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Non-homogeneous semilinear elliptic equations involving critical Sobolev exponent

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Let $\Omega \subset \mathbf{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$ with $N \geq 3$. We consider the existence of multiple positive solutions of the following semilinear elliptic equations

(1.1)
$$\begin{cases} -\Delta u + \kappa u = u^p + \lambda f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\kappa \in \mathbf{R}$, $\lambda > 0$ are parameters, p is the critical Sobolev exponent $p = (N+2)/(N-2)$, and $f(x)$ is a non-homogeneous perturbation satisfying

(1.2)
$$f \in H^{-1}(\Omega), \quad f \geq 0, \ f \not\equiv 0 \quad \text{a.e. in } \Omega.$$

Since p is a critical Sobolev exponent for which the embedding $W^{1,2}(\Omega) \subset L^{2N/(N-2)}(\Omega)$ is not compact, we encounter serious difficulties in applying variational methods to the problem (1.1).

Let us recall the results for the case $f \equiv 0$;

(1.3)
$$\begin{cases} -\Delta u + \kappa u = u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

In this case, by using the Pohozaev identity, it can be shown that (1.3) admits no nontrivial solutions for each $\kappa \geq 0$, provided that Ω is star-shaped. On the other hand, Brezis and Nirenberg [1] obtained the following results when $\kappa < 0$: let κ_1 be the first eigenvalue of $-\Delta$ with zero Dirichlet condition on Ω; then

(i) if $N \geq 4$, then for every $\kappa \in (-\kappa_1, 0)$, there exists a positive solution;

(ii) if $N = 3$ and Ω is a ball, then there exists a positive solution if and only if $\kappa \in (-\kappa_1, -\kappa_1/4)$.

Let us consider the case where f satisfies (1.2). Tarantello [6] considered the problem with $\kappa = 0$;

$$\text{(1.4)} \qquad \begin{cases} -\Delta u = u^p + \lambda f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and showed that (1.4) has at least two positive solution if λ is small enough. The main idea is to divide the Nehari manifold $\Lambda = \{u \in H_0^1(\Omega) : \langle I'(u), u\rangle = 0\}$ into three parts Λ^+, Λ^- and Λ_0, and to use the Ekeland principle to get one solution for Λ^+ and another solution for Λ^-. We note here that no positive solution exists if λ is sufficiently large.

The existence of two nontrivial solutions for more general problem

$$\begin{cases} -\Delta u = u^p + g(x, u) + \lambda f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $g(x, u)$ is a suitable lower-order perturbation of u^p, was proved by Cao and Zhou [2]. These achievements have been extended to the p-Laplace equation by Chabrouski [3] and Zhou [7], and to more general problems by Squassina [5].

In this paper we will consider the problem (1.1) with $\kappa \in \mathbf{R}$ in the case where f satisfies (1.2), and show that, when $\kappa > 0$, the situation is drastically different between the cases $N = 3, 4, 5$ and $N \geq 6$.

We call a positive minimal solution $\underline{u}_\lambda$ of $(1.1)_\lambda$, if $\underline{u}_\lambda$ satisfies $\underline{u}_\lambda \leq u$ in Ω for any positive solution u of $(1.1)_\lambda$. Our main results are stated as following theorems.

Theorem 1. *Assume that $\kappa > -\kappa_1$. Then there exists $\overline{\lambda} \in (0, \infty)$ such that*

(i) *if $0 < \lambda < \overline{\lambda}$ then the problem $(1.1)_\lambda$ has a positive minimal solution $\underline{u}_\lambda \in H_0^1(\Omega)$. Furthermore, if $0 < \lambda < \dot{\lambda} < \overline{\lambda}$ then $\underline{u}_\lambda < \underline{u}_{\dot{\lambda}}$ a.e. in Ω;*

(ii) *if $\lambda > \overline{\lambda}$ then the problem (1.1) has no positive solution $u \in H_0^1(\Omega)$.*

