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# On the principal series representation of $SU(2,2)$

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## 1 Introduction

Let $G$ denote the the special unitary group $SU(2,2)$. In the paper, we will deal with the principal series representations of $G$ which are parabolically induced by the minimal parabolic subgroup $P_{min}$ with Langlands decomposition $P_{min}=MAN$;

$$\pi_{\sigma,\nu}=\mathrm{Ind}_{P_{min}}^{G}(\sigma\otimes e^{\nu+\rho}\otimes 1_N),$$

where $\rho$ is the half sum associated to the root system of the pair $(G,A)$, $\nu$ is a complex valued real linear form on $\mathfrak{a}=Lie(A)$, $\sigma$ is a unitary character of $M$.

Let $\eta$ be a continuous unitary character of $N$. We then have the Jacquet functional $J_{\sigma,\nu}$ on the space of differentiable functions of $L_{\sigma}^{2}(K)$, the representation space of $\pi_{\sigma,\nu}$, such that $J_{\sigma,\nu}(\pi_{\sigma,\nu}(n)f)=\eta(n)J_{\sigma,\nu}(f)$ for any $n\in N$. The functional defines an intertwiner $J$ from $\pi_{\sigma,\nu}|_K$ to $\mathcal{A}_{\eta}(N\backslash G)$ by sending any $v\in\pi_{\sigma,\nu}|_K$ to the function $J_v(g):=J_{\sigma,\nu}(\pi_{\sigma,\nu}(g)v)$, $(g\in G)$. Here the subspace of all $K$-finite vectors of $\pi_{\sigma,\nu}$ is denoted by $\pi_{\sigma,\nu}\mid_K$ and $\mathcal{A}_{\eta}(N\backslash G)$ is the subspace of $C^{\infty}(G)$ consisting of all moderate growth functions $f(g)$ such that $f(ng)=\eta(n)f(g)$ for $n\in N$ and $g\in G$. In fact, $J$ is an intertwiner of $K$ and $\mathfrak{g}$-equivariant, and hence the study of the image of $J$ (the Whittaker model) leads us to the problem of the investigations of the $(\mathfrak{g},K)$-module structure and the functions $J_v(g)$ for certain $K$-types of $\pi_{\sigma,\nu}$.

The main goal of this paper is to describe the above mentioned objects in terms of parameters of the principal series representation $\pi_{\sigma,\nu}$ explicitly. Note that our results are quite similar to that of Ishii [4] and Oda [5], for both $Sp(2,\mathbb{R})$ and $SU(2,2)$ have the same restricted root system.

We also consider a matrix representations of the Knapp-Stein intertwining operator which have been motivated by a result of Goodman-Wallach [2].

## 2 Preliminaries

Let $K$ be the compact group $S(U(2)\times U(2))$. Then $K$ is the maximal compact subgroup of $G$ fixed by the Cartan involution $\theta$ for $G$ given by

$$\theta(g)={}^{t}\bar{g}^{-1},\quad g\in G.$$

We fix the following basis for the 7 dimensional Lie algebra $\mathfrak{k}_{\mathbb{C}}$, the complexification of $\mathfrak{k}=Lie(K)$:

$$h^1=\begin{pmatrix}h&0\\0&0\end{pmatrix},\qquad h^2=\begin{pmatrix}0&0\\0&h\end{pmatrix},\qquad I_{2,2}=\begin{pmatrix}1_2&0\\0&-1_2\end{pmatrix},$$

$$e_{\pm}^{1}=\begin{pmatrix}e_{\pm}&0\\0&0\end{pmatrix},\qquad e_{\pm}^{2}=\begin{pmatrix}0&0\\0&e_{\pm}\end{pmatrix},$$

where $h=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, $e_{+}=\begin{pmatrix}0&1\\0&0\end{pmatrix}$ and $e_{-}=\begin{pmatrix}0&0\\1&0\end{pmatrix}$.

For every $K$-module $V$, it is clear that $I_{2,2} \in \mathfrak{k}_{\mathbb{C}}$ commutes with the action of $K$ on $V$. If $V$ is irreducible, then by Schur's lemma, the operator is a scalar of the identity map.

**Lemma 2.1** *Let $m_1, m_2$ be positive integers and $l$ be an integer. If $m_1 + m_2 + l$ is an even integer, then there is an irreducible $K$-module $(\tau_{[m_1,m_2;\ l]}, V_{m_1 m_2})$ with a basis $\{f_{pq} \mid 0 \le p \le m_1, 0 \le q \le m_2\}$ of $V_{m_1 m_2}$ such that $I_{2,2} f_{pq} = l f_{pq}$ and*

$$h^1(f_{pq}) = (2p - m_1) f_{pq}, \qquad e^1_+(f_{pq}) = (m_1 - p) f_{p+1,q}, \qquad e^1_-(f_{pq}) = p f_{p-1,q},$$
$$h^2(f_{pq}) = (2q - m_2) f_{pq}, \qquad e^2_+(f_{pq}) = (m_2 - q) f_{p,q+1}, \qquad e^2_-(f_{pq}) = q f_{p,q-1}.$$

