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Author(s)	ONO, Takayori
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On solutions of quasilinear elliptic equations with general structure

Takayori ONO (小野太幹)

Fukuyama University (福山大学)

§1. Introduction and preliminaries

Let G be an open set in \mathbf{R}^N ($N \geq 2$) and $1 < p < N$. We consider quasi-linear second order elliptic differential equations of the form

$$(E_T) \quad -\operatorname{div} \mathcal{A}(x, \nabla u) + \mathcal{B}(x, u) = T$$

in G . Here, T is a distribution, $\mathcal{A} : \mathbf{R}^N \times \mathbf{R}^N \rightarrow \mathbf{R}^N$ and $\mathcal{B} : \mathbf{R}^N \times \mathbf{R} \rightarrow \mathbf{R}$ satisfy the following conditions :

- (A.1) $x \mapsto \mathcal{A}(x, \xi)$ is measurable on \mathbf{R}^N for every $\xi \in \mathbf{R}^N$ and $\xi \mapsto \mathcal{A}(x, \xi)$ is continuous for a.e. $x \in \mathbf{R}^N$;
- (A.2) $\mathcal{A}(x, \xi) \cdot \xi \geq \alpha_1 |\xi|^p$ for all $\xi \in \mathbf{R}^N$ and a.e. $x \in \mathbf{R}^N$ with a constant $\alpha_1 > 0$;
- (A.3) $|\mathcal{A}(x, \xi)| \leq \alpha_2 |\xi|^{p-1}$ for all $\xi \in \mathbf{R}^N$ and a.e. $x \in \mathbf{R}^N$ with a constant $\alpha_2 > 0$;
- (B.1) $x \mapsto \mathcal{B}(x, t)$ is measurable on \mathbf{R}^N for every $t \in \mathbf{R}$ and $t \mapsto \mathcal{B}(x, t)$ is continuous for a.e. $x \in \mathbf{R}^N$;
- (B.2) For any bounded open set D in \mathbf{R}^N , there is a constant $\alpha_3(D) \geq 0$ such that $|\mathcal{B}(x, t)| \leq \alpha_3(D)(|t|^{p-1} + 1)$ for all $t \in \mathbf{R}$ and a.e. $x \in D$;

A prototype of the equation (E_T) is

$$-\operatorname{div} (|\nabla u|^{p-2} \nabla u) + b|u|^{p-2}u = T$$

with a locally bounded function b in G .

As a matter of fact, we treat the following two topics: (i) Hölder continuity of a solution of the equation of (E_T) (section 2); (ii) Integrability of the gradients of a solution of the equation of (E_T) (section 3).

Throughout this paper, we use some standard notation without explanation.

§2. Hölder continuity of a solution

In this section, we suppose the following monotoneity conditions on $\mathcal{A} : \mathbf{R}^N \times \mathbf{R}^N \rightarrow \mathbf{R}^N$ and $\mathcal{B} : \mathbf{R}^N \times \mathbf{R} \rightarrow \mathbf{R}$:

$$(A.4) \quad (\mathcal{A}(x, \xi_1) - \mathcal{A}(x, \xi_2)) \cdot (\xi_1 - \xi_2) > 0 \text{ whenever } \xi_1, \xi_2 \in \mathbf{R}^N, \xi_1 \neq \xi_2, \\ \text{for a.e. } x \in \mathbf{R}^N;$$

$$(B.3) \quad t \mapsto \mathcal{B}(x, t) \text{ is nondecreasing on } \mathbf{R} \text{ for a.e. } x \in \mathbf{R}^N.$$

We consider elliptic quasi-linear equations of the form

$$(E_0) \quad -\operatorname{div} \mathcal{A}(x, \nabla u(x)) + \mathcal{B}(x, u(x)) = 0.$$

For an open subset G of \mathbf{R}^N , we consider the Sobolev spaces $W^{1,p}(G)$, $W_0^{1,p}(G)$ and $W_{\text{loc}}^{1,p}(G)$.

Let G be an open subset of \mathbf{R}^N . A function $u \in W_{\text{loc}}^{1,p}(G)$ is said to be a (weak) *solution* of (E_0) in G if

$$\int_G \mathcal{A}(x, \nabla u) \cdot \nabla \varphi \, dx + \int_G \mathcal{B}(x, u) \varphi \, dx = 0$$

for all $\varphi \in C_0^\infty(G)$.

