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# Remarks on extractable submonoids

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**Abstract.** This paper deals with extractable codes. The base of free submonoid of a free monoid is called a code. The code $C$ with the property that $z, xzy \in C^*$ implies $xy \in C^*$ is called an extractable code. Such $C$ is a bifix code, and for example, appears as a Petri net code or a group code. In this paper we examine basic properties of extractable codes.

## 1. INTRODUCTION

Let $A$ be an alphabet, $A^*$ the free monoid over $A$, and 1 the empty word. Let $A^+ = A^* - \{1\}$. A word $v \in A^*$ is a *right factor*(resp. left factor) of a word $u \in A^*$ if there is a word $w \in A^*$ such that $u = wv$(resp. $u = vw$). If $v$ is a right factor of $u$, we write $v <_s u$. Similarly, we write $v <_p u$ if $v$ is a left factor of $u$. For a word $w \in A^*$ and a letter $x \in A$ we let $|w|_x$ denote the number of $x$ in $w$. The length $|w|$ of $w$ is the number of letters in $w$. Therefore $A^n = \{w \in A^* | \ |w| = n\}$, $n \geq 1$. $Alph(w)$ is the set of all letters occurring at least once in $w$.

A non-empty subset $C$ of $A^+$ is said to be a *code* if for $x_1, ..., x_p, y_1, ..., y_q \in C$, $p, q \geq 1$,

$$x_1 \cdots x_p = y_1 \cdots y_q \implies p=q, x_1=y_1, \ldots, x_p=y_p.$$

A subset $M$ of $A^*$ is a *submonoid* of $A^*$ if $M^2 \subseteq M$ and $1 \in M$. Every submonoid $M$ of a free monoid has a unique minimal set of generators $C = (M - \{1\}) - (M - \{1\})^2$. $C$ is called the *base* of $M$. A submonoid $M$ is *right unitary* in $A^*$ if for all $u, v \in A^*$,

$$u, uv \in M \Longrightarrow v \in M.$$

$M$ is called *left unitary* in $A^*$ if it satisfies the dual condition. A submonoid $M$ is *biunitary* if it is both left and right unitary. Let $M$ be a submonoid of a free monoid $A^*$, and $C$ its base. If $CA^+ \cap C = \emptyset$, (resp. $A^+C \cap C = \emptyset$), then $C$ is called a *prefix* (resp. *suffix*) code over $A$. $C$ is called a *bifix* code if it is a prefix and suffix code. A submonoid $M$ of $A^*$ is right unitary (resp. biunitary) if and only if its minimal set of generators is a prefix code (resp. bifix code) ([1, p.46],[3, p.108]).

Let $C$ be a nonempty subset of $A^*$. If $|x| = |y|$ for all $x, y \in C$, then $C$ is a bifix code. We call such a code a *uniform code.* The uniform code $A^n$, $n \geq 1$, is called a *full uniform code.*
Let $M$ be a submonoid of $A^*$. If the condition $z, xzy \in M$ implies $xy \in M$, then $M$ is called an *extractable submonoid* of $A^*$.

If a submonoid $N$ is extractable, then $u, 1uv \in C^*$ implies $1v = v \in C^*$. Similarly $v, uv \in C^*$ implies $u \in C^*$. Consequently $M$ is biunitary. Therefore its minimal set of generators $C$ is a bifix code.

**Definition 1.** Let $C \subset A^*$ be a code. If $C^*$ is extractable, then $C$ is called an extractable code.

## 1. FUNDMENTAL PROPERTIES OF EXTRACTABLE CODES

Let $\mathcal{A} = (Q, A, \delta, 1, F)$ be an automaton, where $Q$, $A$, $\delta : Q \times A \to Q$, 1, and $F$, are the state set, the input set, the next-state function, the initial state, and the final set of $\mathcal{A}$, respectively (For basic concepts of automata, refer to [4] or [1]). If for any $(p, q) \in Q \times Q$ there exists some $w \in A^*$ such that $\delta(p, w) = q$, then $\mathcal{A}$ is called transitive. If $\mathcal{A}$ has a fixed point $s$, and if for every $p, q \in Q, p \neq s$, there exists $w \in A^*$ such that $\delta(p, w) = q$, then $\mathcal{A}$ is called 0−transitive. If an automaton $\mathcal{A}$ is either transitive or 0-transitive, then $\mathcal{A}$ is called a [0]−transitive automaton.
Let $L$ be a subset of $A^*$. For each $x \in A^*$, we define the set of all right contexts of $x$ with respect to $L$ by

$$Cont_L^{(r)}(x) = \{w \in A^* \mid xw \in L\}.$$

The right principal congruence $P_L^{(r)}$ of $L$ is defined by $(x, y) \in P_L^{(r)}$ if and only if $Cont_L^{(r)}(x) = Cont_L^{(r)}(y)$. Let $u \in A^*$. By $[u]_L$ we denote the $P_L^{(r)}$−class of $u$ by $[u]_L$ or simply by $[u]$. That is