It follows from the fact that $SU(2) \times SU(2) \times \mathbb{C}^{(1)}$ is a twofold covering of $K$ with the projection given by

$$pr(g_1, g_2; u) = \mathrm{diag}(u g_1, u^{-1} g_2), \ g_1, g_2 \in SU(2), \ u \in \mathbb{C}^{(1)}.$$

## 3 $K$-finite vectors in the principal series

In this section, for each simple $K$-module $\tau \in \hat{K}$, we associate a matrix function $\mathbf{S}^{(\tau)}_{\sigma,\nu}(k)$, $k \in K$, whose entries give a basis for the $\tau$-isotypic component of $\pi_{\sigma,\nu}$. The main feature of this basis is that the both $\mathfrak{g}$ and $K$-actions on $\pi_{\sigma,\nu}\mid_K$ have simple expressions in terms of parameters of given representation. For more details about this theme, we refer to [5] which is our main reference.

**Proposition 3.1** *Let $H(\tau)$ be the $\tau$-isotypic component of $L^2(K)$, and put $\dim(\tau) = n$. There exists a unique square matrix function $\mathbf{S}^{(\tau)}(k)$, $k \in K$, of size $n$ with entries in $H(\tau)$,*

$$\mathbf{S}^{(\tau)}(k) = \begin{bmatrix} f_{11}(k) & \cdots & f_{n1}(k) \\ \vdots & \ddots & \vdots \\ f_{1n}(k) & \cdots & f_{nn}(k) \end{bmatrix} = \{f_{ij}(k)\}_{1 \le i,j \le n},$$

*satisfying the following two conditions:*

1. $\mathbf{S}^{(\tau)}(1_K) = \mathrm{diag}(1, ..., 1) \in M_n(\mathbb{C})$,
2. *For each $\alpha$ $(1 \le \alpha \le n)$, the set $\{f_{\alpha 1}(k), ..., f_{\alpha n}(k)\}$ is a basis for $\tau$ as in Lemma 2.1. Moreover, we have*

$$H(\tau) = \bigoplus_{\alpha} W_\alpha,$$

*where $W_\alpha$ denotes the space spanned by $f_{\alpha 1}(k), ..., f_{\alpha n}(k)$.*

*Proof.* The existence of the matrix function is similar to that of [5]. We consider the uniqueness. Assume that there exist two matrices $\mathbf{F}^{(\tau)}(k) = \{f_{ij}(k)\}$ and $\mathbf{G}^{(\tau)}(k) = \{g_{ij}(k)\}$ as required. Denote by $F_\alpha$ the isomorphism between $\tau$ and the space spanned by $\{f_{\alpha j}(k), ..., f_{\alpha n}(k)\}$. Similarly, we define $G_\alpha$ for the $\alpha$-th column of $\mathbf{G}^{(\tau)}(k)$. As a result, we obtain two ordered bases $\{F_\alpha\}_\alpha$ and $\{G_\alpha\}_\alpha$ for the $n$-dimensional vector space $\mathrm{Hom}_K(\tau, H(\tau))$. Then we have the $n$ by $n$ matrix $A = \{a_{\alpha\beta}\}$, the change of coordinate matrix, such that

$$F_\alpha = \sum_\beta a_{\alpha\beta} G_\beta.$$

For a basis $\{f_\gamma\}$ of $\tau$, one obtains

$$f_{\alpha\gamma}(k) = F_\alpha(f_\gamma) = \sum_\beta a_{\alpha\beta} G_\beta(f_\gamma) = \sum_\beta a_{\alpha\beta} f_{\beta\gamma}(k).$$

Evaluation at the point $1_K$ shows that

$$a_{\alpha\gamma} = \delta_{\alpha\gamma}.$$

If $v \neq 0 \in W_\alpha \cap W_\beta$, then $Kv = W_\alpha = W_\beta$. Schur's lemma and second condition imply that $\alpha = \beta$. Assume there is a matrix $\mathbf{S}^{(\tau)}(k)$ as required, we then have the direct sum decomposition of $H(\tau)$. □

For each $\tau_m = \tau_{[m_1,m_2;l]} \in \hat{K}$, define a finite set $I(\tau_m)$ to be the collection of indices $\alpha$ such that $W_\alpha$ occurs in $\pi_{\sigma,\nu}\mid_K$ as a $K$-module. Thus, the cardinality of $I(\tau_m)$ is the $K$-multiplicity of $\tau_m$ in $\pi_{\sigma,\nu}$. Let $s$ be the integer parameter corresponding to $\sigma \in \hat{M}$. By setting $n = (m_1 + m_2 + s)/2$, one can see that $p + q = n$ if $\alpha \in I(\tau_m)$ with $\alpha = (m_2+1)p + q + 1, (q \leq m_2)$. We identify the index $\alpha$ with the pair $(p, q)$ defined by $\alpha$.