A continuous solution of (E_0) in an open subset G is called $(\mathcal{A}, \mathcal{B})$ -*harmonic* in G . For any $(\mathcal{A}, \mathcal{B})$ -harmonic functions, the following locally Hölder continuity estimate holds ([7; Theorem 4.7] or [8; Proposition 2.1]) :

Proposition 1. *Let G be an open set. Then there are constants c and $0 < \lambda \leq 1$ such that for $B(x_0, R) \Subset G$ and for every $(\mathcal{A}, \mathcal{B})$ -harmonic function h in G with $|h| \leq L$ in $B(x_0, R)$,*

$$\operatorname{osc}(h, B(x_0, r)) \leq c \left(\frac{r}{R} \right)^\lambda (\operatorname{osc}(h, B(x_0, R)) + R),$$

whenever $0 < r < R \leq 1$. Here c depends only on $N, p, \alpha_1, \alpha_2, \alpha_3(G)$ and L and λ depends only on N, p, α_1, α_2 and $\alpha_3(G)$.

In the case of $\mathcal{A}(x, \xi) = |\xi|^{p-2}\xi$ and $\mathcal{B} = 0$, namely for the p -Laplace equation, we can choose $\lambda = 1$ ([3; Lemma 2.1]).

Suppose that ν is a signed Radon measure on G . Hölder continuity of a solution to the equation of the form

$$(E_\nu) \quad -\operatorname{div} \mathcal{A}(x, \nabla u(x)) + \mathcal{B}(x, u(x)) = \nu$$

was investigated in [9], [2] and [3]. In [6], Kilpeläinen and Zhong showed that, for the equation

$$(1) \quad -\operatorname{div} \mathcal{A}(x, \nabla u(x)) = \nu$$

and for the case ν is a nonnegative Radon measure, if there exist constants $M > 0$ and $0 < \beta < \lambda$ with

$$\nu(B(x_0, r)) \leq M r^{N-p+\beta(p-1)}$$

whenever $B(x, 3r) \subset G$, where λ is the number in Proposition 1 above, then a solution to the equation (1) is Hölder continuous with the same exponent β . We can extend this result to the case of the equation (E_ν) and of the signed Radon measure ([8]).

Theorem 1. *Let G be an open set and $u \in W_{\text{loc}}^{1,p}(G)$ be a solution of (E_ν) in G . If ν is a signed Radon measure on G such that there exist constants $M > 0$ and $0 < \beta < \lambda$, where $\lambda = \lambda(N, p, \alpha_1, \alpha_2, \alpha_3(G)) > 0$ is the number in Proposition 1 above, with*

$$|\nu|(B(x, r)) \leq M r^{N-p+\beta(p-1)}$$

whenever $B(x, 3r) \subset G$, then u is locally Hölder continuous in G with the exponent β .

§3. Global integrability of the gradient of a solution

In this section, we treat the higher integrability of the gradient of a solution of (E_T) in a bounded open set G . In [9], Rakotoson and Ziemer showed the local integrability of the gradient of a solution of (E_T) with $T \in W_{\text{loc}}^{-1,p'+\delta}(G)$ for some $\delta > 0$. In [4], Kilpeläinen and Koskela treated the global integrability of the gradient of a solution of the equation (1) in the previous section under the condition the complement of G satisfies the uniformly thickness.

A set E is said to be *uniformly p -thick* with constants c_0 and $r_0 > 0$, if

$$\text{cap}_p(\overline{B}(x_0, r) \cap E, B(x_0, 2r)) \geq c_0 \text{cap}_p(\overline{B}(x_0, r), B(x_0, 2r))$$

for all $x_0 \in E$ and for all $0 < r < r_0$. For the notion of p -capacity cap_p , we refer to [1; Chapter 2].

We can show the following global integrability of the gradient of a solution of (E_T) .

Theorem 2. *Suppose that G is a bounded open set, $\mathbb{C}G$ is uniformly p -thick with constants $c_0, r_0 > 0$ and u is a solution of (E_T) in G such that $u - \theta \in W_0^{1,p}(G)$. Then there exists $\delta_0 = \delta(N, p, \alpha_1, \alpha_2, \alpha_3(G), c_0)$ such that $|\nabla u| \in L^{p+\delta}(G)$ whenever $T \in W^{-1,p'+\delta}(G)$ and $|\nabla \theta| \in L^{p+\delta}(G)$ for $0 < \delta < \delta_0$.*

Remark 1. The uniformly thickness condition cannot be suppressed in Theorem 2. (see [4; Remark 3.3])

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