$$[u]_L = \{v \mid Cont_{C^*}^{(r)}(v) = Cont_{C^*}^{(r)}(u), v \in A^*\}.$$

We denote by $[w_\phi]$ the class of $P_{C^*}^{(r)}$ consisting of all words $w \in A^*$ such that $wA^* \cap C^* = \phi$.
Let $C$ be a prefic code. We defined the automaton $\mathcal{A}(C^*) = (A^*/P_{C^*}^{(r)}, A, \delta, [1], \{[1]\})$, where $\delta([w], x) = [wx]$ for $[w] \in A^*/P_{C^*}^{(r)}$ and $x \in A^*$. Then the automaton $\mathcal{A}(C^*)$, is minimal and [0]−transitive.

Let $\mathcal{A} = (Q, A, \delta, 1, \{1\})$ be a [0]−transitive minimal automaton recognizing $C^*$ for some prefix code $C$. For each $p \in Q$ we put

$$W_p = \{w \in A^* \mid \delta(p, w) = 1\}.$$

Define the congruence $\rho$ on $\mathcal{A}$ is by

$$p, q \in Q,\ p\rho q \Longleftrightarrow W_p = W_q.$$

Then $\rho$ is the equality(See [2, p.215]).

**Proposition 1.** Let $C$ be a prefix code, and let $\mathcal{A} = (Q, A, \delta, 1, \{1\})$ be a [0]−transitive automaton recognizing $C^*$. The following conditions are equivalent:
(1) $C$ is extractable.
(2) $W_{\delta(p,z)} \subset W_p$ for all $p \in Q$, $z \in C^*$.

**Corollary 2.** Let $C$ be a prefix code, and let A= $(Q, A, \delta, 1, \{1\})$ be a [0]−transitive minimal automaton recognizing $C^*$. If there exists some $z_1$, $z_2 \in C^*$ such that $\delta(p, z_1) = q$ and $\delta(q, z_2) = p$ for some $p, q \in Q$, $p \neq q$, then $C$ is not extractable.

Let $C$ be a prefix code, and let $\mathcal{A}(C^*) = (A^*/P_{C^*}^{(r)}, A, \delta, [1], \{[1]\})$ be the minimal automaton of $C^*$. Then, for $[x] \in A^*/P_{C^*}^{(r)}$ we have

$$u \in W_{[x]} \Leftrightarrow \delta([x], u) = [1] \Leftrightarrow xu \in C^* \Leftrightarrow u \in Cont_{C^*}^{(r)}(x).$$

That is, $W_{[x]} = Cont_{C^*}^{(r)}(x)$. Therefore we have the following rewriting of Proposition 1.

**Proposition 3.** Let $C$ be a prefix code. Then the following two condition are equivalent:
(1) $C$ is extractable.
(2) $Cont_{C^*}^{(r)}(xz) \subset Cont_{C^*}^{(r)}(x)$ for all $x \in A^*$ and $z \in C$.

**Corollary 4.** Let $C \subset A^*$ be a prefix code. If there exists some $z_1, z_2 \in C^*$ and some $[x], [y] \in A^*/P_{C^*}^{(r)}$, $[x] \neq [y]$, such that $[xz_1] = [y]$ and $[yz_2] = [x]$, then $C^*$ is not extractable.

**Corollary 5.** Let $C \subset A^*$ be a prefix code. If either $[xz] = [x]$ or $[xz] = [w_\phi]$ for all $[x] \in A^*/P_{C^*}^{(r)}$ and $z \in C^*$, then $C$ is extractable.

**Proposition 6.** Let $C \subset A^*$ be a prefix code such that $Cont_{C^*}^{(r)}(x) \cap Cont_{C^*}^{(r)}(y) = \phi$ for any distinct $[x], [y] \in A^*/P_{C^*}^{(r)}$. Then the following two conditions are equivalent
(1) $C^*$ is an extractable code.
(2) Either $[xz] = [x]$ or $[xz] = [w_\phi]$ for any $x \in A^*$ and $z \in C$.

## 2. EXTRACTABLE REFLECTIVE CODE

Two words $x$, $y$ are called conjugate if there exists words $u$, $v$ such that $x = uv$, $y = vu$. the conjugacy relation is an equivalence relation. By $cl(x)$ we denote the class of $x$ of this equivalence relation. Let $C \subset A^*$ be a code. If $uv \in C$ implies $vu \in C$, then $C$ is called *reflective*. It is obvious that a reflective $C$ is a union of conjugacy classes of words. Note that $C^*$ is not necessarily a reflective language.
A code $C \subset A^+$ is called *infix* if for all

$$x, y, z \in A, \ xzy \in C \Longrightarrow x = y = 1.$$

**Proposition 7.** The reflective code $C$ is an infix code.

**Example 1.** (1) The set $C = \{ab^2, bab, b^2a, a^3b, a^2ba, aba^2, ba^3\}$ is a union of two coujugacy classes of words in $\{a, b\}^*$. Since $C$ is a prefix set, $C$ is an infix code.
(2) Let $B \subset A$, $B \neq \emptyset$, and $n, k\,(k \leq n)$ be a positive integer. We let $|w|_B$ denote the number of letters of $w$ which are in $B$. Then $U = \{w \in A^n \,||w|_B = k\}$ is an extractable code. Since $uv \in U$ imples $vu \in U$, $U$ is reflective.