We define a matrix function $\mathbf{S}_{\sigma,\nu}^{(\tau_m)}(k)$ attached to the $\tau$-isotypic component of $\pi_{\sigma,\nu}$ by eliminating all the $\alpha$-th columns of $\mathbf{S}^{(\tau_m)}(k)$ when $p + q \neq n$ and change the $\alpha$-th columns by 0 if $\alpha \notin I(\tau_m)$ and $p + q = n$.

# 4 The $(\mathfrak{g}, K)$-module structure on $\pi_{\sigma,\nu}$

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g} = Lie(G)$ corresponding to $\theta$. In this section, we explicitly describe $\mathfrak{p}_{\mathbb{C}}$-action on the space

$$\pi_{\sigma,\nu}\mid_K \cong \bigoplus_{\tau_m \in \hat{K}} \bigoplus_{\alpha \in I(\tau_m)} W_\alpha.$$

Since the adjoint representation of $K$ on $\mathfrak{p}_{\mathbb{C}}$ splits into two irreducible components, the antiholomorphic part $\mathfrak{p}_-$ and the holomorphic part $\mathfrak{p}_+$, it is enough to investigate the $\mathfrak{p}_+$-action for our purpose. Let $E_{ij}$ be the matrix unit of $M_4(\mathbb{R})$ with 1 in the $(i,j)$-entry and zero elsewhere. Then the set $\{E_{ij} \mid i = 1,2\ j = 3,4\}$ forms a basis for $\mathfrak{p}_+$. For a fixed pair $(e_1, e_2)$, $e_j \in \{\pm 1\}$ with $j = 1, 2$, we define $\mathbf{c}_t^j$ by

$$\mathbf{c}_t^j = \frac{t}{m_j + 1}\ (0 \leq t \leq m_j + e_j).$$

Let $(\tau_m, V_m)$ be an irreducible representation of $K$ with parametrization $m = [m_1, m_2; l]$. By the well known Clebsch-Gordan theorem, the irreducible components in the $K$-module $\mathfrak{p}_+ \otimes_{\mathbb{C}} \tau_m$ are precisely the $K$-representations

$$T = \{\ \tau_{[m_1+e_1, m_2+e_2;\ l+2]}\ \mid e_1, e_2 \in \{\pm 1\}\},$$

and we will denote these by $\tau_{[\pm,\pm;+]}$ or $\tau_{[e_1,e_2;+]}$.

For each $K$-isomorphism between $\tau_m$ and $W_\alpha$ in Proposition 3.1, we have the following surjective homomorphism $\mathfrak{p}_+ \otimes_{\mathbb{C}} \tau_m \to \mathfrak{p}_+ W_\alpha$ of $K$-modules. Therefore, we obtain an injection

$$\mathfrak{p}_+ H_{\sigma,\nu}(\tau_m) \hookrightarrow \bigoplus_{\tau_{m'} \in T} H_{\sigma,\nu}(\tau_{m'})$$

which implies the following theorem. Here $H_{\sigma,\nu}(\tau_m)$ stands for the $\tau_m$-isotypic component of $\pi_{\sigma,\nu}$.

**Theorem 4.1** *Let $\tau_{[e_1,e_2;+]}$ be a simple $K$-submodule of the $K$-module $\mathfrak{p}_+ \otimes_{\mathbb{C}} \tau_m$ for a given simple $K$-module $\tau_m$ and the $K$-module* $(\mathrm{Ad}, \mathfrak{p}_+)$. *Then we have that*

$$\mathcal{C}_{[e_1,e_2;+]}\mathbf{S}_{\sigma,\nu}^{(\tau_m)}(k) = \mathbf{S}_{\sigma,\nu}^{(\tau_{[e_1,e_2;+]})}(k)\Gamma_{[e_1,e_2;+]},$$

*where the product of matrices of the left hand side is the differential operation. Here, $r = (s+l)/2$ and*

**1.** $\Gamma_{[-,-;+]} = \{a_{ij}\}_{0\le i\le n-1, 0\le j\le n}$ *is a matrix whose all non zero entries are given by*

$$a_{t-1,t} = \frac{1}{2}(\nu_2 + 1 + m_1 + r - 2t) \qquad \text{if } (t, n-t) \in I(\tau_m),\ (t-1, n-t) \in I(\tau_{m'}),$$
$$a_{t,t} = -\frac{1}{2}(\nu_1 - 1 - m_2 + r - 2t) \qquad \text{if } (t, n-t) \in I(\tau_m),\ (t, n-t-1) \in I(\tau_{m'}).$$

*and* $C_{[-,-;+]} = \{C_{ij}\}$ *is a matrix of size* $(m_1m_2) \times (m_1+1)(m_2+1)$ *with entries given by*

$$\begin{aligned} C_{m_2p+q+1,(m_2+1)p+q+1} &= -E_{14}, \\ C_{m_2p+q+1,(m_2+1)p+q+2} &= -E_{13}, \\ C_{m_2p+q+1,(m_2+1)(p+1)+q+1} &= E_{24}, \\ C_{m_2p+q+1,(m_2+1)(p+1)+q+2} &= E_{23}, \end{aligned}$$

*for each* $0 \le p \le m_1 - 1$ *and* $0 \le q \le m_2 - 1$*, but all other entries are* $0$.