Remark. There is no positive solution of (1.1) with $\kappa \leq -\kappa_1$. Assume to the contrary that there exists a positive solution u of (1.1) with $\kappa \leq -\kappa_1$. Let ϕ_1 be the eigenfucntion of $-\Delta$ corresponding to κ_1 with $\phi_1 > 0$ on Ω. Then we have

$$0 = \int_\Omega \nabla u \cdot \nabla\phi_1 - \kappa_1 u\phi_1 dx \geq \int_\Omega \nabla u \cdot \nabla\phi_1 + \kappa u\phi_1 dx = \int_\Omega u^p\phi_1 + \lambda f\phi_1 dx > 0.$$

This is a contradiction.

We consider the existence of the solutions of (1.1) at the extremal value $\lambda = \overline{\lambda}$, so called extremal solutions.

Theorem 2. *Let $\kappa > -\kappa_1$. If $\lambda = \overline{\lambda}$ then the problem* (1.1) *has a unique positive solution in $H_0^1(\Omega)$.*

Next, let us consider the existence and nonexistence of second positive solutions to (1.1) for $0 < \lambda < \overline{\lambda}$.

Theorem 3. *Assume that either* (i) *or* (ii) *holds.*

$$\text{(i) } \kappa \in (-\kappa_1, 0] \text{ and } N \geq 3; \qquad \text{(ii) } \kappa > 0 \text{ and } N = 3, 4, 5.$$

If $0 < \lambda < \overline{\lambda}$ then (1.1) *has a positive solution $\overline{u}_\lambda \in H_0^1(\Omega)$ satisfying $\overline{u}_\lambda > \underline{u}_\lambda$.*

Theorem 4. *Assume that $\kappa > 0$ and $N \geq 6$.*

(i) *There exists $\lambda^* = \lambda^*(\kappa) \in (0, \overline{\lambda})$ such that if $\lambda^* < \lambda < \overline{\lambda}$ then the problem* (1.1) *has a positive solution $\overline{u}_\lambda \in H_0^1(\Omega)$ satisfying $\overline{u}_\lambda > \underline{u}_\lambda$.*

(i) *Let $\Omega = \{x \in \mathbf{R}^N : |x| < R\}$ with some $R > 0$, and let $f = f(|x|)$ be radially symmetric about the origin. Assume that $f \in C^\alpha([0, R])$ with some $0 < \alpha < 1$, and $f(r)$ is nonincreasing in $r \in (0, R)$. Then there exists $\lambda_* \in (0, \lambda^*)$ such that $(1.1)_\lambda$ has a unique positive solution $\underline{u}_\lambda$ for $\lambda \in (0, \lambda_*]$.*

In the proof of Theorem 1, we will employ the bifurcation results and the comparison argument for solutions of (1.1) to obtain the minimal solutions. We will prove Theorem 2 by establishing a priori bound for the solutions of (1.1) at $\lambda = \overline{\lambda}$.

In order to find a second positive solution of (1.1), we introduce the problem

$$-\Delta v + \kappa v = (v + \underline{u}_\lambda)^p - \underline{u}_\lambda^p \quad \text{in } \Omega, \quad v \in H_0^1(\Omega), \tag{1.5}$$

where $\underline{u}_\lambda$ is the minimal positive solution of (1.1) for $\lambda \in (0, \overline{\lambda})$ obtained in Theorem 1. In fact, assume that (1.5) has a positive solution v, and put $\overline{u}_\lambda = v + \underline{u}_\lambda$. Then $\overline{u}_\lambda \in H_0^1(\Omega)$ and solves (1.1) and satisfies $\overline{u}_\lambda > \underline{u}_\lambda$ in Ω. In the proof of Theorem 3, we will show the existence of solutions of (1.5) by using a variational method. To this end we define the corresponding variational functional of (1.5) by

$$I_\kappa(v) = \frac{1}{2}\int_\Omega \left(|\nabla v|^2 + \kappa v^2\right) dx - \int_{\mathbf{R}^N} G(v, \underline{u}_\lambda) dx$$

