Title	Optimal Execution of Multi-Asset Block Orders under Stochastic Liquidity (Financial Modeling and Analysis)
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Citation	数理解析研究所講究録 (2010), 1675: 185-198
Issue Date	2010-02
URL	http://hdl.handle.net/2433/141229
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

数理解析研究所講究録
第 1675 巻 2010 年 185-198

Optimal Execution of Multi-Asset Block Orders under Stochastic Liquidity*

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abstract

We develop a multivariate market liquidity model and derive the explicit solution of an optimal execution strategy under both liquidity risk and volatility risk. The market liquidity is modeled as the birth-death process of orders flowing into and out of an order book. Given a shape of the order book of each asset, the market impact of the execution is, then, expressed as a recursive impact with geometric recovery accompanied by uncertainty. The optimal execution is derived as the analytical solution to a mean-variance problem which minimizes the tradeoff between the market impact and the volatility/liquidity risk with an investor's risk aversion. Some analyses and implications of the optimal execution strategy are summarized by comparative statics and simulations. We also discuss its relation to the market manipulation.

1 Introduction

Executing a large volume of block securities under fluctuating market liquidity may cause a significant market impact which results in a considerable execution cost. Financial crises have drawn much attention to the execution cost surged under the declining market liquidity. Various models of asset price processes with market impacts have been developed to control the execution cost since the financial crisis in 1990's, from which optimal execution strategies have been constructed. The development began motivating financial institutions to minimize the execution cost even in non-crisis period; the execution cost is considered to hold a large part in the total cost of portfolio management which always facing the needs of trading a large size of block securities.

The minimization of the execution cost is one of the essential tasks currently undertaken in financial institutions. Practitioners try to apply the optimal execution strategy as an algorithmic trading schedule. Owing to the accumulation of high frequency market data, the algorithmic trading strategy, along with technologies that search for off-exchange liquidity such as liquidity

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in dark pools and crossing networks, have been rapidly developed at large financial institutions in both buy side and sell side.

With such motives, we propose a new approach that minimizes the market impact involved in a multi-asset execution while controlling the liquidity risk. We first develop a model of market liquidity as the fluctuating number of limit orders in a limit order book, which is constructed as the birth-death process, or $M/M/\infty$ queue waiting for an execution. We then solve a static problem that minimizes the tradeoff between the market impact and the price's volatility or liquidity risk to obtain an optimal execution strategy. The model is constructed under multi-asset environment and analyzed how the optimal strategy is affected by the price's or liquidity's correlation among assets.

Almgren and Chriss [2000] obtained an optimal execution strategy that minimizes the tradeoff between the market impact and the price's volatility cost by assuming both the short-term market impact that recovers promptly after executions and the permanent one that does not recover. Konishi and Makimoto [2001] extended Almgren and Chriss [2000]'s model to multi-asset environment and obtained optimal slices of block trade by solving a mean and standard deviation problem. Obizhaeva and Wang [2005] studied the optimal execution strategy assuming recursive liquidity in a flat limit order book, where the price diffuses and recovers gradually after executions. Alfonsi *et al.* [2009] extended Obizhaeva and Wang [2005]'s work to derive an optimal strategy where the limit order book has general but continuous and differentiable shape.

Our model is a mixed type of Obizhaeva and Wang [2005] and Almgren and Chriss [2000] with additional uncertainty in liquidity. Our model is constructed from the knowledge of queuing theory, which is different from the Obizhaeva and Wang [2005]'s approach, but our approach induces an expansion of their model. In our model, uncertainty is added to Obizhaeva and Wang [2005] in liquidity recovery after executions; we assume the market impact asymptotically declines with randomness to the fundamental price which is modeled as a stationary process with independent increments. Our problem is similar to that of Almgren and Chriss [2000] which minimizes the tradeoff between the market impact and the volatility risk. But our problem consider the liquidity risk as well. We get an analytical solution to the problem, which is easily analyzed to see how fluctuating liquidity or the fundamental price's volatility affects the optimal execution strategy.

By the analyses of the optimal strategy, we obtain similar results to earlier works such that the execution velocity should increase, as liquidity itself increases, as the investor becomes risk averse, or as the volatility/resilience of liquidity rises. We also detect that the investor should sell/buy a liquid asset faster as the correlation coefficient of the liquidity or the fundamental price increases among assets, while keeping the execution velocity of an illiquid asset unchanged.

The execution strategy should be constructed to avoid market manipulation defined by Huberman and Stanzl [2004]. It is defined mathematically as the round-trip trade with a negative cost, which is conceptually different from the market manipulation regulated by law. Gatheral [2009] pointed out that the model type of Obizhaeva and Wang [2005] may violate no-market-manipulation condition. Alfonsi and Schied [2009], however, showed the optimal strategy does not violate the condition since their strategy is composed of only sell or only buy trades under a simple setting, but they also pointed out that the model may generate a round-trip trade under a special parameter of liquidity decay. We confirm the our optimal strategy does not violate the no-market-manipulation condition in the single asset case, but it may contain round-trip trade in the multi-asset case.

This paper is constructed as follows. Section 2 defines our model of the market price and liquidity recovery after the execution, and Section 3 derives the optimal execution strategy based on our model. Section 4 analyzes the optimal strategy and organizes comparative statics. Section 5 is the summary of the paper.

2 A model of stochastic liquidity and optimal execution problem

2.1 Stochastic liquidity and recursive market impact

We consider a discrete execution schedule at a regular time interval τ. Suppose an investor has N assets and $K+1$ times of execution time slots starting from the current time 0 to a finite time horizon T. Let $t_k = \tau k$ $(k = 0, \dots, K)$ be the time just before the k-th execution where $t_0 = 0$ and $t_K = T$. Given a block of buy orders with size ω_i for an asset i $(i = 1, \dots, N)$[1] , we will determine the volume of slice orders $\xi_0^i, \dots, \xi_K^i$, where ξ_k^i denotes the volume of the asset i to be executed as a market order at k-th time slot, and $\sum_{k=0}^{K} \xi_k^i = \omega_i$. We focus only on the optimal execution of a buy order in this paper. The optimal execution of a sell order can be analyzed in a similar way.

We build the following model that describes the dynamic behavior of asset prices and limit order books. Let P_t^i and A_t^i be the fundamental price and the best ask price of the asset i at time t, respectively. The fundamental price is given by a random walk

$$P_{t_k}^i = P_0^i + \sum_{s=1}^{k} \Delta_s^i, \quad (k = 1, \dots, K,\ i = 1, \dots, N),$$

where P_0^i is the current price. The process of the increment $\{\Delta_s^i\}$ is a stationary process with a finite variance, which satisfies $\mathrm{E}\left(\Delta_s^i\right) = 0$ for any s and i. We also assume the covariance matrix of $\Delta_s^1, \dots, \Delta_s^N$, denoted as $\mathbf{\Sigma}_\Delta = (\sigma_\Delta^{ij})$, is independent from s, i.e., Δ_s^i and Δ_l^j $(l \neq s)$ are uncorrelated.