**2.** $\Gamma_{[+,+;+]} = \{a_{ij}\}_{0\le i\le n+1, 0\le j\le n}$ *is a matrix whose all non zero entries are given by*

$$a_{t,t} = \frac{1}{2}(\nu_2 + 1 + m_1 + r - 2t)(1 - \mathbf{c}_t^1)\mathbf{c}_{\nu-t+1}^2 \qquad \text{if } (t, n-t) \in I(\tau_m),\ (t, n-t+1) \in I(\tau_{m'}),$$
$$a_{t+1,t} = \frac{1}{2}(\nu_1 + 3 + 2m_1 + m_2 + r - 2t)\mathbf{c}_{t+1}^1(\mathbf{c}_{\nu-t}^2 - 1) \qquad \text{if } (t, n-t) \in I(\tau_m),\ (t+1, n-t) \in I(\tau_{m'}).$$

*and* $C_{[+,+;+]} = \{C_{ij}\}$ *is a matrix of size* $(m_1+2)(m_2+2) \times (m_1+1)(m_2+1)$ *with entries cgiven by*

$$\begin{aligned} C_{(m_2+2)p+q+1,(m_2+1)p+q+1} &= -(1-\mathbf{c}_p^1)(1-\mathbf{c}_q^2)E_{23}, \\ C_{(m_2+2)p+q+1,(m_2+1)p+q} &= (1-\mathbf{c}_p^1)\mathbf{c}_q^2E_{24}, \\ C_{(m_2+2)p+q+1,(m_2+1)(p-1)+q+1} &= -\mathbf{c}_p^1(1-\mathbf{c}_q^2)E_{13}, \\ C_{(m_2+2)p+q+1,(m_2+1)(p-1)+q} &= \mathbf{c}_p^1\mathbf{c}_q^2E_{14}, \end{aligned}$$

*for each* $0 \le p \le m_1 + 1$ *and* $0 \le q \le m_2 + 1$*, but all other entries are* $0$.

**3.** $\Gamma_{[-,+;+]} = \{a_{ij}\}_{0\le i\le n, 0\le j\le n}$ *is a square matrix whose all non zero entries are given by*

$$a_{t-1,t} = \frac{1}{2}(\nu_2 + 1 + m_1 + r - 2t)\mathbf{c}_{\nu-t+1}^2 \qquad \text{if } (t, n-t) \in I(\tau_m),\ (t-1, n-t+1) \in I(\tau_{m'})$$
$$a_{t,t} = \frac{1}{2}(\nu_1 + 1 + m_2 + r - 2t)(1 - \mathbf{c}_{\nu-t}^2) \qquad \text{if } (t, n-t) \in I(\tau_m),\ (t, n-t) \in I(\tau_{m'}).$$

*and* $C_{[-,+;+]} = \{C_{ij}\}$ *is a matrix of size* $m_1(m_2+2) \times (m_1+1)(m_2+1)$ *with entries given by*

$$\begin{aligned} C_{(m_2+2)p+q+1,(m_2+1)p+q+1} &= (1-\mathbf{c}_q^2)E_{13}, \\ C_{(m_2+2)p+q+1,(m_2+1)p+q} &= -\mathbf{c}_q^2E_{14}, \\ C_{(m_2+2)p+q+1,(m_2+1)(p+1)+q+1} &= -(1-\mathbf{c}_q^2)E_{23}, \\ C_{(m_2+2)p+q+1,(m_2+1)(p+1)+q} &= \mathbf{c}_q^2E_{24}, \end{aligned}$$

*for* $0 \le p \le m_1 + 1$ *and* $0 \le q \le m_2 - 1$*, but all other entries are* $0$.

**4.** $\Gamma_{[+,-;+]} = \{a_{ij}\}_{0\le i\le n, 0\le j\le n}$ *is a square matrix whose all non zero entries are given by*

$$a_{t,t} = \frac{1}{2}(\nu_2 + 1 + m_1 + r - 2t)(1 - \mathbf{c}_t^1) \qquad \text{if } (t, n-t) \in I(\tau_m),\ (t, n-t) \in I(\tau_{m'}),$$
$$a_{t+1,t} = \frac{1}{2}(\nu_1 + 1 + 2m_1 - m_2 + r - 2t)\mathbf{c}_{t+1}^1 \qquad \text{if } (t, n-t) \in I(\tau_m),\ (t+1, n-t-1) \in I(\tau_{m'}).$$

