,\alpha^{-i},\beta^{-i},\lambda^{-i}}{\text{minimize}} \quad & \frac{1}{2}(x^i)^\top (A_{ii}+\rho_{ii}I)x^i - \sum_{j\in\mathcal{I}_{-i}} \lambda_{ij} \\
\text{subject to} \quad & \begin{bmatrix} P_0^{ij}(x^i) & q^{ij}(x^i,x^j) \\ q^{ij}(x^i,x^j)^\top & r^{ij}(x^i,x^j)-\lambda_{ij} \end{bmatrix} \succeq \alpha_{ij}\begin{bmatrix} P_1^{ij} & 0 \\ 0 & 1 \end{bmatrix} + \beta_{ij}\begin{bmatrix} P_2^{ij} & 0 \\ 0 & 1 \end{bmatrix}, \quad (j\in\mathcal{I}_{-i}) \\
& \alpha^{-i} = (\alpha_{ij})_{j\in\mathcal{I}_{-i}} \in \mathbb{R}_+^{N-1}, \quad \beta^{-i} = (\beta_{ij})_{j\in\mathcal{I}_{-i}} \in \mathbb{R}_+^{N-1}, \\
& \lambda^{-i} = (\lambda_{ij})_{j\in\mathcal{I}_{-i}} \in \mathbb{R}^{N-1}, \\
& \mathbf{1}_{m_i}^\top x^i = 1, \quad x^i \geq 0,
\end{aligned} \tag{3.9}
$$

where

$$
\begin{aligned}
P_0^{ij}(x^i) &= -\frac{1}{2}\begin{bmatrix} 0 & \rho_{ij}(\Xi_{ij}^\top((x^i)^\top\otimes I_{m_j}))^\top \\ \rho_{ij}\Xi_{ij}^\top((x^i)^\top\otimes I_{m_j}) & 0 \end{bmatrix}, \\
q^{ij}(x^i,x^j) &= -\frac{1}{2}\begin{bmatrix} \rho_{ij}((x^i)^\top\otimes I_{m_j})^\top x^j \\ \Xi_{ij}^\top A_{ij}^\top x^i \end{bmatrix}, \quad r^{ij}(x^i,x^j) = -(x^j)^\top A_{ij}^\top x^i, \\
P_1^{ij} &= \begin{bmatrix} -I_{m_i m_j} & 0 \\ 0 & 0 \end{bmatrix}, \quad P_2^{ij} = \begin{bmatrix} 0 & 0 \\ 0 & -I_{m_j-1} \end{bmatrix}, \\
\Xi_{ij} &= \begin{bmatrix} \xi^{ij1} & \cdots & \xi^{ij(m_j-1)} \end{bmatrix}.
\end{aligned} \tag{3.10}
$$

Finally, we show that the robust Nash equilibrium problem reduces to an SDCP. Since the semidefinite constraints in (3.9) are linear with respect to $x^i, \alpha^{-i}, \beta^{-i}$ and λ^{-i}, we can rewrite the constraints as

$$
\sum_{k=1}^{m_i} x_k^i M_k^{ij}(x^j) + \lambda_{ij} M_\lambda^{ij} \succeq \alpha_{ij} M_\alpha^{ij} + \beta_{ij} M_\beta^{ij}, \quad (j\in\mathcal{I}_{-i}),
$$

with $M_k^{ij} \in \mathcal{S}^{m_j(m_i+1)}$ $(k=1,\dots,m_i)$, $M_\lambda^{ij}, M_\alpha^{ij}, M_\beta^{ij} \in \mathcal{S}^{m_j(m_i+1)}$ defined by

$$
M_k^{ij}(x^j) := \begin{bmatrix} P_0^{ij}(e_k^{(m_i)}) & q^{ij}(e_k^{(m_i)},x^j) \\ q^{ij}(e_k^{(m_i)},x^j)^\top & r^{ij}(e_k^{(m_i)},x^j) \end{bmatrix},
$$

$$
M_\lambda^{ij} := -e_{m_j(m_i+1)+1}^{(m_j(m_i+1)+1)}\left(e_{m_j(m_i+1)+1}^{(m_j(m_i+1)+1)}\right)^\top, \quad M_\alpha^{ij} := \begin{bmatrix} P_1^{ij} & 0 \\ 0 & 1 \end{bmatrix}, \quad M_\beta^{ij} := \begin{bmatrix} P_2^{ij} & 0 \\ 0 & 1 \end{bmatrix},
$$

respectively. Then, the Karush-Kuhn-Tucker (KKT) conditions for (3.9) are given by