We then consider limit orders on a limit order book interpreted as the market liquidity of each asset. Let M_t^i denotes the volume of sell limit orders placed above A_t^i in the order book of the asset i. We also consider the volume of potential sell orders V_t^i that are not placed in the order book at time t but will be placed from A_t^i down to P_t^i after time t. Let $f_i(p)$ denotes the volume of limit orders or potential sell orders at the price level $P_t^i + p$ for the asset i. $f_i(p)$ is considered to be a density of limit orders under a continuous price setting. New orders are supposed to be placed in the way that the orders fill up the density down from A_t^i bit by bit; the new order is placed just below A_t^i, then A_t^i moves down as a consequence, but it is not supposed to be placed at a price level apart from A_t^i. The total volume of potential and actual sell orders

[1] We use super-/sub-script i to indicate the variable for the asset i.

between P_t^i and $P_t^i + p$ is given by $F_i(p) = \int_0^p f_i(q)dq$ $(p \geq 0)$. M_t^i and V_t^i can be written in terms of $F_i(p)$ as $M_t^i = F_i(\infty) - V_t^i$ and $V_t^i = F_i(A_t^i - P_t^i)$. Here, we assume the total volume of limit orders on the book is finite, i.e., $F_i(\infty) < \infty$ for any i. P_t^i could be viewed as the mid price of the asset i if the effects of the past trade on the order book are quickly absorbed. However, this is not the case when the past trades, especially large ones, have 'eaten up' the limit orders. We consider this case where the best ask price A_t^i is higher than the fundamental price P_t^i. The concept of the model is described in Figure 1 (a).

Based on the above setting, we also construct the time development of the market liquidity M_t^i or V_t^i. Suppose a new sell limit order arriving at the order book obeys Poisson process at a rate μ_i, while the existing limit orders on the book is supposed to be cancelled at a rate ν times the remaining volume of orders M_t^i. The arriving rate μ_i is supposed to be different asset by asset while the cancel rate ν to be equivalent for all assets. Let $q_j^i(t)$ denotes the probability that M_t^i equals j, i.e., $q_j^i(t) = \Pr(M_t^i = j)$, then the forward Kolmogorov equation of $q_j^i(t)$ is

$$\begin{cases} \dot{q}_j^i(t) = \mu_i q_{j-1}^i(t) + \nu(j+1)q_{j+1}^i(t) - (\mu_i + \nu j)q_j^i(t), & (j > 0) \\ \dot{q}_0^i(t) = \nu q_1^i(t) - \mu_i q_0^i(t), & (j = 0) \end{cases} \quad (2.1)$$

with the initial condition of $q_j^i(0) = \delta_{jM_0^i}$[2]. Solving eq.(2.1) by letting $Q_i(z,t) = \sum_{j=0}^{\infty} q_j^i(t)z^j$ yields

$$Q_i(z,t) = \exp\left[\frac{\mu_i(z-1)t + M_0^i \ln z}{1+\nu t}\right], \quad (i = 1, \ldots, N).$$

From the above equation, the conditional expected value of M_t^i can be computed as $\mathrm{E}\left(M_{t+dt}^i | M_t^i\right) = (Q_i)_z(1, dt)/(n!) = (M_t^i + \mu_i dt)/(1 + \nu dt)$ where dt is a small positive time value. Then, the differential equation of the expected value is derived as $d\mathrm{E}\left(M_t^i\right) = (\mu_i - \nu\mathrm{E}\left(M_t^i\right))dt$. We also compute the stationary state of the liquidity as

$$q_j^i(t) = \frac{1}{j!}\frac{\partial^j Q_i}{\partial z^j}(0,t) \to \frac{1}{j!}\left(\frac{\mu_i}{\nu}\right)^j e^{-\mu_i/\nu} \quad \text{as} \quad t \to \infty.$$

This indicates that the liquidity asymptotically obeys Poisson process at rate μ_i/ν in its stationary state. When μ_i/ν is large enough to approximate Poisson distribution to Normal distribution, the time development of M_t^i is, by the above discussion, simply modeled as

$$dM_t^i = (\mu_i - \nu M_t^i)dt + d\tilde{Z}_t^i, \quad (i = 1, \ldots, N), \quad (2.2)$$

where $d\tilde{Z}_t^i$ is a stationary and independent stochastic increments with a finite variance. By assuming $F_i(\infty)$ equals the stationary level of liquidity, i.e., $F_i(\infty) = \mu_i/\nu$, eq.(2.2) is rewritten in terms of V_t^i as

$$dV_t^i = -\nu V_t^i dt - d\tilde{Z}_t^i, \quad (i = 1, \ldots, N), \quad (2.3)$$

Let $a = e^{-\nu\tau}$ and $Z_{t_k}^i = -\int_{t_k+}^{t_k+\tau} e^{-\nu(t_k+\tau-t)} d\tilde{Z}_t^i$ where t_k+ is the time just after the k-th execution. Integrating eq.(2.3) from t_k+ to the t_{k+1} yields $V_{t_{k+1}}^i = aV_{t_k+}^i + Z_{t_k}^i$ between any execution time slots. Note that $\mathrm{E}\left(Z_t^i\right) = 0$ and $\mathrm{V}\left(Z_t^i\right) < \infty$. We know $V_{t_k+}^i = V_{t_k}^i + \xi_k^i$ since our market order at time slot t_k $(k = 0, \ldots, K)$ eats up existing limit orders M_t^i. In what follows, we

[2] δ_{ab} is Dirac's delta function.

rewrite $V^i_{t_k}$, $Z^i_{t_k}$ and $P^i_{t_k}$ simply as V^i_k, Z^i_k and P^i_k, respectively, for simplicity. In consequence, the time development of V_k is expressed as

$$V^i_{k+1} = a(V^i_k + \xi^i_k) + Z^i_k, \quad (i = 1, \dots, N,\ k = 0, \dots, K-1). \tag{2.4}$$

We also assume the initial state equals the stationary state of the dynamics, hence $V^i_0 = 0$ ($\forall i$). Eq.(2.4) stipulates how the potential liquidity is affected by the market impact of our execution accompanied by a random fluctuation which is inherited to any future market impacts. We refer to this type of market impact as a recursive market impact. The variance-covariance matrix of $Z^1_k, \dots, Z^N_{t_k}$ is denoted as $\mathrm{Cov}\left(Z^i_k, Z^j_k\right) = (\sigma^{ij}_Z) = \mathbf{\Sigma}_Z$ independently from k. Further, we assume that Z^i_k and Z^j_l are uncorrelated for any $k \neq l$, and $\{Z^i_k;\ k = 0, \dots, K\}$ and $\{\Delta^i_k;\ k = 1, \dots, K\}$ are uncorrelated for any i.