*and* $C_{[+,-;+]} = \{C_{ij}\}$ *is a matrix of size* $(m_1+2)m_2 \times (m_1+1)(m_2+1)$ *with entries given by*

$$\begin{aligned} C_{m_2p+q+1,(m_2+1)p+q+1} &= (1-\mathbf{c}_p^1)E_{24}, \\ C_{m_2p+q+1,(m_2+1)p+q+2} &= (1-\mathbf{c}_p^1)E_{23}, \\ C_{m_2p+q+1,(m_2+1)(p-1)+q+1} &= \mathbf{c}_p^1 E_{14}, \\ C_{m_2p+q+1,(m_2+1)(p-1)+q+2} &= \mathbf{c}_p^1 E_{13}, \end{aligned}$$

*for* $0 \le p \le m_1+1$ *and* $0 \le q \le m_2-1$, *but all other entries are* 0.

### 4.0.1 The Knapp-Stein operator

In this subsection, we consider a matrix representation of the Knapp-Stein operator with respect to the basis for $\pi_{\sigma,\nu}\mid_K$. This is motivated by Theorem 6.7 in the paper of Goodman-Wallach [2].

Let us recall the Knapp-Stein intertwining operator $A^s_{\sigma,\nu}$ from the space of all $C^\infty$-vectors of $\pi_{\sigma,\nu}$ to that of $\pi_{s(\sigma),s(\nu)}$ defined by

$$(A^s_{\sigma,\nu}f)(k) = \int_{\bar{N}_s} a(n_s s^* k)^{\nu+\rho} f(k(n_s s^* k))dn_s, \ (f \in \pi^\infty_{\sigma,\nu}).$$

Here $s^* \in K$ such that $s := Ad(s^*) \in W(A)$, $\bar{N}_s = N \cap s^* N s^{*-1}$ and $s(\sigma)$ is a character of $M$ given by $s(\sigma)(m) = \sigma(s^* m s^{*-1})$, $m \in M$. Since it is a linear map from $\pi_{\sigma,\nu}$ to $\pi_{s(\sigma),s(\nu)}$ satisfying

$$A^s_{\sigma,\nu}\pi_{\sigma,\nu}(x)f = \pi_{s(\sigma),s(\nu)}(x)A^s_{\sigma,\nu}f, \ \ x \in G \text{ (or } U(\mathfrak{g})),$$

we have a linear map

$$A^s(\tau) : \mathrm{Hom}_K(\tau, \pi_{\sigma,\nu}\mid_K) \to \mathrm{Hom}_K(\tau, \pi_{s(\sigma),s(\nu)}\mid_K).$$

for any $\tau \in \hat{K}$.

Let $[\alpha_i]$ be the $K$-isomorphism from $\tau$ to $W_{\alpha_i}$ for $\alpha_i \in I(\tau)$. We equip the space $\mathrm{Hom}_K(\tau, \pi_{\sigma,\nu}\mid_K)$ with the basis consisting of the $K$-homomorphisms $[\alpha_i]$. Similarly, we choose a basis for the space $\mathrm{Hom}_K(\tau, \pi_{s(\sigma),s(\nu)}\mid_K)$. Then we want to compute all entries $a_{ij}$ of the matrix $A^s(\tau) = (a_{ij})$ such that

$$A^s(\tau)[\alpha_i] = \sum_{\alpha_j^s \in I} a_{ij}\cdot[\alpha_j^s]$$

where $I = \{\alpha^s \mid W_{\alpha^s} \hookrightarrow \pi_{s(\sigma),s(\nu)}\mid_K\}$. For each basis vector $f_{pq}$ of $\tau$ as in Lemma 2.1, we have that

$$(A^s(\tau)[\alpha_i])(f_{pq}) = \sum_{\alpha_j^s \in I} a_{ij}\cdot[\alpha_j^s](f_{pq}) = \sum_{\alpha_j^s \in I} a_{ij}\cdot f^{(\tau)}_{\alpha_j^s,pq}(k).$$

On the other hand, by definition of the map $A^s(\tau)$, one has

$$(A^s(\tau)[\alpha_i])(f_{pq}) = (A^s_{\sigma,\nu}f^{(\tau)}_{\alpha_i,pq})(k), \ \alpha_i \in I(\pi_{\sigma,\nu},\tau).$$

Thus we have the following formula for the coefficients $a_{ij}$ of the matrix $A^s(\tau)$ for each $\tau \in \hat{K}$.