$$
\begin{aligned}
((A_{ii}+\rho_{ii}I)x^i)_k - \sum_{j\in\mathcal{I}_{-i}} \operatorname{tr}(Z^{ij}M_k^{ij}(x^j)) - (\mu_x^i)_k + \nu^i = 0, \quad (k=1,\dots,m_i),\\
\operatorname{tr}(Z^{ij}M_\alpha^{ij}) - (\mu_\alpha^i)_j = 0, \quad (j\in\mathcal{I}_{-i}),\\
\operatorname{tr}(Z^{ij}M_\beta^{ij}) - (\mu_\beta^i)_j = 0, \quad (j\in\mathcal{I}_{-i}),\\
\operatorname{tr}(Z_{ij}M_\lambda^{ij}) + 1 = 0, \quad (j\in\mathcal{I}_{-i}),\\
\operatorname{tr}\left(Z^{ij}(\sum_{k=1}^{m_1} x_k^i M_k^{ij}(x^j) + \lambda_{ij}M_\lambda^{ij} - \alpha_{ij}M_\alpha^{ij} - \beta_{ij}M_\beta^{ij})\right) = 0,\\
(\mu_\alpha^i)^\top \alpha^{-i} = 0, \quad (\mu_\beta^i)^\top \beta^{-i} = 0, \quad (\mu_x^i)^\top x^i = 0,\\
\sum_{k=1}^{m_i} x_k^i M_k^{ij}(x^j) + \lambda_{ij}M_\lambda^{ij} \succeq \alpha_{ij}M_\alpha^{ij} + \beta_{ij}M_\beta^{ij}, \quad (j\in\mathcal{I}_{-i}),\\
\mathbf{1}_{m_i}^\top x^i = 1, \quad x^i \ge 0, \quad \alpha^{-i} \ge 0, \quad \beta^{-i} \ge 0,\\
Z^{ij} \succeq 0, \quad \mu_x^i \ge 0, \quad \mu_\alpha^i \ge 0, \quad \mu_\beta^i \ge 0,
\end{aligned}
$$

where $Z^{ij} \in \mathcal{S}^{m_j(m_i+1)}$, $\mu_x^i \in \mathbb{R}^{m_i}$, $\mu_\alpha^i, \mu_\beta^i \in \mathbb{R}^{N-1}$ and $\nu^i \in \mathbb{R}$ are Lagrange multipliers. Eliminating μ_x^i, μ_α^i and μ_β^i, we obtain the following conditions for each $i \in \mathcal{I}$:

$$
\begin{aligned}
\mathcal{S}_+^{m_i(m_j+1)} \ni Z^{ij} \perp \sum_{k=1}^{m_i} x_k^i M_k^{ij}(x^j) + \lambda_{ij}M_\lambda^{ij} - \alpha_{ij}M_\alpha^{ij} - \beta_{ij}M_\beta^{ij} \in \mathcal{S}_+^{m_i(m_j+1)}, \quad (j\in\mathcal{I}_{-i}),\\
\mathbb{R}_+^{m_i} \ni x^i \perp \Big(((A_{ii}+\rho_{ii}I)x^i)_k - \sum_{j\in\mathcal{I}_{-i}} \operatorname{tr}(Z^{ij}M_k^{ij}(x^j)) + \nu^i\Big)_{k=1,\dots,m_i} \in \mathbb{R}^{m_i},\\
\mathbb{R}_+^{N-1} \ni \alpha^{-i} \perp \operatorname{tr}(Z^{ij}M_\alpha^{ij})_{j\in\mathcal{I}_{-i}} \in \mathbb{R}_+^{N-1}, \quad \mathbb{R}_+^{N-1} \ni \beta^{-i} \perp \operatorname{tr}(Z^{ij}M_\beta^{ij})_{j\in\mathcal{I}_{-i}} \in \mathbb{R}_+^{N-1},\\
\operatorname{tr}(Z^{ij}M_\lambda^{ij}) = -1, \ (j\in\mathcal{I}_{-i}), \quad \mathbf{1}_{m_i}^\top x^i = 1.
\end{aligned}
\tag{3.11}
$$

Noticing that the above KKT conditions hold for all players simultaneously, the robust Nash equilibrium problem can be reformulated as the problem of finding $(x^i, \alpha^{-i}, \beta^{-i}, \lambda^{-i}, (Z^{ij})_{j\in\mathcal{I}_{-i}}, \nu^i)_{i\in\mathcal{I}}$ such that (3.11) for all $i \in \mathcal{I}$. Thus, we obtain the following theorem.

Theorem 3.3. *Suppose that the cost functions and the strategy sets are given by* (3.5) *and* (3.6), *respectively. Suppose further that Assumption* 4 *holds. Then,* x^* *is a robust Nash equilibrium if and only if* $(x^i, \alpha^{-i}, \beta^{-i}, \lambda^{-i}, (Z^{ij})_{j\in\mathcal{I}_{-i}}, \nu^i)_{i\in\mathcal{I}}$ *is a solution of SDCP*(3.11).

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