By solving eq.(2.4) recursively, we get

$$V^i_k = \sum_{s=0}^{k-1} a^{k-s} \xi^i_s + \sum_{s=1}^{k} a^{k-s} Z^i_s, \tag{2.5}$$

which indicates that the impact of the past execution will decay geometrically at the rate $a \in (0, 1)$. When a is close to 1, the impact decays slowly and V^i_t approximates to a permanent impact. On the other hand, when a is close to 0, V^i_t is considered to be a temporal impact. The dynamics of our model is described in Figure 1 (b).

2.2 Formulation of the optimal execution problem

Firstly, we consider the execution cost. According to Alfonsi *et al.* [2009], the average execution cost, or the average price, of purchasing ξ^i_k of the the asset i is given by

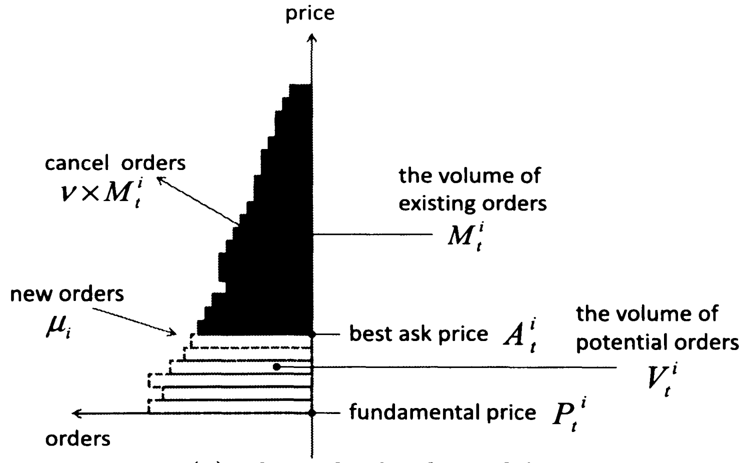
$$P^i_k \xi^i_k + \int_{(F_i)^{-1}(V^i_k)}^{(F_i)^{-1}(V^i_k + \xi^i_k)} x f_i(x) dx = P^i_k \xi^i_k + G_i(V^i_k + \xi^i_k) - G_i(V^i_k), \tag{2.6}$$

where

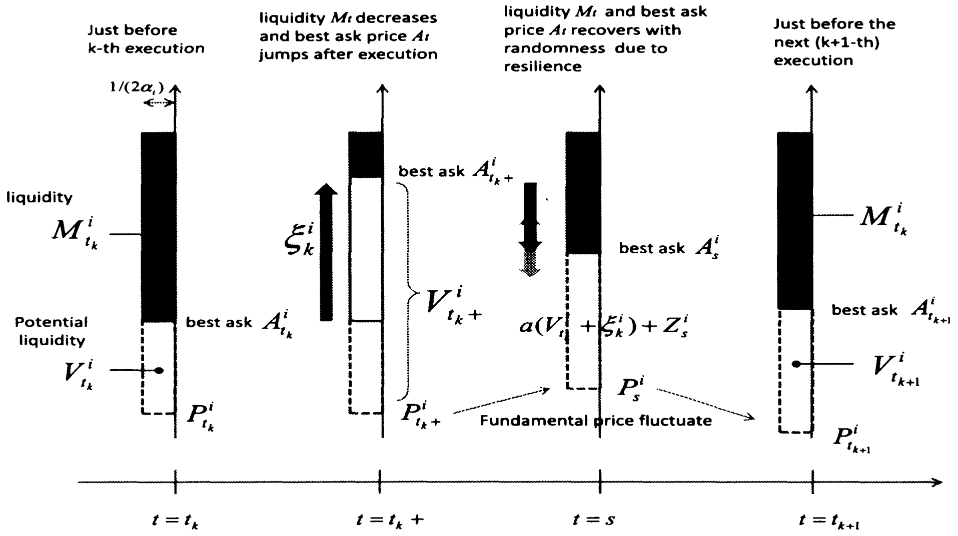
$$G_i(v) = \int_0^{(F_i)^{-1}(v)} x f_i(x) dx$$

denotes the average price of purchasing the asset i above the fundamental price P^i_k. In this paper, we assume the block-shaped order book as in Obizhaeva and Wang [2005]. In this type of a limit order book, $f_i(p) = (2\alpha_i)^{-1}$ for all $p \geq 0$ where α_i is referred to the liquidity parameter. Note that smaller α_i implies higher liquidity of the asset i. Since $G_i(v) = 2\alpha_i v^2$ ($v \geq 0$) for $f_i(p)$ under consideration, the cost of buying ξ^i_k units of asset i at time t_k is given as $P^i_k \xi^i_k + \alpha_i (2V^i_k + \xi^i_k)\xi^i_k$ from eq.(2.6). In what follows, we define $\boldsymbol{\xi}_k = (\xi^1_k, \dots, \xi^N_k)^\top$ and $\boldsymbol{\omega} = (\omega_1, \dots, \omega_N)^\top$ where $\top$ denotes transpose, and the entire execution strategy is represented as $\boldsymbol{\xi} = \{\boldsymbol{\xi}_1, \dots, \boldsymbol{\xi}_K\}$. The total execution cost $TC(\boldsymbol{\xi})$ for strategy $\boldsymbol{\xi}$ is then given by

$$TC(\boldsymbol{\xi}) = \underbrace{\sum_{i=1}^{N} \sum_{k=0}^{K} P^i_k \xi^i_k}_{=I_1(\boldsymbol{\xi})} + \underbrace{\sum_{i=1}^{N} \alpha_i \sum_{k=0}^{K} (2V^i_k + \xi^i_k)\xi^i_k}_{=I_2(\boldsymbol{\xi})}, \tag{2.7}$$



(a) The order book model



(b) The dynamics of liquidity recovery after executions

Figure 1: Dynamics of the model

where $I_1(\boldsymbol{\xi})$ and $I_2(\boldsymbol{\xi})$ are defined as the first and the second term in eq.(2.7), respectively. Since we restrict ourselves to consider static optimization, our objective is to determine the execution schedule $\boldsymbol{\xi}$ at time $t = 0$ so as to minimize $\mathrm{E}\left(TC(\boldsymbol{\xi})\right) + \lambda \mathrm{V}\left(TC(\boldsymbol{\xi})\right)$, where λ denotes the investor's coefficient of risk aversion. The expectations and the variances of $I_1(\boldsymbol{\xi})$ and $I_2(\boldsymbol{\xi})$ are calculated as

$$\mathrm{E}\left(I_1(\boldsymbol{\xi})\right) = \sum_{i=1}^{N} P_0^i \omega_i, \tag{2.8}$$

$$\mathrm{V}\left(I_1(\boldsymbol{\xi})\right) = \sum_{i=1}^{N}\sum_{j=1}^{N} \sigma_\Delta^{ij} \sum_{k=1}^{K} \left(\sum_{s=k}^{K} \xi_s^i\right)\left(\sum_{s=k}^{K} \xi_s^j\right), \tag{2.9}$$

$$\mathrm{E}\left(I_2(\boldsymbol{\xi})\right) = \sum_{i=1}^{N} 2\alpha_i \sum_{k=0}^{K-1}\left(\sum_{s=k+1}^{K} a^{s-k}\xi_s^i\right)\xi_k^i + \sum_{i=1}^{N}\alpha_i\sum_{k=0}^{K}(\xi_k^i)^2, \tag{2.10}$$

$$\mathrm{V}\left(I_2(\boldsymbol{\xi})\right) = \sum_{i=1}^{N}\sum_{j=1}^{N}(2\alpha_i)(2\alpha_j)\sigma_Z^{ij}\sum_{k=1}^{K}\left(\sum_{s=k}^{T} a^{s-k}\xi_s^i\right)\left(\sum_{s=k}^{K} a^{s-k}\xi_s^j\right). \tag{2.11}$$