**Lemma 4.2** *Let* $\alpha_i$ *be in* $I(\pi_{\sigma,\nu},\tau)$ *and* $\alpha_j^s$ *be in* $I(\pi_{s(\sigma),s(\nu)},\tau)$. *Then the* $(i,j)$*-th coefficient of* $A^s(\tau)$

$$a_{ij} = (A^s_{\sigma,\nu}f^{(m)}_{\alpha_i,\alpha_j^s})(1_4).$$

**Example 4.3**

Let $s$ be a generator of $W(A)$ whose image is the matrix diag$(1,-1)$ under the representation of $W(A)$ on $\mathfrak{a}^*$. Then we choose the corresponding $s^* \in K$ as the matrix diag$(1,-i,1,i)$ and hence

$$\bar{N}_s = \exp(\mathfrak{g}_{-2\lambda_2}) = \left\{ n_s(t) = \kappa^{-1} \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & t & & 1 \end{pmatrix} \kappa : t \in \mathbb{R} \right\}.$$

Since $n_s \in \bar{N}_s$, one has ${}^t n_s I_{2,2} n_s = I_{2,2}$ and hence $n_s s^* = I_{2,2} {}^t n_s^{-1} I_{2,2} s^*$. Thus, we have the following.

Assume $n_s = n_s(t) \in \bar{N}_s$. Let $n' \in N$, $a(n_s s^*) \in A$ and $k(n_s s^*) \in K$ be so that $n_s s^* = n' a(n_s s^*) k(n_s s^*)$. Then

$$a(n_s s^*)^{\nu+\rho} = (1+t^2)^{-\frac{\nu_2+1}{2}},$$
$$k(n_s s^*) = \mathrm{diag}(1, -iu, -1, -iu^{-1})$$

where $u = ((1-it)/(1+it))^{\frac{1}{2}}$.

For a fixed $\tau_m \in \hat{K}$, therefore

$$f_{\gamma_i,\beta_j}(k(n_s s^*)) = 0 \text{ when } \gamma_i \neq \beta_j$$

If $\tau = \tau_{[m_1,m_2;l]}$ then we have

$$A^s(\tau) = 2\pi 2^{-\nu_2}\Gamma(\nu_2)\ \mathrm{diag}\left[\frac{(-1)^{(m_1+m_2)/2-p+1} i^{m_2+r}}{\Gamma(\frac{1}{2}\nu_2 + \frac{1}{2} + d)\Gamma(\frac{1}{2}\nu_2 + \frac{1}{2} - d)}\right]_p$$

where $d = \frac{1}{2}(m_1 + r - 2p)$ for $(p, (m_1+m_2)/2 - p) \in I(\pi_{\sigma,\nu}, \tau_m)$.

## 5 Whittaker functions

The main focus of this section is on the integral expressions of Whittaker functions on $G$ related to certain principal series. The results of the section 4.1 lead us to the study of Whittaker functions related to some $K$-types. For this purpose, we focus our investigation on the principal series representations which contain one dimensional $K$-types and apply the method used in [4] to evaluate such Whittaker functions. More precisely, in this setting, the character $\sigma$ of $M$ factors through a character $\chi$ of $\mu_2$. Let $(\pi_{\chi,\nu}, L^2_\chi(K))$ denote the principal representation series corresponding to such character $\sigma$.

For an integer $u$, define a function $f_u(k)$ on $K$ by $f_u(k) := \det(k_2)^u$, $k = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} \in K$.

**Lemma 5.1** *Let $f_u(k)$ be as above. Then $\tau_{[0,0,2u]} \cong \mathbb{C} f_u(k)$ as $K$-modules. Moreover, if $\chi(-1) = (-1)^u$ then $f_u(k) \in L^2_\chi(K)$ and $[\pi_{\chi,\nu} : \tau_{[0,0;2u]}] = 1$.*

### 5.1 The Jacquet integral.

Let $J_{\chi,\nu}$ be the Jacquet functional on the subspace of differentiable functions of $L^2_\chi(K)$ given by

$$J_{\chi,\nu}(f) = \int_N \eta(n)^{-1} a(s^* n)^{\nu+\rho} f(k(s^* n)) dn$$

for a differentiable function $f$ in $L^2_\chi(K)$ and the longest element $s \in W(A)$. Here $W(A)$ is the Weyl group defined as the quotient of $M^* = N_K(\mathfrak{a})$, the normalizer of $\mathfrak{a}$ in $K$, by $M$ and $s^*$ is an element of $M^*$ mapping to the longest element $s \in W(A)$.

Then one has $J_{\chi,\nu}(\pi(n)f) = \eta(n)J_{\chi,\nu}(f)$ and hence

$$J \in \mathrm{Hom}_{(\mathfrak{g},K)}(\pi_{\chi,\nu}|_K, \mathcal{A}_\eta(N\backslash G)), \tag{1}$$

where $J$ associates $v \in \pi_{\chi,\nu}|_K$ to the function $J_v(g) := J_{\chi,\nu}(\pi_{\chi,\nu}(g)v)$, $(g \in G)$. We want to have an explicit formula for the $A$-radial part:

$$J_{f_u}(a) = J_{\chi,\nu}(\pi_{\chi,\nu}(a)f_u) = a^{-\nu+\rho}\int_N \eta(ana)^{-1}a(s^*n)^{\nu+\rho}f_u(k(s^*n))dn.$$

In our case, we can choose $I_{2,2} \in K$ for $s^* \in K$.