for $v \in H_0^1(\Omega)$, where

$$G(t, s) = \frac{1}{p+1}(t_+ + s)^{p+1} - \frac{1}{p+1}s^{p+1} - s^p t_+.$$

It is easy to see that $I_\kappa : H_0^1(\Omega) \to \mathbf{R}$ is C^1 and the critical point $v_0 \in H_0^1(\Omega)$ satisfies

$$\int_\Omega (\nabla v_0 \cdot \nabla \psi + \kappa v_0 \psi + g(v_0, \underline{u}_\lambda)\psi)\, dx = 0$$

for any $\psi \in H_0^1(\Omega)$, where

$$g(t,s) = (t_+ + s)^p - s^p.$$

Denote by S the best Sobolev constant of the embedding $H_0^1(\Omega) \subset L^{p+1}(\Omega)$, which is given by

$$S = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\left(\int_\Omega |u|^{p+1} dx\right)^{2/(p+1)}}.$$

We will obtain Theorem 3 as a consequence of the following two propositions.

Proposition 5. *Let $\lambda \in (0, \lambda^*)$. Assume that there exists $v_0 \in H_0^1(\Omega)$ with $v_0 \geq 0$, $v_0 \not\equiv 0$ such that*

$$\sup_{t>0} I_\kappa(tv_0) < \frac{1}{N} S^{N/2}. \tag{1.6}$$

Then there exists a positive solution $v \in H_0^1(\Omega)$ of (1.5).

Proposition 6. *Assume that either* (i) *or* (ii) *holds.*

(i) $\kappa \in (-\kappa_1, 0]$ *and* $N \geq 3$; (ii) $\kappa > 0$ *and* $N = 3, 4, 5$.

Then there exists a positive function $v_0 \in H_0^1(\Omega)$ such that (1.6) *holds.*

In the proof of Proposition 5, we will derive some estimates to establish inequalities relating certain minimizing sequences. In order to prove Proposition 6, for $\varepsilon > 0$, we will set

$$u_\varepsilon(x) = \frac{\phi(x)}{(\varepsilon + |x|^2)^{(N-2)/2}},$$

where $\phi \in C_0^\infty(\mathbf{R}^N)$, $0 \leq \phi \leq 1$, is a cut off function, and will show that (1.6) holds with $v_0 = u_\varepsilon$ for sufficiently small $\varepsilon > 0$.

In the proof of Theorem 4 (ii), we will verify the nonexistence of positive solutions of (1.5) in the radial case by the Pohozaev type argument for the associated ODE. In fact, by [4], the solution v of (1.5) must be radially symmetric, and $v = v(r)$, $r = |x|$, satisfies the problem of the following ordinary differential equation

$$\begin{cases} (r^{N-1}v_r)_r - \kappa r^{N-1}v + r^{N-1}g(v, \underline{u}_\lambda) = 0, & 0 < r < R, \\ v_r(0) = v(R) = 0. \end{cases} \tag{1.7}$$

For the solution v to (1.7), we will obtain the following Pohozaev type identity:

$$\int_0^R r^{N-1}\left[\frac{2N}{N-2}G(u, \underline{u}_\lambda) - g(u, \underline{u}_\lambda)u\right] dr + \frac{2}{N-2}\int_0^R r^N G_s(u, \underline{u}_\lambda)\underline{u}_\lambda' dr$$
$$+\frac{2\kappa}{N-2}\int_0^\infty r^{N-1}u^2 dr = \frac{1}{N-2}R^N v_r(R)^2.$$

In the proofs of Theorems 2, 3 and 4, the results concerning the eigenvalue problems to the linearized equations around the minimal solutions

$$-\Delta\phi + \phi = \mu p(\underline{u}_\lambda)^{p-1}\phi \quad \text{in } \Omega, \quad \phi \in H_0^1(\Omega).$$

play a crucial role.

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