Noting that eq.(2.8) is a constant independent of $\boldsymbol{\xi}$, we formulate the optimization problem as

$$\min_{\boldsymbol{\xi}} \ \mathrm{E}\left(I_2(\boldsymbol{\xi})\right) + \lambda[\mathrm{V}\left(I_1(\boldsymbol{\xi})\right) + \mathrm{V}\left(I_2(\boldsymbol{\xi})\right)] \qquad \text{s.t. } \boldsymbol{\omega} = \sum_{k=0}^{K}\boldsymbol{\xi}_k. \tag{2.12}$$

3 Optimal execution strategy

3.1 Closed form solution of the optimal execution strategy

The first order optimality condition of the minimization problem in eq.(2.12) with respect to ξ_k^i is given as

$$\frac{\partial \mathrm{E}\left(I_2(\boldsymbol{\xi})\right)}{\partial \xi_k^i} + \lambda\frac{\partial \mathrm{V}\left(I_1(\boldsymbol{\xi})\right)}{\partial \xi_k^i} + \lambda\frac{\partial \mathrm{V}\left(I_2(\boldsymbol{\xi})\right)}{\partial \xi_k^i} = \beta_i, \quad (i = 1, \dots, N, \ k = 0, \dots, K), \tag{3.1}$$

where β_i is a Lagrange multiplier for the constraint of the total volume to execute for the asset i. Here and in what follows, we define $\sum_a^b = 0$ for $a > b$. Eq.(3.1) with the constraint $\boldsymbol{\omega} = \sum_{k=0}^{K}\boldsymbol{\xi}_k$ forms a system of $N(K+2)$ linear equations with the same number of unknown parameters ξ_k^i $(i = 1, \dots, N, k = 0, \dots, K)$ and β_i $(i = 1, \dots, N)$. In general, a solution of the first order condition is just a candidate for the optimal solution. Following a similar manner in Alfonsi *et al.* [2009], however, we can prove that the solution to eq.(3.1) is indeed the optimal solution of eq.(2.12). Our objective is, therefore, reduced to find the explicit solution to eq.(3.1) and the constraint $\boldsymbol{\omega} = \sum_{k=0}^{K}\boldsymbol{\xi}_k$.

From eqs.(2.9)~(2.11), eq.(3.1) can be explicitly expressed in a matrix form as

$$2\left\{\boldsymbol{\alpha}\left(\sum_{s=0}^{k-1}a^{k-s}\boldsymbol{\xi}_s + \sum_{s=k}^{K}a^{s-k}\boldsymbol{\xi}_s\right) + \lambda\boldsymbol{\Sigma}_\Delta\sum_{s=1}^{k}\sum_{u=s}^{K}\boldsymbol{\xi}_u \right.$$
$$\left. + 4\lambda\boldsymbol{\alpha}\boldsymbol{\Sigma}_Z\boldsymbol{\alpha}\sum_{s=1}^{k}a^{k-s}\sum_{u=s}^{K}a^{u-s}\boldsymbol{\xi}_u\right\} = \boldsymbol{\beta}, \quad (k = 0, \dots, K), \tag{3.2}$$

where $\boldsymbol{\beta} = (\beta_1, \dots, \beta_N)^\top$ and $\boldsymbol{\alpha} = \mathrm{Diag}(\alpha_i)$ is a diagonal matrix with α_i's. To represent the optimal solution, we define

$$\boldsymbol{A} = (1-a^2)\boldsymbol{\alpha} + a\lambda\boldsymbol{\Sigma}_\Delta + 4\lambda\boldsymbol{\alpha}\boldsymbol{\Sigma}_Z\boldsymbol{\alpha}, \qquad \boldsymbol{B} = (1-a)^2\lambda\boldsymbol{A}^{-1}\boldsymbol{\Sigma}_\Delta. \tag{3.3}$$

Since $\boldsymbol{\alpha}$, $\boldsymbol{\Sigma}_\Delta$, and $\boldsymbol{\Sigma}_Z$ are positive definite, so is $\boldsymbol{A}$, which ensures the existence of $\boldsymbol{A}^{-1}$. Let $\boldsymbol{C}$ be the Cholesky factorization of $\boldsymbol{\Sigma}_\Delta$, i.e., $\boldsymbol{\Sigma}_\Delta = \boldsymbol{C}^\top\boldsymbol{C}$. From eq.(3.3), we get

$$\boldsymbol{C}\boldsymbol{B}\boldsymbol{C}^{-1} = (1-a)^2\lambda\boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{C}^\top. \tag{3.4}$$

The right hand side of eq.(3.4) is positive definite, and thus diagonalizable as

$$\boldsymbol{C}\boldsymbol{B}\boldsymbol{C}^{-1} = (1-a)^2\lambda \boldsymbol{C}\boldsymbol{A}^{-1}\boldsymbol{C}^{\top} = \boldsymbol{D}^{-1}\boldsymbol{\Gamma}\boldsymbol{D}, \tag{3.5}$$

where $\boldsymbol{\Gamma} = \mathrm{Diag}(\gamma_i)$ with γ_i $(i = 1, \ldots, N)$ being the eigenvalues of both sides of eq.(3.4) and $\boldsymbol{D}$ is the $N \times N$ matrix composed of the associated left eigenvectors. It is noted here that, from the positive definiteness, $\gamma_i > 0$ for all i. Letting $\boldsymbol{R} = (\boldsymbol{D}\boldsymbol{C})^{-1}$, we see from eq.(3.5) that

$$\boldsymbol{B} = \boldsymbol{R}\boldsymbol{\Gamma}\boldsymbol{R}^{-1}, \qquad \boldsymbol{B}\boldsymbol{R} = \boldsymbol{R}\boldsymbol{\Gamma}, \qquad \boldsymbol{R}^{-1}\boldsymbol{B} = \boldsymbol{\Gamma}\boldsymbol{R}^{-1}. \tag{3.6}$$

Since γ_i's are positive, we define

$$\theta_i = \frac{\gamma_i + 2 + \sqrt{\gamma_i^2 + 4\gamma_i}}{2}, \quad (i = 1, \ldots, N).$$

Let $\boldsymbol{\Theta} = \mathrm{Diag}(\theta_i)$, then we obtain the closed form solution of the optimal execution strategy $\boldsymbol{\xi}^*$.