### 5.1.1 The first modification

Let $E_{ij}$ be the usual matrix with 1 in $(i,j)$-entry and zero elsewhere. Put

$$E_0 = \kappa^{-1}(E_{12} - E_{43})\kappa, \qquad E_1 = i\kappa^{-1}(E_{12} + E_{43})\kappa, \qquad E_2 = \kappa^{-1}E_{24}\kappa,$$
$$F_0 = \kappa^{-1}(E_{14} + E_{23})\kappa, \qquad F_1 = i\kappa^{-1}(E_{14} - E_{23})\kappa, \qquad F_2 = \kappa^{-1}E_{13}\kappa,$$

by setting $i = \sqrt{-1}$ and

$$\kappa = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -i & 0 & i & 0 \\ 0 & -i & 0 & i \end{pmatrix}.$$

Then the corresponding root spaces of positive roots in $\Phi(\mathfrak{g},\mathfrak{a})$ are given by

$$\mathfrak{g}_{\lambda_1-\lambda_2} = E_0\cdot\mathbb{R} \oplus E_1\cdot\mathbb{R}, \qquad \mathfrak{g}_{2\lambda_2} = E_2\cdot\mathbb{R},$$
$$\mathfrak{g}_{\lambda_1+\lambda_2} = F_0\cdot\mathbb{R} \oplus F_1\cdot\mathbb{R}, \qquad \mathfrak{g}_{2\lambda_1} = F_2\cdot\mathbb{R}.$$

Let $\mathfrak{n}$ be a subalgebra defined by $\mathfrak{n} = \sum_{\alpha\in\Phi_+}\mathfrak{g}_\alpha$. We now describe elements of a maximal nilpotent subgroup $N$ of $G$ given by $N = \exp(\mathfrak{n})$.

The Killing form $B(X,Y) = tr(\mathrm{ad}X \cdot \mathrm{ad}Y)$, $(X, Y \in \mathfrak{g})$ and Cartan involution $\theta$ of $\mathfrak{g}$ induce an inner product $\langle,\rangle$ of $\mathfrak{g}$ via

$$\langle X, Y\rangle = -B(X, Y^\theta), \ (X, Y \in \mathfrak{g}).$$

Then one has that $\langle\mathfrak{g}_\alpha, \mathfrak{g}_\beta\rangle = 0$ if $\alpha \neq \beta$, because of the involution $\theta$. Moreover, one can see that the set $\{E_i, F_i \mid i = 0,1,2\}$ is an $\langle\ ,\ \rangle$-orthogonal basis for $\mathfrak{n}$ such that a each element $n = n(n_0, n_1, n_2, n_3)$ in the maximal unipotent group $N = \exp(\mathfrak{n})$ is expressed in the form:

$$\kappa^{-1}\begin{pmatrix} 1 & n_0 & & \\ & 1 & & \\ & & 1 & \\ & & -\bar{n}_0 & 1 \end{pmatrix}\begin{pmatrix} 1 & & n_1 & n_2 \\ & 1 & \bar{n}_2 & n_3 \\ & & 1 & \\ & & & 1 \end{pmatrix}\kappa.$$

for $n_1, n_3 \in \mathbb{R}$, $n_0, n_2 \in \mathbb{C}$.

**Lemma 5.2** *We have*

1. *Set* $N_1 = \begin{pmatrix} n_1 & n_2 \\ \bar{n}_2 & n_3 \end{pmatrix}$ *for* $n = n(n_0, n_1, n_2, n_3) \in N$. *Then*

$$f_u(k(I_{2,2}n)) = \left(det(1 - \sqrt{-1}N_1)/det(1 + \sqrt{-1}N_1)\right)^{\frac{k}{2}}.$$

*2. Let $\eta$ be a character of $N$ determined by a real number $c_2$ and $c = c_0 + \sqrt{-1}c_1 \in \mathbb{C}$. Then*

$$\eta(ana^{-1}) = \exp(2\sqrt{-1}\left(\frac{a_1}{a_2}\mathrm{Re}(\bar{c}n_0) + c_2a_2^2n_3\right)),$$

*where $a_i = \exp(t_i)$, $(i = 1, 2)$ for $a = a(t_1, t_2) \in A$.*

*3. For $\nu = (\nu_1, \nu_2) \in \mathrm{Hom}_{\mathbb{R}}(\mathfrak{a}, \mathbb{C})$, we have that $a(I_{2,2}n)^{\nu+\rho} = \Delta_1^{-\frac{\nu_1-\nu_2}{2}-1}\Delta_2^{-\frac{\nu_2+1}{2}}$ where*
*$\Delta_1 = 1 + n_1^2 + \bar{n}_2n_2 + (\bar{n}_0n_2 + n_0\bar{n}_2)(n_1 + n_3) + \bar{n}_0n_0(1 + \bar{n}_2n_2 + n_3^2)$,*
*$\Delta_2 = 1 + n_1^2 + 2n_2\bar{n}_2 + n_3^2 + (n_1n_3 - n_2\bar{n}_2)^2$ for $n = n(n_0, n_1, n_2, n_3) \in N$.*