**Proposition 8.** Let $C \subset A^*$ be a reflective code. The following two conditions are equivalent.
(1) $C$ is extractable.
(2) For any $[x], [y] \in A^*/P_{C^*}^{(r)}$,

$$[x] \neq [y] \implies Cont_{C^*}^{(r)}(x) \cap Cont_{C^*}^{(r)}(y) = \phi.$$

Let $L \subset A^*$ and $n \geq 1$. We set

$$L^{(n)} = \{w^n \mid w \in L\}.$$

If $D$ is a bifix code, then $D^{(n)}$ is a bifix code.

**Proposition 9.** If $C$ is a reflective code, then $C^{(n)}, n \geq 1$, is a reflective code.

**Proposition 10.** Let $D$ be a bifix code, and let $C = D^{(n)}$, $n \geq 2$. Then, for $u, v \in C(A^+)^{-1}$,

$$Cont_{C^*}^{(r)}(u) \cap Cont_{C^*}^{(r)}(v) \neq \phi \quad \implies \quad [u]_{C^*} = [v]_{C^*}.$$

From Proposition 10 and Proposition 8 we have

**Corollary 11.** If $C$ is a reflective code, then $C^{(n)}, n \geq 2$, is an extractable reflective code.

## 3. EXTRACRTABLE UNIFORM CODES

In this section we examine extractable uniform codes.

**Proposition 12.** Let $C \subset A^*$ be a extractable code. Then $C \cap A^n$ is an extractable code.

**Proposition 13.** Let $M$ be an extractable submonoid of $A^*$. Then $M \cap A^n$, $n \geq 1$, is an extractable code.

**Example 2.** Let $S$ be a monoid and $H$ an extractable submonoid. Let $\varphi : A^* \to S$ be a surjective morphism. Then $\varphi^{-1}(H)$ is an extractable submonoid of $A^*$.

**Corollary 14.** Let $G$ be a group and $H$ a normal subgroup of $G$. Let $\varphi : A^* \to G$ be a surjective morphism. Then $\varphi^{-1}(H) \cap A^n$ is an extractable reflective code.

**Proposition 15.** Let $C \subset A^+$ be a finite code with $A = alph(C)$. Then $C$ is a maximal extractable code if and only if $C = A^n$ for some $n \geq 1$.

**Proposition 16.** Let $D \subset A^m$, $m \geq 1$, be a uniform code, then $D^{(n)}$, $n \geq 2$, is extractable.

**Proposition 17.** Let $w = (uv)^n u$, $u, v \in A^*$, $n \geq 2$, and let $C$ be a conjugacy class of $w$.
(1) If $u = 1$ and $v \in A^+$, then $C$ is extractable.
(2) If $u, v \in A^+$, then $C$ is not extractable.

**Proposition 18.** Let $w \in A^+$, and let $C$ be a conjugacy class of $w$.
(1) If $w = uv$, $u, v \in A^+$ and $Alph(u) \cap Alph(v) = \phi$. Then $C$ is extractable.
(2) If $w = (uvu)^n (uv)^m, u, v \in A^+, m \geq 1$, $n \geq 0$ and $Alph(u) \cap Alph(v) = \phi$. Then $C$ is not extractable.

**Lemma 19.** Let $x, y, u, v \in A^+$ with $|u|, |v| < |x| = |y|$. Then the following conditions hold.
(1) $x^2 y = ux^2 v \Rightarrow y = uv$ ($uv = x$ is not necessarily true).
(1') $xy^2 = uy^2 v \Rightarrow x = uv$ ($uv = y$ is not necessarily true).
(2) $x^2 y = uxyv \Rightarrow x = y = uv$ ($uv$ is the power of some primitive word).
(2') $xy^2 = uyxv \Rightarrow x = y = uv$ ($uv$ is the power of some primitive word).
(3) $x^2 y = uyxv \Rightarrow x = y = uv$.
(3') $xy^2 = uyxv \Rightarrow x = y = uv$.
(4) $x^2 y = uy^2 v \Rightarrow y = uv$ ($x$ and $y$ are conjugate).
(4') $xy^2 = ux^2 v \Rightarrow x = uv$ ($x$ and $y$ are conjugate).

**Proposition 20.** Let $x, y \in A^*$ with $|x| = |y| > 0$ and $C = \{x^2, xy, yx, y^2\}$. Then $C$ is extractable.

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