Theorem 1 *The optimal execution strategy at time K is given by*

$$\begin{aligned} \boldsymbol{\xi}_K^* &= (\boldsymbol{I} - \boldsymbol{\Theta}^2)\Big[(\boldsymbol{I} + \frac{\lambda}{1-a}\boldsymbol{\alpha}^{-1}\boldsymbol{\Sigma}_\Delta)\left\{(\boldsymbol{I} - a\boldsymbol{\Theta})\boldsymbol{\Theta}^{-K+1} - (\boldsymbol{\Theta} - a\boldsymbol{I})\boldsymbol{\Theta}^K\right\} \\ &\quad + \left(\boldsymbol{I} + \frac{4\lambda}{1-a}\boldsymbol{\Sigma}_Z\boldsymbol{\alpha}\right)(\boldsymbol{I} - \boldsymbol{\Theta})(\boldsymbol{\Theta}^K + \boldsymbol{\Theta}^{-K+1})\Big]^{-1}\boldsymbol{\omega}. \end{aligned} \tag{3.7}$$

For $k = 0, \ldots, K-1$, the optimal execution strategies are given in terms of $\boldsymbol{\xi}_K^*$ as

$$\begin{aligned} \boldsymbol{\xi}_0^* &= \Big[\frac{\lambda}{1-a}\boldsymbol{\alpha}^{-1}\boldsymbol{\Sigma}_\Delta\left\{(\boldsymbol{I} - a\boldsymbol{\Theta})\boldsymbol{\Theta}^{-K+1} - (\boldsymbol{\Theta} - a\boldsymbol{I})\boldsymbol{\Theta}^K\right\} \\ &\quad + \left(\boldsymbol{I} + \frac{4\lambda}{1-a}\boldsymbol{\Sigma}_Z\boldsymbol{\alpha}\right)(\boldsymbol{I} - \boldsymbol{\Theta})\left(\boldsymbol{\Theta}^K + \boldsymbol{\Theta}^{-K+1}\right)\Big](\boldsymbol{I} - \boldsymbol{\Theta}^2)^{-1}\boldsymbol{\xi}_K^* \end{aligned} \tag{3.8}$$

$$\boldsymbol{\xi}_k^* = (\boldsymbol{I} + \boldsymbol{\Theta})^{-1}\left\{(\boldsymbol{I} - a\boldsymbol{\Theta})\boldsymbol{\Theta}^{-K+k} + (\boldsymbol{\Theta} - a\boldsymbol{I})\boldsymbol{\Theta}^{K-k}\right\}\boldsymbol{\xi}_K^*, \quad (k = 1, \ldots, K-1). \tag{3.9}$$

(Proof) See Appendix A.

3.2 Special cases

To extract the effect of the market impact more explicitly, we set $\boldsymbol{\Sigma}_\Delta = \boldsymbol{O}$ to delete the effect caused by the fundamental price movement. It is easy to check that the optimal execution strategy in Theorem 1 is reduced to the following simplified form.

Theorem 2 *When $\boldsymbol{\Sigma}_\Delta = \boldsymbol{O}$, the optimal execution strategy is given by*

$$\begin{aligned} \boldsymbol{\xi}_K^* &= \left[\{(K-1)(1-a) + 2\}\boldsymbol{I} + \frac{4\lambda}{1-a}\boldsymbol{\Sigma}_Z\boldsymbol{\alpha}\right]^{-1}\boldsymbol{\omega}, \\ \boldsymbol{\xi}_0^* &= \left(\boldsymbol{I} + \frac{4\lambda}{1-a}\boldsymbol{\Sigma}_Z\boldsymbol{\alpha}\right)\boldsymbol{\xi}_K^*, \\ \boldsymbol{\xi}_k^* &= (1-a)\boldsymbol{\xi}_K^*, \quad (k = 1, \ldots, K-1). \end{aligned}$$

A remarkable feature in this case is that the optimal strategy are the same except for $k = 0$ and $k = K$. Optimal strategies of the same type have been found in Alfonsi *et al.* [2009] for single asset case with more general market impact function. See also Obizhaeva and Wang [2005].

When there is only a single asset, the optimal execution strategy can be further simplified. Let

$$\lambda_\Delta = \frac{\lambda\sigma_\Delta^2}{\alpha(1-a)}, \quad \lambda_Z = \frac{4\lambda\alpha\sigma_Z^2}{1-a}, \quad \gamma = \frac{(1-a)^2\lambda_\Delta}{a\lambda_\Delta + \lambda_Z + a + 1}, \quad \theta = \frac{\gamma + 2 + \sqrt{\gamma^2 + 4\gamma}}{2},$$

then, we have the following.

Theorem 3 *For a single asset case, the optimal execution strategy is given as follows:*
(1) When $\lambda\Sigma_\Delta \neq 0$,

$$\begin{aligned}
\xi_0^* &= \{\lambda_\Delta\phi + (\lambda_Z + 1)\psi\}\varphi, \\
\xi_k^* &= \left\{(\theta - a)\theta^{K-k} + (1 - a\theta)\theta^{-K+k}\right\}\varphi, \quad (k = 1, \ldots, K-1), \\
\xi_K^* &= (\theta + 1)\varphi,
\end{aligned}$$

where $\phi = \dfrac{(\theta - a)\theta^K - (1 - a\theta)\theta^{-K+1}}{\theta - 1}$, $\psi = \theta^K + \theta^{-K+1}$, *and* $\varphi = \dfrac{\omega}{(\lambda_\Delta + 1)\phi + (\lambda_Z + 1)\psi}$.
(2) When $\lambda\Sigma_\Delta = 0$,

$$\begin{aligned}
\xi_0^* &= \frac{\lambda_Z + 1}{\lambda_Z + 2 + (K-1)(1-a)}\omega, \\
\xi_k^* &= \frac{1-a}{\lambda_Z + 2 + (K-1)(1-a)}\omega, \quad (k = 1, \ldots, K-1), \\
\xi_K^* &= \frac{1}{\lambda_Z + 2 + (K-1)(1-a)}\omega.
\end{aligned}$$

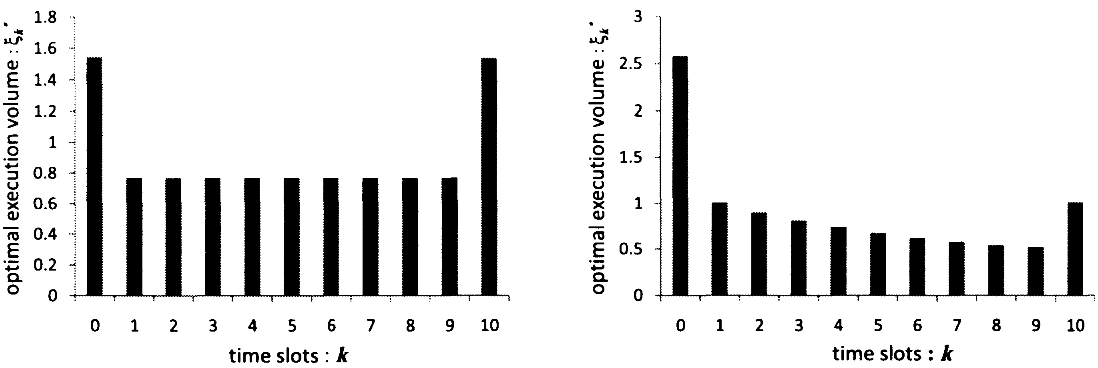
4 Properties of the optimal execution strategy

4.1 Properties in the single asset case

We analyze the marginal properties of the optimal execution strategy in Theorem 3 in this section.