For future convenience, we choose a new coordinate for $A$ by

$$y = (y_1, y_2) = \left(\frac{a_1}{a_2}, a_2^2\right).$$

Since $f \to J_f(g)$ is the Whittaker realization of $\pi_{\chi,\nu}$, $J_{f_u}(a)$ is the radial part of a Whittaker function on $G$ belonging to $\pi_\nu$. Thus, in the new coordinate system, we can summarize that the radial part of Whittaker function associated with the $K$-type $\tau_u$ can be written in the form

$$y_1^{-\nu_1+3}y_2^{-\frac{\nu_1+\nu_2}{2}+2}\int_N \Delta_1^{-\frac{\nu_1-\nu_2}{2}-1}\Delta_2^{-\frac{\nu_2+1}{2}} \times \exp(-2\sqrt{-1}\left(y_1\mathrm{Re}(\bar{c}n_0) + c_2y_2n_3\right)f_u(k(I_{2,2}n))dn,$$

where $dn$ is a multiplicative Haar measure on $N$. Now we shall give a normalization of Haar measure of $N$. Since the exponential map of $\mathfrak{n}$ onto $N$ is an analytic isomorphism, there exists a unique Haar measure $dn$ on $N$ that corresponds to Lebesgue measure on $\mathfrak{n}$.

**Lemma 5.3** *The radial part of the moderate growth Whittaker function $W_{(\nu_1,\nu_2)}(y_1, y_2; u)$ (up to constant) associated with the $K$-type $\tau_u$ can be written in the form*

$$y_1^{-\nu_1+3}y_2^{-\frac{\nu_1+\nu_2}{2}+2}\int_{\mathbb{R}^6} \Delta_1^{-\frac{\nu_1-\nu_2}{2}-1}\Delta_2^{-\frac{\nu_2+1}{2}} \exp(-2\sqrt{-1}(c_0z_0y_1 + c_1t_0y_1 - n_3y_2))f_u(k(I_{2,2}n))$$

*with respect to $dz_0dt_0dn_1dz_2dt_2dn_3$. Here $n_i = z_i + \sqrt{-1}t_i$ $(i = 0, 2)$.*

In fact, it suffices to consider the cases $u = 0$ and $1$ for our purposes. Let $K_\mu(z)$ be the Bessel function.

**Theorem 5.4** *Let $\pi_{\chi,\nu}$ be irreducible and $\eta$ be a nondegenerated unitary character $N$. Then we have the following assertions on the $A$-radial part of the primary Whittaker function $W_{(\nu_1,\nu_2)}(y_1, y_2; u)$.*

*If $\chi$ is trivial then the Whitttaker function $W_{(\nu_1,\nu_2)}(y_1, y_2; 0)$ is identified with $y_1^3y_2^2$ times*

$$\int_0^\infty\int_0^\infty T_{\nu_1,\nu_2}(y_1, y_2, t_1, t_2)\frac{dt_1}{t_1}\frac{dt_2}{t_2}.$$

*If $\chi$ is non-trivial then the Whitttaker function $\tilde{W}_{(\nu_1,\nu_2)}(y_1, y_2; 1)$ is identified with $y_1^4y_2^3/4$ times*

$$\int_0^\infty\int_0^\infty T_{\nu_1,\nu_2}(y_1, y_2, t_1, t_2)(\sqrt{t_1/t_2} - 1/\sqrt{t_1t_2})\frac{dt_1}{t_1}\frac{dt_2}{t_2}$$

*where $T_{\nu_1,\nu_2}(y_1, y_2, t_1, t_2)$ is the function*

$$K_{\frac{\nu_1+\nu_2}{2}}(2\sqrt{t_2/t_1})K_{\frac{\nu_2-\nu_1}{2}}(2\sqrt{t_1t_2})\exp\left(-|c_2|y_2t_1 - \frac{|c_2|y_2}{t_1} - \frac{t_2}{|c_2|y_2} - (c_0^2 + c_1^2)|c_2|\frac{y_1^2y_2}{t_2}\right)$$

Note here that, we need the following formula to reduce the number of integral symbols corresponding to the root spaces $\mathfrak{g}_{\lambda_1-\lambda_2}$ and $\mathfrak{g}_{\lambda_1+\lambda_2}$:

**Formula 5.5** *Let $a, c \in \mathbb{R}_+^*$ and $b, \alpha, \beta \in \mathbb{R}$ such that $\alpha^2 + \beta^2 = 1$. Then we have*

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \exp(-c(x^2 + y^2) - a(\alpha x + \beta y)^2 + 2\sqrt{-1}b(\alpha x + \beta y))dxdy = \frac{\pi}{(c^2 + ac)^{\frac{1}{2}}}\exp(\frac{-b^2}{a + c}).$$

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