First, we consider the case $\lambda\Sigma_\Delta = 0$. Figure 2 (a) displays a typical optimal execution schedule when $K = 10$ and $\omega = 10$. The optimal execution volumes at the first and the last execution are larger than those in the other time slots, while the volumes at the intermediate slots $t_1, \ldots, t_{K-1}$ are all equivalent. This optimal schedule is governed by the following three factors. The first factor is the temporal market impact $\alpha\xi_k^2/2$, from which the equal distribution of orders becomes optimal[3] . The second factor is the recursive market impact, from which a relatively slower execution becomes optimal since an execution in earlier time slots increases the cost of the following executions as we discussed in Section 2.1. The third factor is the assumption that the order book is initially in the stationary state, from which the investor has incentive to

[3] The temporal market impact, $\alpha\xi_k^2/2$, is the difference between the cost of buying ξ of an asset from the price p witout any market impact, $p\xi$, and that with the market impact, $(p + p + \alpha\xi)\xi/2$, based on our model setting.



(a) $\lambda\Sigma_\Delta = 0$ (b) $\lambda\Sigma_\Delta \neq 0$

Figure 2: A typical optimal execution schedule for a single asset

Table 1: Comparative statics when $\lambda\Sigma_\Delta = 0$

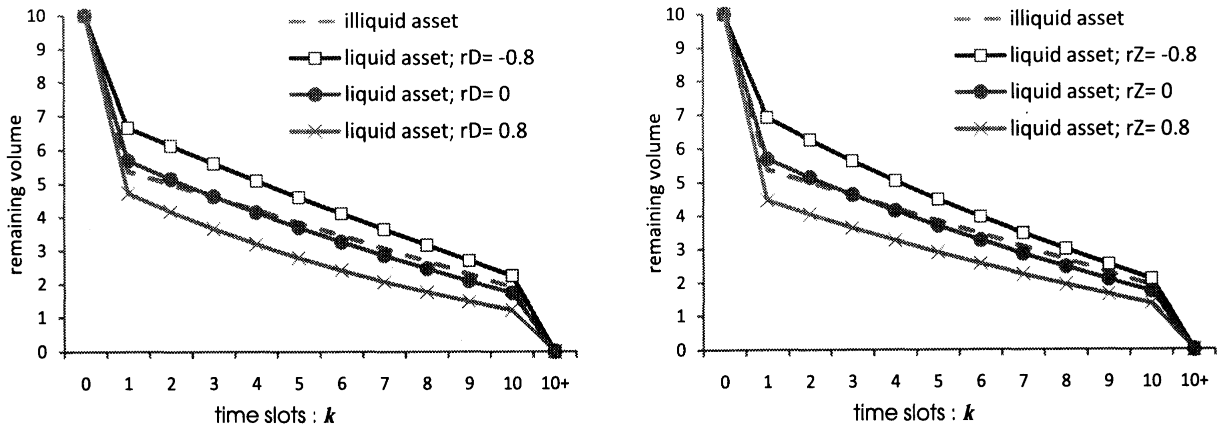
$\partial\xi_i^*/\partial x$	$t = 0$	$t = t_1, \ldots, t_{K-1}$	$t = T$
$x:\ a$	+	−	indefinite
$x:\ \alpha, \lambda, \Sigma_Z$	+	−	−

Note: "±" indicates the sign of $\partial\xi_i^*/\partial x$.

increase the purchase at the first time slot since the execution at time 0 entails the smallest cost. Those factors makes the equal distribution of orders at intermediate time slots, $t = t_1, \ldots, t_{K-1}$, while the execution volume at the first and the last time slots, $t = 0$ and $t = T$, are higher than those at the intermediate slots.

Table 1 summarizes the sign of derivatives of ξ_k^* with respect to each parameter a, α, λ, and σ_Z as comparative statics, showing whether the optimal volume increases or decreases at each time slot when those parameters rise. We see, from the table, that the faster execution becomes optimal as parameters α, λ, or σ_Z rise. This is mainly because the first and the third factors dominate the cost when liquidity decreases or volatility rises. However, the direction of the last execution volume with varying a is indefinite depending on the balance of dominance between the second and the third factors.

Second, we consider the case of $\lambda\Sigma_\Delta \neq 0$. Figure 2 (b) displays a typical optimal schedule in this case. Compare to the previous $\lambda\Sigma_\Delta = 0$ case, the optimal volumes at the intermediate time slots are not flat, hence the optimal execution velocity rises due to the uncertainty in the fundamental price. We consider this to be the fourth factor concerning liquidity fluctuation. We numerically investigate the case since the direction of ξ_k^* is not as simple as that in the previous $\lambda\Sigma_\Delta = 0$ case. We detect that, while ξ_k^* $(k = 0, \ldots, K-1)$ moves down when some parameters are very small, the overall feature is almost the same as $\lambda\Sigma_\Delta = 0$ case; the faster the execution should be, the larger the parameters a, α, λ, σ_Z, or Σ_Δ are. In other words, the investor should, in general, execute faster as the resilience increases, as liquidity drops, as the volatility increases, or as the investor becomes risk averse, that are consistent with intuitive behavior of



(a) Remaining volume vs asset correlation r_D (b) Remaining volume vs liquidity correlation r_Z

Figure 3: Optimal remaining volume with varying correlation parameters

investors.

It is easily seen that the optimal strategy in Theorem 2 does not violate no-market-manipulation condition, because the optimal strategy is composed only of buy orders, i.e., $\xi_k^* > 0$ for $k = 0, \ldots, K$. Since $\theta > 1$,

$$\phi > \frac{(\theta - a) - (1 - a\theta)}{\theta - 1} = 1 + a > 0,$$

hence $\xi_0^* > 0$ is proved. $\xi_k^* > 0$ can be proved in a similar manner.

4.2 Properties in the multi-asset case

We focus on the joint property of the optimal execution strategy in Theorem 1 in this section. The joint property is determined by the four parameters in our model: the asset correlation r_D implicit in $\mathbf{\Sigma}_\Delta$, the liquidity correlation r_Z in $\mathbf{\Sigma}_Z$, the difference in the total volume ω_i, and the difference in the liquidity α_i. Since the correlation parameters mostly reflect the joint property, we analyze numerically the effect of the optimal strategy by varying correlations. We only analyze two assets case here, with one liquid asset where $\alpha = 0.1$ and the other illiquid asset which has relatively low liquidity with $\alpha = 10$, but it can be generalized to the case of more than three assets.

Figure 3 displays the optimal remaining volume of the liquid and the illiquid assets with varying correlations. The left panel (a) shows the optimal remaining volume with varying liquidity correlation, and the right panel (b) shows that with varying asset correlation. Other parameters are set to $T = 10$, $\lambda = 0.7$, $\boldsymbol{\omega} = (10, 10)^\top$, and $a = 0.8$. We also set the marginal standard deviation of both the fundamental price and the liquidity to be 0.1 for both assets.

As seen from Figure 3, it is optimal to buy the liquid asset faster as correlation coefficient increases, while the optimal execution schedule of the illiquid asset stays almost unchanged. This result indicates that the execution cost should be controlled by the liquid asset only. It is intuitively interpreted that investors can reduce the volatility and the liquidity risk by buying or selling a liquid asset which produce a smaller market impact faster than an illiquid asset, when they know a positive correlation among the asset movement in advance. However, this is not the case when assets are negatively correlated, since buying one asset faster negatively affects the price of the other asset, that inherits higher risk afterward.

We have the other type of solutions, which generate a round-trip trade of the liquid asset, while the optimal trade does not include the round trip trade in the single asset case as proved in Section 4.1. This occurs because the joint parameter, the liquidity/volatility correlations, are given constants in our model, in other words, the investor knows the correlation in advance. Suppose the investor knows that the liquidity of the asset i and j are positively correlated. The investor realizes that, if he/she sells the asset i to increase M_t^i, M_t^j is likely to move up hence the price of the asset j move down. This effect results in the reduction of the cost of buying the asset j at later time slots, and the sold volume of the asset i is bought back afterward. This mechanism causes the round trip trade and may violate no-market-manipulation condition discussed in Gatheral [2009].

5 Summary

We developed the multivariate market liquidity model and derived the explicit solution of an optimal execution strategy under both the liquidity risk and the volatility risk. The market liquidity is developed as a queue on an order book waiting for an execution. The mean-variance problem was then solved that minimizes the tradeoff between the market impact and the volatility/liquidity risk, and we got an optimal execution strategy in an analytical form.

Our model and the optimal execution strategy allow us to understand the property of the execution schedule of a risk averse investor under fluctuating liquidity. The investor tries to execute faster as the market become less liquid, as the volatility of liquidity or price increases, or as the investor is more risk averse. Those findings remain consistent with intuitive behavior of investors, and also with typical algorithmic trading strategies such as implementation shortfall, volume peg, or volume weighted average price.

We further continue detailed analysis of the optimal execution strategy in the multi-asset case. Especially, we should examine more the relationship of the optimal strategy to the market manipulation. We should also contemplate the generalization of the shape of the order book. Obtainig a dynamic execution schedule by solving a dynamic problem is also our challenge.

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A Proof of Theorem 1

In this appendix, we will give a concise proof of Theorem 1, though the detailed calculations are omitted due to the space limit. We define an operator $\mathcal{D}$ by

$$\mathcal{D}x_k = ax_{k-1} - (1+a)^2 x_k + 2(1+a+a^2)x_{k+1} - (1+a)^2 x_{k+2} + ax_{k+3}.$$

By some algebras, $\mathcal{D}$ is shown to satisfy

$$\mathcal{D}\left\{\sum_{s=0}^{k-1} a^{k-s}\boldsymbol{\xi}_s + \sum_{s=k}^{K} a^{s-k}\boldsymbol{\xi}_s\right\} = (1-a^2)(-\boldsymbol{\xi}_k + 2\boldsymbol{\xi}_{k+1} - \boldsymbol{\xi}_{k+2}), \tag{A.1}$$

$$\mathcal{D}\left\{\sum_{s=1}^{k}\sum_{u=s}^{K}\boldsymbol{\xi}_u\right\} = -a\boldsymbol{\xi}_k + (1+a^2)\boldsymbol{\xi}_{k+1} - a\boldsymbol{\xi}_{k+2}, \tag{A.2}$$

$$\mathcal{D}\left\{\sum_{s=1}^{k} a^{k-s}\sum_{u=s}^{K} a^{u-s}\boldsymbol{\xi}_u\right\} = -\boldsymbol{\xi}_k + 2\boldsymbol{\xi}_{k+1} - \boldsymbol{\xi}_{k+2}. \tag{A.3}$$

Applying eqs.(A.1)~(A.3) to eq.(3.2), we obtain

$$\boldsymbol{A}\boldsymbol{\xi}_{k+2} - \{2\boldsymbol{A} + (1-a)^2\lambda\boldsymbol{\Sigma}_\Delta\}\boldsymbol{\xi}_{k+1} + \boldsymbol{A}\boldsymbol{\xi}_k = \boldsymbol{0}, \quad (k = 1, \ldots, K-3), \tag{A.4}$$

where $\boldsymbol{0}$ denotes a zero vector. By premultiplying $\boldsymbol{R}^{-1}\boldsymbol{A}^{-1}$ to eq.(A.4), and using eq.(3.6), we get

$$\overline{\boldsymbol{\xi}}_{k+2} - (2\boldsymbol{I} + \boldsymbol{\Gamma})\overline{\boldsymbol{\xi}}_{k+1} + \overline{\boldsymbol{\xi}}_k = \boldsymbol{0}, \quad (k = 1, \ldots, K-3), \tag{A.5}$$

where $\overline{\boldsymbol{\xi}}_k = (\overline{\xi}_k^1, \ldots, \overline{\xi}_K^N)^\top = \boldsymbol{R}^{-1}\boldsymbol{\xi}_k$. Since $\boldsymbol{\Gamma}$ is diagonal, eq.(A.5) can be written in elementwise as

$$\overline{\xi}_{k+2}^i - (\gamma_i + 2)\overline{\xi}_{k+1}^i + \overline{\xi}_k^i = 0, \quad (i = 1, \ldots, N,\ k = 1, \ldots, K-3). \tag{A.6}$$

A general solution to eq.(A.6) is $\overline{\xi}_k^i = c_i\theta_i^k + d_i\theta_i^{-k}$, which is expressed in a matrix form as

$$\overline{\boldsymbol{\xi}}_k = \boldsymbol{\Theta}^k\boldsymbol{c} + \boldsymbol{\Theta}^{-k}\boldsymbol{d}, \quad (k = 1, \ldots, K-1). \tag{A.7}$$

Here, $\boldsymbol{c} = (c_1, \ldots, c_N)^\top$ and $\boldsymbol{d} = (d_1, \ldots, d_N)^\top$ are unknown coefficients determined by boundary conditions.

To obtain $\boldsymbol{c}$ and $\boldsymbol{d}$, we define another difference operator $\widetilde{\mathcal{D}}x_k = -ax_{k-1} + (a+1)x_k - x_{t+1}$. Applying $\widetilde{\mathcal{D}}$ to eq.(3.2) at $k = K - 1$ yields

$$\boldsymbol{A}\boldsymbol{\xi}_{K-1} = (1-a)\{\boldsymbol{A} + (1-a)\lambda\boldsymbol{\Sigma}_\Delta\}\boldsymbol{\xi}_K. \tag{A.8}$$

Thus, premultiplying $\boldsymbol{R}^{-1}\boldsymbol{A}^{-1}$ to eq.(A.8) and rearranging terms, we get the boundary condition

$$\bar{\boldsymbol{\xi}}_{K-1} = \{(1-a)\boldsymbol{I} + \boldsymbol{\Gamma}\}\bar{\boldsymbol{\xi}}_K. \tag{A.9}$$

Similarly, premultiplying $\boldsymbol{R}^{-1}\boldsymbol{A}^{-1}$ to eq.(3.2) at $k = K - 2$ yields

$$\bar{\boldsymbol{\xi}}_{K-2} = (\boldsymbol{I} + \boldsymbol{\Gamma})\bar{\boldsymbol{\xi}}_{K-1} + \boldsymbol{\Gamma}\bar{\boldsymbol{\xi}}_K. \tag{A.10}$$

Substituting eq.(A.7) into eq.(A.9) and eq.(A.10) implies

$$\begin{bmatrix} \boldsymbol{\Theta}^{-1} & \boldsymbol{\Theta} \\ \boldsymbol{\Theta}^{-2} & \boldsymbol{\Theta}^2 \end{bmatrix} \begin{bmatrix} \boldsymbol{\Theta}^K \boldsymbol{c} \\ \boldsymbol{\Theta}^{-K}\boldsymbol{d} \end{bmatrix} = \begin{bmatrix} (1-a)\boldsymbol{I} + \boldsymbol{\Gamma} \\ (\boldsymbol{I}+\boldsymbol{\Gamma})\{(1-a)\boldsymbol{I} + \boldsymbol{\Gamma}\} + \boldsymbol{\Gamma} \end{bmatrix} \bar{\boldsymbol{\xi}}_K.$$

After some manipulations, we obtain

$$\begin{bmatrix} \boldsymbol{c} \\ \boldsymbol{d} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Theta}^{-K}(\boldsymbol{I} - a\boldsymbol{\Theta}) \\ \boldsymbol{\Theta}^K(\boldsymbol{\Theta} - a\boldsymbol{I}) \end{bmatrix} (\boldsymbol{I}+\boldsymbol{\Theta})^{-1}\boldsymbol{R}^{-1}\boldsymbol{\xi}_K,$$

where we use the relation

$$\begin{bmatrix} \boldsymbol{\Theta}^2 & -\boldsymbol{\Theta} \\ -\boldsymbol{\Theta}^{-2} & \boldsymbol{\Theta}^{-1} \end{bmatrix} \begin{bmatrix} (1-a)\boldsymbol{I} + \boldsymbol{\Gamma} \\ (\boldsymbol{I}+\boldsymbol{\Gamma})\{(1-a)\boldsymbol{I} + \boldsymbol{\Gamma}\} + \boldsymbol{\Gamma} \end{bmatrix} = \begin{bmatrix} (\boldsymbol{I} - \boldsymbol{\Theta}^{-1})(\boldsymbol{I} - a\boldsymbol{\Theta}) \\ (\boldsymbol{I} - \boldsymbol{\Theta}^{-1})(\boldsymbol{\Theta} - a\boldsymbol{I}) \end{bmatrix}, \tag{A.11}$$

because $\boldsymbol{\Gamma} = \boldsymbol{\Theta} - 2\boldsymbol{I} + \boldsymbol{\Theta}^{-1}$, and because $\boldsymbol{\Gamma}$ and $\boldsymbol{\Theta}$ are commutative. Eq.(A.7) and eq.(A.11) prove

$$\boldsymbol{\xi}_k = (\boldsymbol{I}+\boldsymbol{\Theta})^{-1}\left\{(\boldsymbol{I} - a\boldsymbol{\Theta})\boldsymbol{\Theta}^{-K+k} + (\boldsymbol{\Theta} - a\boldsymbol{I})\boldsymbol{\Theta}^{K-k}\right\}\boldsymbol{\xi}_K, \quad (k = 1, \ldots, K-1). \tag{A.12}$$

On the other hand, subtracting eq.(3.2) at $k = 1$ from that at $k = 0$, we get another boundary condition

$$\{(1-a)\boldsymbol{\alpha} + \lambda\boldsymbol{\Sigma}_\Delta\}\boldsymbol{\xi}_0 = \lambda\boldsymbol{\Sigma}_\Delta\boldsymbol{\omega} + \{(1-a)\boldsymbol{I} + 4\lambda\boldsymbol{\alpha}\boldsymbol{\Sigma}_Z\boldsymbol{\alpha}\}\sum_{k=1}^{K} a^{k-1}\boldsymbol{\xi}_k. \tag{A.13}$$

Substituting eq.(A.7) with the constraint $\boldsymbol{\omega} = \boldsymbol{\xi}_0 + \sum_{k=1}^K \boldsymbol{\xi}_k$ into eq.(A.13), we obtain

$$\begin{aligned} \boldsymbol{\omega} = & \left[\left(\boldsymbol{I} + \frac{\lambda}{1-a}\boldsymbol{\Sigma}_\Delta\right)\{(\boldsymbol{I} - a\boldsymbol{\Theta})\boldsymbol{\Theta}^{-K+1} - (\boldsymbol{\Theta} - a\boldsymbol{I})\boldsymbol{\Theta}^K\} \right. \\ & \left. + \left(\boldsymbol{I} + \frac{4\lambda}{1-a}\boldsymbol{\Sigma}_Z\boldsymbol{\alpha}\right)(\boldsymbol{I} - \boldsymbol{\Theta})\left(\boldsymbol{\Theta}^K + \boldsymbol{\Theta}^{-K+1}\right)\right](\boldsymbol{I} - \boldsymbol{\Theta}^2)^{-1}\boldsymbol{\xi}_K, \end{aligned}$$

which proves eq.(3.7) for $\boldsymbol{\xi}_K^*$. Eq.(3.9) for $\boldsymbol{\xi}_k^*$ $(k = 1, \ldots, K-1)$ is then easily obtained from eq.(A.12). Finally, we substitute eq.(3.7) and eq.(3.9) into eq.(A.13), then eq.(3.8) for $\boldsymbol{\xi}_0^*$ is obtained after some algebra. This completes the proof of Theorem 1. □